BOUNDING THE TRIPARTITE-CIRCLE CROSSING NUMBER OF COMPLETE TRIPARTITE GRAPHS

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Abstract. A tripartite-circle drawing of a tripartite graph is a drawing in the plane, where each part of a vertex partition is placed on one of three disjoint circles, and the edges do not cross the circles. We present upper and lower bounds on the minimum number of crossings in tripartite-circle drawings of $K_{m,n,p}$ and the exact value for $K_{2,2,n}$. In contrast to 1- and 2-circle drawings, which may attain the Harary-Hill bound, our results imply that balanced restricted 3-circle drawings of the complete graph are not optimal.

1. Introduction

The *crossing number* of a graph G, denoted by $\text{cr}(G)$, is the minimum number of edge-crossings over all drawings of G in the plane. A *good drawing* of G is a drawing where no edge crosses itself, two edges that share a vertex do not cross, and two edges with no shared vertices cross each other at most once. Drawings that minimize the number of crossings are good, so we only consider good drawings.

In the study of crossing-minimal drawings of complete and complete bipartite graphs [11, 19], drawings where the vertices are placed on one or two circles, and edges do not cross the circles, are conjectured to be crossing-minimal and are thus of special interest. Such drawings are known as *2-page book embeddings* [5, 12] and *cylindrical drawings* [4, 7, 13, 23], respectively. (For more details, refer to Section 5.)

As a generalization of 1- and 2-circle drawings, in a *k-circle drawing* of a graph G in the plane, the vertices are placed on k disjoint circles, and the edges do not cross the circles. Analogously, the minimum number of crossings in a k-circle drawing of a graph G is the *k-circle crossing number* of G. For the special case when G is a k-partite graph, we further require that the vertices on each circle form an independent set. We call these drawings *k-partite-circle drawings*. We call the minimum number of crossings in a k-partite-circle drawing the *k-partite-circle crossing number* and denote it by $\text{cr}_{\textcircled{k}}(G)$. In this paper, we investigate the tripartite-circle crossing number of complete tripartite graphs.

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Previous Results and Related Work. A 3-circle drawing can also be understood as a *pair of pants drawing* (an instance of the *map crossing number* [21]), where two circles are enclosed by the third. However, even though a *radial drawing* [9] of two concentric circles is equivalent to a 2-circle drawing (cylindrical drawing), this is not the case for k-circle drawings, $k \geq 3$, since the three or more concentric circles would require that any edges from the outermost to the innermost circle would necessarily cross the middle circle(s).

The bipartite-circle crossing number of complete bipartite graphs, also known as the *bipartite cylindrical crossing number*, is fully understood. In 1997, Richter and Thomassen [23] settled the balanced case by showing that

$$\operatorname{cr}_{②}(K_{n,n}) = n\binom{n}{3}. \tag{1.1}$$

Ábrego, Fernández-Merchant, and Sparks [7] generalized this result to all complete bipartite graphs. For $m \leq n$, the bipartite-circle crossing number is

$$\begin{aligned} \operatorname{cr}_{②}(K_{n,m}) = \binom{n}{2}\binom{m}{2} + &\sum_{1\leq i<j\leq m}\left(\left\lfloor \frac{n}{m}(j-1)\right\rfloor - \left\lfloor \frac{n}{m}(i-1)\right\rfloor\right)^2 \\ -n&\sum_{1\leq i<j\leq m}\left(\left\lfloor \frac{n}{m}(j-1)\right\rfloor - \left\lfloor \frac{n}{m}(i-1)\right\rfloor\right). \end{aligned} \tag{1.2}$$

In particular, if m divides n, then $\operatorname{cr}_{②}(K_{n,m}) = \frac{1}{12}n(m-1)(2mn-3m-n)$.

For the general crossing number, Gethner et al. [14] proved an upper bound for complete tripartite graphs; the bound on $\operatorname{cr}(K_{m,n,p})$ is analogous to the Zarankiewicz Conjecture. Additionally, they provided lower bounds on $\operatorname{cr}(K_{n,n,n})$ and on the *rectilinear crossing number*, $\overline{\operatorname{cr}}(K_{n,n,n})$, in which edges are represented by straight line segments. Asano [8] determined the crossing numbers of $K_{1,3,n}$ and $K_{2,3,n}$, and Ginn and Miller [15] gave bounds on $\operatorname{cr}(K_{3,3,n})$. For other crossing number results and equivalent terminology, see [24].

Our results. We prove several bounds on the tripartite-circle crossing number of complete tripartite graphs.

Theorem 1.1. *Let m, n, and p be natural numbers and $\mathfrak{t} := \{(m,n,p),(n,p,m),(p,m,n)\}$. Then the following bounds hold:*

$$\sum_{(a,b,c)\in\mathfrak{t}}\left(\operatorname{cr}_{②}(K_{a,b}) + ab\left\lfloor\frac{c-1}{2}\right\rfloor\left\lfloor\frac{c}{2}\right\rfloor\right) \leq \operatorname{cr}_{③}(K_{m,n,p}) \leq \sum_{(a,b,c)\in\mathfrak{t}}\left(\binom{a}{2}\binom{b}{2} + ab\left\lfloor\frac{c-1}{2}\right\rfloor\left\lfloor\frac{c}{2}\right\rfloor\right).$$

For $m,n,p \geq 3$ we improve the lower bound by 2 in Corollary 3.5. Using Equation (1.1) and Corollary 3.5, Theorem 1.1 simplifies as follows for the balanced case. Note that the lower bound of order $\sim \frac{5}{4}n^4$ and the upper bound of order $\sim \frac{6}{4}n^4$ are fairly close.

Corollary 1.2. *For any integer $n \geq 3$,*

$$3n\binom{n}{3} + 3n^2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2 \leq \operatorname{cr}_{③}(K_{n,n,n}) \leq 3\binom{n}{2}^2 + 3n^2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor.$$

Finally, we present the exact tripartite-circle crossing number of $K_{2,2,n}$.

Theorem 1.3. *For every integer $n \geq 3$,*

$$\mathrm{cr}_{③}(K_{2,2,n}) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2n - 3$$

In fact, k-partite-circle drawings of complete k-partite graphs easily give rise to drawings of complete graphs, by adding straight-line segments between each pair of vertices on the same circle. If the numbers of vertices on the circles are as close to equal as possible, these drawings are called *balanced restricted k-circle drawings* of K_n. The minimum number of crossings in such a drawing is denoted by $\mathrm{bcr}_{ⓚ}(K_n)$. Indeed, crossing-minimal 1- and 2-circle drawings yield drawings of the complete graph which are conjectured to be optimal [4, 5, 11, 12, 19], i.e., they achieve the Harary-Hill bound

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

Our results imply that this phenomenon does not generalize to 3-circle drawings.

Corollary 1.4. *For $n = 9, 10$ and $n \geq 13$, the number of crossings in any balanced restricted 3-circle drawing of K_n exceeds $H(n)$, i.e., $\mathrm{bcr}_{③}(K_n) > H(n)$.*

However, for $n \leq 7$, the Harary-Hill drawings of K_n coincide with our 3-circle drawings of K_n. For $8 \leq n \leq 11$, there exist crossing-minimal unbalanced restricted 3-circle drawings of K_n.

Organization. The remainder of our paper is organized as follows: In Section 2, we introduce tools to count the number of the crossings, which we then use in Section 3 to prove Theorem 1.1 and Corollary 1.2. In Section 4, we determine the tripartite-circle crossing number of $K_{2,2,n}$ by proving Theorem 1.3. We discuss the connection to the Harary-Hill conjecture in Section 5 and conclude with a list of open problems in Section 6.

2. Tools for Counting the Number of Crossings

In a tripartite-circle drawing of $K_{m,n,p}$, we label the three circles by M, N, and P, and their numbers of vertices are m, n, and p, respectively. Consider Figure 1. Note that this drawing can be transformed by a projective transformation of the

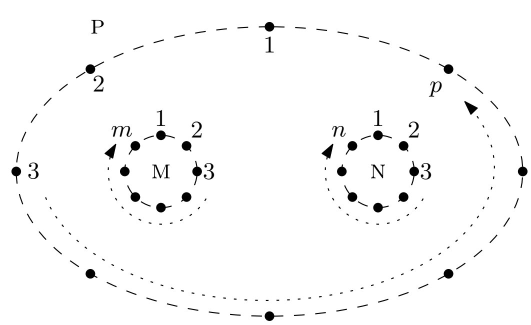
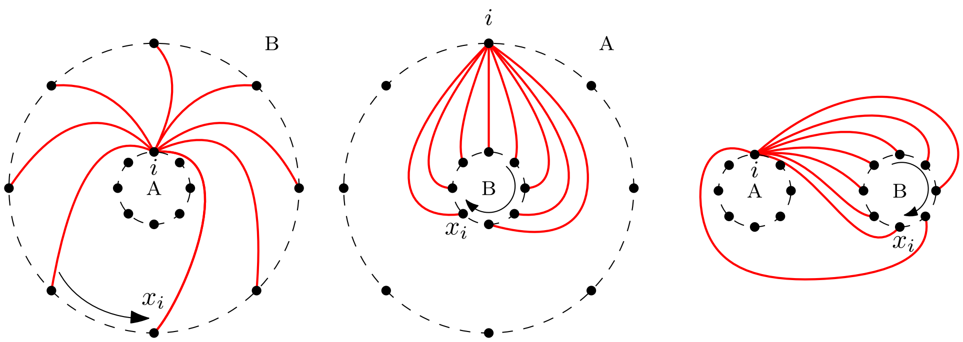


FIGURE 1. The vertices on the circles M and N are labeled clockwise; the vertices of the circle P are labeled counterclockwise.

plane such that any one circle encloses the other two. Therefore, without loss of generality we consider drawings where the *outer* circle P contains the *inner* circles M and N. In such a drawing, we label the vertices on circles M and N in clockwise order and the vertices on circle P in counterclockwise order. Likewise, we read arcs of circles in clockwise order for inner circles and in counterclockwise order for outer circles.

2.1. **Defining the x-Labels.** For simplicity and without loss of generality, both papers [7] and [23] considered good bipartite-circle drawings of the complete bipartite graph where the two circles are assumed to be nested. Their results rely on the assignment of a vertex $x_i(\text{A},\text{B})$ on the outer circle B for each vertex i on the inner circle A. Because we are dealing with three circles and a pair of them is not necessarily nested, we adapt this definition as follows.

Let i be a vertex on circle A. The star formed by all edges from i to B together with circle B partition the plane into several disjoint regions, as shown in Figure 2.



(a) Nested: A inside B (b) Nested: B inside A (c) Non-Nested: A beside B

FIGURE 2. Definition of vertex $x_i(\text{A},\text{B})$.

Exactly one of these regions contains circle A. Such a region is enclosed by two edges from i to B and an arc on B between two consecutive vertices. We define the second of these vertices (in clockwise or counterclockwise order depending on whether B is an inner or outer circle, respectively) as $x_i(\text{A},\text{B})$. If the two circles are clear from the context, we may also write x_i.

Abrego et al. [7] observed that the x-labels are weakly ordered and suffice to describe the drawing up to isomorphism. Because we number the vertices on the outer circle in counterclockwise order (opposite to how it is done in [7]), the ordering of x-labels on the circle is reversed when compared to Lemma 1.4 from [7]. In particular, our weak ordering stated below is achieved, following the proof from [7], by possibly renumbering inner vertices so that $x_1(\text{A},\text{B}) = n$.

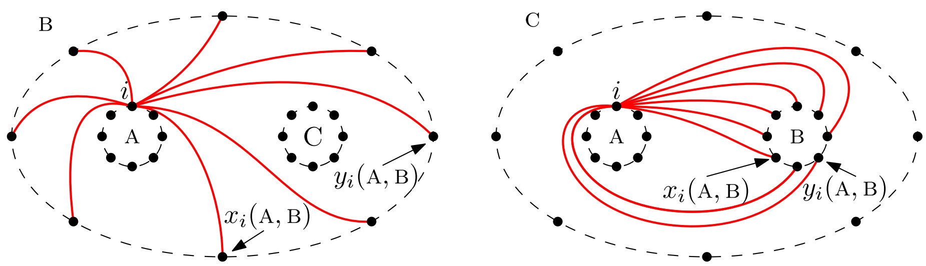
Lemma 2.1 (Lemma 1.4, [7])**.** *Consider a good bipartite-circle drawing of $K_{a,b}$ where the circles* A *and* B *have a and b vertices, respectively and the vertices are labeled so that $x_1 = n$. Then it holds that*

$$x_1 \geq x_2 \geq \cdots \geq x_a.$$

Moreover, for a given sequence $(s_i)_i$ with $s_1 \geq s_2 \geq \cdots \geq s_a$, up to isomorphism, there is a unique good bipartite-circle drawing of $K_{a,b}$ with $x_i = s_i$ for all $i \in \text{A}$.

2.2. **Defining the y-Labels.** As observed in Lemma 2.1, the x-labels are sufficient to describe a bipartite-circle drawing. We now aim to describe tripartite-circle drawings. We therefore introduce a new vertex assignment, $y_i(\text{A},\text{B})$ that depends on all three circles.

Let A, B, and C be the three circles and i be a vertex on A. The star formed by all edges from i to B together with circle B partition the plane into disjoint regions. Exactly one of these regions contains the third circle C. This region is enclosed by two edges incident to i and the arc between two consecutive vertices on B. We define the second of these two vertices (in clockwise or counterclockwise order depending on whether B is an inner or outer circle, respectively) as $y_i(\text{A},\text{B})$. See Figure 3.



(a) Nested: A is inside B (b) Non-Nested: A is beside B

FIGURE 3. The definition of $y_i(\text{A},\text{B})$ illustrated for the cases when (a) A is inside B and (b) when A is besides B.

2.3. **Counting Crossings Using x- and y-labels.** If two edges ab and cd cross, then at least two (nonadjacent) vertices in $\{a, b, c, d\}$ are on the same circle. Hence, there are six total types of crossings between edges ab and cd:

MP/MP-crossings: a and c lie on M, and b and d lie on P;
NP/NP-crossings: a and c lie on N, and b and d lie on P;
MN/MN-crossings: a and c lie on M, and b and d lie on N;
MN/MP-crossings: a and c lie on M, b lies on N, and d lies on P;
MN/NP-crossings: a and c lie on N, b lies on M, and d lies on P;
MP/NP-crossings: a and c lie on P, b lies on M, and d lies on N.

We typically color the edges between each pair of circles with the same color, using three different colors for the different pairs. The first three types of crossings only involve two circles and so we call them *monochromatic* crossings. The last three types involve all three circles and so edges of different colors. So we call them *bichromatic* crossings. We use the x- and y-labels to count the monochromatic and bichromatic crossings, respectively. The following definitions are used throughout the rest of the paper.

For vertices k and ℓ on a circle with n vertices numbered $1, \ldots, n$ clockwise (respectively, counterclockwise), let

$$d_n(k, \ell) := \ell - k \mod n$$

denote the distance from k to ℓ in clockwise (respectively, counterclockwise) order on the circle. Let $[n] := \{1, 2, \ldots, n\}$. For any $u, v \in [n]$, define

$$f_n(u, v) := \binom{d_n(u, v)}{2} + \binom{n - d_n(u, v)}{2}.$$

For vertices i and j on the inner (respectively, outer) circle A, we use $[i, j]$ to denote the arc of A read clockwise (respectively, counterclockwise) from i to j. We include i and j in the interval $[i, j]$, whereas (i, j) does not include i and j. We similarly define $[i, j)$ and $(i, j]$.

2.3.1. *Counting Crossings Involving Two Circles.* We start by stating the following result from [23] to take care of the monochromatic crossings.

Lemma 2.2 ([23])**.** *The number of crossings in a good bipartite-circle drawing of the complete bipartite graph $K_{m,n}$ is*

$$\sum_{1\leq i<j\leq m} f_n(x_i, x_j).$$

2.3.2. *Counting Crossings Involving Three Circles.* The following lemma introduces a means of counting all three types of bichromatic crossings using the y-labels. See Figure 4 for a visual representation of a possible MP/NP-crossing.

Lemma 2.3. *Let* A*,* B*, and* C *be three disjoint circles with the disjoint vertex sets $\{1, \dots, a\}$, $\{1, \dots, b\}$, and $\{1, \dots, c\}$, respectively. Then the number of* AC/BC*-crossings is given by*

$$\sum_{\substack{1\leq i\leq a\\ 1\leq j\leq b}} f_c\big(y_i(\text{A,C}), y_j(\text{B,C})\big).$$

Proof. Fix a vertex i on A and a vertex j on B and consider the corresponding vertices $y_i(\text{A,C})$ and $y_j(\text{B,C})$ on circle C, see Figure 4. For every pair of vertices k and ℓ both in the interval $[y_i(\text{A,C}) + 1, y_j(\text{B,C})] =: I_1$ on C there is exactly one crossing among edges ik and $i\ell$, and jk and $j\ell$. Similarly, there is exactly one crossing among the edges ik and $i\ell$, and edges jk and $j\ell$ when k and ℓ are in $[y_j(\text{B,C}) + 1, y_i(\text{A,C})] =: I_2$. Moreover note that if a vertex k is in I_1 and a vertex ℓ is in I_2 then there are no crossings among edges ik, $i\ell$, jk and $j\ell$. Consequently, there are exactly $f_c\big(y_i(\text{A,C}), y_j(\text{B,C})\big)$ crossings among edges incident with vertices i and j. Therefore the total number of AC/BC-crossing is as claimed. □

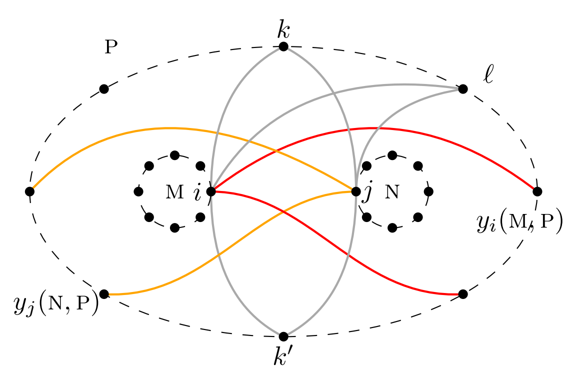


FIGURE 4. Illustration for the case that $(\text{A,B,C}) = (\text{M,N,P})$. Since k' and ℓ are in different intervals, edges of the vertices i, j, k', and ℓ do not cross, but edges of the vertices i, j, k and ℓ do cross since k and ℓ are in the same interval I_2.

2.3.3. *Total Crossing Count.* The number of crossings in a good tripartite-circle drawing of the complete tripartite graph $K_{m,n,p}$ can be found by counting the crossings in the three different, good bipartite-circle drawings of $K_{m,n}$, $K_{m,p}$ and $K_{n,p}$, along with crossings involving all three circles. We count all these crossings in the following theorem.

A *cyclic assignment* of (A,B,C) to (M,N,P) is one triple of the following set $\mathfrak{t} := \{(\text{M,N,P}), (\text{N,P,M}), (\text{P,M,N})\}$, with the number of vertices on the circles A, B, and C denoted by a, b, and c, respectively.

Theorem 2.4. *The number of crossings in a good tripartite-circle drawing of $K_{m,n,p}$ is given by*

$$\sum_{(\text{A,B,C})\in\mathfrak{t}} \left(\sum_{\substack{i<j \\ i,j\in \text{A}}} f_b\big(x_i(\text{A,B}), x_j(\text{A,B})\big) + \sum_{\substack{i\in \text{A} \\ j\in \text{B}}} f_c\big(y_i(\text{A,C}), y_j(\text{B,C})\big) \right).$$

Proof. The monochromatic crossings are counted by the first expressions in the brackets using Lemma 2.2. The second expression corresponds to the bichromatic crossings using Lemma 2.3. □

3. Bounding the tripartite-circle crossing number—Proofs of Theorem 1.1 and Corollary 1.2

In this section, we prove the upper and lower bounds of Theorem 1.1 and Corollary 1.2. We start with the lower bounds and then proceed with the upper bounds.

3.1. **Lower bounds.** To prove the lower bounds, we start with two lemmas.

Lemma 3.1. *The function $f_n(a,b)$ attains its minimum M over the integers if and only if $|a-b| \in \{\lfloor n/2\rfloor, \lceil n/2\rceil\}$. Among pairs (a,b) such that $|a-b| \notin \{\lfloor n/2\rfloor, \lceil n/2\rceil\}$, the minimum of f_n exceeds M by 1 if n is even and by 2 if n is odd.*

Proof. First note that $d_n(a,b) = \lfloor n/2\rfloor$ if and only if $|a-b| \in \{\lfloor n/2\rfloor, \lceil n/2\rceil\}$ because $n = \lfloor n/2\rfloor + \lceil n/2\rceil$.

Consider the auxiliary function $g_n(x) = \binom{x}{2} + \binom{n-x}{2} = (x-\frac{n}{2})^2 + \frac{n^2-2n}{4}$. It is a quadratic function minimized at $\frac{n}{2}$, symmetric about $\frac{n}{2}$, decreasing for $x < \frac{n}{2}$, and increasing for $x > \frac{n}{2}$. Consequently, the minimum value of g_n among integers is attained at $x \in \{\lfloor n/2\rfloor, \lceil n/2\rceil\}$; the next smallest value of g_n among integers is attained at $x \in \{\lfloor n/2\rfloor - 1, \lceil n/2\rceil + 1\}$. A computation shows that

$$g_n(\lfloor n/2\rfloor - 1) - g_n(\lfloor n/2\rfloor) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

and hence, the claim holds. □

For every cyclic assignment of (A,B,C) to (M,N,P), we denote the minimum number of AB/AC-crossings among all good tripartite-circle drawings of $K_{m,n,p}$ by $\mathrm{cr}^{\text{A}}_{③}(K_{m,n,p})$.

Lemma 3.2. *For every cyclic assignment of* (A,B,C) *to* (M,N,P)*, it holds that*

$$\mathrm{cr}^{\text{C}}_{③}(K_{m,n,p}) \geq ab \left\lfloor \frac{c}{2} \right\rfloor \left\lfloor \frac{c-1}{2} \right\rfloor.$$

Proof. From the count in Lemma 2.3, and its minimization in Lemma 3.1, we have the lower bound

$$\sum_{\substack{1\le i\le a\\1\le j\le b}} f_c\big(y_i(\text{A,C}), y_j(\text{B,C})\big) \ge \sum_{\substack{1\le i\le a\\1\le j\le b}} \binom{\lfloor c/2\rfloor}{2} + \binom{\lceil c/2\rceil}{2} = ab\left[\binom{\lfloor c/2\rfloor}{2} + \binom{\lceil c/2\rceil}{2}\right]$$

Adding the last two terms directly yields the claim. □

This allows us to gain a lower bound on $\text{cr}_{③}(K_{m,n,p})$.

Proof of the lower bound of Theorem 1.1. In Theorem 2.4 we count the total number of crossings in a good tripartite-circle drawing of $K_{m,n,p}$.

$$\begin{aligned}\text{cr}_{③}(K_{m,n,p}) \ge\ & \text{cr}_{②}(K_{m,n}) + \text{cr}_{②}(K_{m,p}) + \text{cr}_{②}(K_{n,p})\\ &+ \text{cr}^{\text{M}}_{③}(K_{m,n,p}) + \text{cr}^{\text{N}}_{③}(K_{m,n,p}) + \text{cr}^{\text{P}}_{③}(K_{m,n,p})\end{aligned}$$

Lemma 3.2 gives a lower bound on the last three summands and directly implies the claimed lower bound:

$$\text{cr}_{③}(K_{m,n,p}) \ge \sum_{(\text{A,B,C})\in\mathcal{T}} \left(\text{cr}_{②}(K_{a,b}) + \ ab\left\lfloor\frac{c}{2}\right\rfloor\left\lfloor\frac{c-1}{2}\right\rfloor\right).$$

This finishes the proof of the lemma. □

3.1.1. *Improving the lower bound.* With the help of the following lemmas, the lower bound can be improved by 2.

Lemma 3.3 (Special Inversion Lemma)**.** *Fix the placement of two circles* A *and* B *inside circle* C*. For a tripartite-circle drawing D, $x_i(\text{A,C}) = y_i(\text{A,C})$ for all i on* A *if and only if $y_j(\text{C,A}) = y_k(\text{C,A})$ for all j, k on* C*.*

Proof. Let D^* be the restriction of D to edges between A and C. We refer to the connected components of the complement of D^* as faces.

We start with an observation. By the definitions of $x_i(\text{A,C})$ and $y_i(\text{A,C})$, we have $x_i(\text{A,C}) = y_i(\text{A,C})$ if and only if the pair of incident edges $\{i, x_i(\text{A,C})\}$ and $\{i, x_i(\text{A,C}) + 1\}$ divides the interior of C into two parts where A and B lie in the same part and all other edges from i lie in the other part. This holds for all i if and only if no edge of D^* separates A from B; in other words, the circle B lies in a face F of D^* adjacent to A. Figure 5 shows an example with $x_i(\text{A,C}) = y_i(\text{A,C})$ for all i and with the faces adjacent to A shaded in gray. Circle B lies in one of these faces, say F. This finishes our observation.

Now we prove the lemma. Suppose $x_i(\text{A,C}) = y_i(\text{A,C})$ for all i on A. By the above observation, the circle B lies in a face F of D^* adjacent to A. Let q and $q+1$ be the two vertices on A and on the boundary of F, where $q+1$ comes clockwise directly after q. For each j on C, the triangle $T_j = (j, q, q+1)$ cannot cross F, since it is a face, and shares the side $\{q, q+1\}$ with F. By the properties of good drawings, either F is in the interior of T_j and all other edges from j are on the exterior, or F is on the exterior of T_j and all other edges from j are in the interior of T_j. In either case, $y_j(\text{C,A}) = q$ for all j on C.

For the converse, suppose $y_j(\text{C,A}) = y_k(\text{C,A})$ for all j, k on C. Then some segment of A, say $\{q, q+1\}$, appears in the smallest (by containment) triangle T containing B from each j on C. By the minimality of T, no edge from j in D^* can cross T. The intersection of these triangles is a face F of D^* containing B and adjacent to A at $\{q, q+1\}$. By the observation, $x_i(\text{A,C}) = y_i(\text{A,C})$ for all i on A. □

For vertices x and y on a circle with n vertices, we define

$$\min d_n(x,y) := \min\{d_n(x,y), d_n(y,x)\}.$$

The following lemma analyzes the situation when a term counting bichromatic crossings is minimized.

Lemma 3.4. *In a tripartite-circle drawing with circles* A, B, *and* C *with c vertices on* C, *if the function $f_c(y_i(\text{A,C}), y_j(\text{B,C}))$ attains its minimum for all $i \in \text{A}$ and all $j \in \text{B}$, then there are vertices $u_\text{A}, u_\text{B} \in \text{C}$ with the following properties:*

(i) *For both circles $\text{D} \in \{\text{A,B}\}$, $y_i(\text{D,C}) \in \{u_\text{D}, u_\text{D}+1\}$ for all $i \in \text{D}$.*
(ii) *For some circle $\text{D} \in \{\text{A,B}\}$, $y_i(\text{D,C}) = u_\text{D}$ for all $i \in \text{D}$. If c is even, then for both circles $\text{D} \in \{\text{A,B}\}$, $y_i(\text{D,C}) = u_\text{D}$ for all $i \in \text{D}$.*

Proof. Suppose that $f_c(y_i(\text{A,C}), y_j(\text{B,C}))$ attains its minimum value for every $i \in \text{A}$ and $j \in \text{B}$. By Lemma 3.1, it holds that

$$\left|y_i(\text{A,C}) - y_j(\text{B,C})\right| \in \left\{\lfloor c/2 \rfloor, \lceil c/2 \rceil\right\} \quad \text{and} \quad \min d_c\big(y_i(\text{A,C}), y_j(\text{B,C})\big) = \lfloor c/2 \rfloor.$$

First we prove property (i). Without loss of generality, we assume that $\text{D} = \text{A}$. Let $v := y_j(\text{B, C})$ for some $j \in \text{B}$. The vertex v on C can have only one vertex at distance $\lfloor c/2 \rfloor = c/2$ if c is even and only two vertices at distance $\lfloor c/2 \rfloor$, adjacent to one another, if c is odd. Since $\min d_c(y_i(\text{A,C}), v) = \lfloor c/2 \rfloor$ for all $i \in \text{A}$, we get that the $y_i(\text{A,C})$ labels are all the same in the even case, or all on one of two adjacent vertices in the odd case.

Next we prove property (ii). Suppose for contradiction that $y_i(\text{A,C}) = u_\text{A}$, $y_j(\text{A,C}) = u_\text{A} + 1$, $y_k(\text{B,C}) = u_\text{B}$, and $y_\ell(\text{B,C}) = u_\text{B} + 1$ for some $i, j \in \text{A}$ and $k, \ell \in \text{B}$. Since $\min d_c(y_i(\text{A,C}), y_j(\text{B,C})) = \lfloor c/2 \rfloor$ for every $i \in \text{A}$ and $j \in \text{B}$, we get that $\min d_c(u_\text{A}, u_\text{B}) = \min d_c(u_\text{A}, u_\text{B}+1) = \lfloor c/2 \rfloor$. No vertex on C other than u_A is at distance $\lfloor c/2 \rfloor$ from both u_B and $u_\text{B}+1$, but $u_\text{A}+1 \neq u_\text{A}$ must also be at distance $\lfloor c/2 \rfloor$ from both. □

Now, we use the gained insights in order to improve the lower bound of Theorem 1.1.

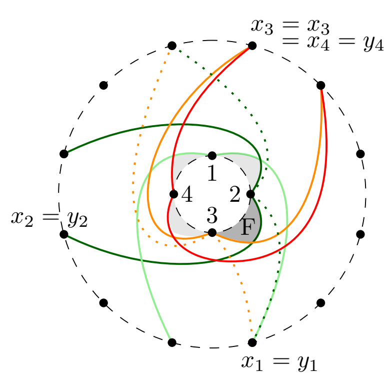


FIGURE 5. Depiction of special y-inversion.

Corollary 3.5 (Improvement of lower bound of Theorem 1.1)**.** *For any integers* $m, n, p \geq 3$*, let* $\mathfrak{t} := \{(m,n,p),(n,p,m),(p,m,n)\}$*. Then*

$$\mathrm{cr}_{③}(K_{m,n,p}) \geq \sum_{(a,b,c)\in\mathfrak{t}} \left(\mathrm{cr}_{②}(K_{a,b}) + \ ab \left\lfloor \frac{c}{2} \right\rfloor \left\lfloor \frac{c-1}{2} \right\rfloor \right) + 2.$$

Proof. Recall that the lower bound in Theorem 1.1 was obtained by simultaneously minimizing all six terms in the formula of Theorem 2.4.

Suppose a bichromatic crossing count $\sum_{\substack{i\in\mathrm{A}\\ j\in\mathrm{B}}} f_c(y_i(\mathrm{A,C}), y_j(\mathrm{B,C}))$ attains its minimum. Lemma 3.4 (ii) implies without loss of generality that all $y_i(\mathrm{A,C})$ labels are equal. Lemma 3.3 then implies that $x_j(\mathrm{C,A}) = y_j(\mathrm{C,A})$ for all j on C (and $x_j(\mathrm{C,B}) = y_j(\mathrm{C,B})$ for all $j \in \mathrm{C}$ if c is even). To achieve the minimum number of monochromatic crossings between A and C, the $x_j(\mathrm{C,A})$ labels must be equally spaced around A as already observed in [23]. Then, since $y_j(\mathrm{C,A}) = x_j(\mathrm{C,A})$, the $y_j(\mathrm{C,A})$ labels are also equally spaced. Since $a \geq 3$, the $y_j(\mathrm{C,A})$ labels are on more than two points. By Lemma 3.4, the term $\sum_{\substack{i\in\mathrm{B}\\ j\in\mathrm{C}}} f_a(y_i(\mathrm{B,A}), y_j(\mathrm{C,A}))$ does not attain its minimum.

For c even, Lemma 3.4 (ii) further implies that all the $y_i(\mathrm{B,C})$ labels are equal. By Lemma 3.3, $x_j(\mathrm{C,B}) = y_j(\mathrm{C,B})$ for all $j \in \mathrm{C}$. If the minimum number of monochromatic crossings between B and C is achieved, then the bichromatic crossings term $\sum_{\substack{i\in\mathrm{A}\\ j\in\mathrm{C}}} f_b(y_i(\mathrm{A,B}), y_j(\mathrm{C,B}))$ also does not attain its minimum.

By Lemma 3.1, if c is odd then at least one of the six terms is at least 2 more than its minimum. If c is even then at least two of the six terms are at least 1 more than their minima. Regardless of the parity of c, the lower bound given by minimizing all six terms simultaneously can be improved by 2. □

3.2. **Upper bounds.** In this subsection, we provide drawings that settle the upper bounds of Theorem 1.1 and Corollary 1.2. We define the drawing within a small stripe around the equator of the sphere and visualize it by a rectangle where the left and right boundaries are identified. Consider Figure 6 for an illustration. In contrast to before, in the following drawings the three circles appear in cyclic order.

Definition (Linear Description)**.** We start by defining the subdrawing D' induced by the vertices of two distinct circles A and B. Let A be a circle with a vertices and B a disjoint circle with b vertices. We assume that A is strictly left of B. The vertices are placed on the circles such that $\lceil \frac{a}{2} \rceil$ and $\lceil \frac{b}{2} \rceil$ vertices lie in the closed top half of A and B, respectively; the vertices on A are labeled clockwise by $\{1, \ldots, a\}$ starting with the clockwise first vertex in the closed top half, while the vertices on B are labeled counterclockwise by $\{1, \ldots, b\}$ starting with the counterclockwise first vertex in the closed top half. Let ℓ_1 and ℓ_2 be two vertical lines separating A and B where ℓ_1 is strictly left of ℓ_2.

(i) On ℓ_1 we mark $a \cdot b$ points, which are labelled by $a_{i,j}$ for $i \in [a]$ and $j \in [b]$ such that the indices increase lexicographically from top to bottom. Each $a_{i,j}$ belongs to an edge of vertex i on A; between vertex i and $a_{i,j}$ the edge is realized by some x- and y-monotone curve $e^1_{i,j}$. Moreover, no two curves $e^1_{i,j}$ intersect.

(ii) On ℓ_2 we mark $a \cdot b$ points, which are labelled by $b_{i,j}$ for $i \in [b]$ and $j \in [a]$ such that the indices increase lexicographically from top to bottom. Each $b_{i,j}$ belongs to an edge of vertex i on B; the edge between these two points is realized by some x- and y-monotone curve $e^2_{i,j}$. Moreover, no two curves $e^2_{i,j}$ intersect.

(iii) Between ℓ_1 and ℓ_2, we connect $a_{i,j}$ and $b_{j,i}$ by a straight-line segment.

The drawing D is obtained by constructing a drawing D' for each pair of circles and overlaying them.

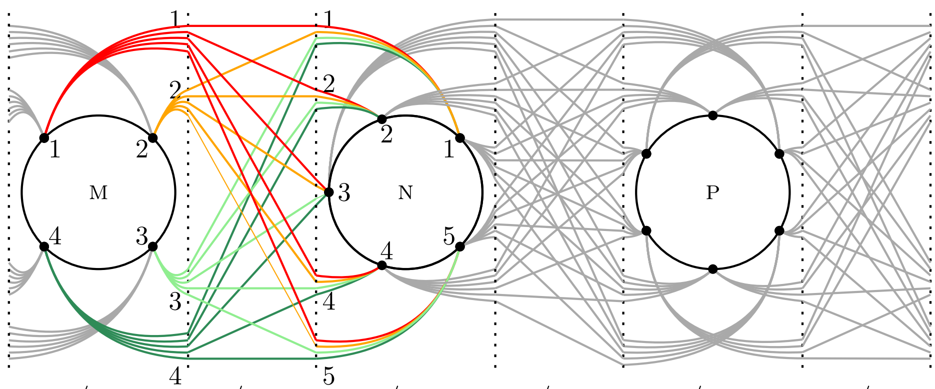


FIGURE 6. Illustration of the construction of a tripartite-circle drawing of $K_{m,n,p}$ with $m = 4, n = 5, p = 6$. The subdrawing induced by the circles A := M and B := N is highlighted by colors.

By construction, the drawing D has the following properties:

(i) each edge is x-monotone,
(ii) the drawing is partitioned into six vertical stripes; within each stripe every edge is x- and y-monotone,
(iii) there exist two types of stripes, either containing AB/AB-crossings or AB/BC-crossings, and
(iv) each edge is contained in three stripes.

These properties imply the following.

Proposition 3.6. *The drawing D is good.*

Proof. For each pair of edges, there exists a unique stripe where the two edges potentially cross. Since, by property (ii), the edges are x- and y-monotone within each stripe, any pair of edges crosses at most once. Thus, D is a good drawing. □

It remains to analyze the number of crossings.

Proposition 3.7. *For any integers $m, n, p \geq 3$, let $\mathfrak{t} := \{(m,n,p),(n,p,m),(p,m,n)\}$. The number of crossings in the drawing D is*

$$\sum_{(a,b,c)\in\mathfrak{t}} \left(\binom{a}{2}\binom{b}{2} + ac\left\lfloor\frac{b}{2}\right\rfloor\left\lfloor\frac{b-1}{2}\right\rfloor\right)$$

Proof. We present two proofs for the number of crossings. In the first, we use our derived formulas; in the second, we count the crossings directly.

Proof 1: It is easy to see from the construction that

$$x_i(\text{A,B}) = y_i(\text{A,B})$$
$$x_i(\text{B,A}) = y_i(\text{B,A})$$
$$d_b(y_i(\text{A,B}), y_j(\text{C,B})) = \left\lfloor \frac{b}{2} \right\rfloor.$$

Consequently, by Lemma 2.2, the number of crossings of type AB/AB is

$$\sum_{1\le i<j\le n} f_n(x_i(\text{A,B}), x_j(\text{A,B})) = \binom{a}{2}\binom{b}{2}.$$

By Lemma 2.3, the number of type AB/BC is

$$\sum_{\substack{1\le i\le a\\ 1\le j\le c}} f_n\big(y_i(\text{A,B}), y_j(\text{C,B})\big) = ac\left(\binom{\lfloor \frac{b}{2}\rfloor}{2} + \binom{\lceil \frac{b}{2}\rceil}{2}\right) = ac\left\lfloor \frac{b}{2}\right\rfloor \left\lfloor \frac{b-1}{2}\right\rfloor.$$

This finishes the first proof.

Proof 2: Alternatively, we count the number of crossings directly. By definition, the AB/AB crossings occur between ℓ_1 and ℓ_2; in this part of the drawing the edges are straight-line segments. Any pair of vertices on circle A and any pair of vertices on circle B together form exactly one crossing. We have $\binom{a}{2}\binom{b}{2}$ crossings.

For the crossings of type AB/BC, it suffices to count the bundle crossings. If two bundles cross they add ac crossings. Moreover, it follows from the construction that two bundles cross if they are both in the top or both in the bottom half. Consequently, the number of crossings is

$$ac\left(\binom{\lfloor \frac{b}{2}\rfloor}{2} + \binom{\lceil \frac{b}{2}\rceil}{2}\right).$$

As shown above, this evaluates to $ac\lfloor \frac{b}{2}\rfloor \lfloor \frac{b-1}{2}\rfloor$ and therefore finishes the second proof of the proposition. □

As the proposition was the last missing item, this finishes the proof of the upper bound and thus of Theorem 1.1. Note that this construction achieves the minimum possible number of bichromatic crossings by Lemma 3.2.

3.3. **Balanced case.** Theorem 1.1 and Corollary 3.5 imply Corollary 1.2 for the special case of $n = m = p$.

Proof of Corollary 1.2. For the lower bound, Theorem 1.1 and Corollary 3.5 give that $\text{cr}_{③}(K_{n,n,n}) \ge 3\,\text{cr}_{②}(K_{n,n}) + 3n^2 \lfloor \frac{n}{2}\rfloor \lfloor \frac{n-1}{2}\rfloor + 2$. With the bipartite cylindrical crossing number from Equation 1.1 we have

$$\text{cr}_{③}(K_{n,n,n}) \ge 3n\binom{n}{3} + 3n^2\left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor + 2$$

For the upper bound, the construction includes drawings for $K_{n,n,n}$. In this case, we obtain highly symmetric drawings, which are especially appealing. In particular, such a drawing can be defined by two consecutive stripes, see Figure 10(c). The formula simplifies to $3\binom{n}{2}^2 + 3n^2 \lfloor \frac{n}{2}\rfloor \lfloor \frac{n-1}{2}\rfloor$. □

Note that the lower bound order is $\frac{5}{4}n^4$, while the upper bound order is $\frac{6}{4}n^4$. Consequently, the bounds are fairly close. Moreover, instead of a linear representation, similar drawings can be defined in a *cyclic* way, as shown in Figure 7.

Remark 3.8. By a slight modification, we can improve the upper bound. To do so, we place at least one vertex on the intersection of the closed top and bottom half of the circle and route half of its incident edges via the upper half and the other half of its edges via the bottom half. This idea has been used to construct the drawings in Figures 8, 9 and 10(c).

Depending on the parity of n, the number of monochromatic crossings between two circles is

$$\begin{cases} \binom{n-1}{2}\binom{n}{2} + (n-1)\left(\binom{\lfloor n/2\rfloor}{2} + \binom{\lceil n/2\rceil}{2}\right) & n \text{ odd,} \\ 2\left(\binom{n/2-1}{2}n^2 + 1/2(n-2)n^2 + 1/4n^2\right) & n \text{ even.} \end{cases}$$

while the number of bichromatic crossings is

$$\begin{cases} n^2\left(\binom{\lfloor n/2\rfloor}{2} + \binom{\lceil n/2\rceil}{2}\right) & n \text{ odd,} \\ \left(\binom{n-1}{2}\right)^2 + 2(n-1)\left(\binom{n/2-1}{2} + \binom{n/2}{2}\right) + (n/2)^2 + (n/2-1)^2 & n \text{ even.} \end{cases}$$

Consequently, multiplying by three and summing both terms, the number of crossings evaluates to

$$\begin{cases} 3/4(2n^4 - 5n^3 + 3n^2 + n - 1) & n \text{ odd,} \\ 3/4(2n^4 - 6n^3 + 7n^2) & n \text{ even.} \end{cases}$$

Unfortunately, this improves only lower order terms, i.e., the number of saved crossings is

$$\begin{cases} \frac{3}{4}(n^3 - n^2 - n + 1) & n \text{ odd,} \\ \frac{3}{2}(n^3 - 3n^2) & n \text{ even.} \end{cases}$$

3.3.1. *Balanced case with few vertices.* In this section, we present numerical results and improved drawings of $K_{n,n,n}$ for small values of n. The values are summarized in Table 1. We improve the upper bounds with concrete drawings and the lower bounds with Corollary 3.5 and the following fact:

TABLE 1. Bounds of $\text{cr}_{③}(K_{n,n,n})$ for small n.

n	lower bound Corollary 1.2	improved lower bound	improved upper bound	upper bound Corollary 1.2
2	–	3	3	–
3	38	–	42	54
4	146	147	175	204
5	452	–	528	600
6	1010	–	1161	1323
7	2060	–	2430	2646
8	3650	–	4176	4656
9	6158	–	7296	7776
10	9602	–	11025	12075

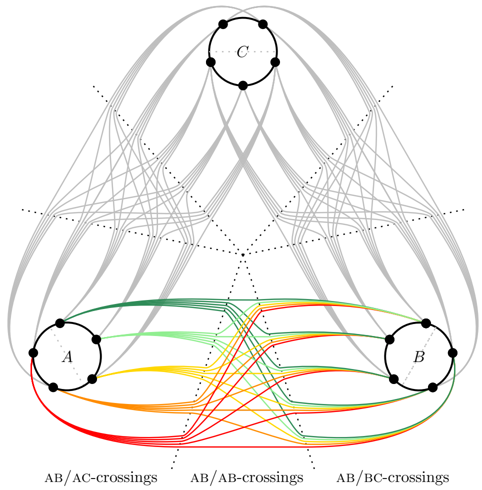


FIGURE 7. A tripartite-circle drawing of $K_{n,n,n}$ for $n = 5$.

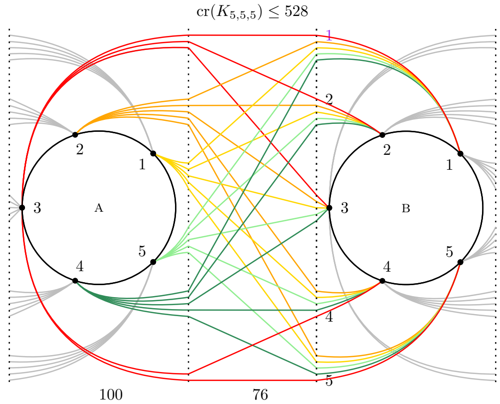


FIGURE 8. A drawing of $K_{5,5,5}$ with 528 crossings.

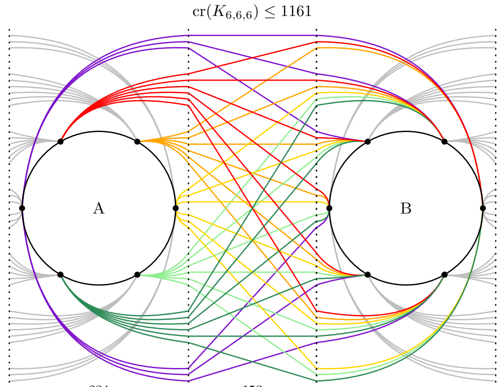


FIGURE 9. A drawing of $K_{6,6,6}$ with 1161 crossings.

Proposition 3.9. *For any integers m, n, and p,*

$$\mathrm{cr}_{③}(K_{m,n,p}) \geq cr(K_{m+n+p}) - \binom{m}{4} - \binom{n}{4} - \binom{p}{4}$$

The proposition follows from the fact that a tripartite circle drawing of K_{m+n+p} yields a drawing of the complete graph K_{m+n+p} by adding all straight-line segments within the three circles, see also Section 5.

In the following we explain how to obtain the bounds displayed in Table 1. For $n = 2$, it holds that $\mathrm{cr}_{③}(K_{2,2,2}) \geq 3$ since $cr(K_6) = 3$ and Figure 10(a) shows that three crossings can be attained. Note that this case is not covered by Theorem 1.3.

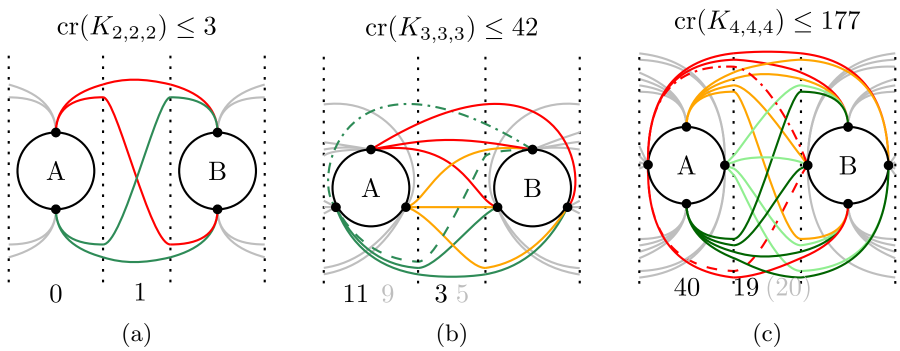


FIGURE 10. (a) An optimal drawing of $K_{2,2,2}$ with three crossings. (b) Two drawings of $K_{3,3,3}$ with 42 crossings. (c) Two drawings of $K_{4,4,4}$ one with 177 and one with 180 crossings. In (b) and (c), the two drawings are obtained by considering either the dash dotted or the dotted edge.

For $n = 3$, a lower bound of 38 follows by Corollary 3.5 and the upper bound of 42 by the drawing in Figure 10(b).

In case $n = 4$, we use Proposition 3.9 for the lower bound. Since $cr(K_{12}) = 150$, we obtain $\mathrm{cr}_{③}(K_{4,4,4}) \geq cr(K_{12}) - 3 = 147$. For the upper bound, Figure 11 presents a drawing with 175 crossings. This drawing is obtained by a slight modification of the drawing corresponding to Figure 10(c) with the dash dotted edge. In particular, in the middle copy (orange), the long edge between the leftmost vertex of B and the rightmost vertex of C is drawn in the top half, while its corresponding edges

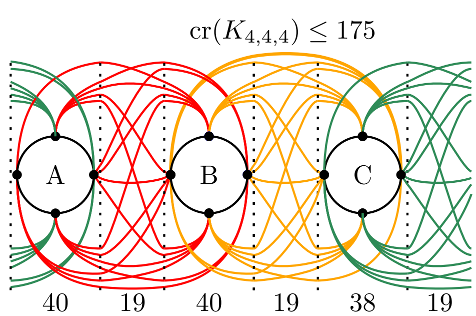


FIGURE 11. A 3-circle drawing of $K_{4,4,4}$ with 175 crossings.

in the other two copies are drawn in the bottom half. This saves the two crossings between the middle long edge and the left and right long edges.

For $n \geq 5$, we use the ideas of Remark 3.8 to improve the upper bounds.

4. Tripartite-circle crossing number of $K_{2,2,n}$—Proof of Theorem 1.3

In this section, we determine the tripartite-circle crossing number of $K_{2,2,n}$. First, we enumerate the good tripartite-circle drawings of $K_{2,2,0}$. Then, we give a construction to show the upper bound. Finally, we consider each case to minimize the number of crossings of $K_{2,2,n}$. Combining the upper bound and lower bound yields the desired result.

Lemma 4.1. *Up to topological equivalence, there are exactly four good tripartite-circle drawings of* $K_{2,2,0}$.

Proof. Up to topological equivalence, there are two different good bipartite-circle drawings of $K_{2,2}$ on the sphere, namely the ones depicted in Figure 12(a).

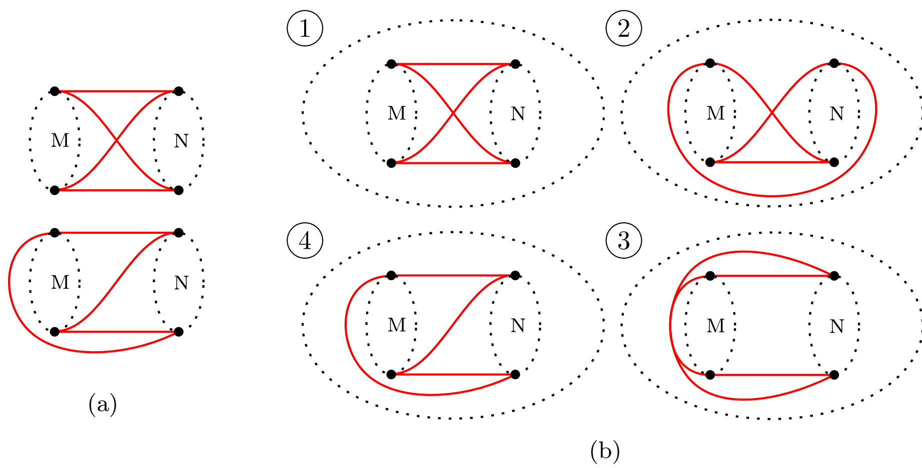


FIGURE 12. (a) The two good bipartite-circle drawings of $K_{2,2}$. (b) The four good tripartite-circle drawings of $K_{2,2,0}$.

Placing the third circle in different 'cells' of these two drawings yields the four drawings in Figure 12(b), where the third circle is placed in the outer cell. □

Lemma 4.2. *For any integer* $n \geq 3$,

$$\mathrm{cr}_{③}(K_{2,2,n}) \leq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2n - 3.$$

Proof. As depicted in Figure 13, we color the edges between circles M and P orange, the edges between circles N and P green, and the edges between circles M and N red. Moreover, we label the vertices on N and M with elements from [4], so that $\{1,2\} = V(\mathrm{N})$ and $\{3,4\} = V(\mathrm{M})$. Note that in this proof, there are n vertices on circle P.

For all $i \in [4]$, let $x_i := x_i(\mathrm{A},\mathrm{P})$ for $\mathrm{A} \in \{\mathrm{M},\mathrm{N}\}$ such that $i \in V(\mathrm{A})$, and likewise for $y_i := y_i(\mathrm{A},\mathrm{P})$. The red edges are drawn as in Type 1 in Lemma 4.1. For the orange and green edges, we define the parameters as follows:

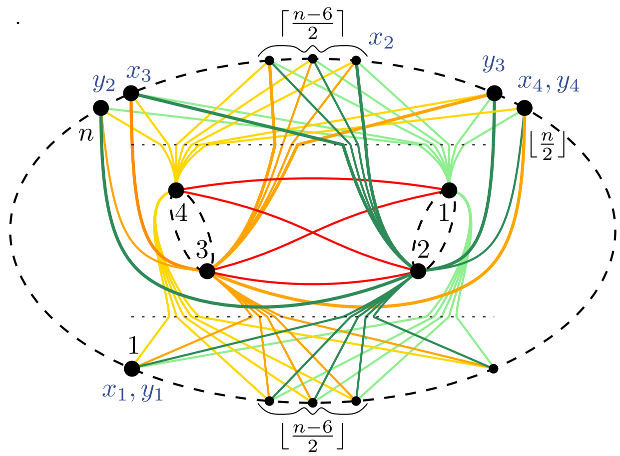


FIGURE 13. A crossing-minimal tripartite-circle drawing of $K_{2,2,n}$.

$$x_1 = y_1 = 1, \qquad x_2 = \left\lfloor \frac{n}{2} \right\rfloor + 2 \quad \text{and} \quad y_2 = n,$$
$$x_4 = y_4 = \left\lfloor \frac{n}{2} \right\rfloor, \qquad x_3 = n - 1 \quad \text{and} \quad y_3 = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

By Theorem 2.4, there are $f_n(x_1, x_2)$ monochromatic green (NP/NP-), $f_n(x_3, x_4)$ monochromatic orange (MP/MP-), and $f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4)$ orange-green (MP/NP-) crossings. It remains to analyze the crossings with red edges, that is, the MN/MN-, MN/MP-, and MN/NP-crossings.

There is one monochromatic red (MN/MN-) crossing. Note that the green edges from vertex 1 on N and the orange edges from vertex 4 on M do not intersect red edges. Moreover, a green edge e from vertex 2 on N intersects two red edges if and only if the edge e goes to the interval $[x_2, y_2)$ (counterclockwise); and an orange edge e from vertex 3 on M intersects two red edges if and only if the edge e goes to the interval $[y_3, x_3)$ (counterclockwise). Hence the number of crossings with red edges is

$$1 + 2 \cdot d_n(x_2, y_2) + 2 \cdot d_n(y_3, x_3) = 4 \left\lceil \frac{n}{2} \right\rceil - 7.$$

Therefore, in this drawing, the number of crossings can be computed by the following formula:

$$\sum_{i=1,3} f_n(x_i, x_{i+1}) + \sum_{\substack{i=1,2 \\ j=3,4}} f_n(y_i, y_j) + 4 \left\lceil \frac{n}{2} \right\rceil - 7.$$

With the chosen parameters, this evaluates to

$$\operatorname{cr}_{③}(K_{2,2,n}) \leq 3 \left\lfloor \frac{n}{2} \right\rfloor^2 + 3 \left\lceil \frac{n}{2} \right\rceil^2 - \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor - 3 = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 3.$$

and finishes the proof of the upper bound. □

Now, we turn our attention towards proving the lower bound, i.e., we want to show that this drawing has a minimal number of crossings. To do so, we partition the set of drawings of $K_{2,2,n}$ by the induced subdrawings of $K_{2,2,0}$ depicted in Figure 12(a).

Any good tripartite-circle drawing of $K_{2,2,n}$ can be seen as an extension of one of the four drawings of $K_{2,2,0}$ in Figure 12(b). We say that a drawing of $K_{2,2,n}$ is of type i if it is an extension of drawing i in Lemma 4.1. In our figures, we color the edges of $K_{2,2,0}$ red and the remaining $4n$ edges green. We first count the number of crossings with red edges. For an illustration consider Figure 14; note that some edges are omitted for clarity.

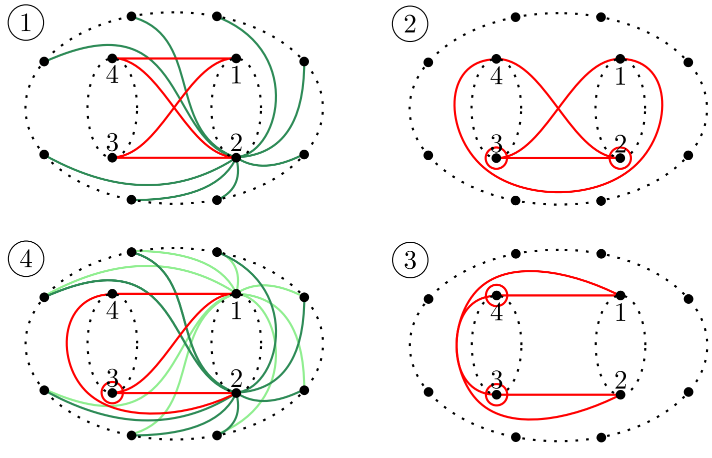


FIGURE 14. The four good tripartite-circle drawings of $K_{2,2,n}$. Red edges connect vertices on the two inner circles, all other edges are green (some of them omitted for clarity). The two drawings in the right-hand column have two enclosed vertices. A green edge from these vertices must cross a red edge.

There is exactly one monochromatic red crossing for types 1, 2, and 3 and none for type 4. *Enclosed* vertices in drawings of types 2, 3, and 4 are those separated from the outer circle by red edges; the green edges incident to enclosed vertices must cross at least one red edge. In a drawing of type 1, a green edge from vertex $i \in \{1, 3\}$ crosses two red edges if the other vertex lies in the interval $[y_i, x_i)$; otherwise it does not cross any edge. Note that the number of these vertices is $d_n(y_i, x_i)$.

Likewise, a green edge from a vertex $i \in \{2, 4\}$ crosses two red edges if the other vertex lies in the interval $[x_i, y_i)$. Recall that we consider the vertices on P in counterclockwise order.

The same holds for green edges incident to vertices 1 or 2 in a drawing of type 4. Hence, for each type of drawing, the number of crossings with red edges is at least

$$(4.1)\quad \begin{cases} 2 \cdot (d_n(y_1, x_1) + d_n(x_2, y_2) + d_n(y_3, x_3) + d_n(x_4, y_4)) + 1 & \text{type 1,} \\ 2n + 1 & \text{type 2 or 3,} \\ 2 \cdot (d_n(y_1, x_1) + d_n(x_2, y_2)) + n & \text{type 4.} \end{cases}$$

We partially use Theorem 2.4 to count the monochromatic green crossings as

$$(4.2)\quad f_n(x_1, x_2) + f_n(x_3, x_4) + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4).$$

It remains to show that the sum of (4.1) and (4.2) is at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n - 3$ for any of the four types of drawings. By Lemma 3.1, each term of (4.2) is minimized when the corresponding pair of points has a distance of roughly $\frac{n}{2}$. Hence, for all a and b it holds that

$$f_n(a,b) \geq \binom{\lfloor \frac{n}{2} \rfloor}{2} + \binom{\lceil \frac{n}{2} \rceil}{2} = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{4.3}$$

We start with two lemmas bounding some of the terms in (4.1) and (4.2). In the following lemmas, we write $Z_i = d_n(y_{i+1}, y_i)$ and $z_i = \min\{d_n(y_{i+1}, y_i), d_n(y_i, y_{i+1})\}$ for $i = 1, 3$ for notational convenience.

Lemma 4.3. *Let $Z_i = d_n(y_{i+1}, y_i)$. The following inequality holds:*

$$2d_n(y_i, x_i) + 2d_n(x_{i+1}, y_{i+1}) + f_n(x_i, x_{i+1}) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 - 2Z_i.$$

Moreover, the inequality can be strengthened in the following cases:

$$2d_n(y_i, x_i) + 2d_n(x_{i+1}, y_{i+1}) + f_n(x_i, x_{i+1})$$
$$\geq \begin{cases} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n - 1 - 2Z_i + 2n; & \text{if ccw order } \neq x_{i+1}y_{i+1}y_ix_i \\ \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n - 1 - 2Z_i + \left\lfloor \left(Z_i - \frac{n-2}{2}\right)^2 \right\rfloor & \text{if ccw order is } x_{i+1}y_{i+1}y_ix_i \\ & \text{and } Z_i \geq n/2 \end{cases}$$

Proof. Note that the (general) statement is equivalent to showing

$$2\big(d_n(y_i, x_i) + d_n(x_{i+1}, y_{i+1}) + d_n(y_{i+1}, y_i)\big) + f_n(x_i, x_{i+1}) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1$$

A short case analysis (by considering the six counterclockwise orders) verifies that for every four points a, b, c, d, the following holds:

$$d_n(a,b) + d_n(b,c) + d_n(c,d) = \begin{cases} d_n(a,d) & \text{if the ccw order is } abcd \\ d_n(a,d) + 2n & \text{if the ccw order is } adcb \\ d_n(a,d) + n & \text{otherwise.} \end{cases} \tag{4.4}$$

With $a = x_{i+1}, b = y_{i+1}, c = y_i, d = x_i$, Equation (4.4) implies that

$$2\big(d_n(x_{i+1}, y_{i+1}) + d_n(y_{i+1}, y_i) + d_n(y_i, x_i)\big) + f_n(x_i, x_{i+1}) \geq 2d_n(x_{i+1}, x_i) + f_n(x_i, x_{i+1}) \tag{4.5}$$

The right side of Inequality 4.5 can be expressed as the quadratic function

$$2d_n(x_{i+1}, x_i) + f_n(x_i, x_{i+1}) = d_n(x_{i+1}, x_i)^2 + (2-n)d_n(x_{i+1}, x_i) + \binom{n}{2}, \tag{4.6}$$

which is minimized for $d_n(x_{i+1}, x_i) = \lfloor \frac{n-2}{2} \rfloor$. Evaluation at $d_n(x_{i+1}, x_i) = \lfloor \frac{n-2}{2} \rfloor$ yields

$$2d_n(x_{i+1}, x_i) + f_n(x_i, x_{i+1}) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1.$$

This finishes the proof of the general statement.

For the strengthening, we consider the two cases. If the counterclockwise order is different from $x_{i+1}y_{i+1}y_ix_i$, note that we can add a $2n$ term to the right hand side of the Inequality 4.5. If the counterclockwise order is $x_{i+1}y_{i+1}y_ix_i$ and $d_n(x_{i+1}, x_i) \geq d_n(y_{i+1}, y_i) \geq n/2$, then the expression in Equation (4.6) is minimized for $d_n(x_{i+1}, x_i) = d_n(y_{i+1}, y_i) = Z_i$. ◻

Now, we show a lower bound on the remaining four f_n-terms in (4.2). To do so, we define

$$\Delta_k := \begin{cases} 0 & k \text{ even} \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.4. *Let $z_1 = \min d_n(y_1, y_2)$, $z_3 = \min d_n(y_3, y_4)$ and $S = f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4) - 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.*

i) *If $y_1, y_2 \in [y_3, y_4]$ or $y_1, y_2 \in [y_4, y_3]$, then it holds that*

$$S \geq z_1^2 + z_3^2 - \Delta_n \Delta_{z_1+z_3}$$

ii) *If $y_1 \in [y_3, y_4]$ and $y_2 \in [y_4, y_3]$ (or vice versa), then it holds that*

$$S \geq \frac{1}{4}\left(z_1^2 + (n - z_1)^2 + z_3^2 + (n - z_3)^2\right) - \frac{1}{2}\Delta_n$$

Moreover, in all cases it holds that

$$S \geq z_1^2 - \Delta_n \Delta_{z_1}.$$

Proof. First note that exchanging y_1 and y_2 (or y_3 and y_4) does not influence $f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4)$. Consequently, by swapping y_1 and y_2 or y_3 and y_4, all counterclockwise orders can be transformed to one of the following two: $y_1y_2y_3y_4$ (non-alternating, i.e., i)) and $y_1y_3y_2y_4$ (alternating, i.e., ii)).

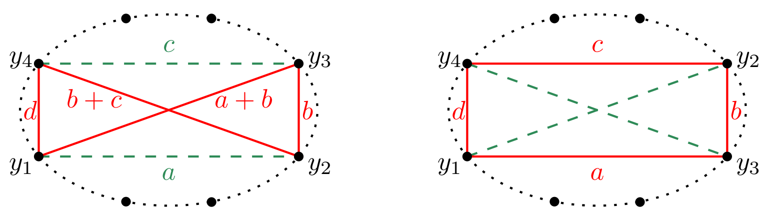


FIGURE 15. a) non-alternating b) alternating

Case i): Without loss of generality, we consider the counterclockwise order y_1, y_2, y_3, y_4 and define $a := d_n(y_1, y_2)$, $b := d_n(y_2, y_3)$, $c := d_n(y_3, y_4)$, and $d := d_n(y_4, y_1)$, see also Figure 15. Then, it holds that

$$\begin{aligned}
f_n(y_1, y_3)+&f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4) \\
&= 4\binom{n}{2} - (a+b)(n-(a+b)) - d(n-d) - b(n-b) - (b+c)(n-(b+c)) \\
&= 2n^2 - 2n + a^2 + b^2 + c^2 + d^2 - (a+b+c+d)n + 2b(a+b+c-n) \\
&= 2n^2 - 2n + a^2 + b^2 + c^2 + d^2 - n^2 - 2bd \\
&= n^2 - 2n + a^2 + c^2 + (b-d)^2 \\
&\geq n^2 - 2n + z_1^2 + z_3^2 + \Delta_n \Delta_{z_1+z_3+1} \\
&\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + z_1^2 + z_3^2 - \Delta_n \Delta_{z_1+z_3}
\end{aligned}$$

Note that $a^2 + c^2 + (b-d)^2 \geq z_1^2 + z_3^2 + \Delta_n \Delta_{z_1+z_3+1}$ trivially holds when n is even. Suppose n is odd and $z_1 + z_3$ is even. It holds that either $z_1 = a$,

$z_3 = c$ or $z_1 = b + c + d$, $z_3 = c$ (or symmetrically $z_1 = a$, $z_3 = a + b + d$). If $z_1 = a$, $z_3 = c$ and $z_1 + z_3$ is even, then $b \neq d$, and hence $a^2 + c^2 = z_1^2 + z_3^2$ and $(b - d)^2 \geq 1 \geq \Delta_n \Delta_{z_1+z_3+1}$. If $z_1 = b + c + d < a$, $z_3 = c$ then it holds that $a^2 + c^2 \geq (z_1 + 1)^2 + z_3^3 \geq z_1^2 + z_3^3 + 1 \geq z_1^2 + z_3^2 + \Delta_n \Delta_{z_1+z_3+1}$. Therefore, we obtain the term $+\Delta_n \Delta_{z_1+z_3+1}$ in the first inequality.

For the second inequality, note that $+\Delta_n \Delta_{z_1+z_3+1} - \Delta_n = -\Delta_n \Delta_{z_1+z_3}$. This finishes the proof of part i).

Case ii): Without loss of generality, we consider the order y_1, y_3, y_2, y_4 and define $a := d_n(y_1, y_3)$, $b := d_n(y_3, y_2)$, $c := d_n(y_2, y_4)$, and $d := d_n(y_4, y_1)$. By symmetry, we may assume that $z_1 = a + b$ (so $n - z_1 = c + d$). Then

$$2(\frac{z_1}{2})^2 = 2(\frac{a+b}{2})^2 = \frac{1}{2}(a^2 + b^2 + 2ab) = a^2 + b^2 - \frac{1}{2}(a - b)^2 \leq a^2 + b^2 - \frac{1}{2}\Delta_{z_1}$$

since $(a - b)^2 \geq 1$ when z_1 is odd. Similarly, $c^2 + d^2 \geq 2(\frac{n-z_1}{2})^2 + \frac{1}{2}\Delta_{n-z_1}$, so we find $a^2 + b^2 + c^2 + d^2 \geq \frac{1}{2}(z_1^2 + (n - z_1)^2 + \Delta_{z_1} + \Delta_{n-z_1})$. By a similar argument, we have $a^2 + d^2 \geq 2(\frac{z_3}{2})^2 + \frac{1}{2}\Delta_{z_3}$ and $b^2 + c^2 \geq 2(\frac{n-z_3}{2})^2 + \frac{1}{2}\Delta_{n-z_3}$ (or vice versa), so $a^2 + b^2 + c^2 + d^2 \geq \frac{1}{2}(z_3^2 + (n - z_3)^2 + \Delta_{z_3} + \Delta_{n-z_3})$. Therefore $a^2 + b^2 + c^2 + d^2 \geq \frac{1}{4}(z_1^2 + (n - z_1)^2 + z_3^2 + (n - z_3)^2 + \Delta_{z_1} + \Delta_{n-z_1} + \Delta_{z_3} + \Delta_{n-z_3})$. We thus have

$$\begin{aligned}
f_n(y_1, y_3)+&f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4) \\
&= 4\binom{n}{2} - a(n - a) - b(n - b) - c(n - c) - d(n - d) \\
&= n^2 - 2n + a^2 + b^2 + c^2 + d^2 \\
&\geq n^2 - 2n + \frac{1}{4}(z_1^2 + (n - z_1)^2 + z_3^2 + (n - z_3)^2) \\
&\qquad + \frac{1}{4}(\Delta_{z_1} + \Delta_{n-z_1} + \Delta_{z_3} + \Delta_{n-z_3}) \\
&= 4\left\lfloor\frac{n}{2}\right\rfloor \left\lfloor\frac{n-1}{2}\right\rfloor + \frac{1}{4}(z_1^2 + (n - z_1)^2 + z_3^2 + (n - z_3)^2) \\
&\qquad + \frac{1}{4}(\Delta_{z_1} + \Delta_{n-z_1} + \Delta_{z_3} + \Delta_{n-z_3} - 4\Delta_n)
\end{aligned}$$

When n is odd, then for $i = 1, 3$ one of z_i and $n - z_i$ is even. Therefore, it holds that $\frac{1}{4}(\Delta_{z_1} + \Delta_{n-z_1} + \Delta_{z_3} + \Delta_{n-z_3} - 4\Delta_n) \geq -\frac{1}{2}\Delta_n$. Hence, part ii) is proved.

Note that in case ii) it holds that

$$a^2 + b^2 + c^2 + d^2 \geq \frac{1}{2}(z_1^2 + (n - z_1)^2) \geq z_1^2.$$

Therefore, we obtain in both cases i) and ii) that

$$f_n(y_1, y_3)+f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4) \geq 4\left\lfloor\frac{n}{2}\right\rfloor \left\lfloor\frac{n-1}{2}\right\rfloor + z_1^2 - \Delta_n \Delta_{z_1}$$

which finishes the proof. □

Now, we are ready to prove the lower bound.

Lemma 4.5. *For any integer $n \geq 3$,*

$$\mathrm{cr}_{③}(K_{2,2,n}) \geq 6\left\lfloor\frac{n}{2}\right\rfloor \left\lfloor\frac{n-1}{2}\right\rfloor + 2n - 3.$$

Proof. We consider the drawings depending on their type. We start with the types for which there exist simple arguments and end with the more complicated ones, namely we consider the drawing types in the order 2 or 3, 4, and 1.

For a **drawing of type 2 or 3**, by (4.1) and (4.2), the number of crossings is

$$1 + 2n + f_n(x_1, x_2) + f_n(x_3, x_4) + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4).$$

Using Equation (4.3), this is bounded from below by $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + 1 > 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n - 3$, as desired. Note that drawings of these types do may not attain the minimum number of crossings.

Next, we consider **drawings of type 4**. The number of crossings is

$$\begin{aligned} &n + 2(d_n(y_1, x_1) + d_n(x_2, y_2)) + f_n(x_1, x_2) + f_n(x_3, x_4) \\ &\quad + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4). \end{aligned}$$

We show the lower bound by considering two cases for Z_1. In each of the cases, we use Lemma 3.1 to bound $f_n(x_3, x_4)$.

In the first case, it holds that $Z_1 \leq (n-1)/2$. By Lemmas 4.3 and 4.4, the number of crossings is at least

$$n + 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 - 2Z_1 + Z_1^2 - \Delta_n \Delta_{Z_1}$$

and $2n - 1 - 2Z_1 + Z_1^2 - \Delta_n \Delta_{Z_1} \geq 2n - 2 + (Z_1 - 1)^2 - \Delta_n \Delta_{Z_1} \geq 2n - 3$. This shows the claim.

In the second case, it holds that $Z_1 \geq n/2$. Here we distinguish the subcases whether or not $[x_1, x_2] \subseteq [y_1, y_2]$. If $[x_1, x_2] \subseteq [y_1, y_2]$. Using the third inequality of Lemma 4.3, the number of crossings is at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor +$

$$\begin{aligned} &n + n - 1 - 2Z_1 + \left\lfloor \left(Z_1 - \frac{n-2}{2} \right)^2 \right\rfloor + (n - Z_1)^2 - \Delta_n \Delta_{n-Z_1} \\ &\geq 2n - 3 + \left\lfloor \frac{1}{8} \left((4Z_1 - 3n)^2 + (n-4)^2 \right) \right\rfloor \qquad \geq 2n - 3. \end{aligned}$$

If $Z_1 \geq n/2$ and $[x_1, x_2] \not\subseteq [y_1, y_2]$, then the second inequality of Lemma 4.3 shows that the number of crossings is at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor +$

$$\begin{aligned} n + n - 1 - 2Z_1 + 2n + (n - Z_1)^2 - \Delta_n \Delta_{n-Z_1} & \\ &\geq 2n - 1 + 2(n - Z_1) + (n - Z_1)^2 - 1 \\ &= 2n - 3 + (n - Z_1 + 1)^2 \\ &\geq 2n - 3. \end{aligned}$$

This finishes the proof for drawings of type 4.

It remains to consider **drawings of type 1**, which have the following number of crossings:

$$\begin{aligned} &2(d_n(y_1, x_1) + d_n(x_2, y_2) + d_n(y_3, x_3) + d_n(x_4, y_4)) + f_n(x_1, x_2) + f_n(x_3, x_4) \\ &\quad + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4) + 1. \end{aligned}$$

Without loss of generality, we assume that $Z_1 \leq Z_3$.

Case A: First,we consider the case that Lemma 4.4 part i) applies and distinguish three cases depending on whether Z_1 and Z_3 are small ($\leq (n-1)/2$) or big ($\geq n/2$).

Case A1: if $Z_1, Z_3 \leq (n-1)/2$. By Lemmas 4.3 and 4.4 part i, the number of crossings is at least

$$6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 - 2Z_1 + n - 1 - 2Z_3 + Z_1^2 + Z_3^2 + -\Delta_n \Delta_{Z_1+Z_3} + 1$$

Then, it holds that:

$$\begin{aligned} &2n - 1 - 2Z_1 - 2Z_3 + Z_1^2 + Z_3^2 - \Delta_n \Delta_{Z_1+Z_3} \\ &\geq 2n - 3 + (Z_1 - 1)^2 + (Z_3 - 1)^2 - \Delta_n \Delta_{Z_1+Z_3} \\ &\geq 2n - 3 \end{aligned}$$

Note that in the case $Z_1 = Z_3 = 1$, it holds that $Z_1 + Z_3$ is even and thus $(-1\Delta_n \Delta_{Z_1+Z_3}) = 0$.

Case A2: if $Z_1 \leq (n-1)/2$, $Z_3 \geq n/2$. Suppose we do not have the counterclockwise ordering $x_4 y_4 y_3 x_3$. By Lemmas 4.3 and 4.4 i), the number of crossings is at least

$$\begin{aligned} 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor &+ (n - 1 - 2Z_1) + (n - 1 - 2Z_3) + Z_1^2 + (n - Z_3)^2 \\ &+ 2n - \Delta_n \Delta_{Z_1+n-Z_3} + 1. \end{aligned}$$

Then, since $Z_3 \leq n-1$ it holds that $(n - Z_3 + 1)^2 \geq 2^2$, which implies that

$$\begin{aligned} 2n - 1 - 2Z_1 + Z_1^2 - 2Z_3 + 2n + (n - Z_3)^2 - \Delta_n \Delta_{Z_1+n-Z_3} \\ = 2n - 3 + (Z_1 - 1)^2 + (n - Z_3 + 1)^2 - \Delta_n \Delta_{Z_1+n-Z_3} \\ > 2n - 3. \end{aligned}$$

Now, suppose we have the counterclockwise ordering $x_4 y_4 y_3 x_3$. By Lemmas 4.3 and 4.4 i), the number of crossings is at least

$$\begin{aligned} &6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 - 2Z_1 + n - 1 - 2Z_3 + Z_1^2 + (n - Z_3)^2 \\ &+ \left\lfloor \left(Z_3 - \frac{n-2}{2} \right)^2 \right\rfloor - \Delta_n \Delta_{Z_1+n-Z_3} + 1 \end{aligned}$$

Then,

$$\begin{aligned} 2n - 1 - 2Z_1 - 2Z_3 + Z_1^2 + (n - Z_3)^2 + \left\lfloor \left(Z_3 - \frac{n-2}{2} \right)^2 \right\rfloor - \Delta_n \Delta_{Z_1+n-Z_3} \\ \geq 2n - 3 + (Z_1 - 1)^2 + \left\lfloor \frac{1}{8}((3n - 4Z_3)^2 + (n-4)^2) \right\rfloor - \Delta_n \Delta_{Z_1+n-Z_3}. \end{aligned}$$

This is at least $2n-3$ if $(Z_1 - 1)^2 \geq 1$, so we may assume that $Z_1 = 1$. This is also at least $2n-3$ if n is even or if $Z_1 + n - Z_3$ is even, and so we may assume that n is odd, and further that Z_3 is odd, since $Z_1 = 1$. In this case, we have that

$$(Z_1 - 1)^2 + \left\lfloor \frac{1}{8}((3n - 4Z_3)^2 + (n-4)^2) \right\rfloor - \Delta_n \Delta_{Z_1+n-Z_3} = \frac{1}{8}((3n - 4Z_3)^2 + (n-4)^2) - \frac{5}{4}.$$

Toward contradiction, assume that $\frac{1}{8}((3n-4Z_3)^2+(n-4)^2)-\frac{5}{4}\leq -1$. By rearranging, we obtain $(3n-4Z_3)^2+(n-4)^2\leq 2$, which implies that $n\in\{3,5\}$ since n is odd and $(n-4)^2\leq 2$.

If $n=3$, since $\frac{n}{2}\leq Z_3<n$, this means that $Z_3=2$, which contradicts the assumption that Z_3 is odd.

If $n=5$, since $\frac{n}{2}\leq Z_3<n$ and Z_3 is odd, $Z_3=3$. With these values,

$$(3n-4Z_3)^2+(n-4)^2=(15-12)^2+(5-1)^2\not\leq 2,$$

a contradiction. Therefore, $\frac{1}{8}((3n-4Z_3)^2+(n-4)^2)-\frac{5}{4}\geq 0$, completing this subcase.

Case A3: if $Z_1,Z_3\geq n/2$. By Lemmas 4.3 and 4.4, the number of crossings is at least

$$1+6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n-1-2Z_1+B(1)+n-1-2Z_3+B(3)$$
$$+(n-Z_1)^2+(n-Z_3)^2-\Delta_n\Delta_{Z_1+Z_3}.$$

where, for $i=1,3$, $B(i)=\begin{cases}2n & \text{if ccw order } \neq x_{i+1}y_{i+1}y_ix_i\\ \left\lfloor\left(Z_i-\frac{n-2}{2}\right)^2\right\rfloor & \text{if ccw order is } x_{i+1}y_{i+1}y_ix_i\\ & \text{and } Z_i\geq n/2\end{cases}$.

Let $A(i)=1-2Z_i+(n-Z_i)^2+B(i)$. Then the number of crossings in this case is at least

$$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n-3+A(1)+A(3)-\Delta_n\Delta_{Z_1+Z_3}. \tag{4.7}$$

If the ccw order is not $x_{i+1}y_{i+1}y_ix_i$ for either $i=1$ or $i=3$, then

$$A(i)=(n-Z_i)^2+2(n-Z_i)+1\geq 4$$

because $n-1\geq Z_i$. On the other hand, if the ccw order is $x_{i+1}y_{i+1}y_ix_i$ for either $i=1$ or $i=3$, then after simplifying,

$$A(i)=\frac{1}{8}\left((4Z_i-3n)^2+(n-4)^2\right)-\frac{1}{4}\Delta_n\geq-\frac{1}{4}.$$

In particular, if at least one of $i=1$ or $i=3$ does not have the ccw order $x_{i+1}y_{i+1}y_ix_i$, then (4.7) becomes

$$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n-3+4-\frac{1}{4}-1>6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n-3,$$

which would yield the desired lower bound and complete this subcase.

Therefore, assume we have the ccw order $x_{i+1}y_{i+1}y_ix_i$ for both $i=1,3$. Define $f(x)=\frac{1}{8}\left((4x-3n)^2+(n-4)^2\right)$. Then (4.7) becomes

$$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n-3+f(Z_1)+f(Z_3)-\frac{1}{2}\Delta_n-\Delta_n\Delta_{Z_1+Z_3}. \tag{4.8}$$

If either n or Z_1+Z_3 is even, then (4.8) is at least

$$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n-3-\frac{1}{2}.$$

Since (4.8) is an integer, it follows that (4.8) is in fact bounded below by $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2n-3$, as desired. Hence, assume that both n and Z_1+Z_3 are odd. As (4.8) is an integer, it suffices to show that

$$f(Z_1)+f(Z_3)-\frac{3}{2} > -1.$$

Suppose to the contrary that $f(Z_1)+f(Z_3)\leq 1/2$. Then one of $f(Z_1)$ or $f(Z_3)$ is at most $1/4$. Assume without loss of generality that $f(Z_1)\leq 1/4$. In particular, this means $(4Z_1-3n)^2+(n-4)^2\leq 2$. Since n is odd, we must have $n\in\{3,5\}$. If $n=3$, then $Z_1=2$ (as we saw at the end of Case A2), while if $n=5$, then $Z_1=3$ or $Z_1=4$. Because $(4Z_1-3n)^2+(n-4)^2\leq 2$, we must have $Z_1=4$ when $n=5$. Thus, in both cases of n, we have $f(Z_1)=1/4$. This implies that $f(Z_3)\leq 1/4$. The same argument gives us that $Z_3=2$ if $n=3$ or $Z_3=4$ if $n=5$. However, in both cases of n, the sum Z_1+Z_3 is even, contradicting the assumption that Z_1+Z_3 is odd. Therefore, $f(Z_1)+f(Z_3)-3/2>-1$. We conclude that (4.8) is at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2n-3$. This completes Case A.

Case B: Now, we suppose that we are in the case that Lemma 4.4 part ii applies; note that this implies $n\geq 4$.

Before we consider different cases, note that for $n\geq 10$, the desired bound follows easily from Lemma 4.4 part ii:

$$\begin{aligned}
&2n-1-2Z_1-2Z_3+1/4(Z_1^2+Z_3^2+(n-Z_1)^2+(n-Z_3)^2)-\Delta_n/2\\
&\geq -5+1/4((Z_1-2)^2+(Z_3-2)^2+(n-Z_1+2)^2+(n-Z_3+2)^2)-\Delta_n/2\\
&\geq -5+1/4n^2-\Delta_n/2\\
&\geq 2n-3
\end{aligned}$$

if $n\geq 10$. It remains to consider the cases that $4\leq n\leq 9$. We consider the same three cases as in Case A.

Case B1: if $Z_1,Z_3\leq (n-1)/2$. Then, by Lemmas 4.3 and 4.4 ii), the number of crossings is at least

$$\begin{aligned}
1+&6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+(n-1-2Z_1)+(n-1-2Z_3)\\
&\qquad\qquad+1/4(Z_1^2+Z_3^2+(n-Z_1)^2+(n-Z_3)^2)-\Delta_n/2\\
=&6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n-1-2Z_1-2Z_3\\
&\qquad\qquad+1/4(Z_1^2+Z_3^2+(n-Z_1)^2+(n-Z_3)^2)-\Delta_n/2
\end{aligned}$$

Note that if $n\geq 4$, then $n(n-Z_1-Z_3)-\Delta_n\geq n-\Delta_n\geq 4$ since $Z_1,Z_3\leq(n-1)/2$ implies that $n-Z_1-Z_3\geq 1$. Here we use the fact that $\Delta_4=0$. Therefore, it follows that

$$\begin{aligned}
-2Z_1-2Z_3+1/4(Z_1^2+Z_3^2+(n-Z_1)^2+(n-Z_3)^2)-\Delta_n/2&\geq -2\\
\iff 2(Z_1-2)^2+2(Z_3-2)^2+2n^2-2nZ_1-2nZ_3-2\Delta_n&\geq 8\\
\iff (Z_1-2)^2+(Z_3-2)^2+n(n-Z_1-Z_3)-\Delta_n&\geq 4\\
\Longleftarrow n(n-Z_1-Z_3)-\Delta_n&\geq 4\\
\iff n&\geq 4
\end{aligned}$$

Case B2: if $Z_1 \leq (n-1)/2$ and $Z_3 \geq n/2$. Firstly, we consider the case that we have the counterclockwise order $x_{i+1}y_{i+1}y_ix_i$ for $i=3$. By Lemmas 4.3 and 4.4 (ii), the number of crossings is at least

$$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2n - 1 - 2Z_1 - 2Z_3 + \left\lfloor\left(Z_3 - \frac{n-2}{2}\right)^2\right\rfloor + \frac{1}{4}(Z_1^2 + (n-Z_1)^2 + Z_3^2 + (n-Z_3)^2) - \frac{1}{2}\Delta_n.$$

We show that

$$-2Z_1 - 2Z_3 + \left\lfloor\left(Z_3 - \frac{n-2}{2}\right)^2\right\rfloor + \frac{1}{4}(Z_1^2 + (n-Z_1)^2 + Z_3^2 + (n-Z_3)^2) - \frac{1}{2}\Delta_n \geq -2.$$

By simplifying and rearranging, it is equivalent to show that

$$(Z_1-2)^2 + (Z_3-2)^2 + n(n-Z_1-Z_3) + 2\left\lfloor\left(Z_3 - \frac{n-2}{2}\right)^2\right\rfloor \geq 4 + \Delta_n.$$

The left-hand side of the claimed inequality is minimized when $Z_1 = n/2+2$ and $Z_3 = n/2$. Moreover, as $Z_1 \leq (n-1)/2$ and $Z_3 \geq n/2$, we have that the left-hand side of the inequality decreases as Z_1 increases and Z_3 decreases. This is seen from the partial derivatives. Since $Z_1 \leq (n-1)/2$, we take $Z_1 = \lfloor (n-1)/2 \rfloor$ and $Z_3 = \lceil n/2 \rceil$ for the location of the minimum of the left-hand side in this case. When n is even, it follows that $Z_1 + Z_3 \leq n-1$ and thus

$$n(n-Z_1-Z_3) \geq n \geq 4 + \Delta_n \text{ if } n \geq 4.$$

When $n = 2k+1$, the minimum is attained for $Z_1 = k$ and $Z_3 = k+1$ and the left-hand side of the claimed inequality simplifies to $2k^2 - 6k + 9$, which is at least $5 = 4 + \Delta_{2k+1}$ for $k \geq 1$.

Secondly, if we do not have the counterclockwise order $x_{i+1}y_{i+1}y_ix_i$ for $i=3$, then the number of crossings is at least

$$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4n - 1 - 2Z_1 - 2Z_3 + \frac{1}{4}(Z_1^2 + (n-Z_1)^2 + Z_3^2 + (n-Z_3)^2) - \frac{1}{2}\Delta_n.$$

And

$$\begin{aligned}
&2n - 2Z_1 - 2Z_3 + \frac{1}{4}(Z_1^2 + (n-Z_1)^2 + Z_3^2 + (n-Z_3)^2) - \frac{1}{2}\Delta_n \geq -2 \\
\iff &4n + (Z_1-2)^2 + (Z_3-2)^2 + n(n-Z_1-Z_3) \geq 4 + \Delta_n.
\end{aligned}$$

The left-hand side of the claimed inequality is minimized when $Z_1 = Z_3 = n/2+2$. Therefore, for $n \geq 4$, it holds that

$$\begin{aligned}
&4n + (Z_1-2)^2 + (Z_3-2)^2 + n(n-Z_1-Z_3) \\
\geq &2(n/2)^2 + n(n - \lfloor (n-1)/2 \rfloor - \lfloor n/2 \rfloor - 2 + 4) \\
&\geq n^2/2 \geq 8 \geq 4 + \Delta_n.
\end{aligned}$$

This finishes the proof of B2.

Case B3: By the last argument of case B2, we may assume that we do have the counterclockwise order $x_{i+1}y_{i+1}y_ix_i$ for $i = 1, 3$. Thus, the number of crossings is

$$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2n - 1 - 2Z_1 - 2Z_3 + \left\lfloor\left(Z_1 - \frac{n-2}{2}\right)^2\right\rfloor + \left\lfloor\left(Z_3 - \frac{n-2}{2}\right)^2\right\rfloor$$
$$+ \frac{1}{4}(Z_1^2 + (n - Z_1)^2 + Z_3^2 + (n - Z_3)^2) - \frac{1}{2}\Delta_n,$$

which is minimized if $Z_1 = Z_3 = n/2$ In this case, $Z_1 \geq n/2$ and $Z_3 \geq n/2$, so the minimum occurs when $Z_1 = Z_3 = \lceil n/2 \rceil$. Consequently, we obtain

$$\begin{aligned}
2(-Z_1 - Z_3) + \left\lfloor\left(Z_1 - \frac{n-2}{2}\right)^2\right\rfloor + \left\lfloor\left(Z_3 - \frac{n-2}{2}\right)^2\right\rfloor& \\
+ \frac{1}{4}(Z_1^2 + (n - Z_1)^2 + Z_3^2 + (n - Z_3)^2) - \frac{1}{2}\Delta_n& \\
\geq -2(n + \Delta_n) + (1 + \Delta_n) + (1 + \Delta_n) + \frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^2 + \frac{1}{2}\left\lceil\frac{n}{2}\right\rceil^2 - \frac{1}{2}\Delta_n& \\
\geq -2n + 2 + \frac{1}{4}(n^2 + \Delta_n) - \frac{1}{2}\Delta_n& \\
= 1/4n^2 - 2n + 2 - 1/4\Delta_n = 1/4(n-4)^2 - 2 - 1/4\Delta_n \geq -2&
\end{aligned}$$

if $n \geq 4$. This finishes the proof of B3 as well as of the lemma. □

5. Connection to the Harary-Hill Conjecture

The Harary-Hill Conjecture [11, 19] states that the number of crossings in any drawing (in the plane) of the complete graph K_n is at least

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

Drawings with exactly $H(n)$ crossings [12, 19] show that $\text{cr}(K_n) \leq H(n)$. The Harary-Hill conjecture has been confirmed for $n \leq 12$ (see [17] for $n \leq 10$ and [22] for $n = 11, 12$), and $\text{cr}(K_{13}) \leq H(13)$ or $H(13) - 2$ [1]. Very recently, the conjecture has been proved when restricted to certain families of graphs [2–5, 10, 20].

For decades, only two families of drawings of K_n with $H(n)$ crossings were known, shown in Figure 16: the Blažek-Koman construction [12], which is an instance

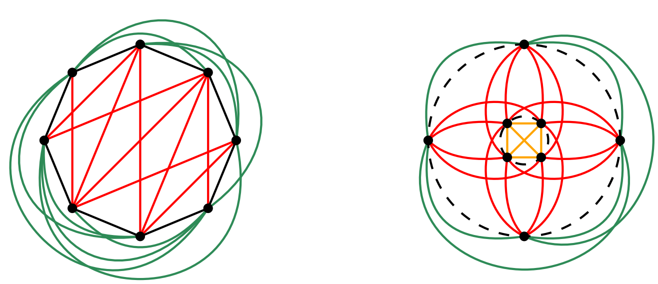


FIGURE 16. Drawings of K_n with $H(n)$ crossings (here $n = 8$):
a) Construction of Blažek and Koman; a restricted 1-circle drawing
b) Construction of Harary and Hill; a balanced restricted 2-circle drawing.

of a restricted 1-circle drawing (or 2-page book-embedding), and the Harary-Hill construction [19], which is an instance of a balanced restricted 2-circle drawing (or cylindrical drawing).

Ábrego et al. recently announced in [6] a new family of drawings of K_n having the property that each edge is crossed at least once, so these are not k-circle drawings. Kynčl and others [18] naturally asked about the existence of alternative k-circle constructions of K_n with $H(n)$ crossings: Is $H(n) = \mathrm{bcr}_{ⓚ}(K_n)$ for some $k \geq 3$? (Recall $\mathrm{bcr}_{ⓚ}$ is the minimum number of crossings in a balanced restricted k-circle crossing drawing.) Figure 17 shows crossing-optimal balanced restricted 1-circle, 2-circle, and 3-circle drawings of K_6. We prove that balanced restricted 3-circle

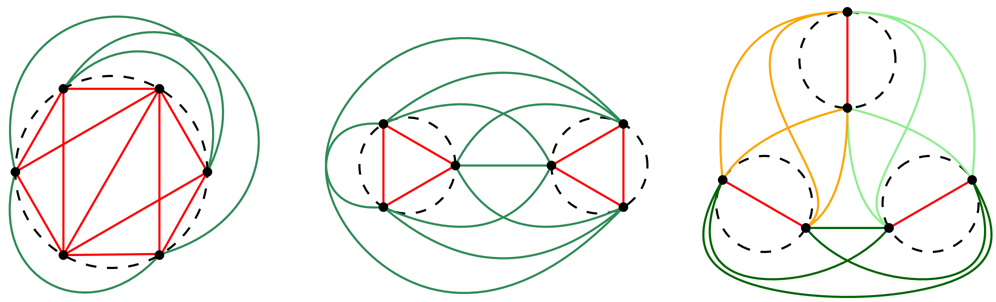


FIGURE 17. Balanced 1-circle, 2-circle and 3-circle drawings of K_6; each with $3 = H(6)$ crossings.

drawings are suboptimal for n large enough.

Corollary 1.4. *For $n = 9, 10$ and $n \geq 13$, the number of crossings in any balanced restricted 3-circle drawing of K_n exceeds $H(n)$, i.e., $\mathrm{bcr}_{③}(K_n) > H(n)$.*

Proof. Suppose $n \geq 14$. Let $q \geq 5$ and $r \in \{-1, 0, 1\}$ be the unique integers such that $n = 3q + r$. We want to show that $\mathrm{bcr}_{③}(K_n) - H(n) > 0$. Consider a balanced restricted 3-circle drawing of K_n with q, q, and $q + r$ vertices on the 3 circles. Then

$$\mathrm{bcr}_{③}(K_n) = \mathrm{cr}_{③}(K_{q,q,q+r}) + 2\binom{q}{4} + \binom{q+r}{4}. \tag{5.1}$$

We use Theorem 1.1 to bound $\mathrm{cr}_{③}(K_{q,q,q+r})$ and $\lfloor q/2 \rfloor \lfloor (q-1)/2 \rfloor \geq q(q-2)/4$ to remove the floor function.

$$\begin{aligned}
&\mathrm{cr}_{③}(K_{q,q,q+r}) \\
&\geq \mathrm{cr}_{②}(K_{q,q}) + 2\,\mathrm{cr}_{②}(K_{q,q+r}) + q^2 \left\lfloor \frac{q+r}{2} \right\rfloor \left\lfloor \frac{q+r-1}{2} \right\rfloor + 2q(q+r) \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \\
&\geq \mathrm{cr}_{②}(K_{q,q}) + 2\,\mathrm{cr}_{②}(K_{q,q+r}) + \frac{1}{4} q^2 (q+r)(3q+r-6).
\end{aligned}$$

By Equation 1.1, it holds $\mathrm{cr}_{②}(K_{q,q}) = q\binom{q}{3}$. By Equation 1.2, we obtain $\mathrm{cr}_{②}(K_{q,q-1}) = (q-2)\binom{q}{3}$ and $\mathrm{cr}_{②}(K_{q,q+1}) = (q-1)\binom{q+1}{3}$. Thus

$$\mathrm{cr}_{③}(K_{q,q,q+r}) \geq \begin{cases} 3q\binom{q}{3} + \frac{1}{4}q^3(3q-6) & \text{if } r = 0, \\ (3q-4)\binom{q}{3} + \frac{1}{4}q^2(q-1)(3q-7) & \text{if } r = -1, \\ q\binom{q}{3} + 2(q-1)\binom{q+1}{3} + \frac{1}{4}q^2(q+1)(3q-5) & \text{if } r = 1. \end{cases} \tag{5.2}$$

The result holds for $n \geq 14$ ($q \geq 5$) by (5.1), (5.2), and $H(n) \leq \frac{1}{64}(n-1)^2(n-3)^2$:

$$\mathrm{bcr}_{③}(K_n) \; - \; H(n) \quad \geq \quad \begin{cases} \frac{1}{64}(7q^4 - 24q^3 - 46q^2 + 24q - 9) > 0 & \text{if } r = 0, \\ \frac{q}{192}(21q^3 - 100q^2 - 36q + 112) > 0 & \text{if } r = -1, \\ \frac{q}{192}(q+2)(21q^2 - 86q - 8) > 0 & \text{if } r = 1. \end{cases}$$

Finally, by (5.1) and Corollary 3.5, $\mathrm{cr}_{③}(K_9) = \mathrm{cr}_{③}(K_{3,3,3}) \geq 38 > 36 = H(9)$, $\mathrm{cr}_{③}(K_{10}) = \mathrm{cr}_{③}(K_{3,3,4}) + 1 \geq 62 > 60 = H(10)$, and $\mathrm{cr}_{③}(K_{13}) = \mathrm{cr}_{③}(K_{4,4,5}) + 7 \geq 227 > 225 = H(13)$. □

Our previous argument does not settle the cases $n = 11$ and 12. However, the Harary-Hill constructions for $n \leq 8$ are in fact balanced restricted 3-circle drawings. If we allowed for unbalanced constructions for $n \geq 8$, Theorem 1.3 implies that

$$\mathrm{cr}_{③}(K_{2,2,n-4}) + \binom{n-4}{4} = \binom{n-4}{4} + \frac{3}{2}(n-4)^2 - (n-4) - \begin{cases} 3 & n \text{ even} \\ 3/2 & n \text{ odd} \end{cases} \geq H(n)$$

for $n \geq 8$, with equality if and only if $8 \leq n \leq 11$. That is, for $n \geq 8$ the drawings of K_n given by a crossing-optimal 3-circle drawing of $K_{2,2,n-4}$ together with the straight-line drawings inside of the three circles achieve $H(n)$ crossings if and only if $8 \leq n \leq 11$.

6. Conclusion and open problems

In this paper, we prove upper and lower bounds on the tripartite-circle crossing number of complete tripartite graphs. For the lower bound, we introduce formulas describing the number of crossings in a tripartite-circle drawing. For the upper bounds, we present beautiful drawings. While there are drawings of K_n achieving the Harary-Hill bound that are obtained in a simple way from balanced 1- and 2-partite-circle drawings, our results imply that this is not the case for balanced 3-partite-circle drawings for $n \geq 13$. It remains open for future work whether the same holds for k-partite circle drawings when $k > 3$. We have made progress in the direction of extending our work to $k > 3$ and plan to return to this question in a subsequent paper. We conclude with a list of interesting open problems for future work:

- Do there exist k-circle drawings achieving the Harary-Hill bound for $k > 3$?
- Can the number of crossings of k-circle drawings generally be described by labels analogous to x-,y-labels?
- How are crossing-minimal k-circle drawings characterized?
- What are the exact values for small graphs? Is $\mathrm{cr}_{③}(K_{3,3,3}) = 42$? For the remaining displayed values in Table 1, we believe that the truth lies closer to the presented upper bounds. In particular, it remains to develop better tools in order to improve the lower bounds.

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