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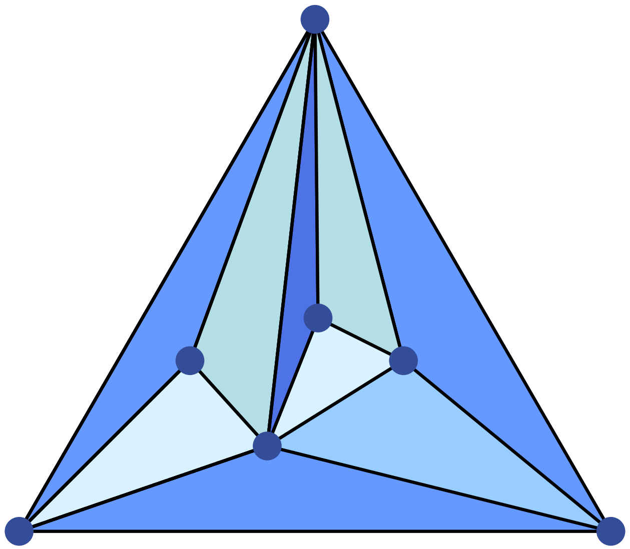


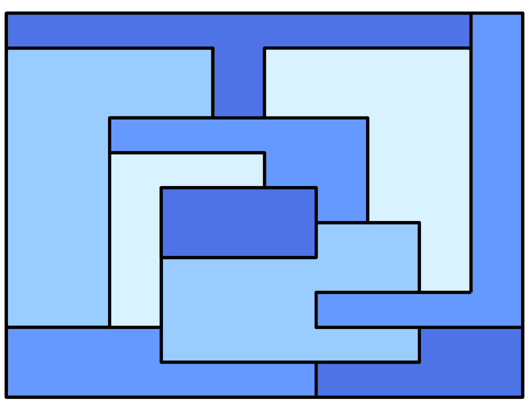
Drawings of planar graphs with prescribed face area

Linda Kleist | WG 2016

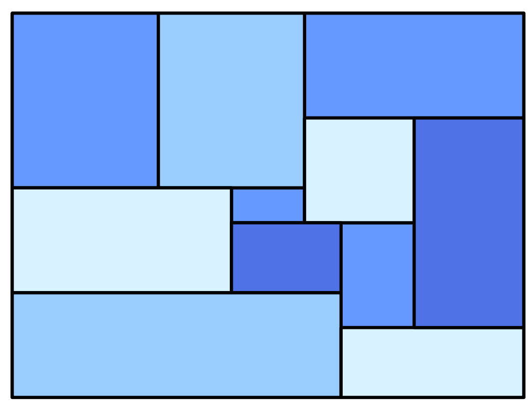
Planar graphs and areas

Cartograms

- contact representations
- weights on the vertices
- complexity of polygons
- restricted shapes



rectilinear dual



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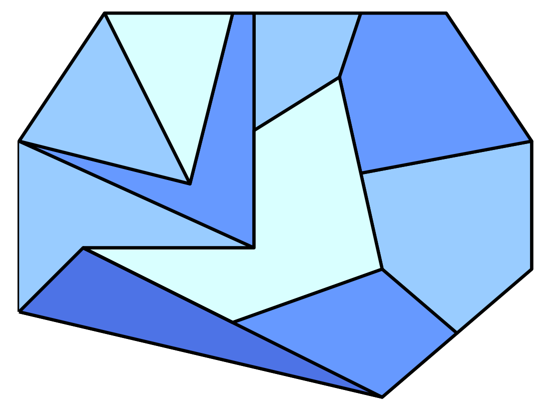

Planar graphs and areas

- weights on the faces
- straight-line drawings

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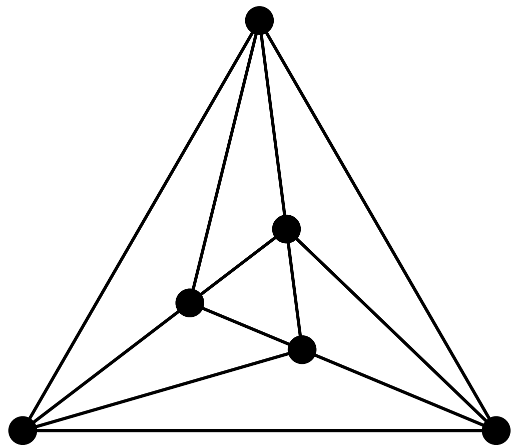
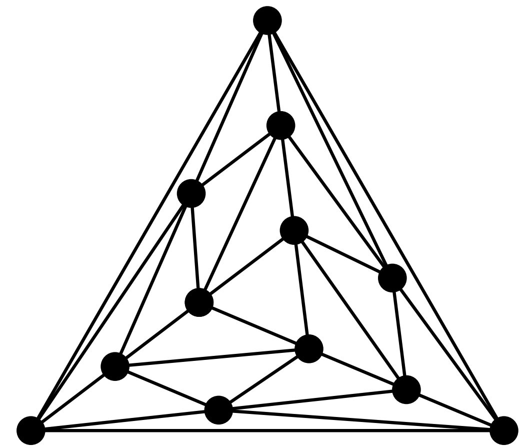
[Ringel, 1990] Octahedron and icosahedron are equi-areal.



octahedron graph



icosahedron graph

Area-Universality

G plane graph, F' set of inner faces

G is area-universal if

- for all $A\colon F' \to \mathbb{R}^+$ there exists
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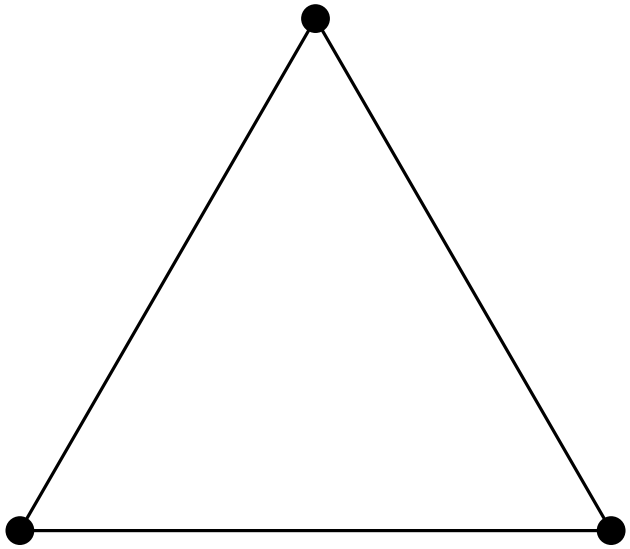

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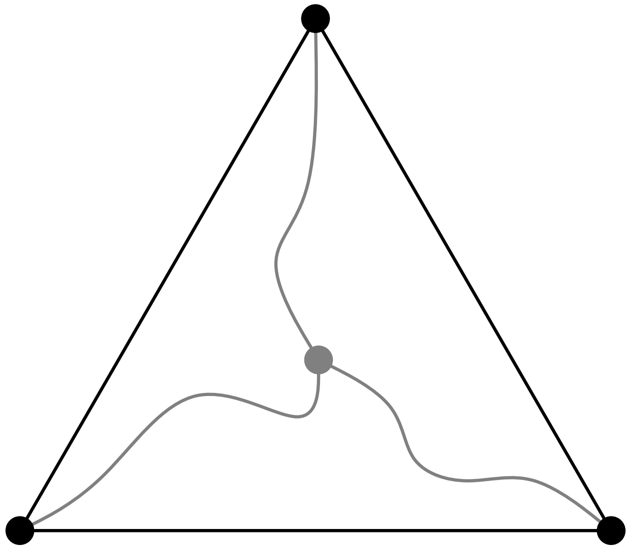

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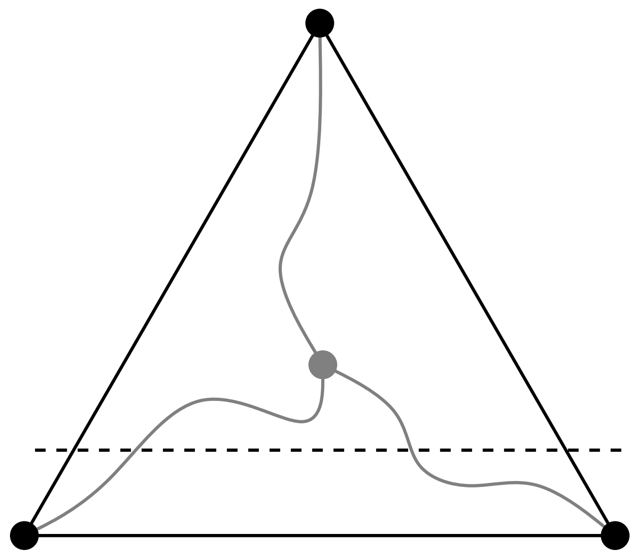

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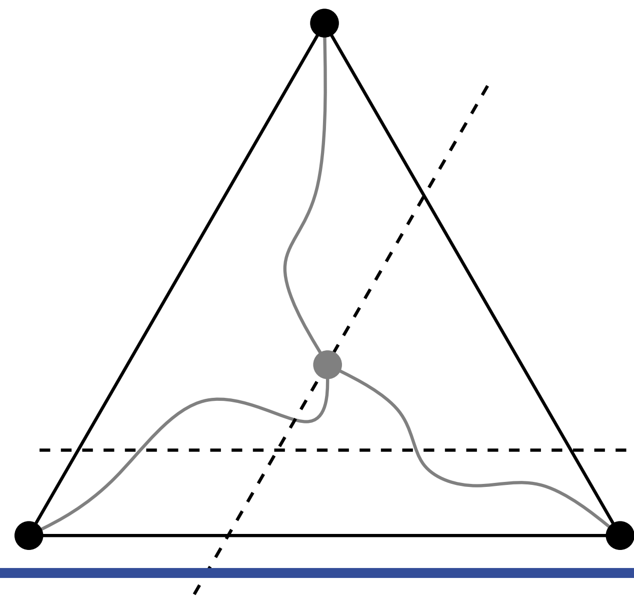

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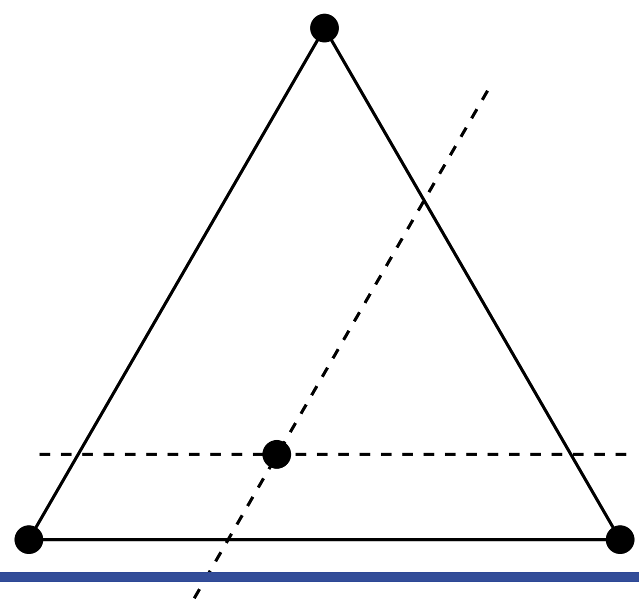

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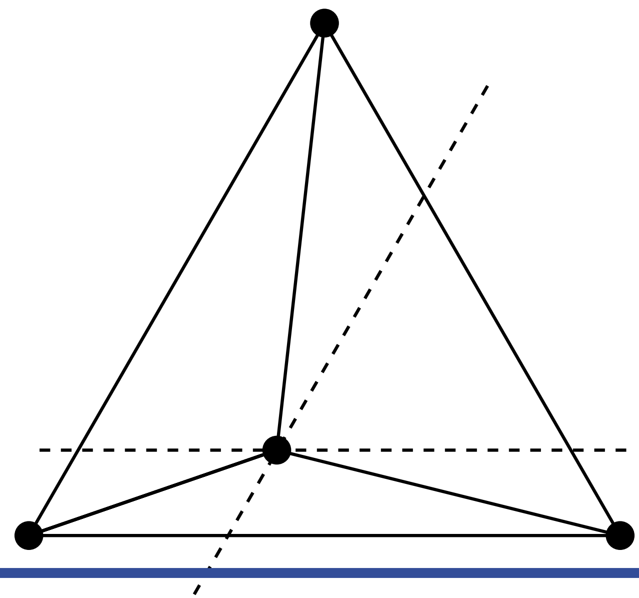

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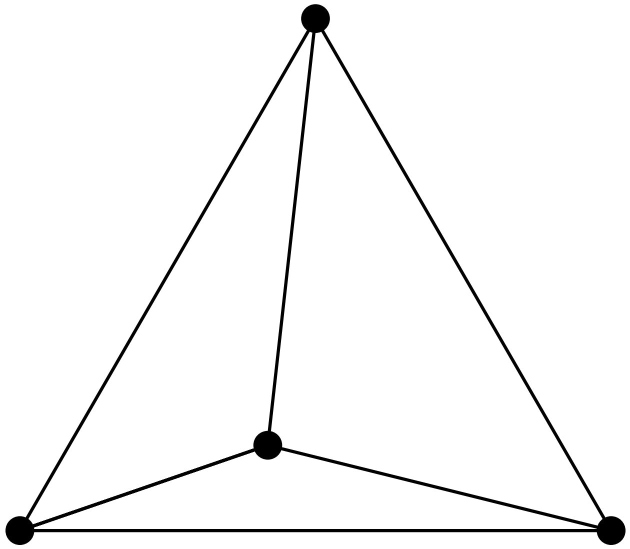

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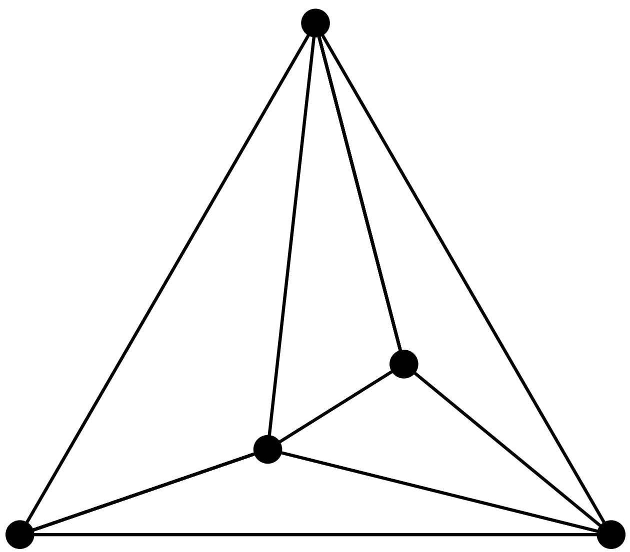

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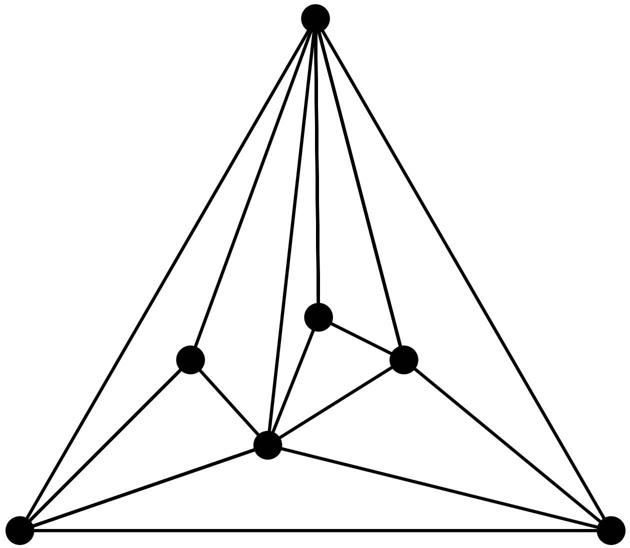

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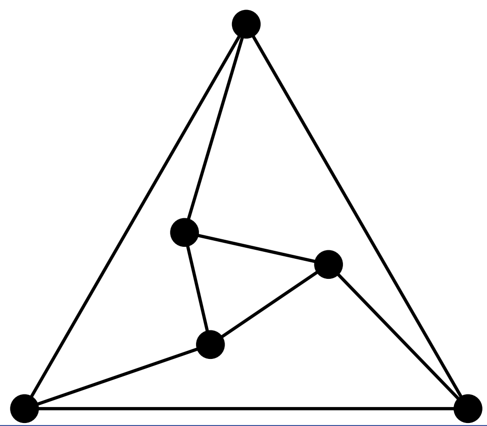

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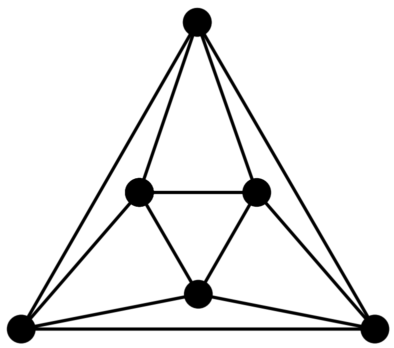

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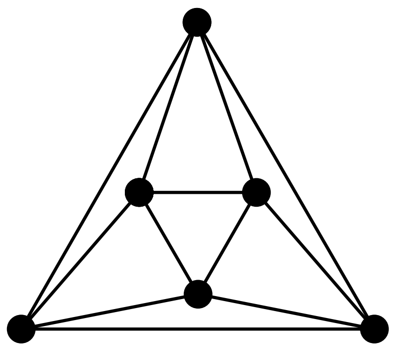
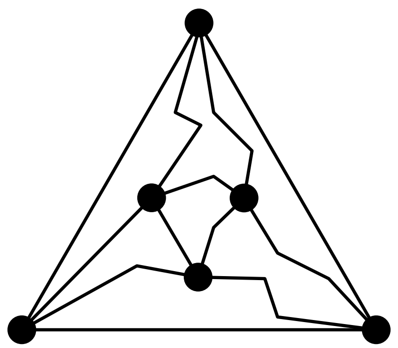


- Realizing all faces areas
  $\rightarrow$ Drawings with bends



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Proof-Sketch:

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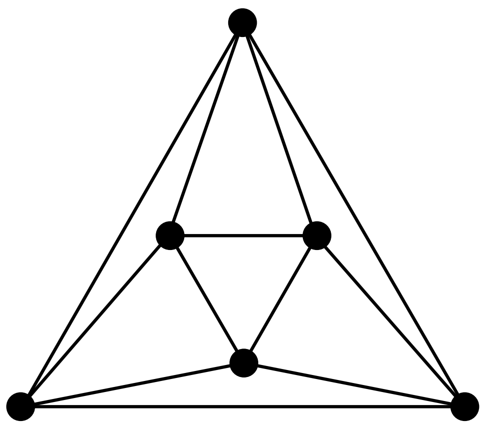

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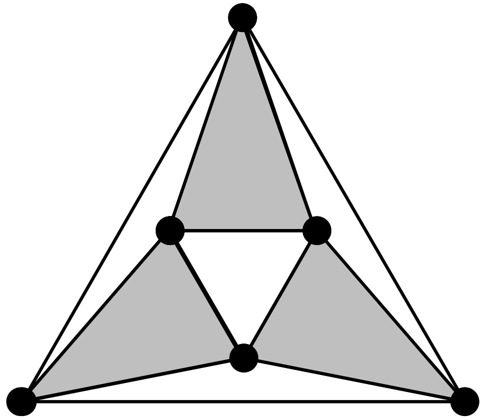

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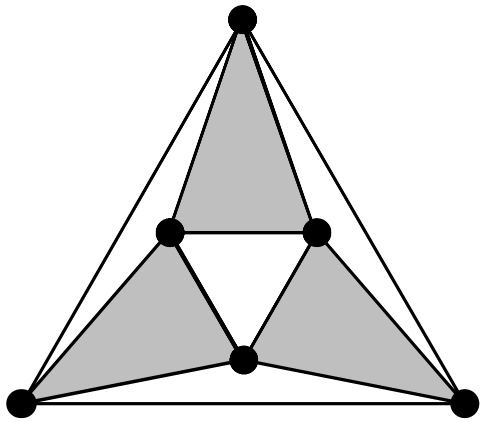

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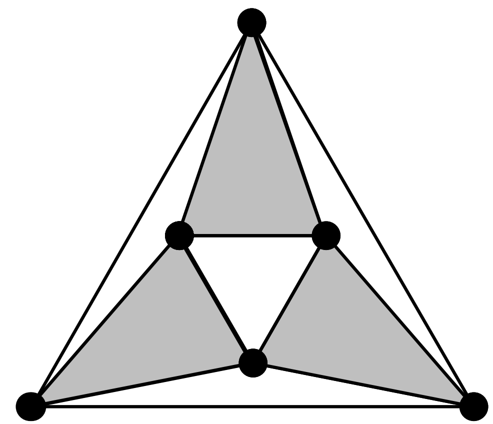

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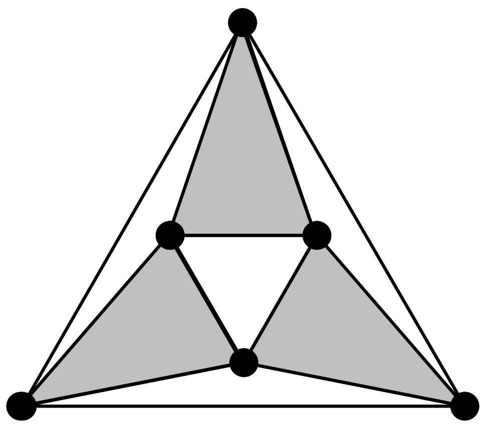

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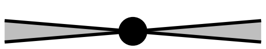
- each inner vertex has at most one flat angle 

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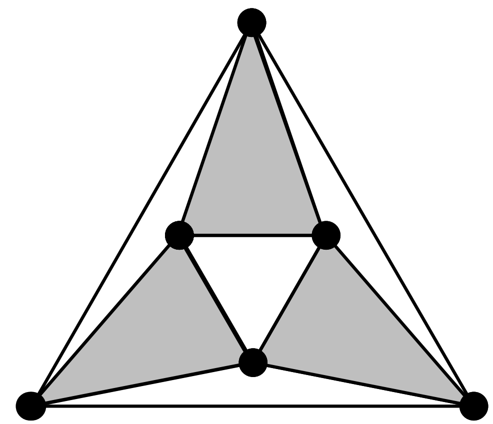

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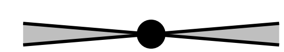
- each inner vertex has at most one flat angle 

number of white faces $>$ inner vertices ↯ $\square$

Realizing all face areas

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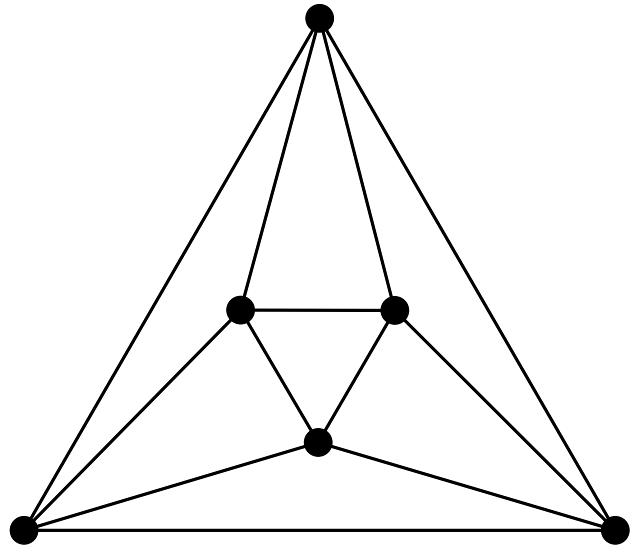

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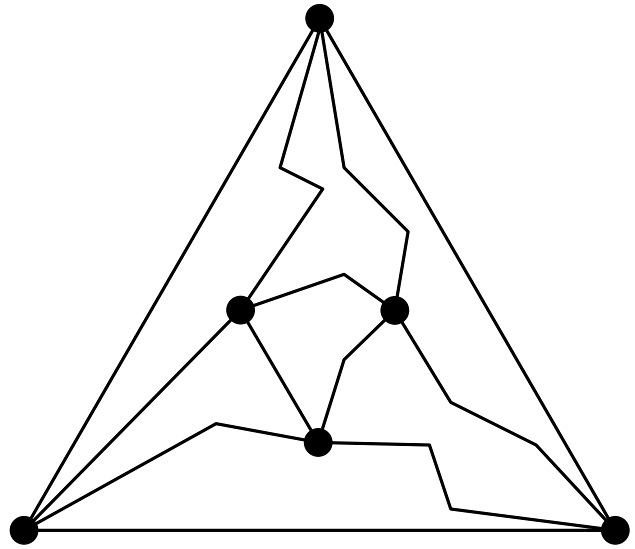

$\rightsquigarrow$ allow bends

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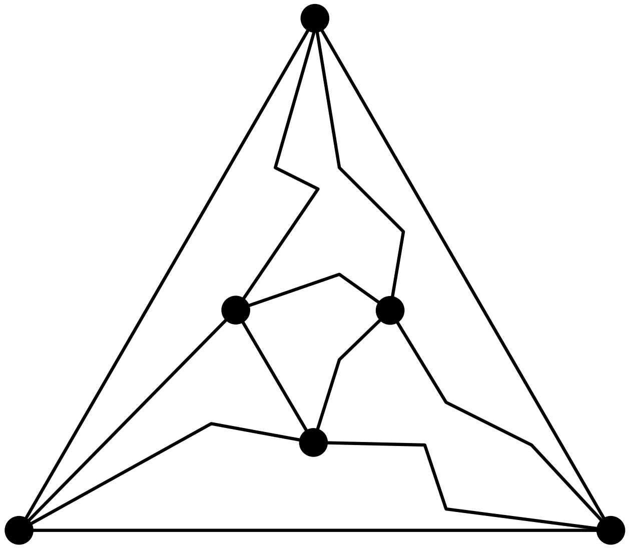

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How many bends are sufficient?

One bend per edge is enough!

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$\exists$ 1-bend-drawing of G s.t. $\text{area}(f) = A(f)\ \forall\ f \in F'$

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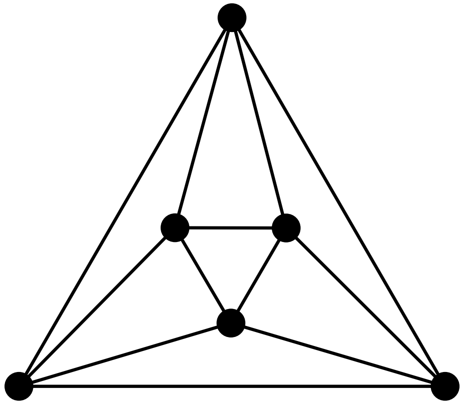

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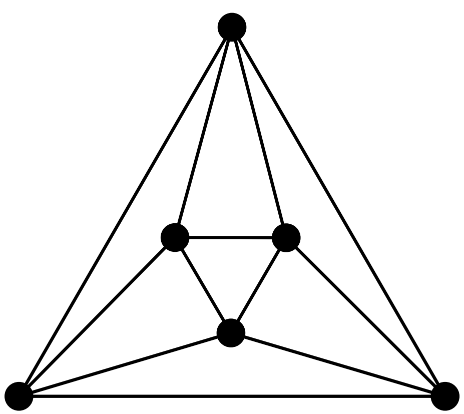

Schnyder wood

coloring / orientation of inner edges

outer vertex   inner vertex

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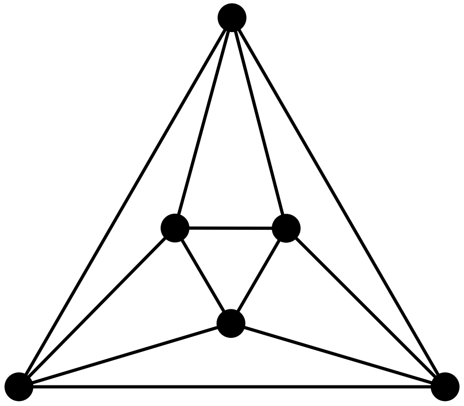

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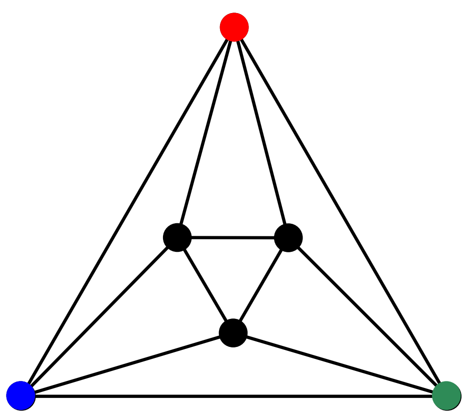

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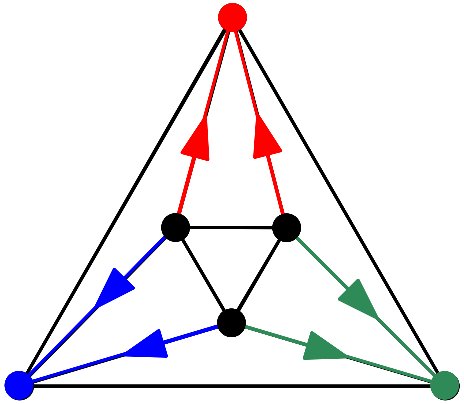

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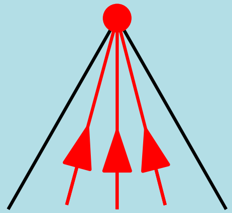
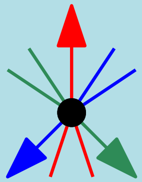


outer vertex



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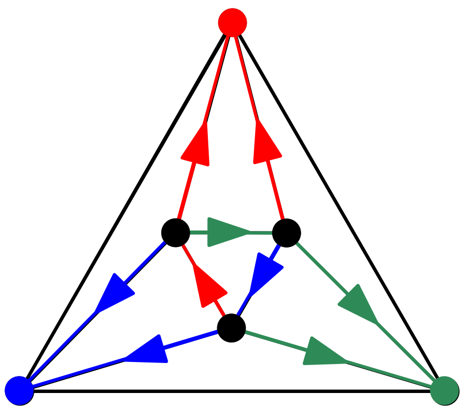

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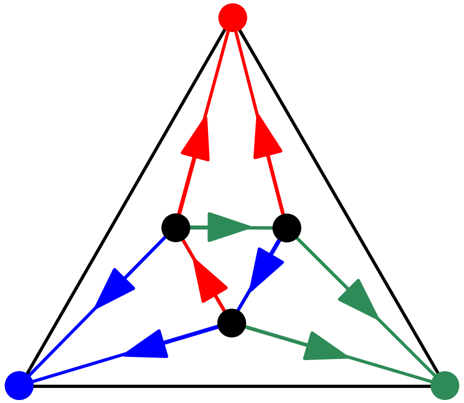

Triangulation
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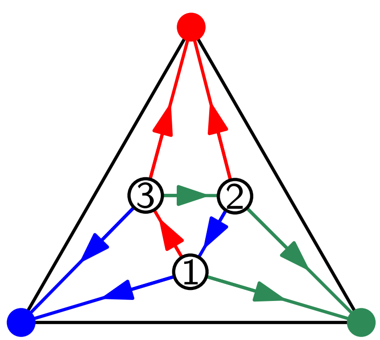

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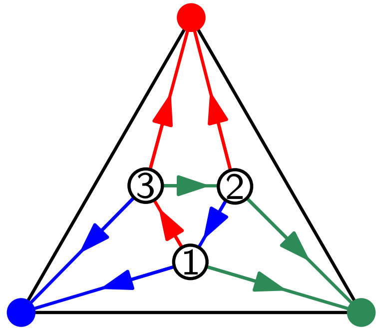

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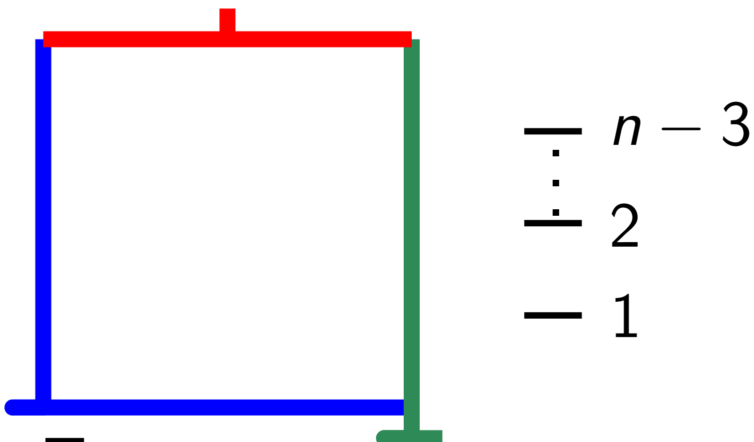

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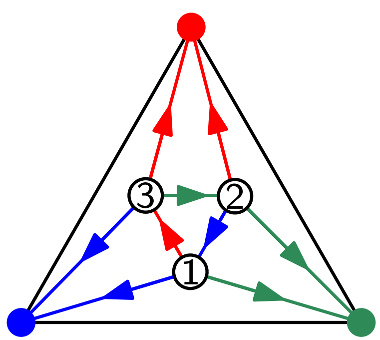

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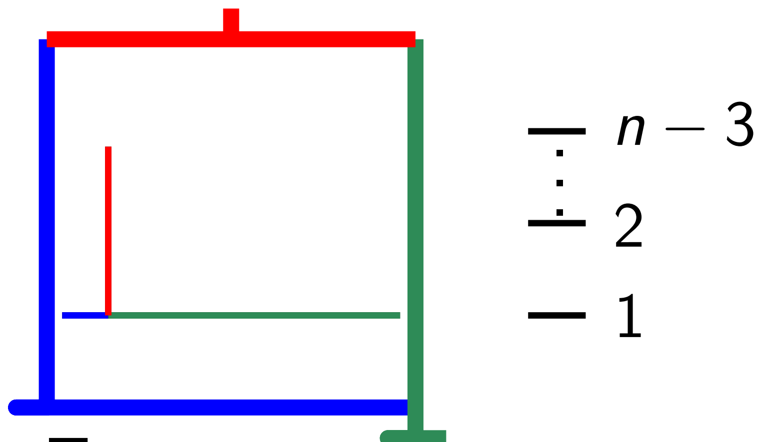

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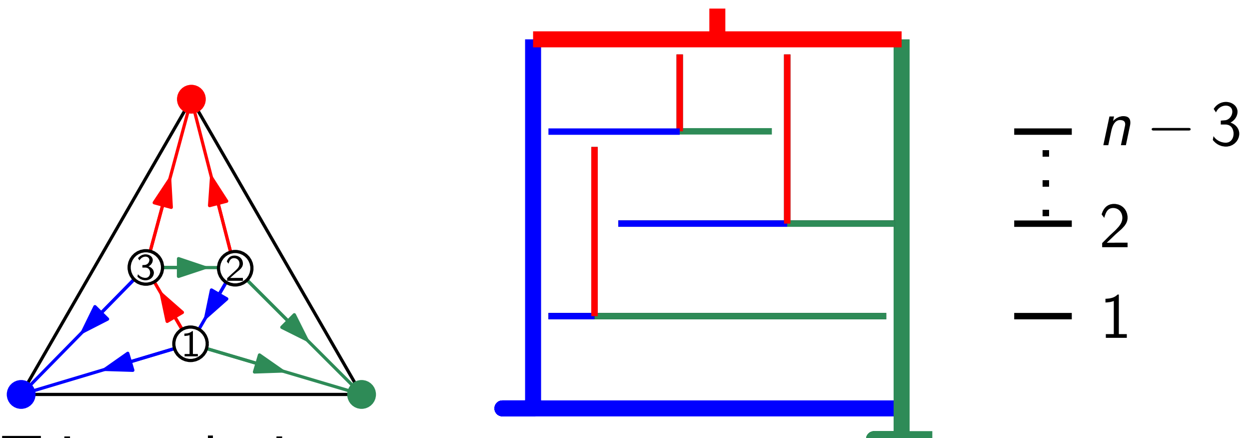

Triangulation with Schnyder Wood

T-contact representation

One bend per edge is enough!

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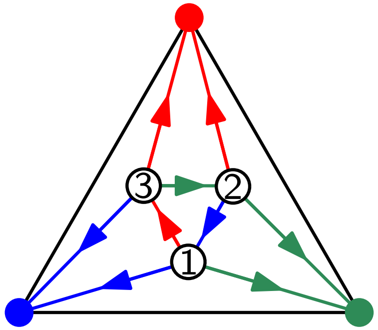

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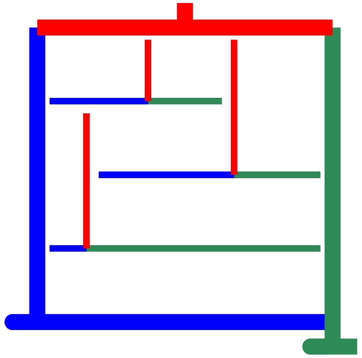

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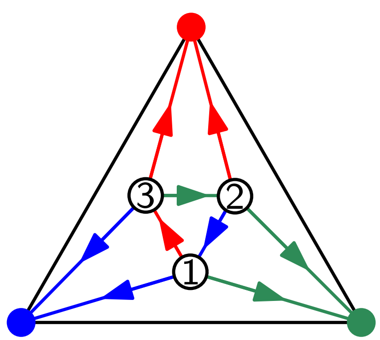

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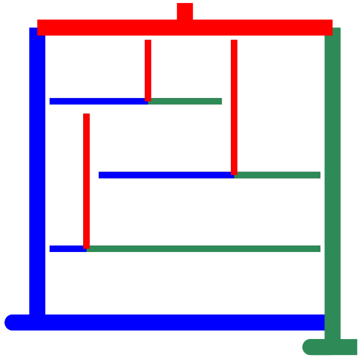

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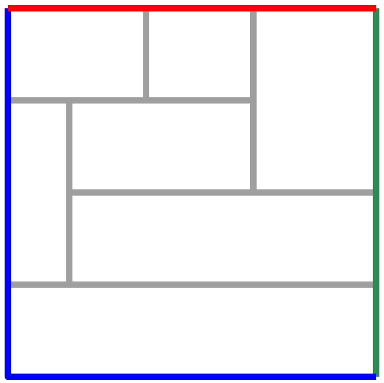

$\to$ rectangular layout $\mathcal{L}$

One bend per edge i

[Eppstein et al., 2009]
$\forall \mathcal{L}$ and $\forall w$:
$\exists$ weak-eq $\mathcal{L}'$ realizing w.

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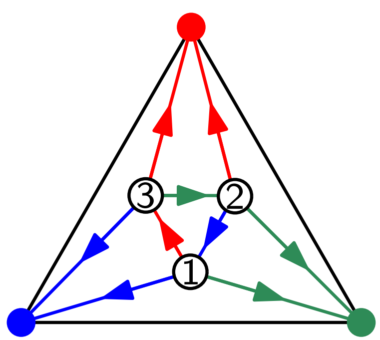

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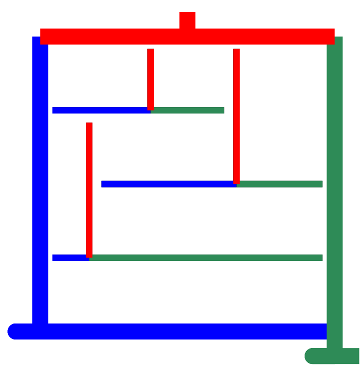

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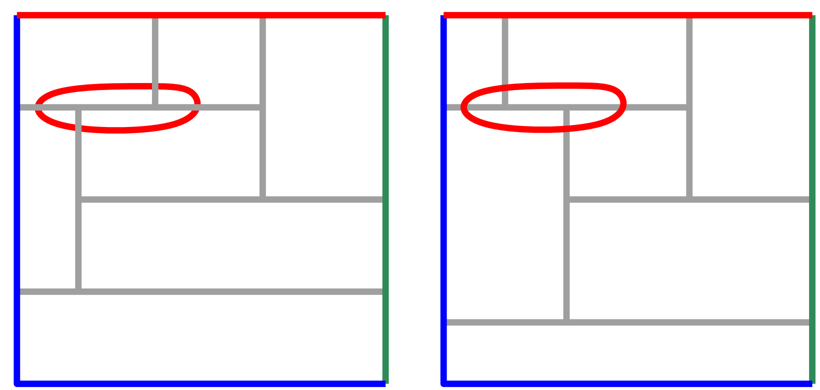

$\to$ rectangular layout $\mathcal{L}$

realized areas

weak equivalent $\mathcal{L}'$

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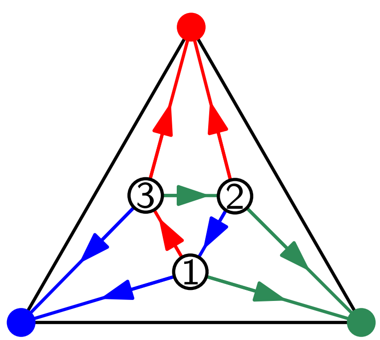

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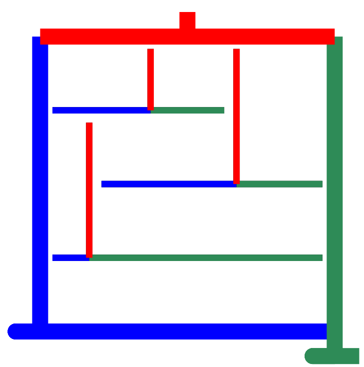

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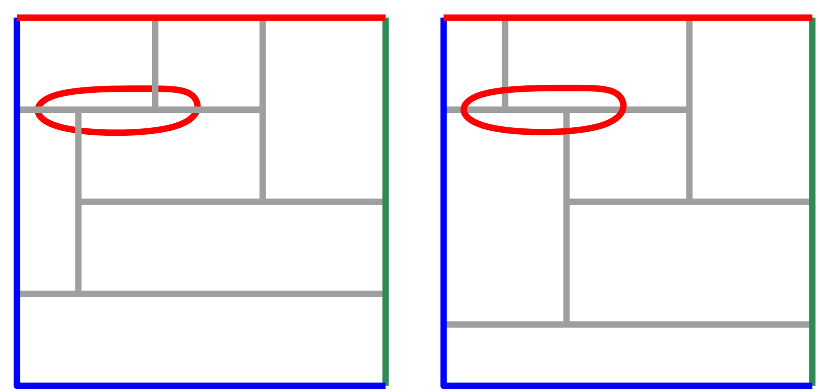

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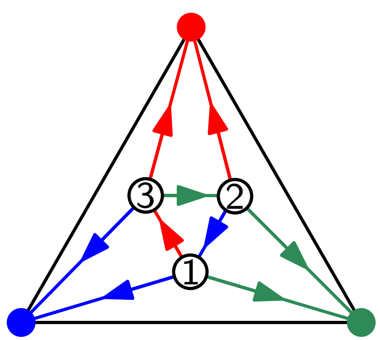

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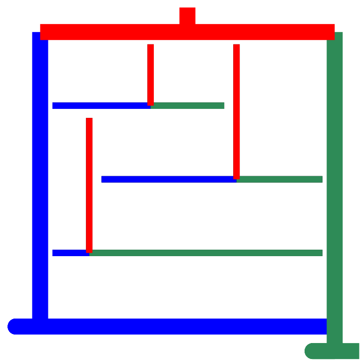

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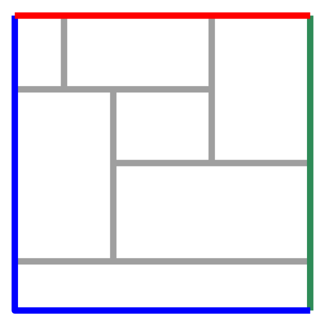

rect. layout realizing areas

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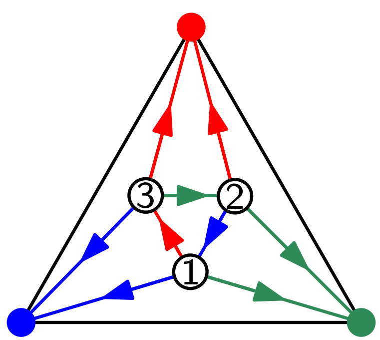

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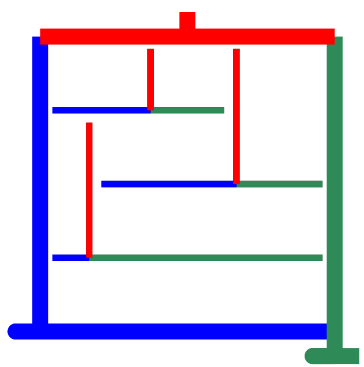

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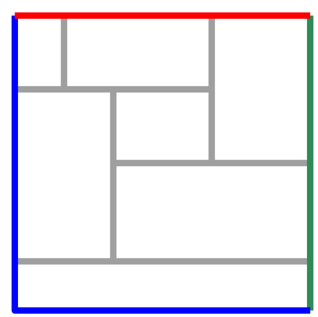

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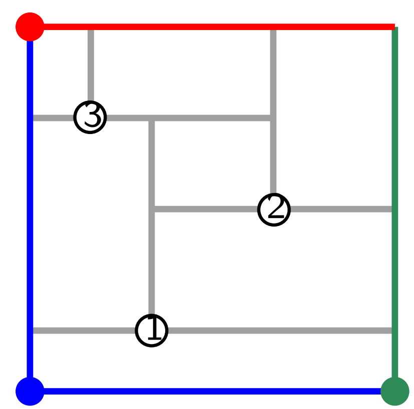

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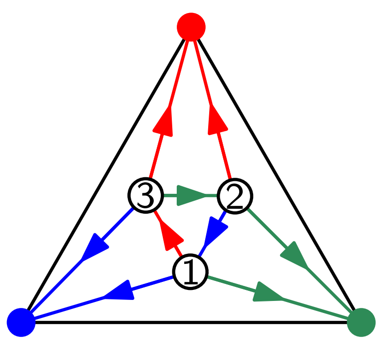

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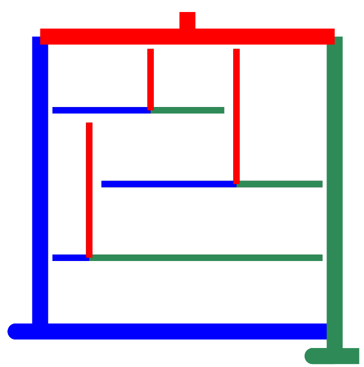

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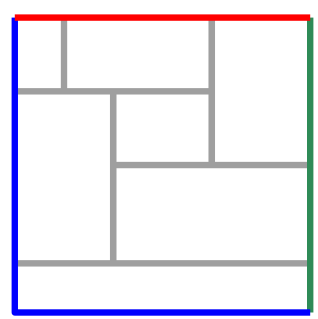

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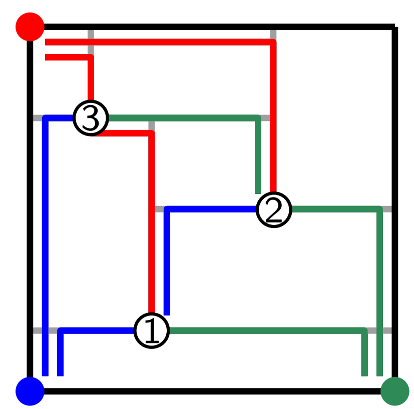

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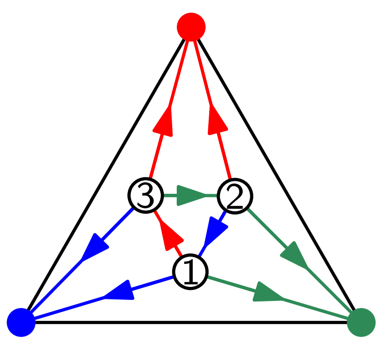

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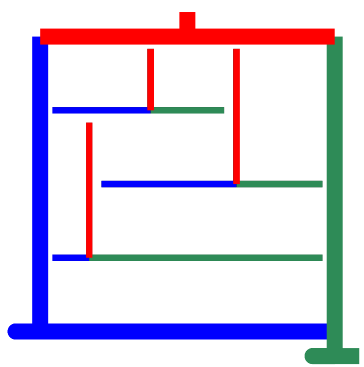

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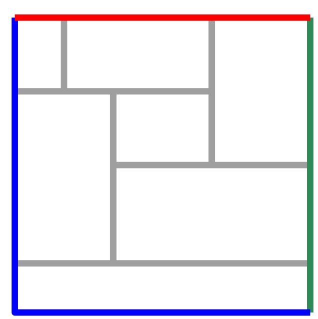

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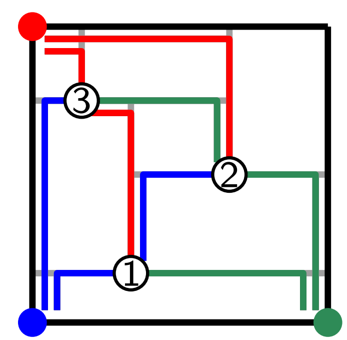

1-bend-draw. with degeneracies

One bend per edge is enough!

Technische Universität Berlin

G plane graph, F' set of inner f
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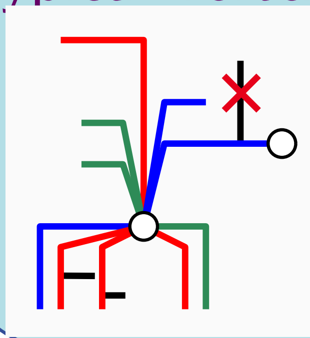
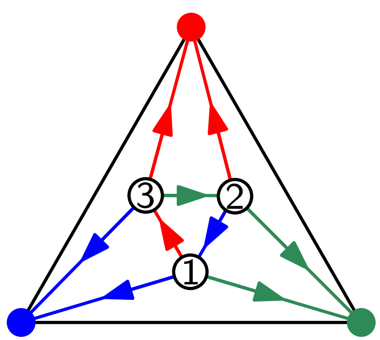


1. green /blue (bottom-up) $\forall v$
2. red (middle-out)

proof:



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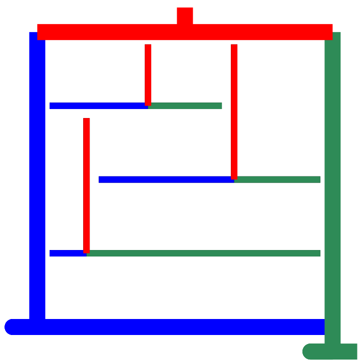

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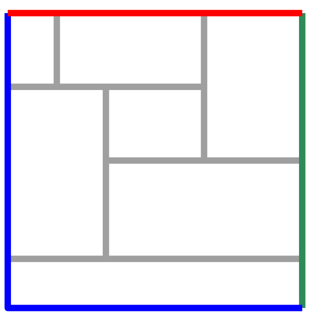
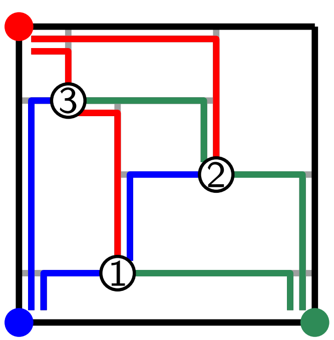


rect. layout realizing areas



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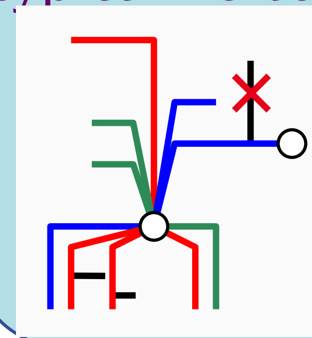

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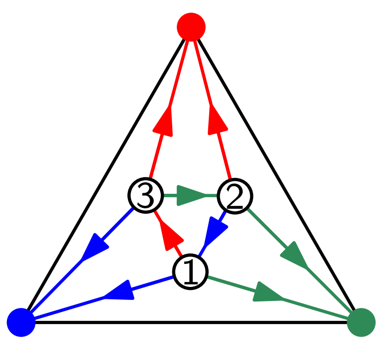

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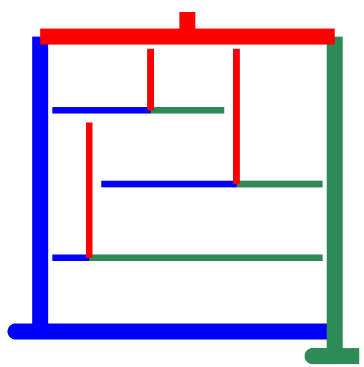

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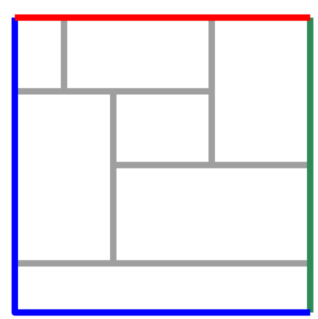

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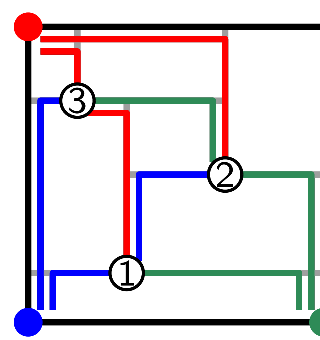

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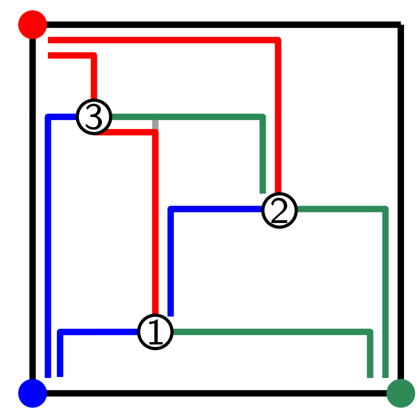

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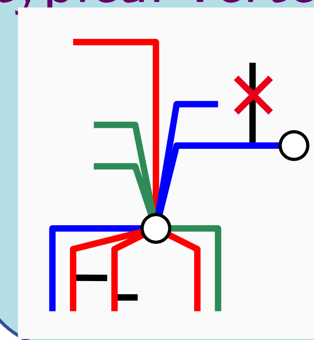

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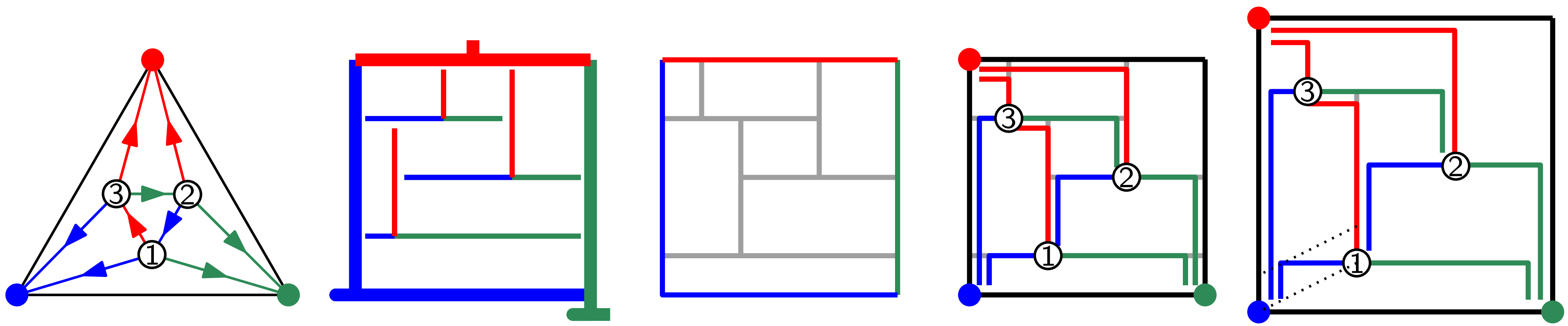

Triangulation with Schnyder Wood | T-contact representation | rect. layout realizing areas | 1-bend-draw. with degeneracies

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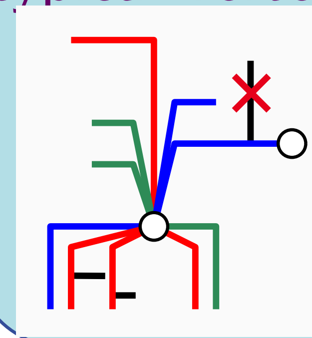

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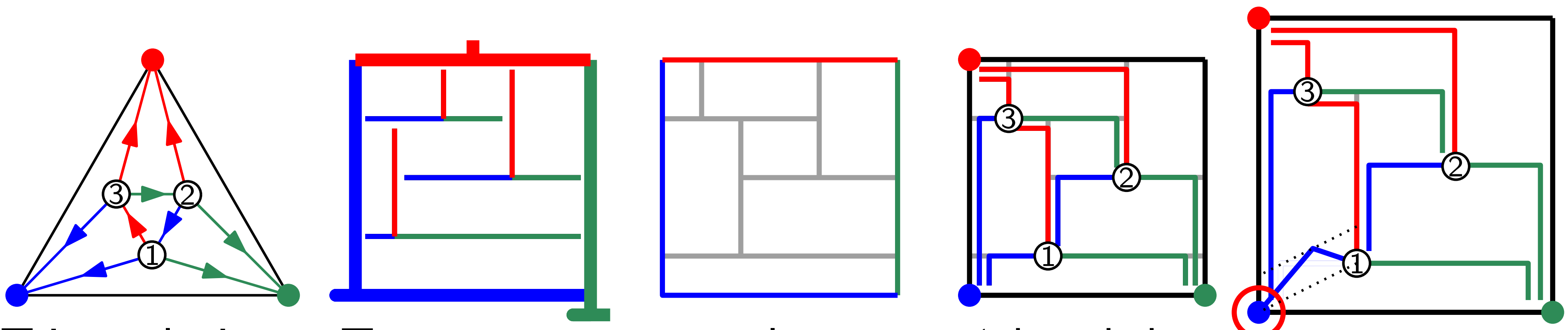

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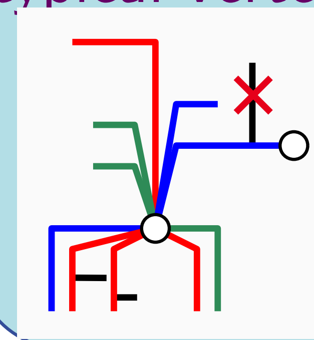

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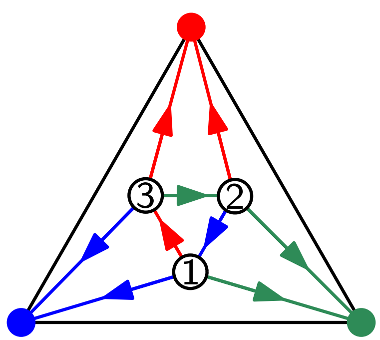

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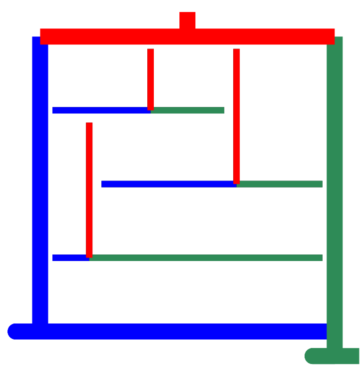

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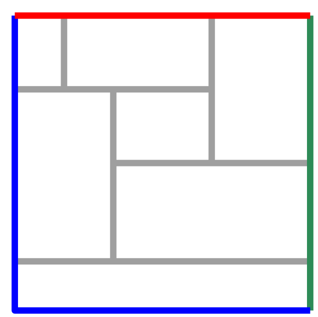

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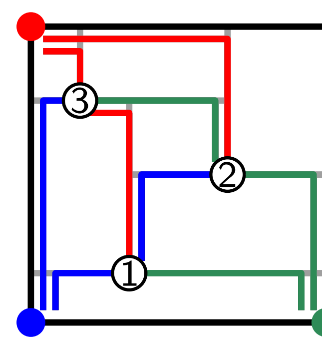

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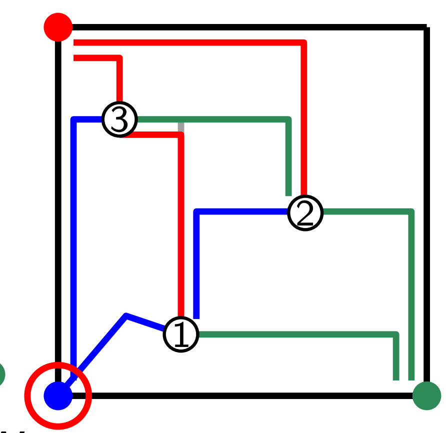

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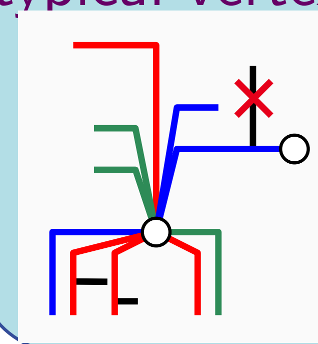

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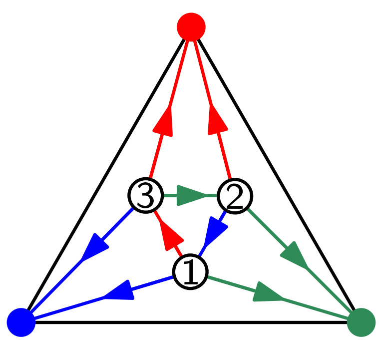

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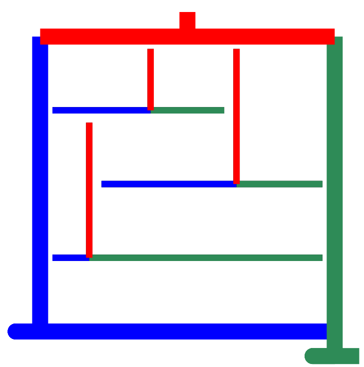

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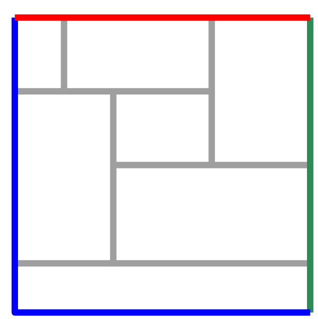
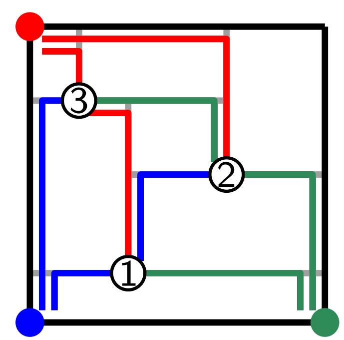


rect. layout realizing areas



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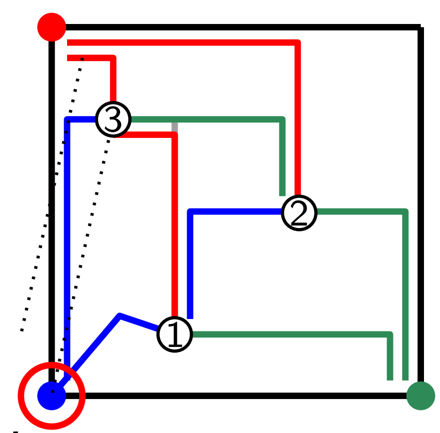

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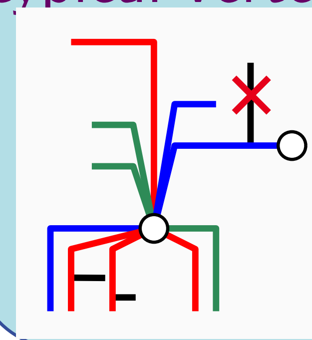

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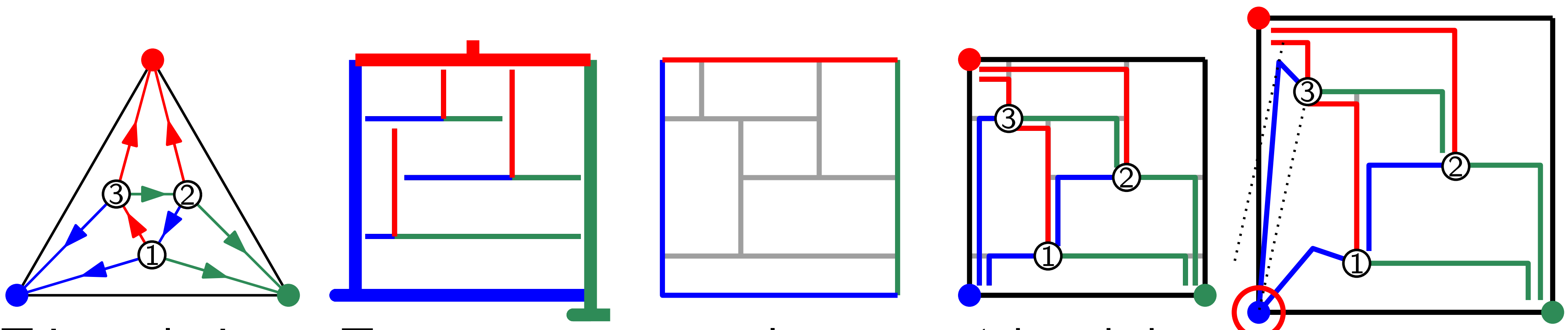

Triangulation with Schnyder Wood | T-contact representation | rect. layout realizing areas | 1-bend-draw. with degeneracies

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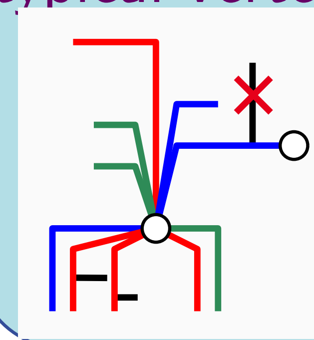

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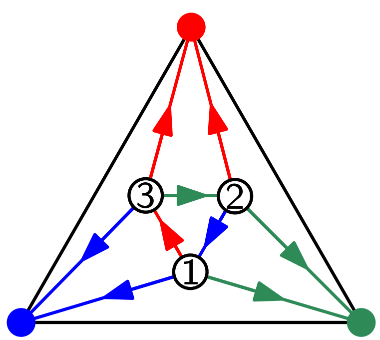

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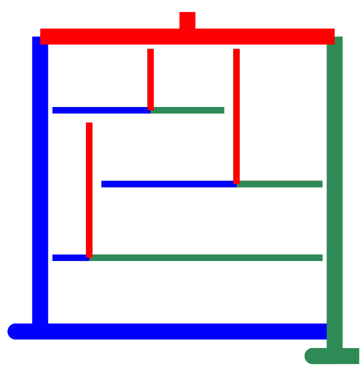
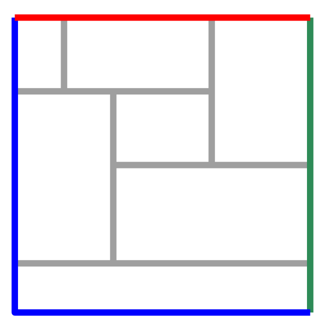
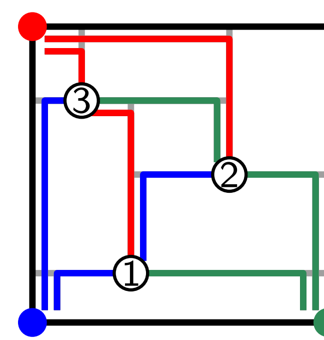


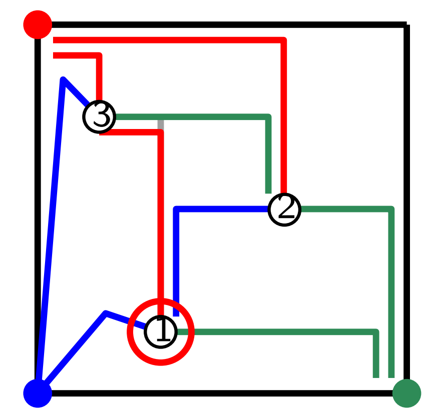
T-contact representation



rect. layout realizing areas



1-bend-draw. with degeneracies



One bend per edge is enough!

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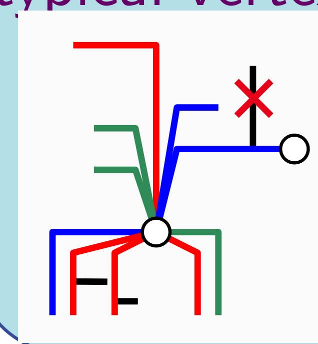
G plane graph, F' set of inner f

A: $F' \to \mathbb{R}^+$ face-area assignm

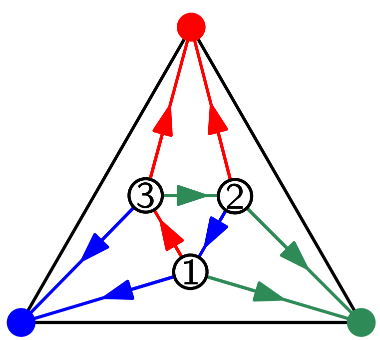
$\exists$ 1-bend-drawing of G s.t. are

typical vertex

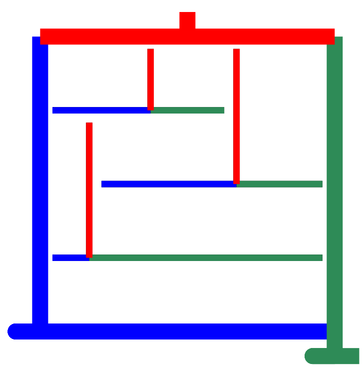


1. green / blue (bottom-up) $\forall v$
2. red (middle-out)

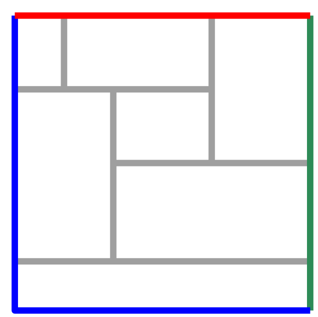
proof:



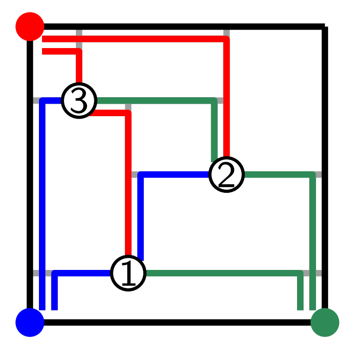
Triangulation with Schnyder Wood



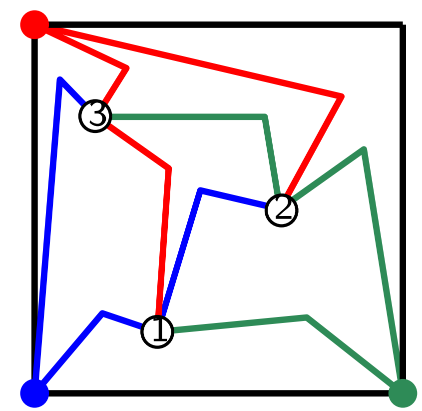
T-contact representation



rect. layout realizing areas



1-bend-draw. with degeneracies

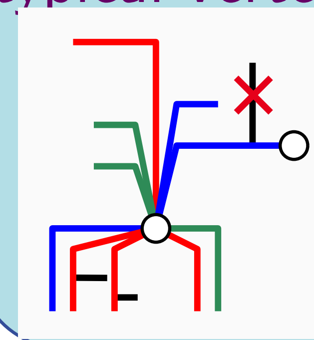


1-bend-draw.

One bend per edge is enough!

G plane graph, *F′* set of inner f
A: $F' \to \mathbb{R}^+$ face-area assignm
∃ 1-bend-drawing of *G* s.t. are

typical vertex



1. green / blue (bottom-up) $\forall v$
2. red (middle-out)

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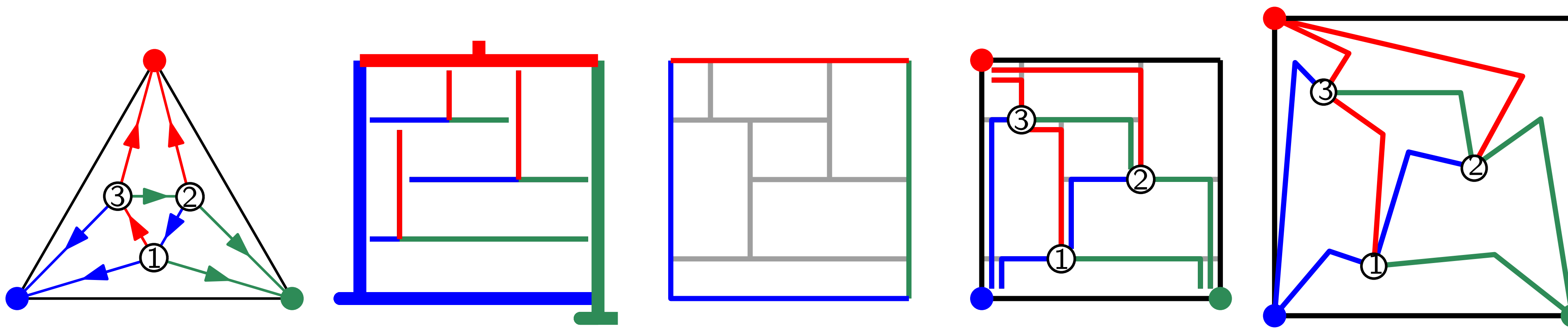


Triangulation with Schnyder Wood

T-contact representation

rect. layout realizing areas

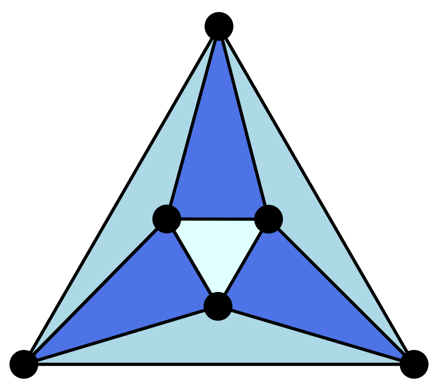
1-bend-draw. with degeneracies

1-bend-draw.

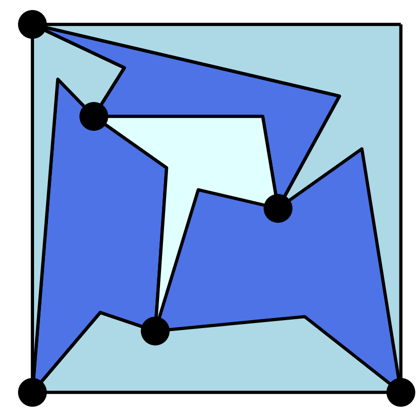
□

Summary & Questions

Eulerian triangulations are not area-universal.



All planar graphs have realizing 1-bend-drawings.



Open Questions:

- How many bends are really necessary and sufficient?

  $\frac{1}{12}|E| \leq \#\text{bends} \leq |E|$

- Are bipartite graphs area-universal?
- How hard is testing the realizability of an area-assignment?