Folding Polyominoes with Holes into a Cube★

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Abstract

When can a polyomino piece of paper be folded into a unit cube? Prior work studied tree-like polyominoes, but polyominoes with holes remain an intriguing open problem. We present sufficient conditions for a polyomino with one or several holes to fold into a cube, and conditions under which cube folding is impossible. In particular, we show that all but five special *simple* holes guarantee foldability.

Keywords: folding, origami folding, cube, polyomino, polyomino with holes, non-simple polyomino

1. Introduction

Given a piece of paper in the shape of a polyomino, i.e., a polygon in the plane formed by unit squares on the square lattice that are connected edge-to-edge, does it have a folded state in the shape of a unit cube? The standard rules of origami apply; in particular, we allow each unit square face to be covered by multiple layers of paper. Examples of this decision problem are given by the three puzzles

★A preliminary extended abstract appears in the Proceedings of the 31st Canadian Conference on Computational Geometry [1].

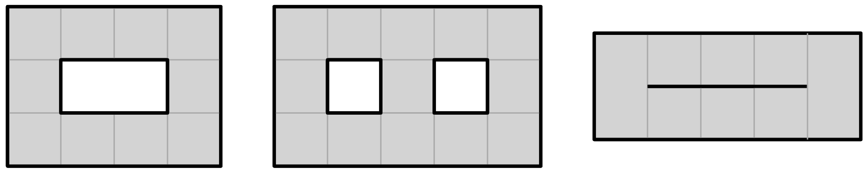

Figure 1: Three polyominoes that fold along grid lines into a unit cube, from puzzles by Nikolai Beluhov [2].

by Nikolai Beluhov [2] shown in Figure 1. We encourage the reader to print out the puzzles and try folding them.

Prior work [3] studied this decision problem extensively, introducing and solving several different models of folding. This gave rise to a model that matches the puzzles in Figure 1: Fold only along grid lines of the polyomino; allow only orthogonal folding angles (±90° and ±180°); and forbid folding material strictly interior to the cube. In this model, the prior work [3] characterizes which tree-shaped polyominoes lying within a $3 \times n$ strip can fold into a unit cube.

Notably, however, the polyominoes in Figure 1 are not tree-shaped or even simple: One puzzle has a hole, another puzzle has two holes, and a third puzzle has a degenerate hole, namely a slit. Arguably, these holes are what makes the puzzles fun and challenging. Therefore, in this paper, we embark on characterizing which polyominoes with hole(s) fold into a unit cube in this model. Although we do not obtain a complete characterization, we give many interesting conditions under which a polyomino does or does not fold into a unit cube.

The problem is sensitive to the choice of model. In the more flexible model allowing half-grid folds and 45° diagonal folds between grid points, the prior work [3] shows that *all* polyominoes of at least ten unit squares can fold into a unit cube, and lists all smaller polyominoes that fold into a cube. Thus this model already has a complete characterization of polyominoes that fold into a cube, including those with holes. Therefore, we focus on the grid-fold model described above.

Specific to polyominoes and polycubes, there is extensive work in this model on finding polyominoes that fold into many different polycubes [4] and into multiple different boxes [5, 6, 7, 8, 9].

Our Results

1. We show that all but five *simple* holes always guarantee that a polyomino containing the hole folds into a cube; see Theorem 1, Section 3.1. Four of the five remaining holes only sometimes allow for foldability, and we conjecture that one hole never helps for foldability.

2. We identify combinations of two (of the remaining five) holes that allow the polyomino to fold into a cube; see Section 3.2.
3. We show that certain of the remaining five simple holes or their combinations do not allow a foldable polyomino; see Section 4.
4. We present an algorithm that checks a necessary local condition for foldability; see Section 4.3.

2. Notation

A *polyomino* is a polygon P in the plane formed by a union of $|P| = n$ *unit squares* on the square lattice that are connected edge-to-edge. We do not require a connection between every pair of adjacent squares; that is, we allow *slits* along the edges of the lattice subject to the condition that the polyomino is connected.

We call a maximal set h of connected missing squares and slits a *hole* if the dual has a cycle containing h in its interior. We call a hole of a polyomino *simple* if it is one of the following: a unit square, a slit of size 1, slits of size 2 (L- or straight), or a U-slit of size 3, see Figure 2 for an illustration.

A *unit cube* $\mathcal{C}$ is a three-dimensional polyhedron with six unit-square faces and volume of 1.

In this paper, we study the problem of folding a given polyomino P with holes to form $\mathcal{C}$, allowing only 90° and 180° folds along the lattice. We illustrate mountain folds in red, and valley folds in blue. Whenever we show numbers on faces in crease patterns these refer to a "real" die, i.e., opposite faces sum up to 7.

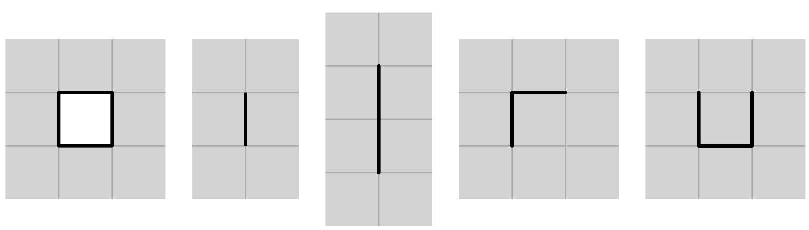

Figure 2: The five simple holes: a unit square, a slit of size 1, a straight slit of size 2, a L-slit of size 2, and a U-slit of size 3.

3. Polyominoes That Do Fold

In this section, we present polyominoes that fold. We start with polyominoes that contain a hole guaranteeing foldability.

3.1. Polyominoes with Single Holes

In this section, we show that all holes different from a simple hole guarantee foldability.

Theorem 1. *If a polyomino P contains a hole h that is not simple, then P folds into a cube.*

Proof. It is easy to see that because the hole h is non-simple, it must be a superset of one of the holes in Figure 4, that is, we distinguish the cases where h contains

- two unit squares sharing an edge,
- two unit squares sharing a vertex,
- a unit square and an incident slit,
- a slit of length at least 3 (straight, zigzagged, L-shaped, or T-shaped).

In a first step, we show that if h contains one of the four above holes, we may assume that it contains exactly this hole. Let h be a hole containing a hole h' of the above type. By definition of a hole, h needs to be enclosed by solid squares. Thus we can sequentially fold the squares of P in columns to the left and right of h' on top of the columns directly left and right of h', respectively, as illustrated in Figure 3. Afterwards, we fold the squares of P in rows to the top and bottom of h' on top of the rows directly top and bottom of h', respectively. We call the resulting polyomino P'. Note that because h is a hole of P, all neighbouring squares of h' exist in P'. Thus we may assume that we are given one of the seven polyominoes depicted in Figure 4, where striped squares may or may not be present.

Secondly, we present folding strategies. Note that the case if h' consists of two squares sharing only a vertex, we can fold the top row on its neighboring row and obtain the case where h' consist of a square and an incident slit. For an illustration of the folding strategies for the remaining six cases consider Figure 5. □

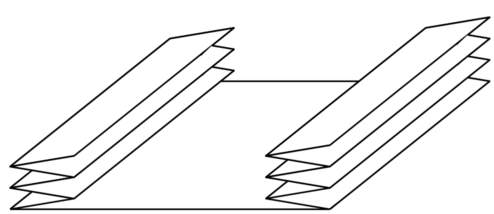

Figure 3: Folding strategy to reduce to seven cases.

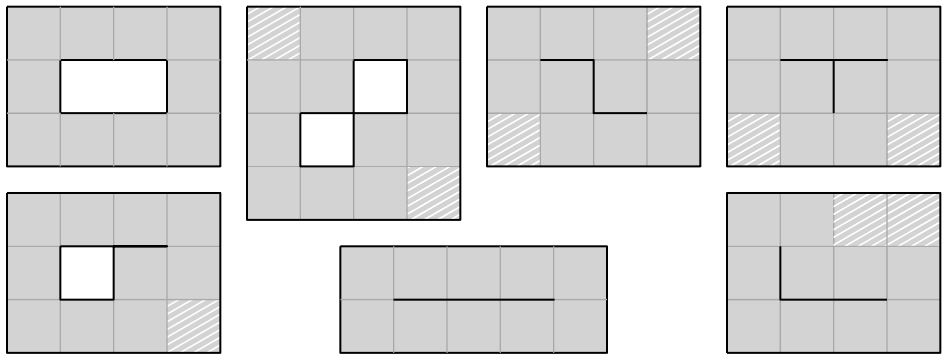

Figure 4: Any polyomino with a hole that is not simple can be reduced to one of the seven illustrated cases; striped squares may or may not be present.

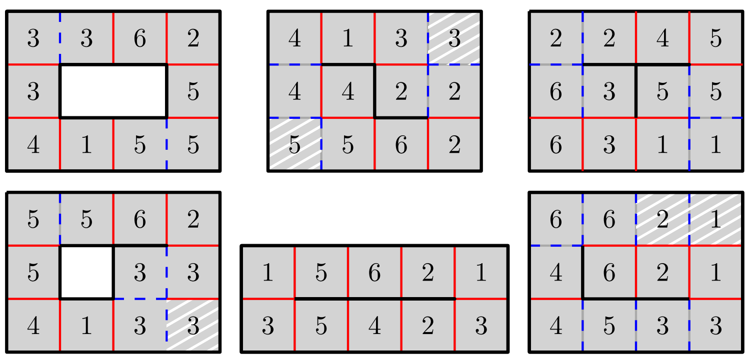

Figure 5: Crease pattern of cube foldings; mountain folds (solid red), valley folds (dashed blue). Squares with the same number cover the same face of the cube.

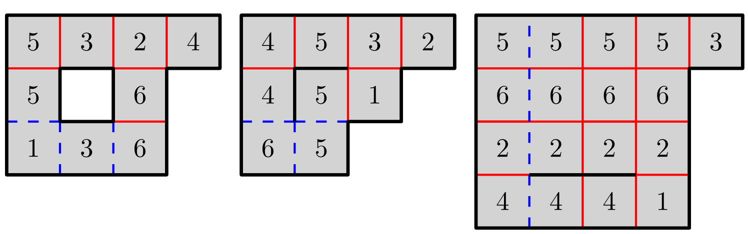

Figure 6: Four simple holes may be helpful. Mountain folds are shown in solid red, valley folds in dashed blue.

Are simple holes ever helpful?

In fact, four of the five simple holes sometimes allow foldability, as illustrated in Figure 6. Note that the U-slit of size 3 reduces to the square hole.

In Theorem 15, we show that the slit of size 1 never helps to fold a rectangular polyomino. In fact, we conjecture that the slit of size 1 never helps to fold a polyomino into $\mathcal{C}$. Corollary 1 implies that the polyominoes without the holes cannot be folded.

3.2. Combinations of Two Simple Holes

In this section we consider combinations of two simple holes that fold.

Theorem 2. *A polyomino with two vertical straight size-2 slits with at least two columns and an odd number of rows between them folds.*

Proof. As in the previous section, we first fold all rows between the slits together to one row; this is possible because there is an odd number of rows between the slits. Then, all the surrounding rows and columns are folded towards the slits. Finally, we fold the columns between the slits to reduce their number to two or three. Depending on whether the number of columns between the slits was even or odd, this yields a shape as shown in Figure 7 (a) and (b), respectively, where the striped squares may be (partially) present. In all cases, the two shapes fold as indicated by the illustrated crease pattern. Note that in Figure 7 (b) the polyomino is of course connected, which implies that at least one square of the central column is part of the polyomino, i.e., a square with label 6 is used. □

If the two slits have only one or no column between them, then the shape cannot be folded as can be verified by the algorithm of Section 4.3. In the following theorems we call a U-slit which has the open part at the bottom an A-slit. If the orientation of the U-slit is not relevant, then we call it a C-slit.

Theorem 3. *A polyomino with an A-slit and a unit square hole/C-slit in the same column above it, having an even number of rows between them, folds.*

Proof. We can assume that the upper hole is a unit square, as the flaps generated by a C-slit can always be folded away. Similar to before we fold away all surrounding rows and columns and reduce the number of rows between the A-slit and the unit square hole to two. This yields the shape of Figure 7 (c), which can be folded as indicated by the crease pattern. □

Note that if the bottom slit is a U-slit, then the shape of Figure 7 (c) does not fold, again verified by the algorithm of Section 4.3.

Theorem 4. *A polyomino with an A-slit and a unit square hole/C-slit below it and separated by an odd number of rows, folds, regardless in which columns they are.*

Proof. As before, we assume that the lower hole is a unit square, fold away the surrounding rows and columns, and reduce the number of rows between the two slits/holes to one. Furthermore, we fold the columns between the slits/holes such that most two columns remain between the two slits/holes. Consequently, we obtain one of the shapes shown in Figure 7 (d) to (g). All of them fold, with or without the striped region. Note that the upper unit square holes in Figure 7 (d) and (e) can be replaced by an A-slit which can be folded away. □

Note that if the two holes are in the same or neighboured column(s) (Figure 7 (d) and (e)), then independent of the orientation of the U-slits or whether

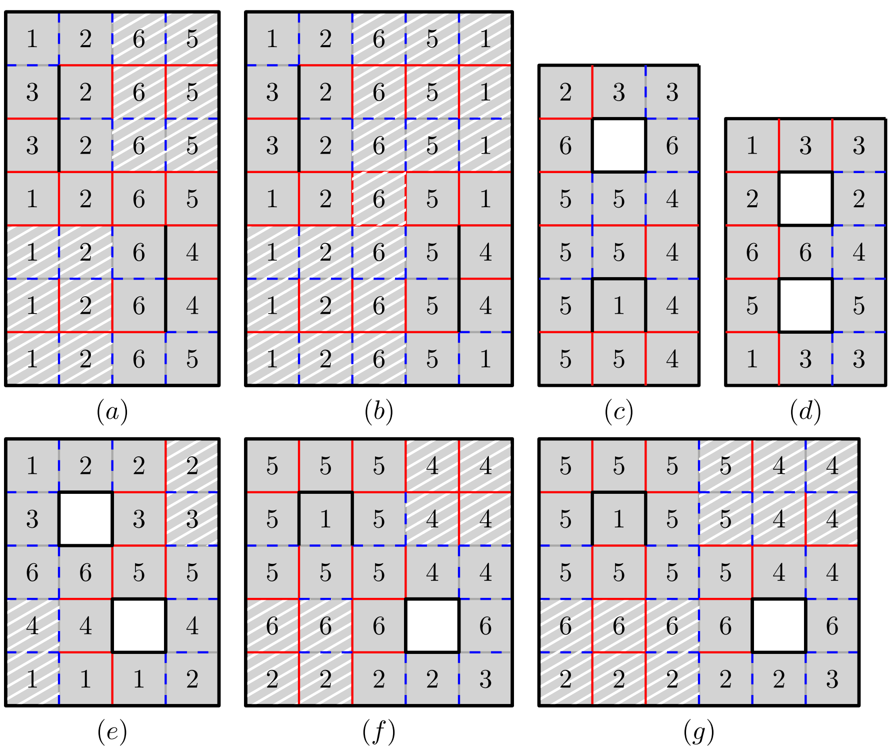

Figure 7: Combinations of two simple holes that are foldable with and without (part of) the striped region. Mountain folds are shown in solid red, valley folds in dashed blue.

they are unit square holes, any combination folds, yielding the following fact. In the other cases, the unit square incident to all three slit edges constitutes the only unit square that covers the face '1' in the unit cube.

Theorem 5. *A polyomino with two unit square holes which are in the same or in neighboured column(s) and have an odd number of rows between them folds.*

4. Polyominoes That Do Not Fold

In this section, we identify simple holes and combinations of simple holes that do not allow the polyomino to fold. First, we present some results that show how the paper is constrained around an interior vertex.

Lemma 6. *Four faces around a polyomino vertex v for which the dual graph is connected cannot cover more than three faces of $\mathcal{C}$.*

Proof. The vertex v is incident to four faces in P. As vertices of P are mapped to vertices of $\mathcal{C}$ and all vertices of $\mathcal{C}$ are incident to 3 faces, v is incident to only 3 faces in $\mathcal{C}$. □

Lemma 7. *Four faces around a vertex v not in the boundary of P cannot cover more than two faces of $\mathcal{C}$. In particular, at least two collinear incident creases are folded by* $180°$.

Proof. Let A, B, C, and D be the faces around v in circular order, see the left of Figure 8. By Lemma 6, A, B, C, and D cover at most three faces of $\mathcal{C}$. Hence, at least two faces map to the same face of $\mathcal{C}$; these can be edge-adjacent or diagonal.

In the first case, let without loss of generality A and B map to the same face. Hence, the crease between them must be folded by $180°$. Then C and D must also map to the same face of $\mathcal{C}$ to maintain the paper connected. Consequently, the crease between C and D is folded by $180°$.

In the latter case, let without loss of generality A and C map to the same face of $\mathcal{C}$. As they are both incident to v, only two options of folding those two faces on top of each other exist. Either the edge between A and B gets folded on top of the edge between B and C, this leaves a diagonal fold on B, a contradiction, or the edge between A and D gets folded on top of the edge between B and C, which results in D being mapped to C, and those are two adjacent faces, in which case we already argued that two collinear incident creases must be folded by $180°$. □

Figure 8: Illustration of Lemmas 7 and 8. 180° creases are illustrated in orange; they could be mountain or valley folds.

Lemma 8. *Consider a vertex v that is not in the boundary of a polyomino P that folds into $\mathcal{C}$. If one crease of v is folded by* 180°*, then the incident collinear crease is also folded by* 180°.

Proof. Without loss of generality, we show that if the left horizontal crease of v is folded by 180°, the same holds for the right horizontal crease. We denote the left and right adjacent vertices of v by a and b, respectively, as indicated in Figure 8, right.

Suppose for a contradiction, that the right crease is not folded by 180°. Then, by Lemma 7, both vertical creases are folded by 180°. In particular, a and b are mapped to the same vertex of $\mathcal{C}$ and thus the edges av and bv coincide. Hence, since av is folded by 180°, bv is also folded by 180°. □

Lemmas 7 and 8 imply that:

Corollary 1. *Let $k,n \geq 2$ and let P be polyomino containing a rectangular $k \times n$-subpolyomino P' whose interior does not contain any boundary of P. Then, in every folding of P into $\mathcal{C}$, all collinear creases of P' are either folded by* 90° *or by* 180°. *Moreover, either all horizontal or all vertical creases of P' are folded by* 180°*, see Figure 9.*

Proof. First, suppose for a contradiction that there exist two collinear creases, one of which is folded by 90° and the other by 180°. Then there also exists an interior vertex of P' where the crease type of the two collinear edges changes from 90° to 180°. However, by Lemma 8, if one is folded by 180°, then both are. A contradiction.

Second, suppose that not all horizontal creases are folded by 180°. Then, by the first statement, there exists a row in which no vertex has a horizontal edge that is folded by 180°. By Lemma 7, all vertical creases incident to the vertices of this row are folded by 180°. Since all collinear edges behave alike, it follows that all vertical creases are folded by 180°. □

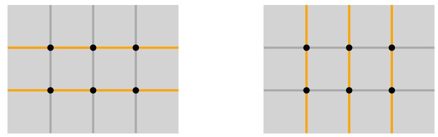

Figure 9: Illustration of Corollary 1.

4.1. Polyominoes with Unit Square, L-Shaped, and U-Shaped Holes

We begin by examining the possible foldings of a polyomino containing a unit square hole. Suppose that a given polyomino P with a unit square hole h folds into a cube. Furthermore, let the shape of h no longer be a square in the folded state. That is, the hole h is folded in a *non-trivial* way. Then, in the folded state, either all edges of h are mapped to the same edge of $\mathcal{C}$, or two pairs of edges are glued forming an L-shape. In the following, we show that if P folds into $\mathcal{C}$, the first case is impossible, while the second produces a specific crease pattern around h.

Lemma 9. *The four edges of a unit square hole h of a polyomino P that folds into $\mathcal{C}$ are not mapped to the same edge of $\mathcal{C}$ in the folded state.*

Proof. We denote the four faces of the polyomino edge-adjacent to h by A, B, C, and D, and the four faces vertex-adjacent to h be F_1, F_2, F_3, and F_4, as illustrated in Figure 10. Assume for a contradiction that all edges of h are mapped to the same edge of $\mathcal{C}$. Consider A, F_1, and B in the folded state. As the two corresponding edges of h are glued together, the three faces must be pairwise perpendicular. The similar statement holds for the triples $\{B, F_2, C\}$, $\{C, F_3, D\}$, and $\{D, F_4, A\}$. This results in a configuration as illustrated in the right of Figure 10.

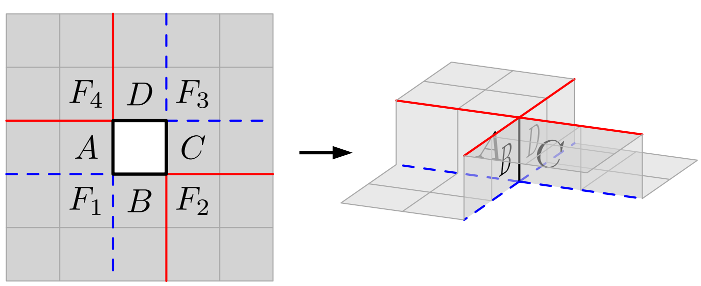

Figure 10: Four edges of a square hole glued together.

Since the faces A, B, C share an edge of $\mathcal{C}$ in the folded state such that A and B, as well as B and C are perpendicular, A and C must cover the same face of $\mathcal{C}$. Likewise, B and D cover the same face of $\mathcal{C}$. If P folds into $\mathcal{C}$, then F_1 and F_3, as well as F_2 and F_4 are mapped to same faces of $\mathcal{C}$. Suppose, without loss of generality, that in the folded state A lies in a more outer layer than C. Then, F_1 and F_4 are in a more outer layer than F_3 and F_2, respectively. Thus, face B connects the more inner layer of F_2 to the more outer layer of F_1, and at the same time D connects the inner layer of F_3 to the outer layer of F_4. Hence, faces B and D intersect, which is impossible. Therefore, if the polyomino folds into a cube, the four edges of a square hole cannot all be mapped to the same edge of $\mathcal{C}$. □

It follows that the only non-trivial way to glue the edges of a square hole h of a polyomino folded into a cube is to form an L-shape. We use this to show the following fact:

Lemma 10. *Let P be a polyomino with a unit square hole that folds into $\mathcal{C}$. In every folding of P into $\mathcal{C}$ where h is folded non-trivially (i.e., h is not a square), the crease pattern of the faces incident to h is as illustrated in the right image of Figure 10 (up to rotation and reflection).*

Proof. Suppose the four edges of h are not mapped to distinct edges of $\mathcal{C}$. Then, by Lemma 9, the four edges are not mapped to the same edge, but to two edges forming an L-shape. This effectively amounts to gluing a pair of diagonal vertices of the hole.

Let a, b, c, and d be the four vertices of h, and suppose a and c are mapped to the same vertex of $\mathcal{C}$ when folding P into $\mathcal{C}$, see also the left image of Figure 11.

Consider the crease pattern around h. We shall only focus on the angles of the creases and not the type of the fold, as there may be (and will be) other creases in P affecting the type of the creases under our consideration. Observe that the three faces incident to each of the vertices b and d are pairwise perpendicular, they form a corner of a cube. Thus, the creases emanating from b and d are all 90°. Further

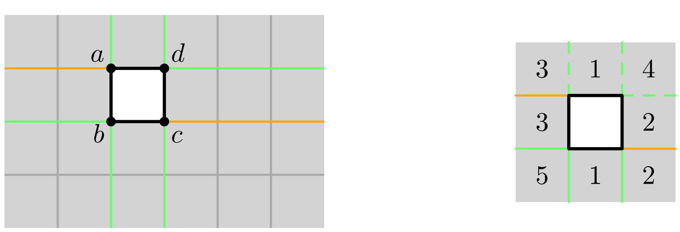

Figure 11: Left: crease pattern around a unit square hole folding into an L-shape when vertices a and c are mapped to the same vertex of $\mathcal{C}$; 90° creases are shown in green, and 180° creases in orange. Right: numbers indicate the face of the cube in the folded state; mountain folds are shown in solid, and valley folds as dashed lines.

observe that the three faces around each of the vertices a and c fold into two faces of a cube, thus leading to one of the creases being 90° and the other 180°. Finally, the two 180° creases are parallel to each other. Indeed, consider the right side of Figure 10. For a crease to form an L-shape one of the two dashed blue lines must fold to 180°, which corresponds to two parallel creases in the unfolded state. Therefore, the crease pattern in Figure 11 (left) is the only pattern of creases (up to rotation and reflection) around a non-trivially folded square hole. Figure 11 (right) shows the faces of the corresponding crease pattern. □

With the help of Lemma 10, we can show that several types of polyominoes with unit square holes do not fold into $\mathcal{C}$.

Theorem 11. *If P is a rectangle with a square hole h, then P does not fold into $\mathcal{C}$.*

Proof. First note that h is folded non-trivially, otherwise P corresponds to a rectangle which does not fold into $\mathcal{C}$. Therefore, by Lemma 10, the crease pattern around h is as depicted in Figure 11. Note that on each side of h, there exists a 90° fold.

Consider the rectangle R obtained by cutting P by the top edge of h and deleting the part below. If R has a height of at least 2, then by Corollary 1, either all vertical or all horizontal creases are folded by 180°. In the first case, in particular the creases incident to h are folded by 180°. However, this is a contradiction to the crease pattern around h in which each side of h has 90° fold. Consequently, all horizontal edges are folded by 180°. This corresponds to folding R on top of the row above h. In particular, we may assume that if P is foldable into $\mathcal{C}$ then only this row exists.

Likewise, we treat all other sides of P and obtain the polyomino P' consisting of a 3×3-rectangle with a central unit square hole, see also Figure 11 (right). In particular, P is foldable (if and) only if P' is foldable into $\mathcal{C}$.

Since h is folded non-trivially, the crease pattern of P' is given by Figure 11. Note that in the folded state P' covers only 5 faces and hence, P' does not fold into $\mathcal{C}$. □

A similar result holds for rectangular polyominoes with two unit square holes.

Theorem 12. *A rectangle with two unit square holes in the same row does not fold into $\mathcal{C}$ if the number of columns between the holes is even.*

Proof. Note that if the polyomino can be folded into $\mathcal{C}$, both holes must be folded non-trivially: If one hole behaves as a square in the folded state, i.e., is folded

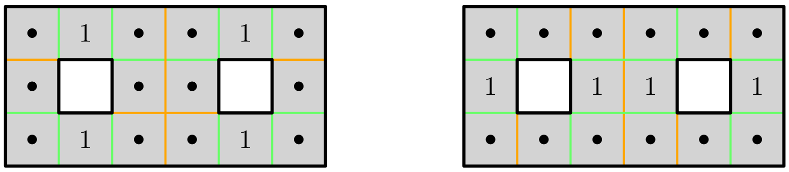

Figure 12: A polyomino that does not fold into a cube.

trivially, the polyomino is effectively reduced to a rectangle with one simple hole. However, by Theorem 11, this does not fold into $\mathcal{C}$. Consequently, both holes are folded non-trivially.

Therefore, by Lemma 10, the crease pattern around the two holes is as depicted in Figure 11. Consider the $3 \times 2k$-rectangle R between the two holes (with $k \geq 1$). By the above observation, at least one horizontal edge of R is folded by 90°. Consequently, Corollary 1 implies that all vertical edges are folded by 180°. In particular, every square of R is mapped to the same face of $\mathcal{C}$ as the leftmost (or rightmost) square in the same row of R. This reduces the polyomino to one with R being a 3×2-rectangle. We will show that the squares of P neighbouring the two holes are not able to cover $\mathcal{C}$, that is, it remains to show that the polyomino P, depicted in Figure 12, does not fold into $\mathcal{C}$.

Consider the left 3×3 block of P. If the two parallel 90° creases of it are vertical, then the right 3×3 block will also have the two parallel 90° creases run vertical, see Figure 12 (left). Then, the four faces above and below the two holes match to the same face on $\mathcal{C}$. Denote it as '1'. Observe that the rest of the faces share a vertex with '1' and thus cannot cover the face on $\mathcal{C}$ opposite to '1'.

In the second case, when the two parallel 90° creases of the left block are horizontal, then they extend into the right 3×3 block by Corollary 1. Refer to Figure 12 (right). Then, the four faces to the left and to the right of the two holes match to the same face on $\mathcal{C}$, which we denote by '1'. As before, every square of P shares a vertex with '1' and thus the face opposite to '1' on $\mathcal{C}$ cannot be covered. □

Remark. Note that the arguments of Lemma 10 and Theorems 11 and 12 extend to an L-slit of size 2, and a U-slit of size 3. The resulting crease patterns are illustrated in Figure 13.

These insights help to obtain the following result:

Theorem 13. *Let P be polyomino with two holes, which are both either a unit square, or an L-slit of size* 2*, or a U-slit of size* 3*, such that (1) P contains all the*

Figure 13: Crease pattern around an L-slit (left) and a U-slit (right). Numbers indicate the face of the cube in the folded state; 90° creases are shown in green, 180° creases in orange, mountain folds are shown in solid, and valley folds as dashed lines.

other cells of the bounding box of the two holes and (2) the number of rows and the number of columns between the holes is at least 1. *In every folding of P into $\mathcal{C}$, the two holes are not both folded non-trivially.*

Proof. If P contains a unit square holes that is not folded non-trivially, then, by Lemma 10, the crease pattern in the neighborhood the hole is as depicted in Figure 11. Likewise, if P contains an L-slit of size 2 or a U-slit of size 3 that is folded non-trivially, the crease pattern in the neighborhood the hole is as depicted in Figure 13. Note that on each side of the crease patterns in the neighborhood of the holes, there exists a 90° crease

We turn the paper such that the left hole is above the right hole as in Figure 14 and consider the rectangular region R to the right of the left hole and above the right hole.

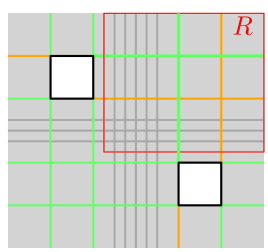

Figure 14: Two unit square holes with at least one row and column in between, if folded non-trivially imply two perpendicular 90° creases (in green).

Since on each side of the crease patterns in the neighborhood of the holes, there exists a 90° crease, R contains a vertical and a horizontal 90° crease. By Corollary 1, all collinear creases are also folded by 90°. Hence, there exists a vertex in R for which all incident creases are folded by 90°, yielding a contradiction to Lemma 7. □

4.2. Polyominoes with a Single Slit of Size 1

In the following, we show that a slit hole of size 1 does not help in folding a rectangular polyomino into $\mathcal{C}$. We start with a lemma:

Lemma 14. *In every folding of a polyomino P with a slit hole of size 1, the crease pattern behaves as if the slit hole was nonexistent.*

Proof. Consider the six faces A, B, C, D, E and F of P that are incident to the slit hole of size 1 as illustrated in Figure 15. We distinguish two cases: The crease between A and F is of 90° or of 180°.

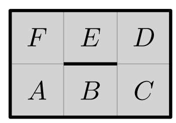

Figure 15: A polyomino with a slit hole of size one.

If the AF-crease is of 90°, we must further distinguish if the EF-crease is of 90° or of 180°. If the EF-crease is of 180°, then the slit edge is mapped to the edge between AF, fixing that B maps to A. Hence, this corresponds to a 90° crease of the slit-edge.

By symmetry, we may assume that both the AB-crease and the EF-crease is of 180°. This implies that B and E cover the same face in such a way that the top edge of B is mapped to the left edge of E. However, then the bottom left corner of D is also mapped to the top left corner of E. A contradiction. Consequently, this is impossible.

If the AF-crease is of 180°, then A and F cover the same face and in particular, their left edges are mapped to the same edge such that the top edge of F and the bottom edge of A coincide. This implies that the left edge of E and the left edge of B also coincide such that the top edge of E and the bottom edge of B coincide. This corresponds to a 180° crease of the slit-edge.

This shows that the slit edge is a crease of 90° or of 180°. Hence, the crease pattern behaves as if the slit hole was nonexistent. □

Theorem 15. *If P is a rectangle with a slit of size 1, then P does not fold into $\mathcal{C}$.*

Proof. By Lemma 14, the crease pattern behaves as if the slit was nonexistent, i.e., as if P was a rectangle. By Corollary 1, all horizontal or vertical creases are folded by 180°, reducing P to a rectangle of height or width 1, which does not fold into $\mathcal{C}$. □

Furthermore, we conjecture that the slit of size 1 never is the deciding factor for foldability.

Conjecture 1. *Let polyomino P' be obtained from a polyomino P by adding a slit s of size 1. If P' folds into $\mathcal{C}$, then P folds into $\mathcal{C}$ as well.*

4.3. An Algorithm to Check a Necessary Local Condition for Foldability

Consider the following local condition: let s be a square in a polyomino P such that the mapping between vertices of s and vertices of a face of $\mathcal{C}$ has been fixed. Then, for every adjacent square s' of s, there are two possibilities how to map its four vertices onto $\mathcal{C}$: the two vertices shared by s and s' must be mapped consistently and for the other two vertices there are two options depending on whether s' is folded at 90° angle to an adjacent face of $\mathcal{C}$, or whether it is folded at 180° to the same face of $\mathcal{C}$.

The algorithm below checks whether there exists a mapping between all vertices of squares of P to vertices of $\mathcal{C}$ such that the above condition holds for every pair of adjacent polyomino squares of P.

1. Run a breadth-first-search on the polyomino squares, starting with the leftmost square in the top row of P and continue via adjacent squares. This produces a numbering of polyomino squares in which each but the first square is adjacent to at least one square with smaller number.
2. Map vertices of the first square to the bottom face of $\mathcal{C}$. Extend the mapping one square at a time according to the numbering, respecting the local condition (that is, in up to two ways). Track all such partial mappings.

The algorithm is exponential, because unless inconsistencies are produced, the number of possible partial mappings doubles with every polyomino square. Nevertheless, it can be used to show non-foldability for small polyominoes: if no consistent mapping exists for a polyomino, then the polyomino cannot be folded onto $\mathcal{C}$. On the other hand, any consistent vertex mapping covering all faces of $\mathcal{C}$ obtained by the algorithm that we tried could in practice be turned into a folding. However, we have not been able to prove that this is always the case.

The algorithm above was used to prove that polyominoes in Figure 16 do not fold, as well as it aided us to find the foldings of polyominoes in Figure 7. An implementation of the algorithm is available at the following site `http://github.com/zuzana-masarova/cube-folding`.

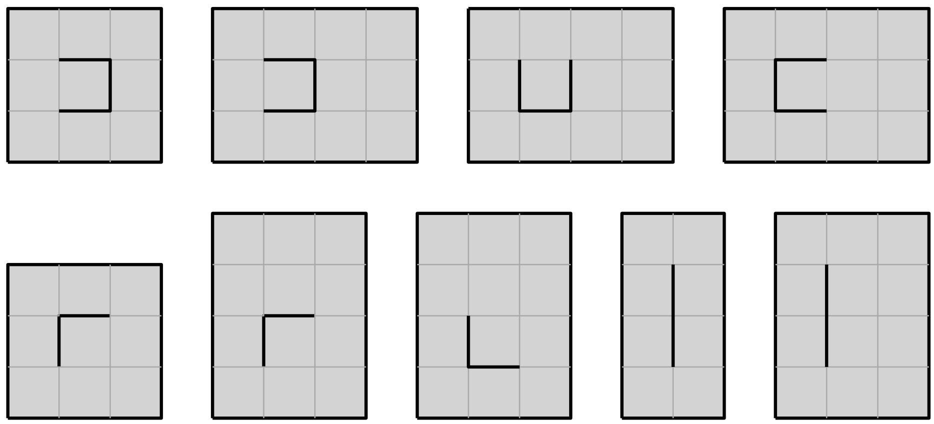

Figure 16: These polyominoes with single L-, U- and straight size-2 slits do not fold.

5. Conclusion and Open Problems

We showed that, if a polyomino P does contain a non-simple hole, then P folds into $\mathcal{C}$. Moreover, we showed that a unit square hole, size 2 slits (straight or L), and a size-3 U-slit sometimes allow for foldability.

Based on the presented results, we created a font of 26 polyominoes with slits that look like each letter of the alphabet, and each fold into $\mathcal{C}$. See Figure 17, and `http://erikdemaine.org/fonts/cubefolding/` for a web app.

We conclude with a list of interesting open problems:

- Does a consistent vertex mapping output by the algorithm in Section 4.3 imply that the polyomino is foldable? If so, is the folding uniquely determined?
- Is any rectangular polyomino with one L-slit, U-slit or straight slit of size 2 foldable? Currently, we only know that the small polyominoes in Figure 16 do not fold.

Acknowledgments

This research was performed in part at the 33rd Bellairs Winter Workshop on Computational Geometry. We thank all other participants for a fruitful atmosphere. H. Akitaya was supported by NSF CCF-1422311 & 1423615.

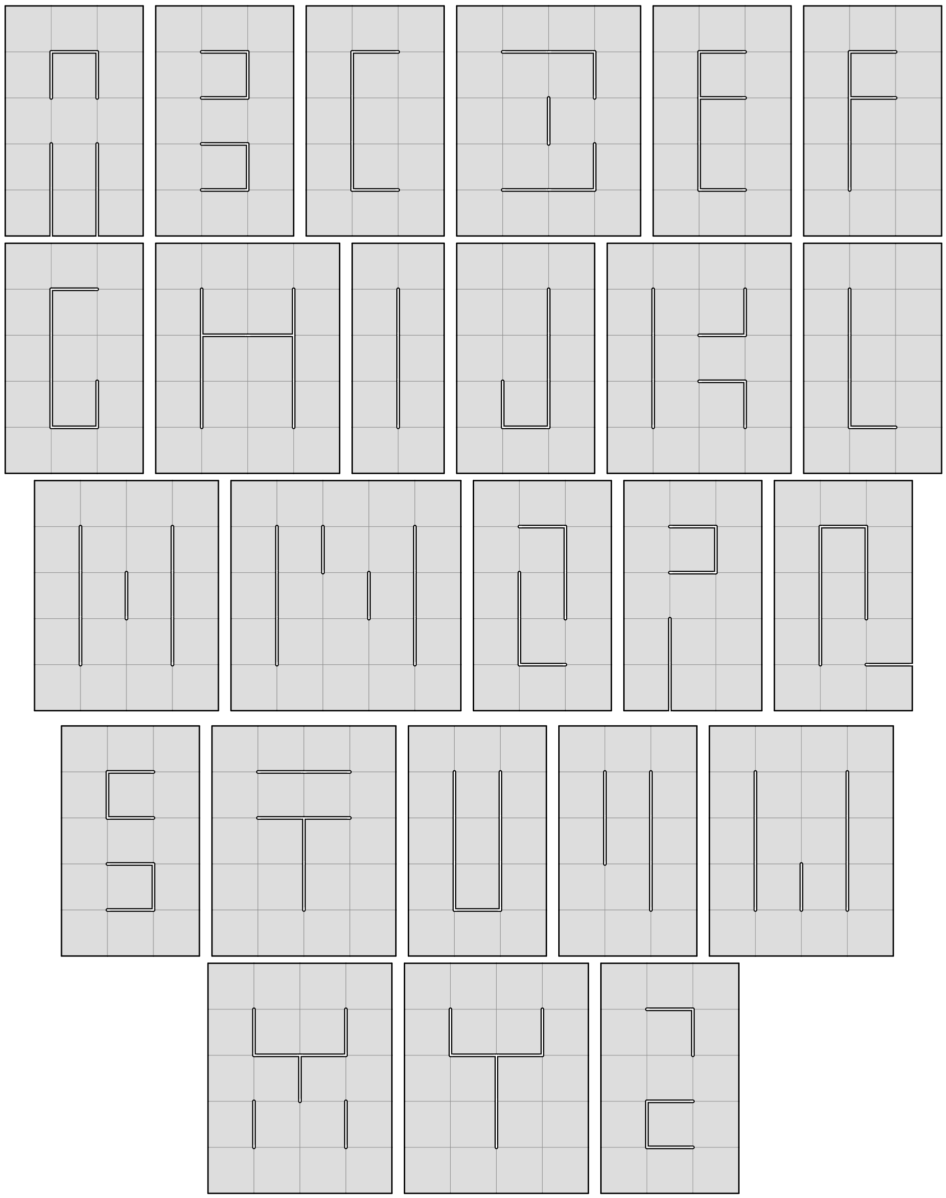

Figure 17: Cube-folding font: the slits representing each letter enable each rectangular puzzle to fold into a cube.

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URL `http://dx.doi.org/10.1007/978-3-319-17142-5_21`