Drawing Planar Graphs with Prescribed Face Areas

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Abstract. We study drawings of planar graphs such that every inner face has a prescribed area. A plane graph is *area-universal* if for every area assignment on the inner faces, there exists a straight-line drawing realizing the assigned areas. It is known that not all plane graphs are area-universal. The only counterexample in literature is the octahedron graph.
We give a counting argument that allows to prove non-area-universality for a large class of triangulations. Moreover, we relax the straight-line property of the drawings, namely we allow the edges to bend. We show that one bend per edge is enough to realize any face area assignment of every plane graph. For plane bipartite graphs, it suffices that half of the edges have a bend.

1 Introduction

Planar graphs link graph theory and geometry. Since various real-life problems are connected to embedded graphs on the surface of our planet, planar graphs and their representations have many practical applications: in the manufacture of chips and electrical circuits, in the design of network infrastructure such as roads, subway, and utility lines. Other applications are in cartography, geography, and visualization. Consequently, there is a large body of theoretical and applied work on representations of planar graphs with special features [11].

One direction is the representation of planar graphs with given areas. Proportional contact representations, so-called cartograms, are studied when areas are assigned to the vertices [2, 5].

We are interested in drawings of plane graphs such that the inner faces have prescribed face areas. Let $G = (V, E)$ be a plane graph and F' the set of its inner faces. A *redrawing* of G is a drawing such that the set of inner faces remains. We denote the set of all redrawings of G by $\mathcal{D}$.

A *face area assignment* is a function $A : F' \to \mathbb{R}^+$. Let $a : F' \times \mathcal{D} \to \mathbb{R}^+$ be a function measuring the area of an inner face f in a specified redrawing D of G. A redrawing D of G is A-*realizing* if $a(f, D) = A(f)$ for each face $f \in F'$. The graph G is *area-universal* if for every face area assignment A there exists an A-realizing straight-line redrawing of G. The graph G is *equiareal* if there exists a straight-line redrawing D of G such that $a(f, D) = 1$ for each face $f \in F'$.

1.1 Previous Work & Our Contribution

Ringel [10] can be seen as the initiator of the study of drawings of plane graphs with prescribed face areas. He gave examples of equiareal graphs and proved that not every plane graph is equiareal. Moreover, Ringel conjectured that every plane 3-regular graph is equiareal. This turned out to be true when Thomassen [12] proved something stronger: Every plane graph with maximum degree 3 is area-universal. There are further area-universal graph classes. A straight-forward result is the class of *planar 3-trees*, also known as *stacked triangulations*. Biedl & Ruiz Velázquez [3] proved that planar 3-trees have realizing drawings with rational coordinates if the face areas are rational.

To the best of our knowledge, this is the state of the art. Clearly, many interesting questions remain open. In this paper, we study two directions, namely negative and positive results. In terms of negative results, all non-area-universal graphs in literature contain the octahedron graph as a subgraph. This leads to the following questions:

- ► Is the octahedron graph the only minimal counterexample (by taking subgraphs)? Is it the only 4-connected counterexample?
- ► Are highly connected planar graphs area-universal?

The answers to these questions are negative. In Section 2 we give a broad class of triangulations which are not area-universal. This class contains many 4-connected graphs. Additionally, we show that the 5-connected icosahedron graph is not area-universal. Hence, high connectivity does not imply area-universality.

In terms of positive results, we investigate relaxations:

- ► What drawings can realize all face area assignments for every plane graph?

In Section 3 we show that every plane graph has a drawing realizing any face area assignment such that each edge has at most one bend. Moreover, every plane bipartite graph has a drawing realizing any face area assignment such that at most half of the edges have a bend.

2 Non-Area-Universal Graphs

In this section we discuss non-area-universality. It is known that the octahedron graph is not area-universal [10]. There exist two proofs. In both proofs, the used area assignment is similar to ours; however, the concepts behind the proofs are different. Ringel [10] shows that the system of equations has no rational solution. A different proof relies on a classical geometric result on the area of a triangle inscribed into a triangle. The proof is similar to [1] (for details on the geometric result see [4]).

In contrast, we give a simple counting argument. Interestingly, the ideas can be extended to show that every plane Eulerian triangulation is not area-universal. Indeed, we use the fact that the dual graph of a plane triangulation is bipartite, and hence has an inner face 2-coloring. An *inner face 2-coloring* of a triangulation is a coloring of the inner faces with white and black such that every inner edge is incident to a white and black face. For an inner face 2-coloring of the octahedron graph, see Figure 1.

Theorem 1. *Every plane Eulerian triangulation (on more than 3 vertices) is not area-universal.*

Proof. Let T be a plane triangulation on n vertices with an inner face 2-coloring. We denote the set of white faces by W and the set of black faces by B. Recall that the number of inner faces of a triangulation with n vertices is $2n-5$. Without loss of generality $|W| > |B|$, that is $|W| \geq n-2$.

We show that for sufficiently small $\varepsilon > 0$, T has no straight-line drawing realizing the following area assignment with a total area of 1:

$$A(f) := \begin{cases} \varepsilon & \text{if } f \in W, \\ \delta_\varepsilon := \frac{1-\varepsilon|W|}{|B|} & \text{else.} \end{cases}$$

The idea is to show that an A-realizing drawing has contradicting properties. One property is that every white face needs a big angle of almost π. Another property is that every inner vertex can have at most one such big angle, and every outer vertex has none. Since the number of white faces exceeds the number of inner vertices this gives a contradiction. To illustrate the idea of the proof, we start with the degenerate case of $\varepsilon = 0$.

We suppose, for the purpose of contradiction, that there exists an A-realizing drawing D of T. Since every inner edge e is incident to a black face, e has positive length in D; otherwise, the area of the black face cannot be realized. This yields the main property of D: One angle of every white face is of size π, i.e., a *big* angle. The area of a white face vanishes only if a vertex lies on a non-incident edge. Since all edges have positive length, the vertex must lie on an inner point of the non-incident edge. We *assign* (the big angle of) this white face to the vertex with the big angle. Clearly, only inner vertices may have big angles. Recall that T has $n-2$ white faces and $n-3$ inner vertices. By the pigeonhole principle, in every assignment of big angles to inner vertices, there exists a vertex v which is assigned to two big angles, see Figure 1. Vertex v is also incident to at least 2 black faces which are separating the white faces. However, due to the two big angles, no space remains in order to realize the area of the black faces incident to v. Consequently, D is not a realizing drawing, thus a contradiction.

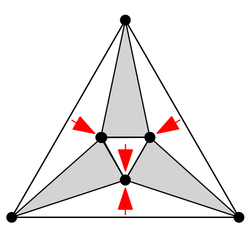

Fig. 1. The octahedron graph with an inner face 2-coloring indicating the area assignment. In every assignment of white faces to inner vertices, one vertex is assigned to two faces.

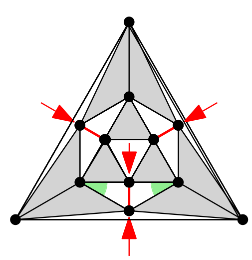

Fig. 2. The icosahedron graph with an inner face 2-coloring indicating an area assignment which cannot be realized by any straight-line drawing.

For the case $\varepsilon > 0$, we suppose that there exists a realizing straight-line drawing. Since triangles are affine equivalent, any two realizing straight-line drawings are affine equivalent. Hence, we suppose that there exists a straight-line drawing within an equilateral triangle with area 1 and sidelength L. It has the following properties:

- The longest edge is of length L.
- Each edge is incident to a black face and, hence, at least of length $s := {}^{2\delta_\varepsilon}/_{L}$.
- The height of a white face is bounded by ${}^{2\varepsilon}/_{s}$.
- There exists a small α'_ε, continuously decreasing with ε, such that a white face angle is either *tiny* (at most α'_ε) or *big* (at least $\alpha_\varepsilon := \pi - 2\alpha'_\varepsilon$).
- Each white face has a big angle.
- There exists β_ε such that a black face angle is at least of size β_ε.

Recall that $2\alpha_0 + 2\beta_0 > 2\pi$. By continuity of α_ε and β_ε, the intermediate value theorem implies the existence of $\epsilon > 0$ with

$$2\alpha_\varepsilon + 2\beta_\varepsilon > 2\pi.$$

Consequently, in every realizing drawing no inner vertex may realize the big angles of two white faces. However, the number of white faces exceeds the number of inner vertices. This is a contradiction and, hence, establishes the proof. □

Remark 1. It suffices to choose ε in the order of n^{-3}.

Remark 2. Indeed, a triangulation has an inner face 2-coloring if and only if it has a face 2-coloring. This stems from the fact that the number of edges of a Eulerian inner triangulation of a quadrangle is divisible by 3.

Remark 3. This construction implies that one cannot hope for drawings realizing the areas up to a constant factor. If ε is small enough, then there is no c such that a drawing of an Eulerian triangulation fulfills ${}^{1}/_{c} \cdot A(f) \leq a(f, D) \leq c \cdot A(f)$ for all inner faces f.

Note that the graphs of Theorem 1 are at most 4-connected. With similar ideas (but more work) we can show that the 5-connected icosahedron graph is not area-universal. Here, we only sketch the proof idea.

Theorem 2. *The icosahedron graph is not area-universal.*

Proof (Sketch). We can show that for $\varepsilon > 0$ small enough, there is no straight-line drawing of the icosahedron in which the white faces in Figure 2 have area ε and the black faces have area $\delta_\varepsilon := \frac{1-10\varepsilon}{9}$. Note that there are three (red) edges adjacent to two white faces. These edges may become small, therefore, the adjacent white triangles do not necessarily need a big angle in a realizing drawing. Hence, a simple counting argument is not sufficient. However, as before, each of the four white triangles with three black edges needs a big angle. In every big angle assignment of these four white triangles, there is a red *special* edge whose vertices are both assigned. Moreover, the angle between two black edges in a white face, is either *tiny* (almost 0) or *big* (almost π). A case distinction on the size of the (green) angles opposite to the red special edge yields the result. □

3 Drawing Planar Graphs with Bends

We now aim for realizing each face area assignment of every plane graph. As discussed in Section 2, this is impossible for straight-line drawings. Therefore, we relax the straight-line property by allowing the edges to have bends. A drawing of a plane graph is a *k-bend drawing* if each edge is a concatenation of at most $k+1$ segments. These drawings are also called *polyline* drawings. In this section we show that one bend per edge is sufficient.

Theorem 3. *Let G be a plane graph and $A\colon F' \to \mathbb{R}^+$ a face area assignment. Then, there exists an A-realizing 1-bend redrawing of G.*

Proof. Without loss of generality, we assume that G is a plane triangulation: If G is not a triangulation, there exists a triangulation T such that G is an induced subgraph. For each face of G, partition the assigned area between its subfaces in T and obtain the area assignment A' of T. Given an A'-realizing 1-bend redrawing of T, delete the artificial vertices and edges. The result is an A-realizing 1-bend redrawing of G.

We construct the final drawing of G in four steps (see definitions below):

1. Take a $\perp$-contact representation $\mathcal{C}$ which yields a rectangular layout $\mathcal{L}$.
2. Obtain a weak equivalent rectangular layout $\mathcal{L}'$ realizing the areas.
3. Define a degenerate drawing $D_\perp$.
4. Construct a non-degenerate drawing from $D_\perp$.

The steps are visualized for the octahedron graph in Figure 3.

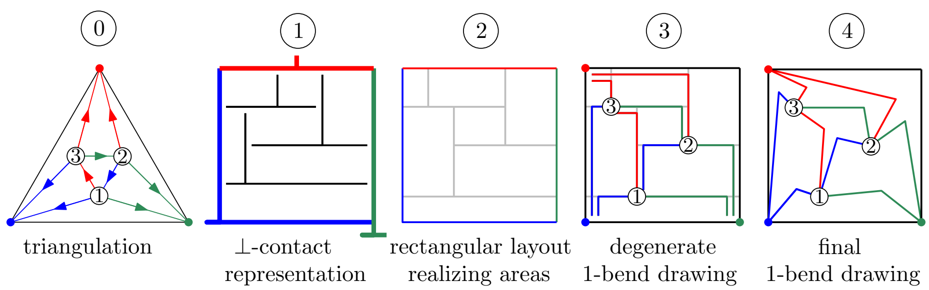

Fig. 3. Construction of a 1-bend drawing realizing the prescribed areas in 4 steps.

In the first step, we construct a $\perp$-contact representation $\mathcal{C}$ of G. A $\perp$-shape is the union of a horizontal and vertical segment such that the lower end of the vertical segment lies in the horizontal segment. We call this point of intersection the *heart* of the $\perp$-shape. Each of the other three ends of the segments is an *end* of the $\perp$-shape. A $\perp$*-contact representation* of a graph $G=(V,E)$ is a family of $\perp$-shapes $\{\perp_v : v \in V\}$ where $\perp_u$ and $\perp_v$ intersect if and only if $(u,v) \in E$. Moreover, if $\perp_u$ and $\perp_v$ intersect then the intersection must consist of a single point which is an end of $\perp_u$ or $\perp_v$. The point is the *contact point* of $\perp_u$ and $\perp_v$.

Lemma 1 ([8]). *Every plane triangulation has a $\perp$-contact representation $\mathcal{C}$ such that each inner face is represented by a rectangular region.*

Following the ideas of [8], $\mathcal{C}$ can be constructed as described in detail in Section 3.1 in [2]. Observe that the $\perp$-shapes of two outer vertices may be pruned to segments such that the outer face is the complement of a rectangle. The segments of the $\perp$-shapes of inner vertices partition the rectangle into finitely many rectangles; such a partition is called a *rectangular layout.* By Lemma 1, $\mathcal{C}$ yields a rectangular layout $\mathcal{L}$ in which every rectangle r corresponds to a face f_r of G.

In the second step, we want to achieve correct areas in a weak-equivalent layout. The maximal segments of a rectangular layout yield a segment contact graph. Two rectangular layouts are *weak-equivalent* if their segment contact graphs are isomorphic. We apply the following lemma.

Lemma 2. *For every rectangular layout with area assignment w on the inner rectangles, there exists a weak-equivalent layout realizing the areas of w.*

This lemma has several variants and proofs; we refer to [5, 6, 13]. For each rectangle r corresponding to the face f_r, we set $w(r) := A(f_r)$. By Lemma 2, we obtain a weak-equivalent rectangular layout $\mathcal{L}'$ in which the area of each rectangle r is $w(r)$. Due to the weak-equivalence of $\mathcal{L}$ and $\mathcal{L}'$, the layout $\mathcal{L}'$ can be viewed as a $\perp$-contact representation $\mathcal{C}'$, which now realizes the areas.

In the third step, we obtain a (degenerate) 1-bend drawing $D_\perp$ of G from $\mathcal{C}'$:

- Place each vertex v in the heart of $\perp_v$; (for a pruned $\perp$-shape, the heart coincides with the bottom or left end of the remaining segment),
- The edges are supported by the segments of C' in the following way: If two vertices u and v share an edge, $\perp_u$ and $\perp_v$ have a point of contact in which a vertical and a horizontal segment meet. We define the edges to run from the heart of one $\perp$-shape (along the horizontal segment) to the contact point and then (along the vertical segment) to the heart of the other $\perp$-shape.

Up to the fact that two edges may intersect interiorly (but do not cross), the properties of a plane redrawing of G are fulfilled. We call such an embedding *degenerate drawing.* By construction, each edge consists of a horizontal and a vertical segment and, hence, has at most (and in general exactly) one bend.

Observation 1 *$D_\perp$ is a degenerate 1-bend redrawing of G realizing A.*

For the fourth step, it remains to remove the degeneracies of $D_\perp$. A bend of an edge e can be interpreted as a vertex w_e of degree 2, which we call *bend vertex.* Moreover, we refer to the two incident edges of w_e as its horizontal and vertical segments. As part of the degeneracies, bend vertices intersect non-incident edges. We handle this issue by parallel shifts. A *parallel shift* of a bend vertex w in a drawing D yields a (planar) redrawing D' of D in which only vertex w has a new special position: Let p and p' denote the positions of w in D and D', respectively. Let ℓ be the line through the two neighbors of w in D. A redrawing D' of D is obtained by a parallel shift of w if p' lies on the line ℓ' parallel to ℓ through p, see Figure 4. A bend vertex w is *shiftable* if there exists a parallel shift of w.

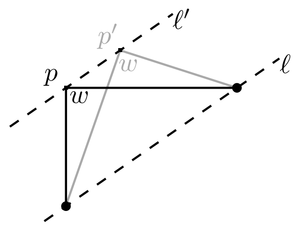

Fig. 4. A parallel shift of w.

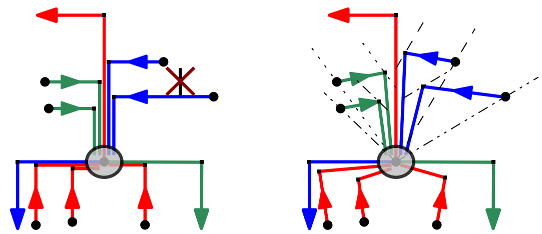

Fig. 5. The neighborhood of a typical vertex in the degenerate drawing $D_\perp$ and in the final 1-bend drawing.

Observation 2 *A parallel shift of a bend vertex keeps all face areas invariant.*

The $\perp$-contact representation induces a coloring and an orientation of the inner edges: each edge corresponds to a contact point of two $\perp$-shapes. Orient the edge such that it is an outgoing edge for the vertex belonging to the $\perp$-shape whose end is the contact point. Color the edge red, blue, or green, if the contact point is the top end, left end, or right end of the $\perp$-shape, respectively. (Such a coloring and orientation is a *Schnyder wood* of G.) We analyze the typical situation for a vertex v in $D_\perp$, see Figure 5. By construction, v has three outgoing edges such that all incoming edges partially run on one of these outgoing edges. Observe that the vertical segments of the incoming blue and green edges are free of segments touching it from the top: Due to the $\perp$-contact representation, every horizontal segment has exactly one vertical segment touching it from above. Moreover, the vertex is placed on this intersection point. Thus, the bend vertex of the lowest green and blue incoming edges is shiftable topwards. Due to the fact that every rectangle has positive area, some space is guaranteed. Therefore, we can parallel shift the bend vertex such that the edge is free of degeneracies. In particular, the bend vertex does not intersect non-incident edges anymore. Hence, the bend vertex of the second lowest incoming edge of v becomes shiftable. We iterate this process for all v such that all blue and green bend vertices do not intersect non-incident edges. Afterwards, only red bend vertices are involved in degeneracies.

For every vertex, we consider the incoming red edges, which have either a left or a right bend with respect to the orientation. Consider the rightmost right bend (and likewise the leftmost left bend) vertex. Its horizontal segment is free to the bottom since it is rightmost (leftmost) and the vertical segment is free to both sides, since by the first step there is no green or blue bend vertex on a red segment. Consequently, the rightmost right bend (leftmost left) vertex is shiftable to the bottom. We shift it parallel downwards such that no new degeneracies are introduced. Hence, the number of degeneracies decreased. Moreover, this process achieved that the second rightmost (leftmost) bend vertex becomes shiftable. By iterating, we remove all degeneracies. Finally, we have a 1-bend drawing realizing the areas prescribed by A. □

Knowing that one bend per edge is always sufficient, we wonder how many bends may be necessary to realize a face area assignment of a plane graph? We construct a family of lower bound examples with the octahedron graph.

Lemma 3. *For every $k \geq 3$, there is a graph G on $n := 4k - 6$ vertices and a face area assignment A, such that in every A-realizing polyline drawing of G at least $1/12$ of the edges have a bend.*

Proof. Recall that, by Theorem 1, the octahedron graph is not area-universal. Hence, there are area assignments such that at least one of its twelve edges needs a bend. We call such an area assignment *bad*.

Now, we construct a graph for each $k \geq 3$. Take a triangulation on k vertices with an inner face 2-coloring. Without loss of generality, assume that $k-2$ inner faces are white. *Stack* an octahedron graph into each white face f, i.e., identify three outer vertices of a plane octahedron graph and with the vertices of f. This yields a graph G on $k+3(k-2) = n$ vertices, consisting of $k-2$ octahedron graphs which are pairwisely edge-disjoint. Let A be a face area assignment such that the inner faces of the stacked octahedron graphs obtain a bad area assignment and the remaining black faces receive some arbitrary value.

Consider an A-realizing polyline drawing D of G. By Theorem 1, each of the octahedron-subgraphs has least one edge with a bend in D. By the edge-disjointness, every edge with a bend can satisfy only one of the octahedron graphs. Hence, in each of the octahedron graphs at least one of the twelve edges has a bend. This implies the claim. □

Corollary 1. *For a plane graph G with area assignment A, let $B_k(G, A)$ denote the minimum number of bends in a k-bend drawing realizing A. For all $k \geq 1$*

$$1/12|E| \leq \max_G \max_A \{B_k(G, A)\} \leq |E|.$$

Note that the octahedron graph may not yield a better lower bound.

Proposition 1. *For every face area assignment of the octahedron graph, there exists a realizing drawing with at most one bend in total.*

Proof (Sketch). The proof consists of two ideas. Firstly, let G be the octahedron graph and e an edge of the triangle of inner vertices, see Figure 1. Deleting e gives a quadrangle Q. Replacing e by the other diagonal $\overline{e}$ of Q yields a planar 3-tree. Planar 3-trees are area-universal.

Secondly, the intermediate value theorem asserts a position for a bend vertex of e in Q such that the area of Q is arbitrarily split among the two adjacent faces of e. Hence, each area assignment of G is realizable with a bend on e. □

Observation 3 *The last argument shows that every 2-degenerate quadrangulation is area-universal since it can be constructed by iteratively inserting a degree 2 vertex in a quadrangle.*

3.1 Fewer Bends for Planar Bipartite Graphs

Now, we improve the number of sufficient bends for plane bipartite graphs. We show that at most half of the edges need a bend. Interestingly, no plane bipartite graph is known that needs a bend.

Theorem 4. *Let $G = (X \cup Y, E)$ be a plane bipartite graph and $A\colon F' \to \mathbb{R}^+$ a face area assignment. Then, there exists an A-realizing 1-bend redrawing of G with at most $|E|/2$ bends.*

Proof. First, we assume that G is a quadrangulation. The proof consists of four steps, illustrated in Figure 6. The main difference to the proof of Theorem 3 lies in Step 1. Steps 3 and 4 are relatively more involved.

1. Take a segment contact representation $\mathcal{C}$ yielding a rectangular layout $\mathcal{L}$.
2. Obtain a weak equivalent rectangular layout $\mathcal{L}'$ realizing the areas.
3. Define a degenerate drawing D.
4. Construct a non-degenerate drawing from D by parallel shifts.

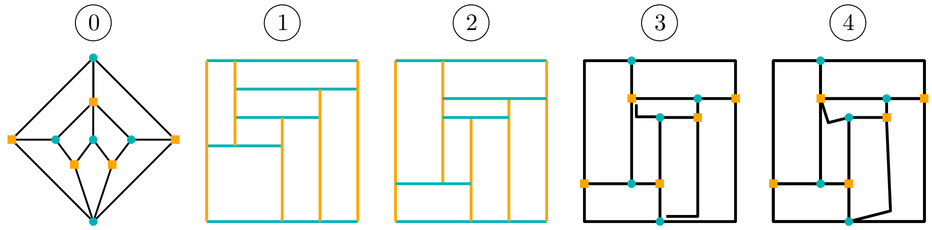

Fig. 6. Construction of a 1-bend drawing realizing the prescribed areas in 4 steps.

The first step is to take a *segment contact representation* of $G = (X \cup Y, E)$. This is a family $\{s_v | v \in V\}$ of horizontal and vertical segments where s_v and s_u intersect if and only if $(u, v) \in E$. Moreover, each non-empty intersection consists of a single point which is an endpoint of at least one of the segments.

Lemma 4 ([7, 9]). *Every plane quadrangulation has a segment contact representation such that each inner face is represented by a rectangle.*

Let $\mathcal{C}$ be a segment contact representation of G. We assume that the vertices of X are represented by vertical segments. Segments of $\mathcal{C}$ partition a rectangle into rectangles, and hence, $\mathcal{C}$ yields a rectangular layout $\mathcal{L}$. In the second step, we obtain a weak equivalent rectangular layout $\mathcal{L}'$ realizing the areas by Lemma 2; let $\mathcal{C}'$ denote the corresponding segment contact representation. In the third step, we define a degenerate drawing D from $\mathcal{C}'$. The challenge is to place the vertices such that, firstly, we save one bend per vertex, and secondly, the degeneracies can be removed by parallel shifts. We distinguish two cases depending on the minimal degree δ of G. Note that in a quadrangulation δ is 2 or 3.

<u>Case 1:</u> $\delta(G) = 3$. Since every segment has only two endpoints but at least three contacts, every segment has an inner contact point. We construct D as follows:

- $v \in X$ is placed on topmost inner contact point of the vertical segment s_v.
- $v \in Y$ is placed on leftmost inner contact point of the horizontal segment s_v.
- $e = (v, w) \in E$ is supported by the segments s_v and s_w in C: e runs from v along s_v to the contact point of s_v and s_w, and then along s_w to w.

Observation 4 *D is a degenerate (orthogonal) 1-bend redrawing of G realizing the areas prescribed by A. The number of bends is at most $|E| - |V|$.*

For the number of bends, observe that by placing the vertex on an inner contact point, the corresponding edge has no bend. Hence, we save one bend per vertex and the number of bends is at most $|E| - |V|$.

In the fourth step, we remove the degeneracies; again, by parallel shifts of bend vertices. Indeed, we iterate twice through all vertices. For a vertex $v \in X$ ($v \in Y$) with a vertical (horizontal) segment s_v, let $\mathcal{B}_1(s_v)$ denote the set of bend vertices on s_v with a horizontal (vertical) segment touching s_v from the right (bottom); we exclude the endpoints of s_v. Likewise $\mathcal{B}_2(s_v)$, denotes the set of bend vertices on s_v not in $\mathcal{B}_1(s_v)$.

Loop 1: For each $v \in X$ ($v \in Y$), do: while $\mathcal{B}_1(s_v)$ is not empty, choose the topmost (leftmost) bend vertex $b \in \mathcal{B}_1(s_v)$, parallel shift b. Delete b from $\mathcal{B}_1(s_v)$.
Loop 2: For each $v \in X$ ($v \in Y$), do: while $\mathcal{B}_2(s_v)$ is not empty, choose the topmost (leftmost) bend vertex $b \in \mathcal{B}_2(s_v)$, parallel shift b. Delete b from $\mathcal{B}_2(s_v)$.

In order to prove that Loop 1 is possible, consider a vertex $v \in X$ with vertical segment s_v, see Figure 7. We need to argue that the topmost $b \in \mathcal{B}_1(s_v)$ is shiftable. By definition, each $w \in Y$ is placed on the leftmost contact point of s_w. Hence, the horizontal segment of b is the leftmost part of some s_w and therefore free of bend vertices. Since b is the leftmost bend vertex in $\mathcal{B}_1(s_v)$, the vertical segment of s_v is free to the right. Therefore, b is shiftable down-rightwards. Moreover, after shifting b, the second topmost bend vertex becomes shiftable. Consequently, by the order from top to bottom, the horizontal segment of each bend vertex is free to the right if considered. The argument for $v \in Y$ is analogous. After Loop 1, every segment is free to one side in the following way: Every vertical segment is free to the right and every horizontal segment is free to the bottom.

In Loop 2, a considered bend vertex $b \in \mathcal{B}_2(s_v)$ is shiftable due to the order and the 1-side-freeness guaranteed by Loop 1. Now, no bend vertex sits on a non-incident edge. Consequently, this process yields a non-degenerate 1-bend drawing of G which realizes the prescribed areas and has at most $|E|-|V|$ bends.

<u>Case 2:</u> $\delta(G) = 2$. If it exists, choose an inner vertex of degree 2 and remove the segment s_v in $\mathcal{C}'$. This results in a quadrangulation where two old faces are unified to a new face. Assign the sum of the two old face areas to the new face. Delete inner vertices of degree 2 until all inner vertices are of degree at least 3. This yields a graph G' with area assignment A'. Proceed with G' as in Case 1 with some extra care. If an outer vertex is of degree 2, we injectively place the outer vertices of degree 2 on incident endpoints of their segments. Moreover, we make the parallel shifts small enough, such that the following *special* property is fulfilled in an A'-realizing drawing of G': up to a tiny ε with $2^n\varepsilon << A_{\min}$, each face f of G' contains an axis-aligned rectangle with area $A'(f) - \varepsilon$, where $A_{\min} := \min_{f \in F'(G)} A(f)$. We use the special property to reinsert the degree 2 vertices in reverse order of deletion and obtain a sequence of drawings G'_i. We

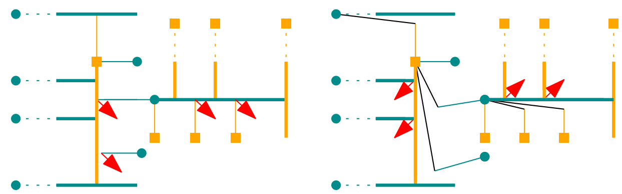

Fig. 7. Schematic neighborhood of vertices in D and after Loop 1.

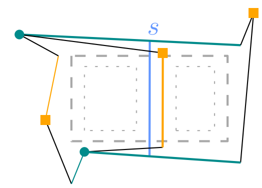

Fig. 8. Inserting degree 2 vertices in Case 2.

use the invariant that G'_k is a non-degenerate drawing where each face area is realized by an axis-aligned rectangle up to $2^k\varepsilon$. Consider the $(k+1)$th vertex v of degree 2 and the face f in G_k where v must be inserted. Assume f has area $a_1 + a_2$ and must be split into two faces f_1 and f_2 with area a_1 and a_2, where $a_1, a_2 \geq A_{\min}$. By the invariant, f contains an axis-aligned rectangle R of area $a_1 + a_2 - 2^k\varepsilon$, see Figure 8. Assume that $v \in X$. By the intermediate value theorem, there exists a vertical segment s within R such that s dissects f into two parts of area a_1 and a_2, respectively. Place v on one endpoint of s and a bend vertex b on the other endpoint of s. Note that the areas of f_1 and f_2 are realized by a rectangle up to $2^k\varepsilon$. In order to remove the degeneracies, use parallel shifts of v and b which are small enough to guarantee that f_i contains a rectangle of area $a_i - 2^{k+1}\varepsilon$. This ends Case 2.

If G is not a quadrangulation, then we consider a quadrangulation Q with G as an induced subgraph. For each face in G, dividing its area assignment among its subfaces in Q yields A'. Clearly, an A'-realizing 1-bend drawing of Q induces an A-realizing 1-bend drawing of G. However, the number of bends may exceed $|E|/2$. Therefore, we ensure to save one bend per vertex by placing the vertices on inner contact points which belong to edges of G. To do so, delete all segments belonging to artificial vertices in $\mathcal{C}$. If necessary, remove vertices of low degree iteratively as in Case 2. Afterwards, place vertices, remove degeneracies and reinsert vertices of G with low degree as in Case 1 and Case 2. Note that degree 1 vertices may also appear, but are no problem to be reinserted.

A planar bipartite graph has at most $(2|V|-4)$ edges. Therefore, the number of edges with bends is at most $|E|-|V| \leq |V|-4$ and without bends at least $|V|$. Consequently, in all cases the number of bends is less than $|E|/2$. □

If two adjacent contact points around a corner are chosen, one can save an additional bend. For the cube graph this saves 4 bends and shows that it is not only area-universal (partial 3-tree, cubic), but also *convex area-universal*, i.e. for every face area assignment there exists a realizing drawing with convex faces.

Proposition 2. *The cube graph is convex area-universal.*

Proof. Assign the doubled area to the four boundary faces. Theorem 4 and Figure 9 show the existence of a 1-bend drawing realizing the perturbed face areas with one bend on each outer edge. Replacing the boundary edges by segments halves the area of the boundary faces and gives a realizing straight-line drawing with convex faces. □

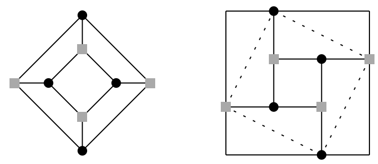

Fig. 9. The cube graph is convex area-universal.

4 Open Questions

Various interesting questions remain. We want to emphasize three of them:

- ▶ Are plane bipartite graphs area-universal?
- ▶ Are 3-connected plane bipartite graphs convex area-universal?
- ▶ How many bends are necessary and sufficient to realize arbitrary prescribed areas for all planar graphs?

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