

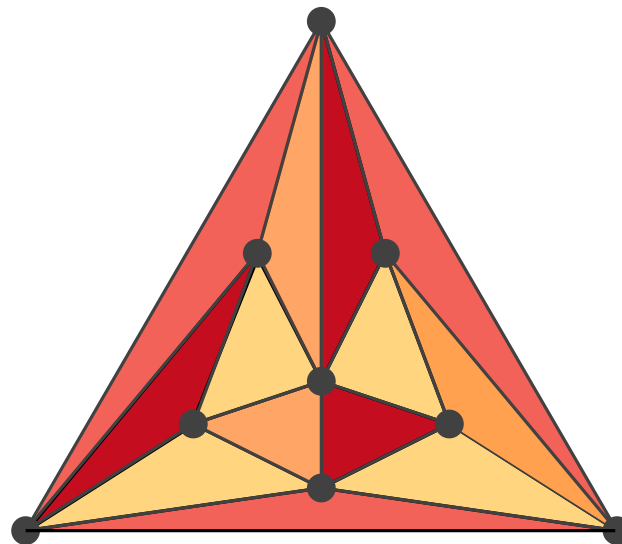
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# Is area-universality $\forall\exists\mathbb{R}$ -complete?

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Linda Kleist, Tillmann Miltzow, Paweł Rzążewski | EuroCG 2017

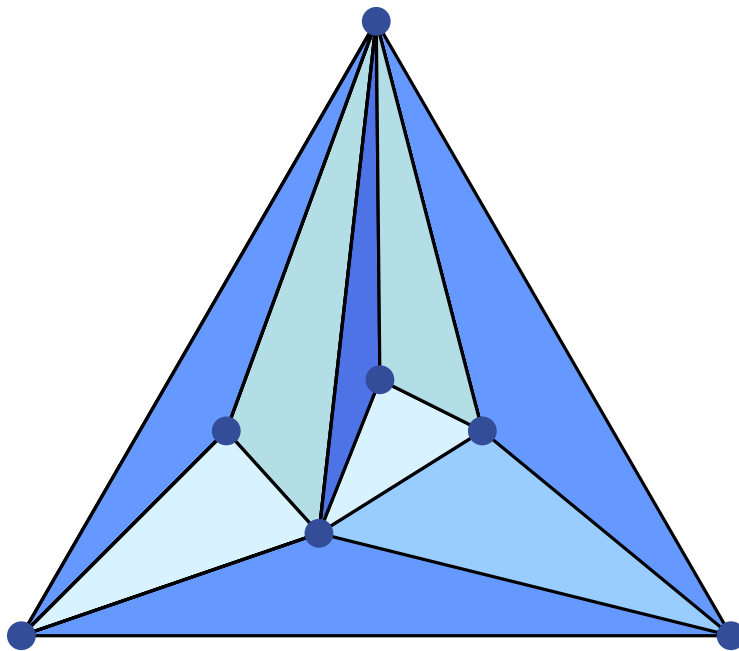


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# Area-Universality

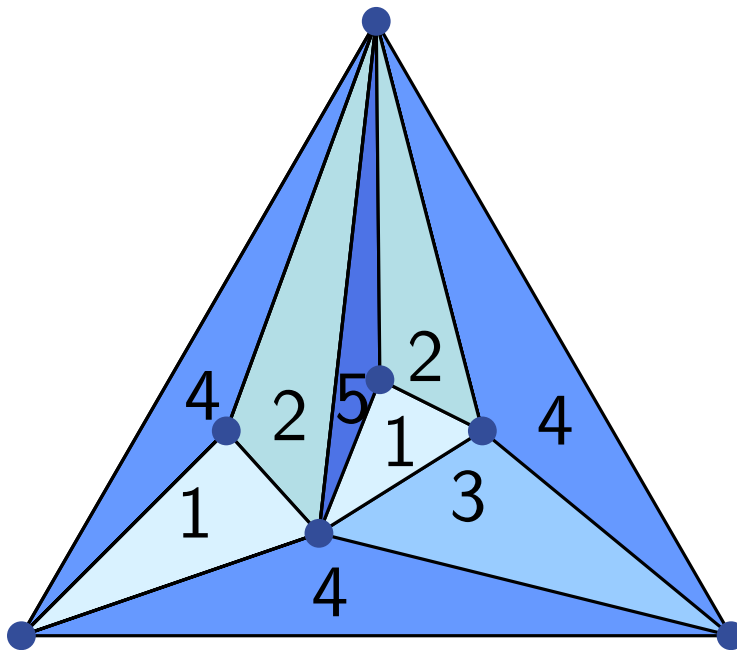
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plane graph  $G$



plane graph  $G$

area assignment  $\mathcal{A}$



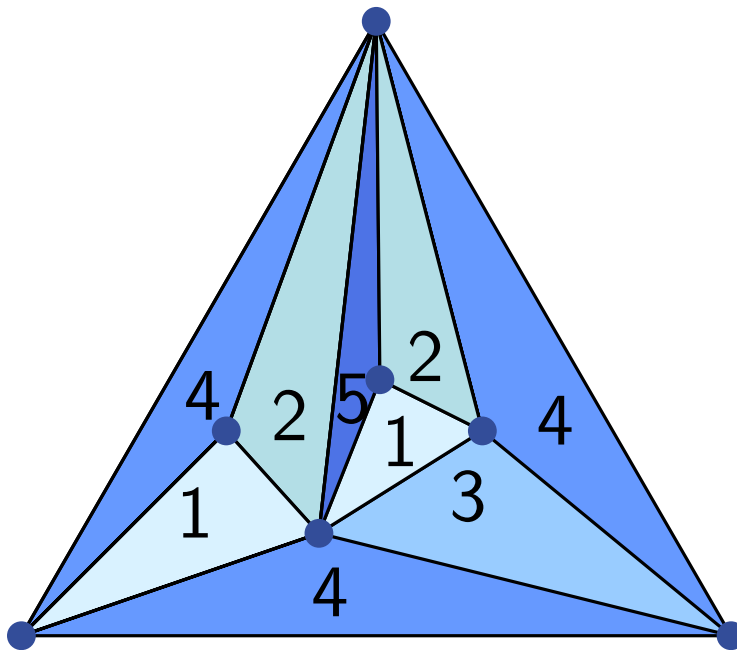
# Area-Universality



plane graph  $G$

$\exists$  realizing straight-line drawing?

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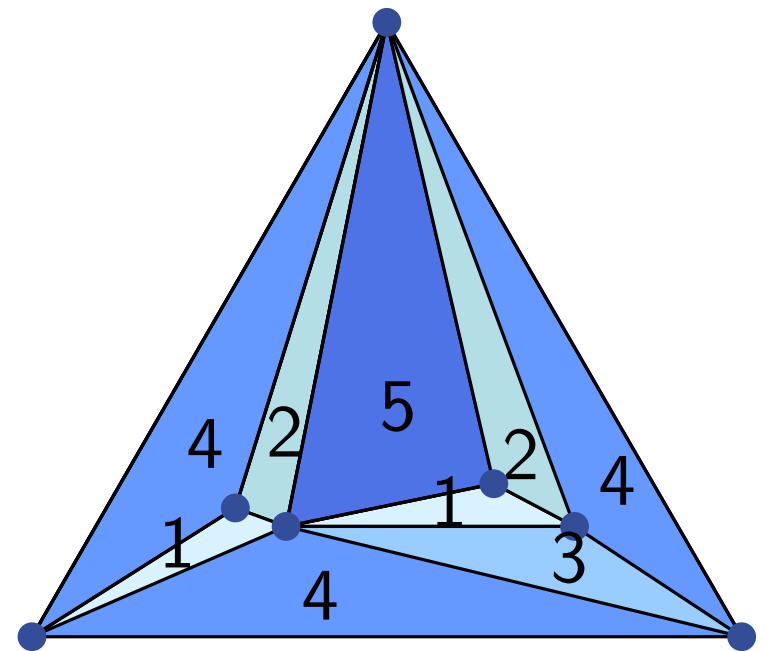
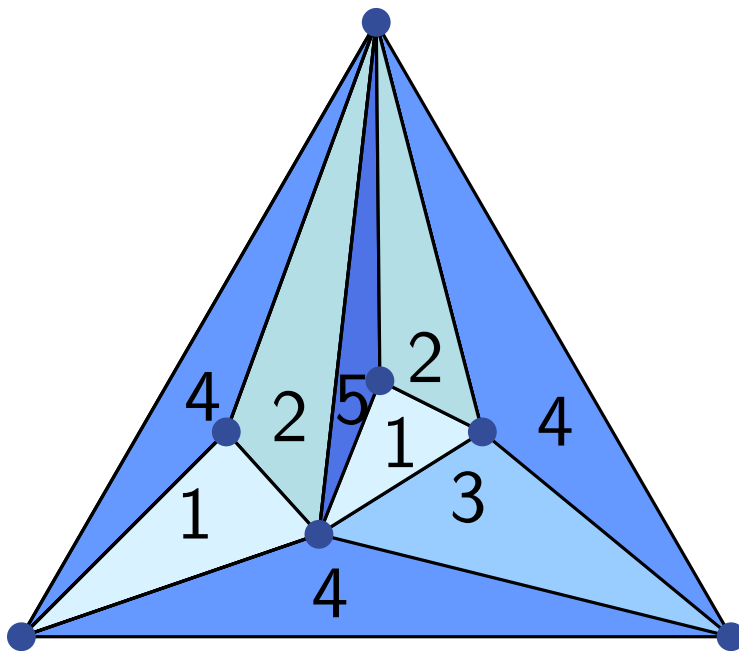


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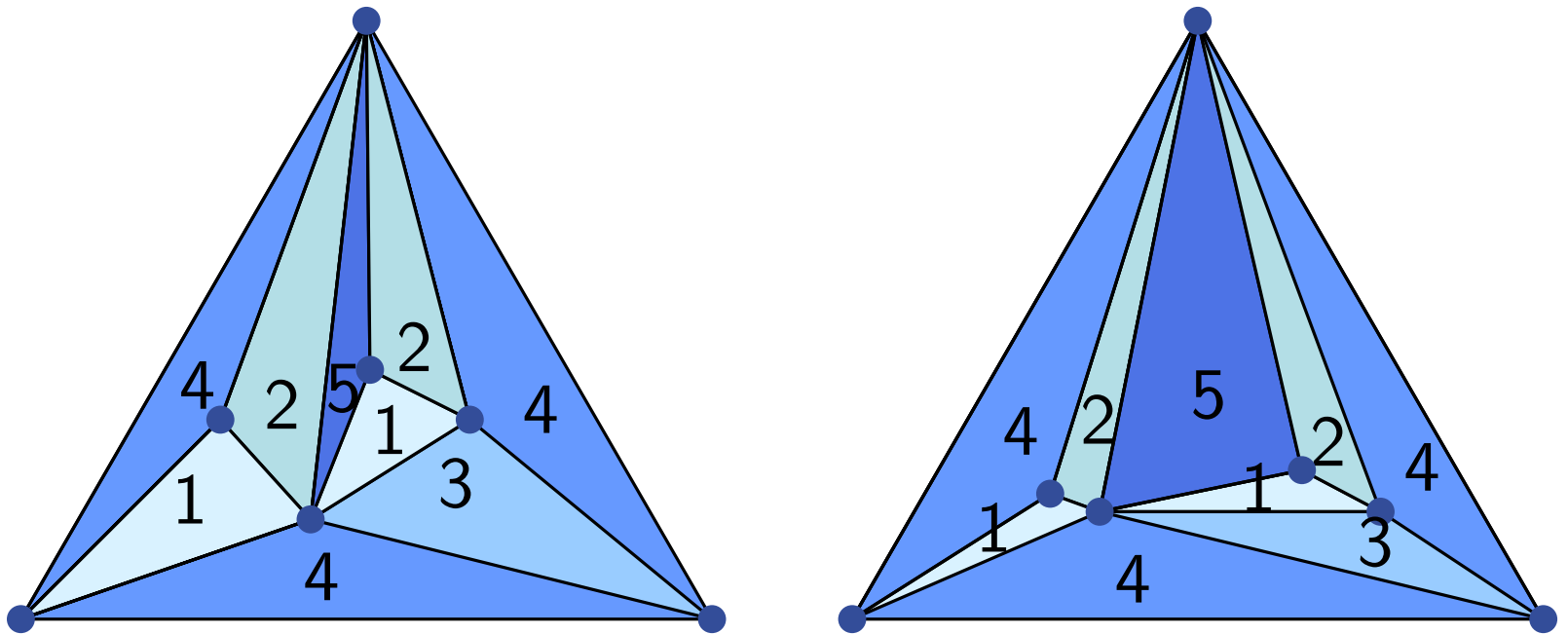


# Area-Universality

plane graph  $G$

$\exists$  realizing straight-line drawing?

area assignment  $\mathcal{A}$



$\forall \mathcal{A}$  the answer is 'yes'  $\rightarrow G$  is **area-universal**

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# Area-universality – results

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positive area-universal graphs:

- ▶ planar 3-trees

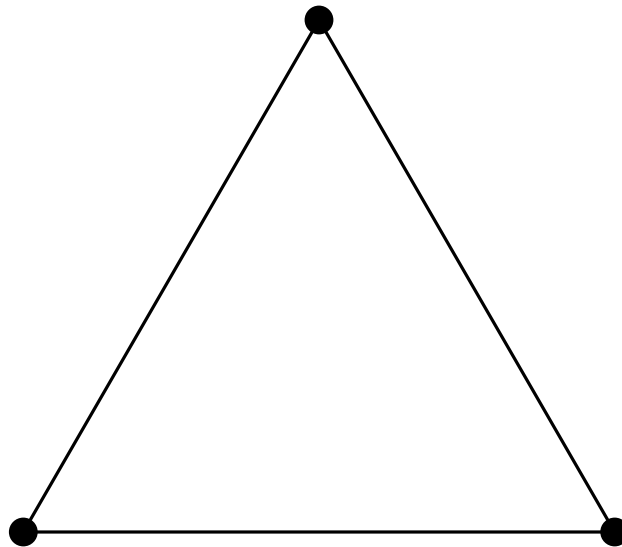
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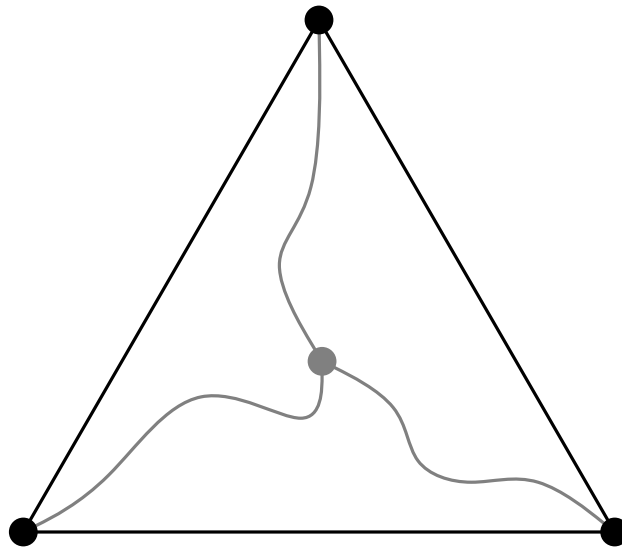
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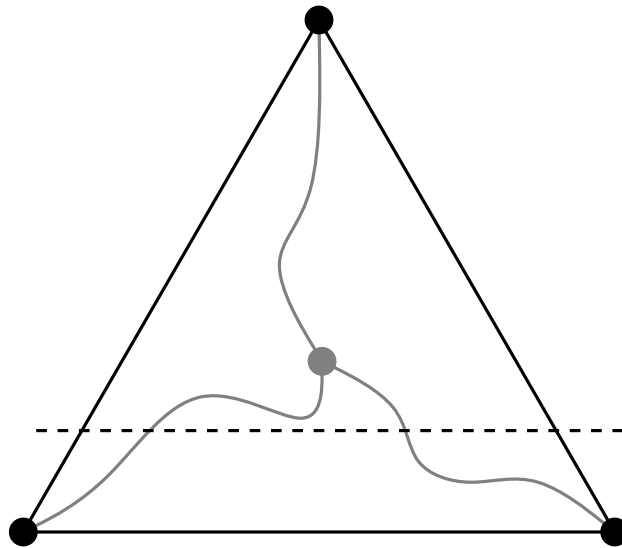
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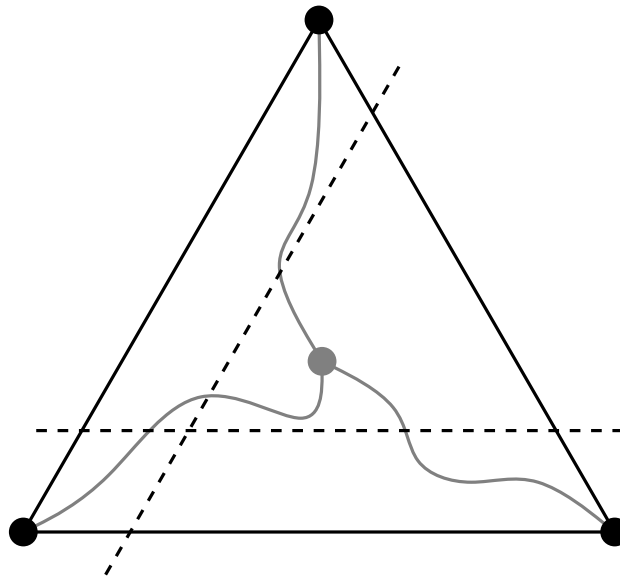
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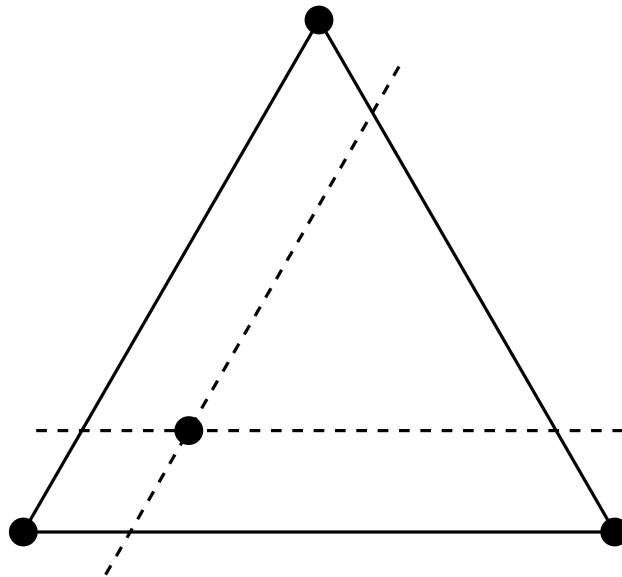
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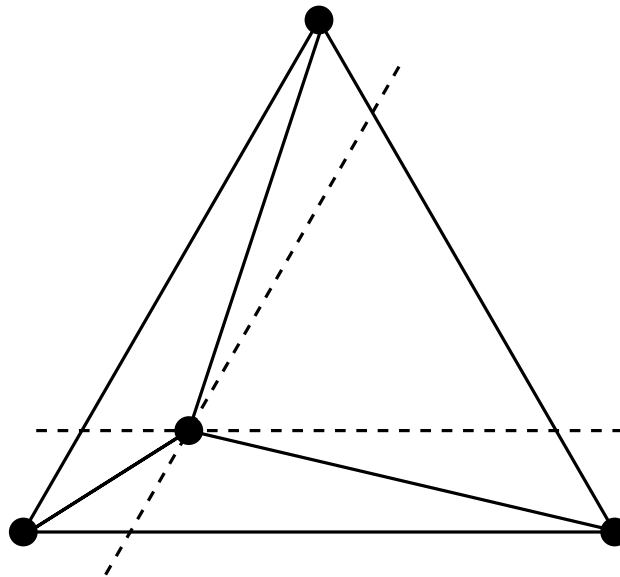
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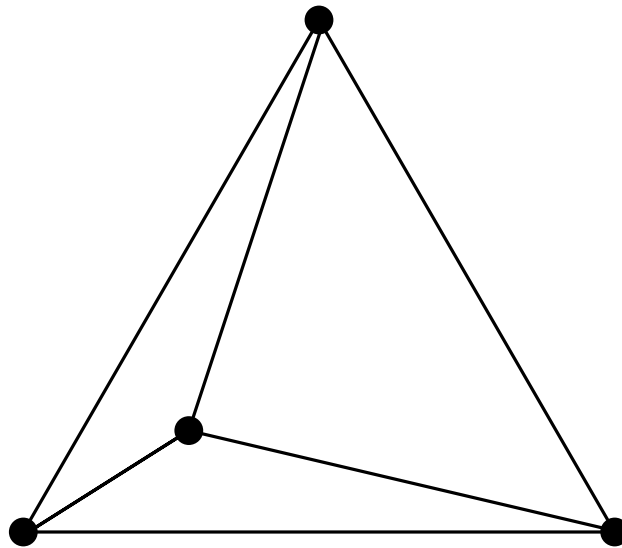
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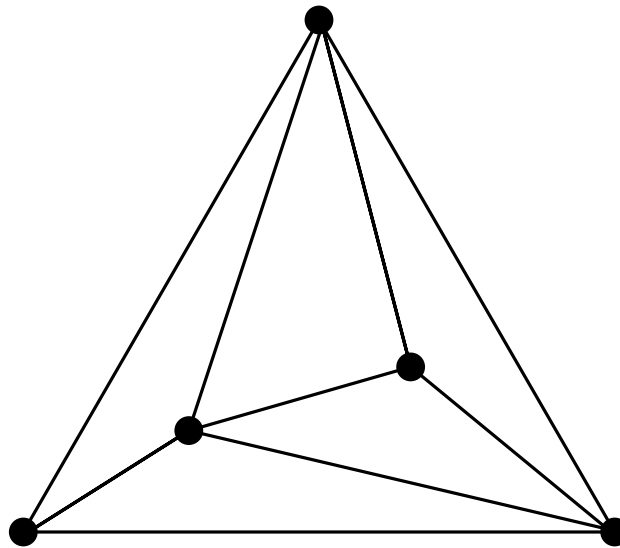
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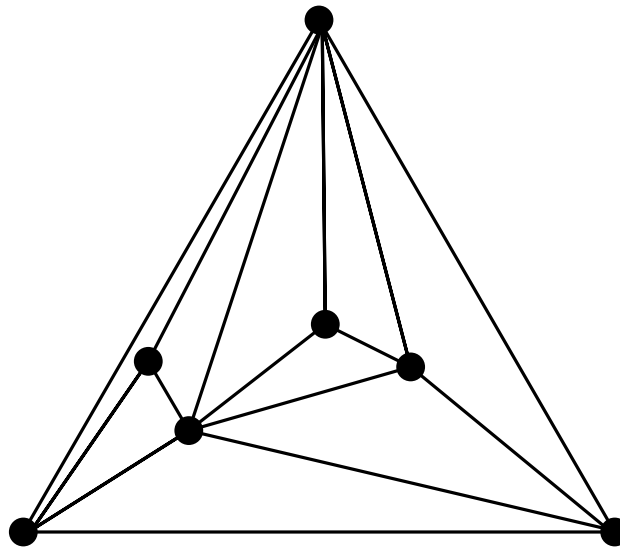
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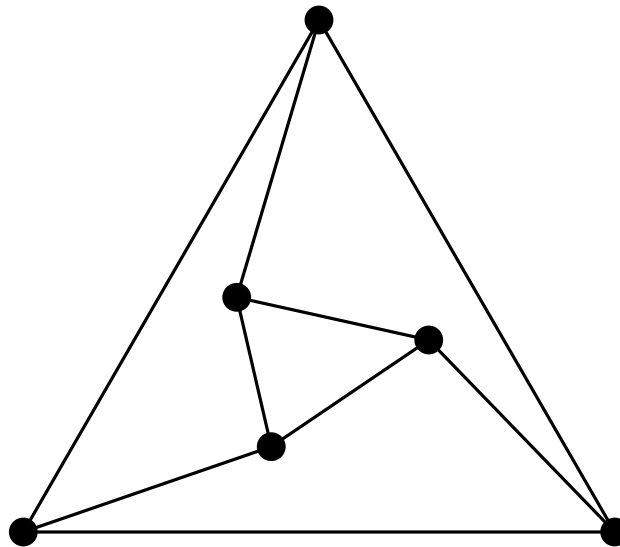
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# Area-universality – results

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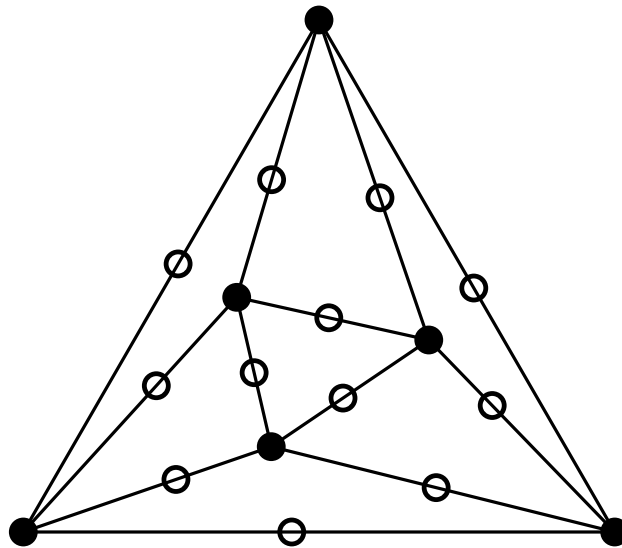
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# Area-universality – results

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- ▶ 1-subdivisions of plane graphs [LK, 2016]



---

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negative not area-universal graphs

---

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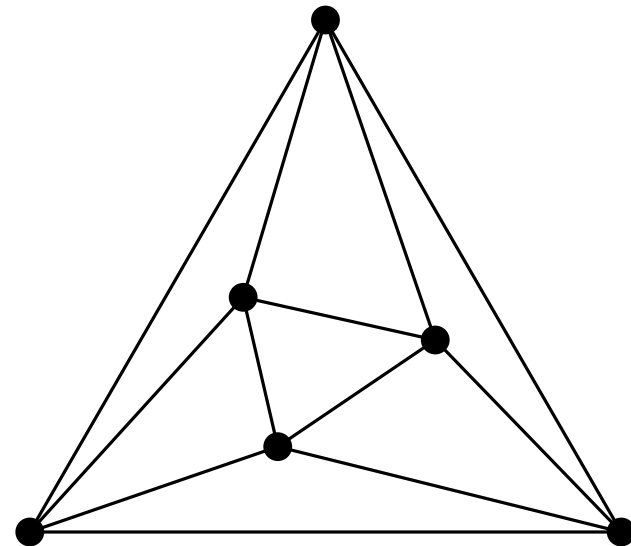
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# Area-universality – results

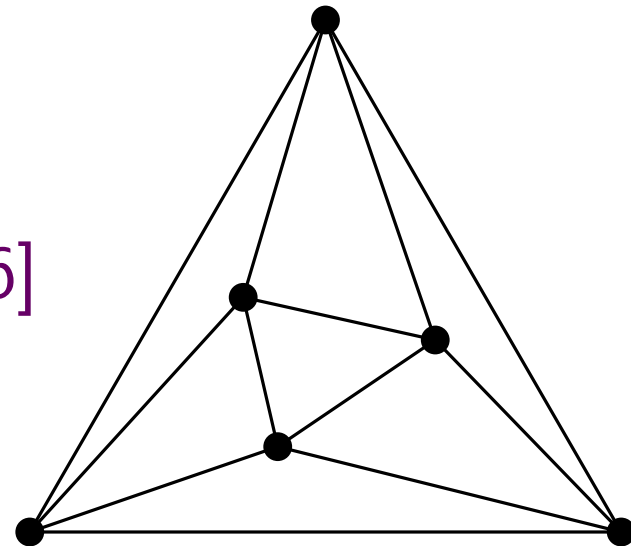
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- ▶ planar 3-trees
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- ▶ octahedron graph [Ringel, 1990]
- ▶ Eulerian triangulations [LK, 2016]



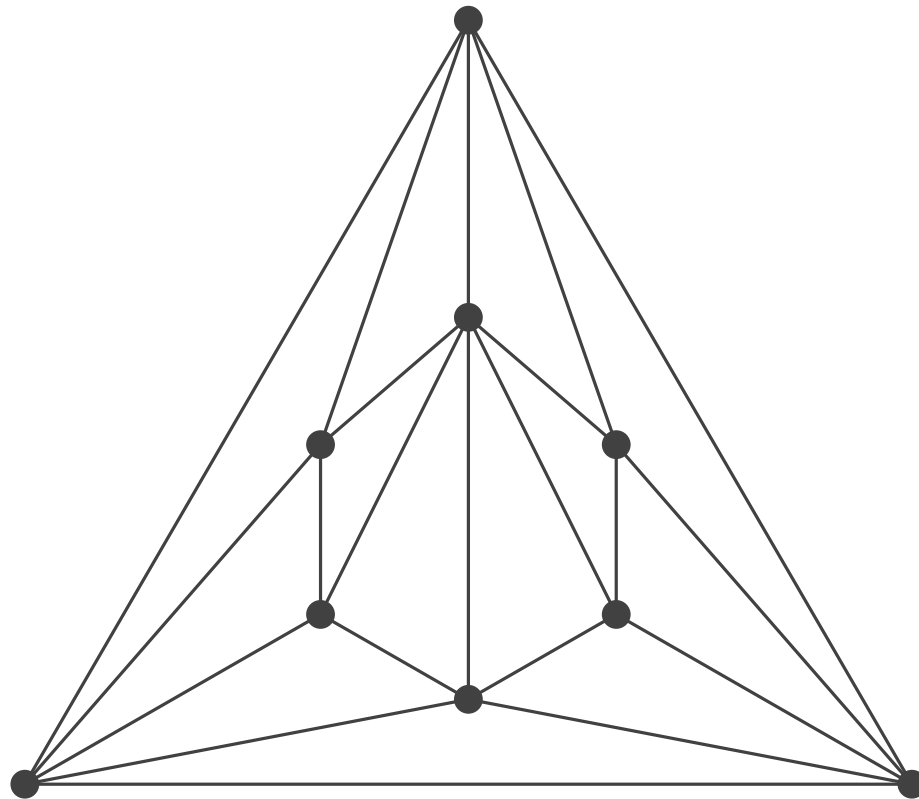
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Win a mug! :)

---

Is this graph area-universal?



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# Today's agenda

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- ▶ computational complexity
- ▶  $\exists\text{IR}$
- ▶  $\forall\exists\text{IR}$

- ▶ problems reducing to ETR



---

# $\exists\mathbb{R}$

---

- ▶ problems reducing to ETR
- ▶ Existential theory of the reals (ETR)

$$\exists x_1, \dots, x_n: \phi(x_1, \dots, x_n)$$

- $\phi$  has symbols:  $0, 1, x_1, \dots, x_n, +, \cdot, =, \leq, <, \wedge, \vee, \neg$
- Example:  $\exists x, y: (x^2 + y = 7) \wedge (y > 5)$

---

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- ▶  $\exists\mathbb{R}$ -hard problems
  - stretchability of pseudoline arrangements
  - recognition of segment intersection graphs

- ▶ problems reducing to **UETR**

- ▶ problems reducing to **UETR**
- ▶ **Universal** Existential theory of the reals (UETR)

$$\forall y_1, \dots, y_k \exists x_1, \dots, x_n: \Phi(x_1, \dots, x_n, y_1, \dots, y_k)$$

-  $\Phi$  has symbols:  $0, 1, x_1, \dots, x_n, +, \cdot, =, \leq, <, \wedge, \vee, \neg$

$$\forall z \exists x \exists y: (x^2 + y^2 > z) \wedge (3x + 2y = 10z)$$

# $\forall\exists\mathbb{R}$

- ▶ problems reducing to **UETR**
- ▶ **Universal** Existential theory of the reals (UETR)

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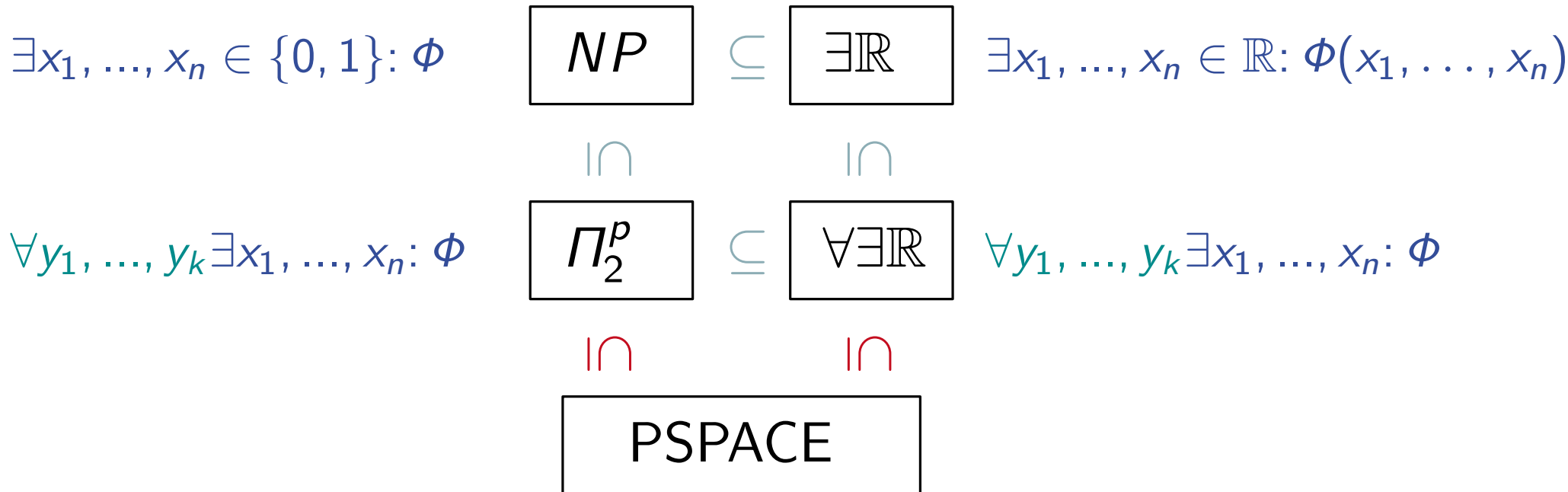
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- ▶ problems that are  $\forall\exists\mathbb{R}$ -hard

Conjecture: **Area-universality is  $\forall\exists\mathbb{R}$ -complete!**

# Polytime Hierachy



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# Our results: Variants are hard

---

▶ area-universality is in  $\forall\exists\mathbb{R}$

▶ Part. Ext. Prescribed Area is  $\exists\mathbb{R}$ -complete.

▶ Area-universality in hypergraphs is  $\forall\exists\mathbb{R}$ -complete.

---

# Part. Ext. Prescribed Area

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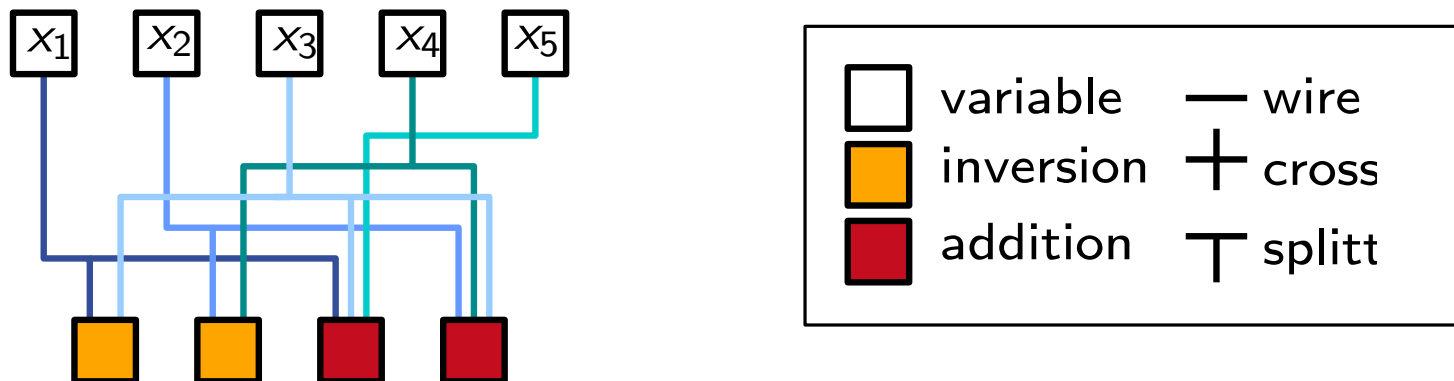
- ▶ **Input:**  $G$  graph,  $\mathcal{A}$  areas,  $V' \subset V$  fixed
- ▶ **Question:** realizable?



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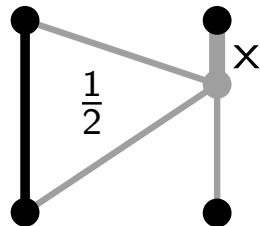
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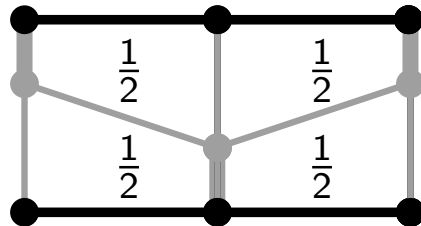
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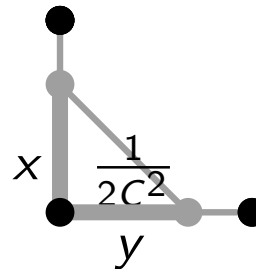
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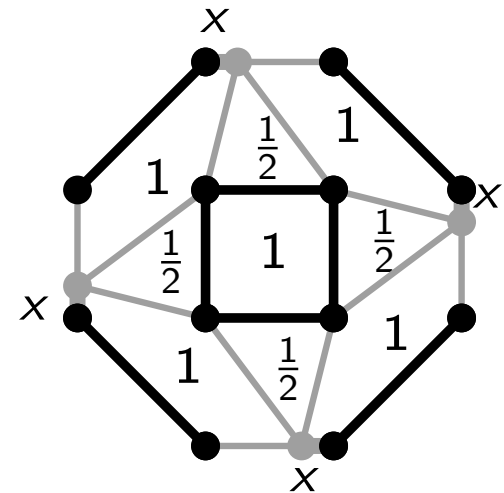
variable



edge



inversion

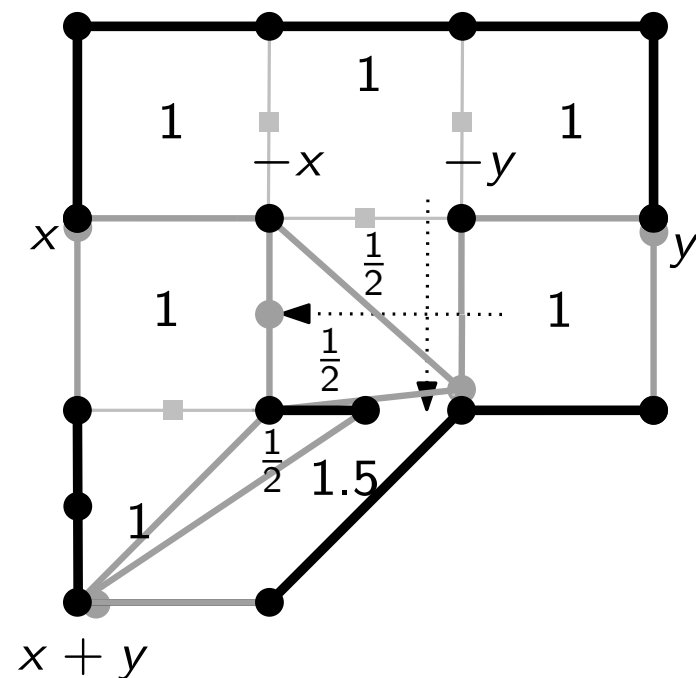
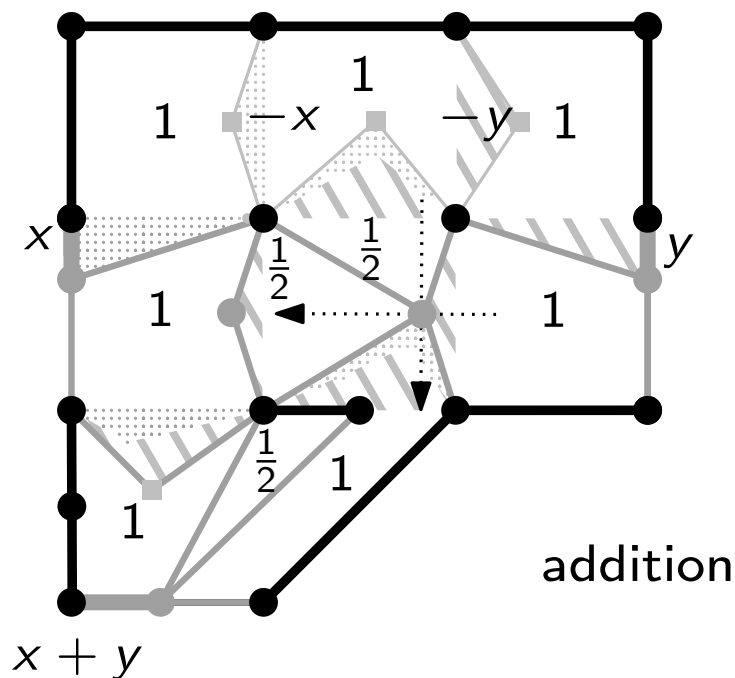


splitter

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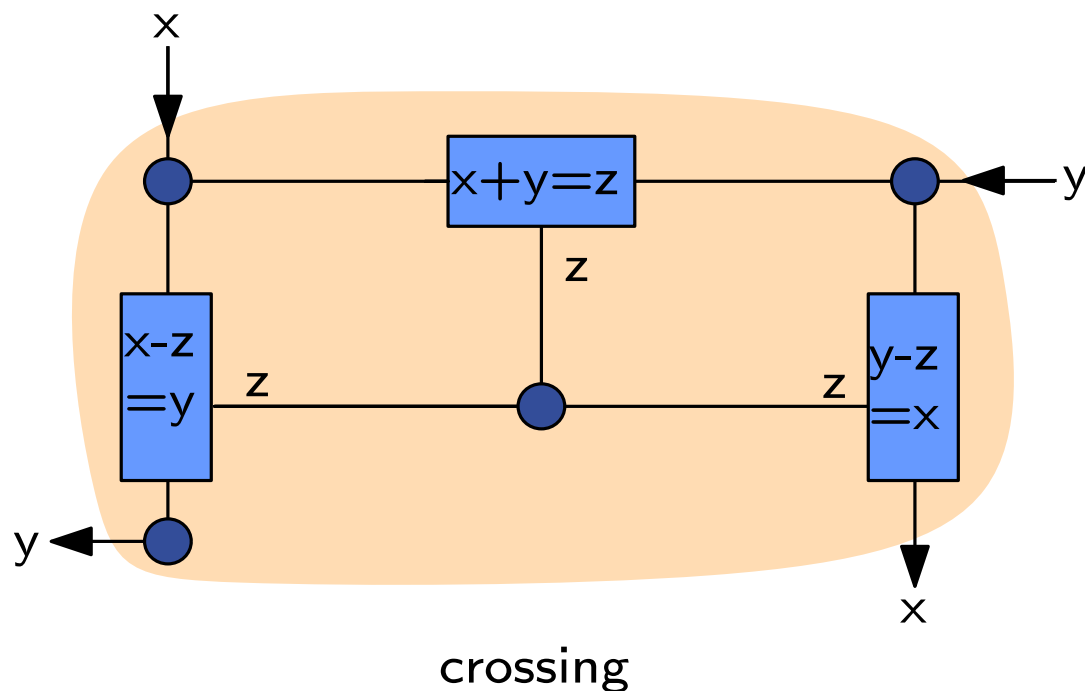
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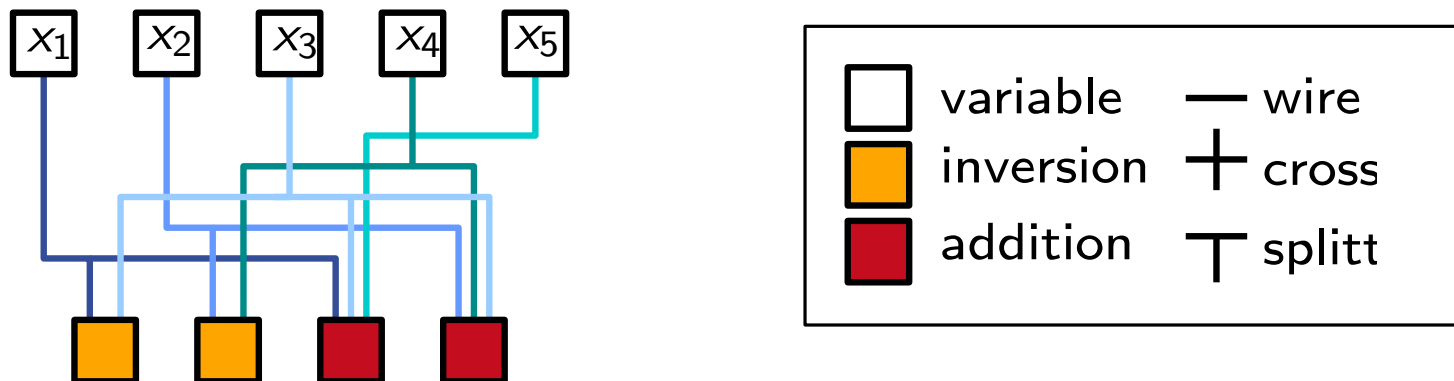
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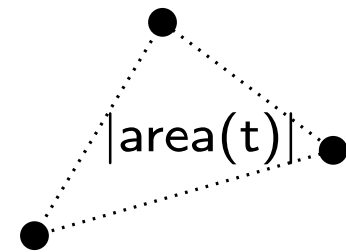


# Area-universality in hypergraphs

Area-universality in hypergraphs is  $(\forall)\exists\mathbb{R}$ -complete.

► **Input:**  $\mathcal{P}$  abstract point set,  $\mathcal{T} \subset \binom{\mathcal{P}}{3}$  set of triples  
 $\mathcal{A}: \mathcal{T} \rightarrow \mathbb{R}^+$  area assignment

► **Question:**  $\exists$  realizing point set?

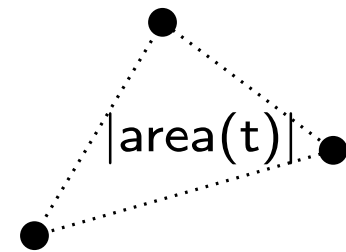
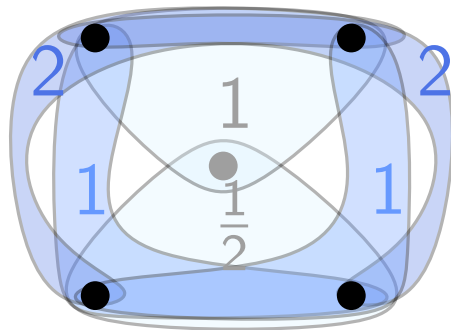


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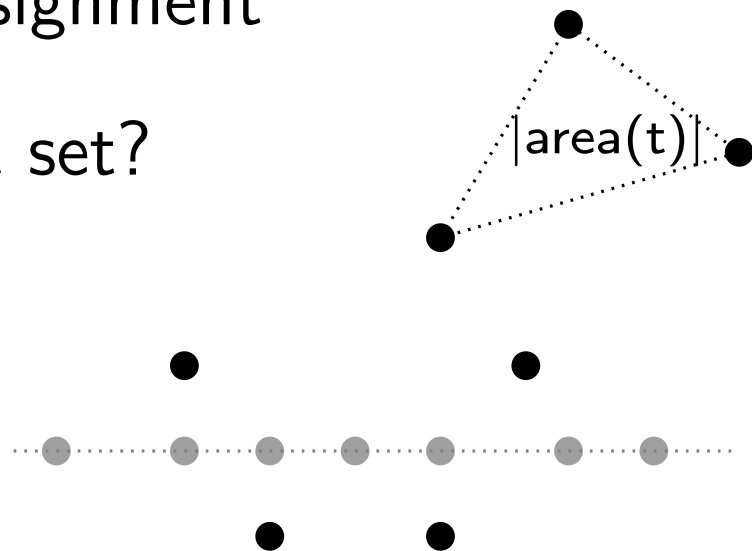
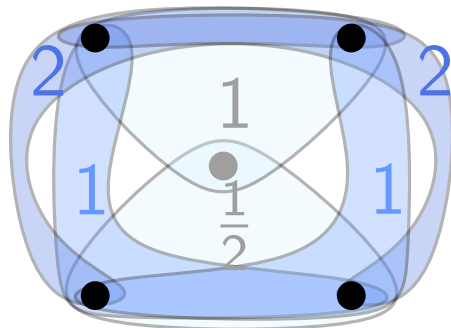


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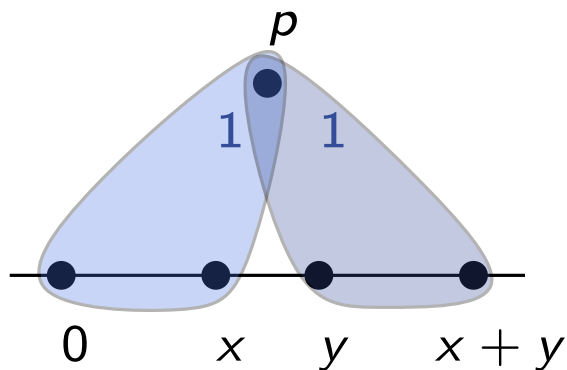
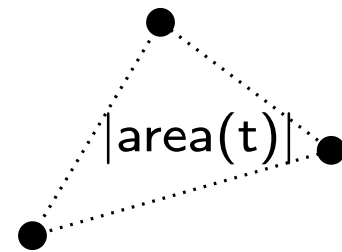


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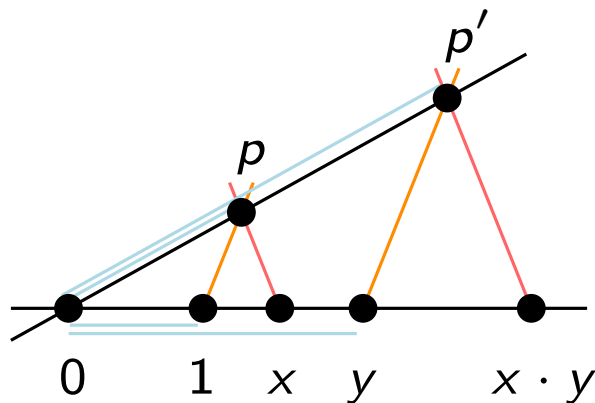
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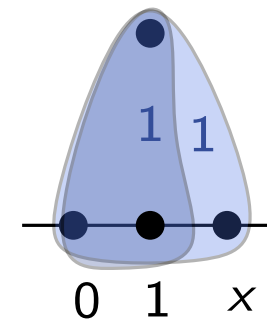
- ▶ **Question:**  $\exists$  realizing point set?



addition



multiplication



checking equality

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# Summary & Questions

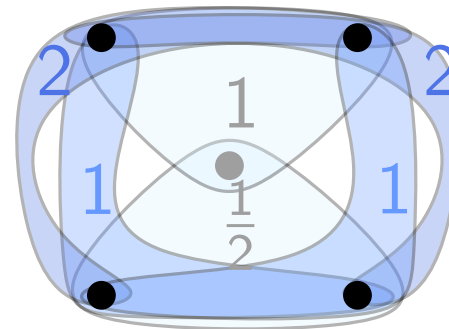
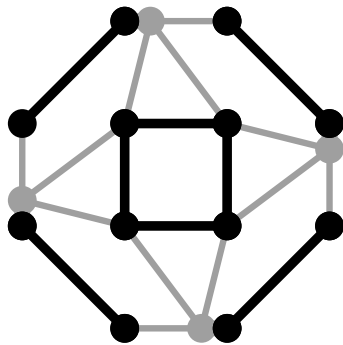
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Conjecture: Area-universality is  $\forall\exists\mathbb{R}$ -complete!

# Summary & Questions

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Results: Some variants are hard



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