# $\forall\exists\mathbb{R}$-completeness and area-universality*


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**Abstract**

In the study of geometric problems, the complexity class $\exists\mathbb{R}$ turned out to play a crucial role. It exhibits a deep connection between purely geometric problems and real algebra, and is sometimes referred to as the "real analogue" to the class NP. While NP can be considered as a class of computational problems that deals with existentially quantified *boolean* variables, $\exists\mathbb{R}$ deals with existentially quantified *real* variables.

In analogy to $\Pi_2^p$ and $\Sigma_2^p$ in the famous polynomial hierarchy, we introduce and motivate the complexity classes $\forall\exists\mathbb{R}$ and $\exists\forall\mathbb{R}$ with *real* variables. Our main interest is focused on the Area Universality problem, where we are given a plane graph $G$, and ask if for each assignment of areas to the inner faces of $G$ there is an area-realizing straight-line drawing of $G$. We conjecture that the problem Area Universality is $\forall\exists\mathbb{R}$-complete and support this conjecture by a series of partial results, where we prove $\exists\mathbb{R}$- and $\forall\exists\mathbb{R}$-completeness of variants of Area Universality. To do so, we also introduce first tools to study $\forall\exists\mathbb{R}$, such as restricted variants of $\forall\exists\mathbb{R}$, which are $\forall\exists\mathbb{R}$-complete. Finally, we present geometric problems as candidates for $\forall\exists\mathbb{R}$-complete problems. These problems have connections to the concepts of imprecision, robustness, and extendability.


## 1 Introduction

In this paper we investigate problems related to face areas in drawings of planar graphs. We consider two drawings of a planar graph as *equivalent* if they have the same outer face and rotation system, i.e., for each vertex the cyclic ordering of the incident edges is the same. A plane graph is a planar graph together with a crossing-free drawing. Let $G$ be a plane graph and let $F$ be the set of inner faces of $G$. A *face area assignment* is a function $\mathcal{A}\colon F \to \mathbb{R}_0^+$. We say that $G'$ is an *$\mathcal{A}$-realizing drawing*, if $G'$ is an equivalent straight-line drawing of $G$ in which the area of each $f \in F$ is exactly $\mathcal{A}(f)$. If $\mathcal{A}$ has an area-realizing drawing, we say that $\mathcal{A}$ is *realizable*. A plane graph $G$ is *area-universal* if every face area assignment is realizable. Since we only consider straight-line drawings, we simply call them drawings from now on.

The questions on plane graph with prescribed face areas were initiated in 1990 by Ringel [23], who presented drawings of the octahedron and the icosahedron graph, where all faces have the same area. It is straightforward to observe that *stacked triangulations*, also known as *planar 3-trees* or *Apollonian networks*, are area-universal: stacked triangulations can be obtained by recursively subdividing a triangle $t$ into three smaller triangles $t_1, t_2, t_3$. For each $i \in \{1, 2, 3\}$, the prescribed area of the small triangle $t_i$ yields the line $\ell_i$ of valid positions for the central vertex such that the area of $t_i$ is realized. If the triangle $t$ has the correct total area, then the three lines $\ell_1, \ell_2, \ell_3$ have a common intersection point.

*A video presenting this paper is available at https://youtu.be/OQkACiNS66o


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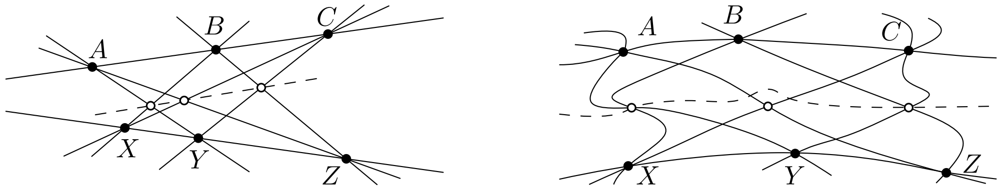
Moreover, it is easy to see that if a graph is area-universal, then each of its subgraphs is also area-universal. These two observations together imply that partial planar 3-trees are area-universal. Biedl and Velázquez [5] studied the grid size of area-realizing drawings of partial planar 3-trees and plane cubic graphs. Since area-universality seems to be a strong property, it is somewhat surprising that many graphs indeed are easily seen to be area-universal. In 1992, Thomassen [27] proved that plane cubic graphs are area-universal. More recently, Kleist [13] showed that all 1-subdivisions of plane graphs are area-universal. In other words, every plane graph does not necesarily allow for a realizing straight-line drawing but for a polyline drawing where every edge has at most one bend. For a long time, the only graph known not to be area-universal, was the octahedron graph, which was already proven by Ringel [23]. Kleist [13] introduced the first infinite family of non-area-universal graphs (not counting the trivial family of all graphs containing the octahedron as a subgraph). In particular, she showed that all Eulerian triangulations and also the icosahedron graph are not area-universal. This implies for instance that high connectivity of a graph does not imply area-universality. Moreover, area-universality is not a minor-closed property, as every grid graph is area-universal [12], but the octahedron graph is not area-universal, although it is a minor of the grid. Note that all known graphs that are not area-universal are plane triangulations (or contain them as a subgraph).

In this paper we are interested in the computational problem of deciding if a given plane graph is area-universal. We denote this problem by Area Universality.

**Area Universality**
**Input:** Plane graph $G = (V, E)$.
**Question:** Is every face area assignment realizable?

When investigating natural geometric problems, one often discovers that an instance of such a problem can be described as a first-order formula over the reals containing only existential quantifiers: $\exists X_1, X_1, \ldots, X_n \colon \Phi(X_1, X_2, \ldots, X_n)$. The variables encode the configuration of geometric objects and the quatifier-free formula $\Phi$ describes the relations between them. A value assignment that satisfies $\Phi$ corresponds to a solution of the original geometric problem. Existential Theory of the Reals (ETR) is a computational problem that takes such a formula as an input and asks whether it is true or not. The complexity class $\exists\mathbb{R}$ consists of all problems that reduce in polynomial time to ETR. Surprisingly many natural geometric problems appear to be $\exists\mathbb{R}$-complete. A prominent example is the stretchability of a pseudoline arrangement (see [15, 18, 26]).

A *pseudoline arrangement* in the plane is a set of unbounded $x$-monotone Jordan curves where every pair of curves intersects in exactly one crossing point. A pseudoline arrangement is *stretchable* if there exists an arrangement of straight lines with the same face structure. Stretchability is a computational problem which asks whether a given pseudoline arrangement is stretchable. The $\exists\mathbb{R}$-completeness of Stretchability reflects the deep algebraic connections between line arrangements and real algebra. This connection has important consequences. The first consequence is that there is little hope to find a simple combinatorial algorithm for Stretchability, since simple combinatorial algorithms to decide ETR are not known. In fact, without certain algebraic tools, bounding the length of integer coordinate representations, we would not even know if Stretchability is *decidable* (see [15]). However, it is not just difficult to decide whether a given pseudoline arrangement is strechable. It requires algebraic arguments eventually. For instance, the non-strechability of the (smallest non-strechable) pseudoline arrangement depicted in Figure 1b is based on Pappus's Hexagon Theorem [14], dating back to the 4th century. It




(a) Illustration of Pappus's Hexagon Theorem. (b) A non stretchable pseudoline arrangement.

Figure 1: Pappus's Hexagon Theorem and Stretchability.

considers two different lines with three points each, the points are denoted by $A, B, C$ and $X, Y, Z$, see Figure 1a . If the lines $\overline{AY}, \overline{BZ}, \overline{CX}$ intersect the lines $\overline{BX}, \overline{CY}, \overline{AZ}$, respectively, then the three points of intersection are collinear. Although the statement is intrinsically geometric, it is non-trivial to prove and most known proofs have some algebraic flavor, see [21, Chapter 1].

As mentioned before, geometric problems that are ∃ℝ-complete usually ask for existence of certain objects, satisfying some easily checkable properties. However, the nature of Area Universality seems to be different. We therefore define the new complexity class ∀∃ℝ as the set of all problems that reduce in polynomial time to Universal Existential Theory of the Reals (UETR). The input of UETR is a first order formula over the reals in prenex form, which starts with a block of universal quantifiers followed by a block of existential quantifiers and is otherwise quantifier-free. We ask if the formula is true.

**Universal Existential Theory of the Reals (UETR)**
**Input:** A formula over the reals of the form $(\forall\, Y_1, Y_2, \ldots, Y_m)\,(\exists\, X_1, X_2, \ldots, X_n) : \Phi$,
where $\Phi$ is a quantifier-free formula with variables $Y_1, Y_2, \ldots, Y_m, X_1, X_2, \ldots, X_n$. **Question:** Is the input formula true?

The class ∃∀ℝ is defined analogously. Clearly ∃ℝ is contained in ∀∃ℝ. It is easy to observe and well-known that NP is contained in ∃ℝ. Highly non-trivial is the containment of ∀∃ℝ in PSPACE (see [2]). For all we know, all these complexity classes could collapse, as we do not know whether NP and PSPACE constitute two different or the same complexity class, see Figure 2. However, $\exists\mathbb{R} \neq \forall\exists\mathbb{R}$ can be believed with similar confidence as $\text{NP} \neq \Pi_2^p$. In addition, the expressibility of ∀∃ℝ is larger than ∃ℝ in an algebraic sense, see [10].

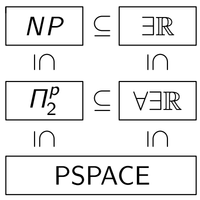



Figure 2: Containment relations of complexity classes.

It is worth mentioning that Blum et. al. [6] also introduce a hierarchy of complexity classes analogous to the complexity class NP, but over the reals (this also generalizes to other rings). Their canonical model of computation is the so-called Blum-Shub-Smale machine (BSS). The main difference between this approach and ours is that BSS accepts real numbers as input. However, the classes ∃ℝ, ∀∃ℝ, ∃∀ℝ work with ordinary Turing machines, which only take strings over a finite (say, binary) alphabet as input.

## Our Results

It is straightforward to show that Area Universality belongs to ∀∃ℝ:

**Proposition 1.** Area Universality *is in* ∀∃ℝ*.*

The idea is to use a block of universal quantifiers to describe the face area assignment and the block of existential quantifiers to describe the placement of the vertices of the drawing of $G$. A proof can be found in Section 5. We believe that a stronger statement holds.

**Conjecture.** Area Universality *is* ∀∃ℝ*-complete.*

While this conjecture, if true, would show that Area Universality is a really difficult problem in an algebraic and combinatorial sense, it would also give the first known natural geometric problem that is complete for ∀∃ℝ.

As a first step towards proving our conjecture, we consider three variants of Area Universality, each approaching the conjecture from a different direction. To do so, we introduce restricted variants of ETR and UETR which are still complete. These are presented in Section 2 and may be useful to show hardness for other problems.

As a starting point we drop the planarity restriction. For a plane graph $G$ with vertex set $V$, the *face hypergraph* of $G$ has vertex set $V$, and its edges correspond to sets of vertices forming the faces (see e.g. [11]). Observe that the face hypergraph of a plane triangulation is 3-uniform. Moreover, we define the area of a triple of points in the plane to be the area of the triangle formed by the points. It is clear that Area Universality can be equivalently formulated in the language of face hypergraphs. This relation motivates the following relaxation of the problem.

**Area Universality for Triples***
**Input:** A set $V$ of vertices, a collection of vertex-triples $F \subseteq \binom{V}{3}$, and a partial area assignment $\mathcal{A}'\colon F' \to \mathbb{R}_0^+$ for some $F' \subseteq F$.
**Question:** Is it true that for every $\mathcal{A}\colon F \to \mathbb{R}_0^+$, such that $\mathcal{A}(f) = \mathcal{A}'(f)$ for all $f \in F'$, there exist a placement of $V$ in the plane, such that the area for each $f \in F$ is $\mathcal{A}(f)$?

**Theorem 2.** Area Universality for Triples* *is* $\forall\exists\mathbb{R}$*-complete.*

For the proof of Theorem 2 we use gadgets similar to the *von Staudt constructions* used to show the $\exists\mathbb{R}$-hardness of order-types, see [15]. The proof can be found in Section 3.

Our second result concerns a variant, where we investigate the complexity of realizing a specific area-assignment. Prescribed Area denotes the following problem: Given a plane graph $G$ with an area assignment $\mathcal{A}$, does there exist a crossing-free drawing of $G$ that realizes $\mathcal{A}$? We study a partial extension version of Prescribed Area, where some vertex positions are fixed and we seek for an area-realizing placement of the remaining vertices.

**Fixed Prescribed Area**
**Input:** Plane graph $G = (V, E)$, a face area assignment $\mathcal{A}$, fixed positions for $V' \subseteq V$.
**Question:** Does there exist an $\mathcal{A}$-realizing drawing of $G$ respecting the positions of all $v \in V'$?

**Theorem 3.** Fixed Prescribed Area *is* $\exists\mathbb{R}$*-complete.*

A first hint of Theorem 3 was given by the fact that the octahedron graph has an integral face area assignment, whose unique realizing drawing requires irrational vertex coordinates [5]. A proof of Theorem 3 is presented in in Section 4.

The last two results consider the analogous question for simplicial complexes in three dimensions. Recall that an *abstract simplicial complex* is a family $\Sigma$ of non-empty finite sets over a ground set $V = \bigcup \Sigma$, which is closed under taking non-empty subsets. We say $\Sigma$ is *pure* when the inclusion-wise maximal sets of $\Sigma$ all have the same number of elements. We say $\Sigma$ is *realizable* when there is a simplicial complex $\mathcal{S}$ in $\mathbb{R}^3$ that has a vertex for each element of $V$ and a simplex corresponding to each set in $\Sigma$.

A *crossing-free drawing* of $\Sigma$ is a mapping of every $i \in V$ to a point $p_i \in \mathbb{R}^3$, such that the following holds. For any pair of sets $\sigma_1, \sigma_2 \in \Sigma$ there is a separating hyperplane $h = \{x \in \mathbb{R}^3 : \langle a, x\rangle = b\}$ such that $\langle a, p_i\rangle \leq b$ for all $i \in \sigma_1$ and $\langle a, p_i\rangle \geq b$ for all $i \in \sigma_2$. A *volume assignment* for $\Sigma$ is a non-negative-valued function on the collection $T$ of all 4-element sets in $\Sigma$, and a crossing-free drawing of $\Sigma$ *realizes* a volume assignment $\mathcal{V} : T \to \mathbb{R}_0^+$ when for each $\tau \in T$, the convex hull of the points $\{p_i : i \in \tau\}$ has volume $\mathcal{V}(\tau)$.

The analogous questions can be stated as follows:

**Prescribed Volume**
**Input:** A pure abstract simplicial complex $\Sigma$ realizable in $\mathbb{R}^3$; a volume assignment $\mathcal{V}$.
**Question:** Does there exist a crossing-free drawing of $\Sigma$ that realizes $\mathcal{V}$?

**Volume Universality***
**Input:** A pure abstract simplicial complex $\Sigma$ realizable in $\mathbb{R}^3$; volume assignment $\mathcal{V}' : T' \to \mathbb{R}_0^+$ for some of the 4-element sets $T' \subseteq T$ of $\Sigma$.
**Question:** Is it true that for every $\mathcal{V}\colon T \to \mathbb{R}_0^+$, such that $\mathcal{V}(\tau) = \mathcal{V}'(\tau)$ for all $\tau \in T'$, there exists a crossing-free drawing of $\Sigma$ that realizes $\mathcal{V}$?

**Proposition 4.** Volume Universality* *is in* $\forall\exists\mathbb{R}$.

Note that 3-dimensional simplicial complexes are the analogue to planar triangulations. The fact that the tetrahedra only intersect in common faces corresponds to the non-crossing condition in planar drawings. Indeed, Prescribed Volume generalizes area-realizibility for triangulations in the following sense:

**Proposition 5.** *There is a polynomial time algorithm that takes as input any plane triangulation $G$ with positive area-assignment $\mathcal{A}$ and outputs a simplicial complex $\mathcal{S}$ with volume assignment $\mathcal{V}$ such that $\mathcal{A}$ is realizable for $G$ if and only if $\mathcal{V}$ is realizable for $\mathcal{S}$.*

Moreover, the 3-dimensional analogues of Prescribed Area and Area Universality are hard. The proofs are presented in Section 6. The two versions read as follows:

**Theorem 6.** Prescribed Volume *is* $\exists\mathbb{R}$*-complete.*

**Theorem 7.** Volume Universality* *is* $\forall\exists\mathbb{R}$*-complete.*

To conclude we present problems that are interesting candidates for $\forall\exists\mathbb{R}$-complete problems in Section 7.

## 2 Toolbox: Hard variants of ETR and UETR

In this section we introduce some restricted variants of ETR and UETR which enable us to show hardness. Recently, Abrahamsen et al. showed that the following problem is also $\exists\mathbb{R}$-complete [1]. In particular, note that we can restrict multiplication to inversion and assume that a yes-instance has a solution within a bounded positive interval.

**ETRINV**
**Input:** A formula over the reals of the form $(\exists\, X_1, X_2, \ldots, X_n) : \Phi$,
where $\Phi$ is a conjunction of constraints of the following form: $X = 1$ (introducing a constant 1), $X + Y = Z$ (addition), $X \cdot Y = 1$ (inversion), with $X, Y, Z \in \{X_1, \ldots, X_n\}$. Additionally, $\Phi$ is either unsatisfiable, or has a solution, where each variable is within $[1/2, 2]$.
**Question:** Is the input formula true?

In order to define some even more restricted variant of ETRINV, we need one more definition. Consider a formula $\Phi$ of the form $\Phi = \Phi_1 \wedge \Phi_2 \wedge \ldots \wedge \Phi_m$, where each $\Phi_i$ is a quantifier-free formula of the first order-theory of the reals with variables $X_1, X_2, \ldots, X_n$, which uses arithmetic operators and comparisons but no logic symbols. An *incidence graph* of $\Phi$ is the bipartite graph with vertex set $\{X_1, X_2, \ldots, X_n\} \cup \{\Phi_1, \Phi_2, \ldots, \Phi_m\}$ that has an edge $X_i\Phi_j$ if and only if the variable $X_i$ appears in the subformula $\Phi_j$.

By Planar-ETRINV we denote the restriction of ETRINV, where the incidence graph of $\Phi$ is planar, and $\Phi$ is either unsatisfiable, or has a solution with all variables within $(0, 5)$.

**Theorem 8.** Planar-ETRINV *is* $\exists\mathbb{R}$*-complete.*

*Proof.* Consider an instance $(\exists\, X_1, X_2, \ldots, X_n) : \Phi$ of ETRINV. Let $G$ be some embedding of $G(\Phi)$ in $\mathbb{R}^2$. Suppose that $G$ is not crossing-free and consider a pair of crossing edges. Let $X$ and $Y$ denote the variables corresponding to (one endpoint of) these edges. We introduce three new existential variables $X', Y', Z$ and three constraints:

$X + Y = Z$; $X + Y' = Z$; $X' + Y = Z$. Observe that these constraints ensure that $X = X'$ and $Y = Y'$. Moreover, as shown in Figure 3, the embedding of $G$ can be modified such that the new incidence graph $G'$ has strictly fewer crossings. In particular, $G'$ looses the considered crossing and no new crossing is introduced. We repeat this procedure until the incidence graph of the obtained formula is planar. Finally, note that $0 < 1 \leq Z = X + Y \leq 2 + 2 = 4 < 5$ whenever $1/2 \leq X, Y \leq 2$. Note that the number of new variables and constraints is at most $O(|\Phi|^4)$, since the degree of each subformula in ETRINV is at most three. □

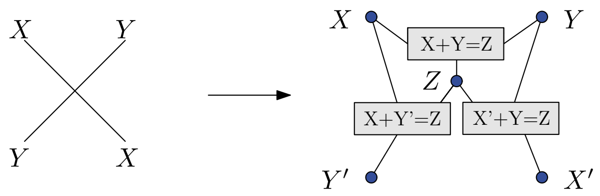



Figure 3: The crossing gadget.

Now we introduce a restricted variant of UETR.

> **Constrained-UETR**
> **Input:** A formula over the reals $(\forall\, Y_1, \ldots, Y_m \in \mathbb{R}^+)\,(\exists\, X_1, \ldots, X_n \in \mathbb{R}^+) : \Phi(X, Y)$, where $\Phi$ is a conjunction of constraints of the form: $X = 1$ (introducing constant 1), $X + Y = Z$ (addition), $X \cdot Y = Z$ (multiplication), with $X, Y, Z \in \{X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$.
> **Question:** Is the input formula true?

It can be seen as a simplified $\forall\exists\mathbb{R}$-version of a variant of ETR called Ineq, where we ask if a conjunction of polynomial equations and inequalities has a real solution. Ineq is known to be $\exists\mathbb{R}$-complete [15, 26]. Similarly, we will show that Constrained-UETR is $\forall\exists\mathbb{R}$-complete. The proof can be found in Appendix A.

**Theorem 9.** Constrained-UETR *is* $\forall\exists\mathbb{R}$*-complete.*

# 3 Hardness of Area Universality for Triples*

In this section, we prove that Area Universality for Triples* is $\forall\exists\mathbb{R}$-complete.

**Theorem 2.** Area Universality for Triples* *is* $\forall\exists\mathbb{R}$*-complete.*

*Proof.* We reduce from Constrained-UETR. For every instance $\Psi$ of Constrained-UETR, we give a collection of points $V$ and unordered triples $T$, along with a partial area assignment $\mathcal{A}'$. Let $\Psi$ be a formula of the form:

$$\Psi = (\forall Y_1, \ldots, Y_m \in \mathbb{R}^+)(\exists X_1, \ldots, X_n \in \mathbb{R}^+) \colon \Phi(Y_1, \ldots, Y_m, X_1, \ldots, X_n).$$

Recall that $\Phi$ is a conjunction of constraints of the form $X = 1$, $X + Y = Z$, and $X \cdot Y = Z$. First, we will show how to express $\Phi$. We use gadgets similar to the ones used to show $\exists\mathbb{R}$-hardness of Order Type (see Matoušek [15]). We represent all variables as points on one line; we denote the line by $\ell$ for the rest of the proof. We start to enforce points to be on $\ell$ and then we construct gadgets mimicking addition and multiplication.

To force points on a line, introduce three points $p_0$, $p_1$, and $r$ and set the area of the triangle $(p_0, p_1, r)$ to 1, i.e., $\mathcal{A}'(p_0, p_1, r) := 1$. Note that the positive area ensures that the points are pairwise different and that they are not collinear. Without loss of generality we can assume that $\|p_0 p_1\| = 1$. Denoting a line through two points $a$ and $b$ by $\ell_{a,b}$, we set $\ell := \ell_{p_0,p_1}$. We interpret the position of $p_0$ as 0 and the position of $p_1$ as 1. To force a point $x$ on $\ell$, we set $\mathcal{A}'(x, p_0, p_1) := 0$. Note that this does not introduce any other constraints on the position of $x$.

Each variable $X$ will be represented by a point $x$ on $\ell$. Additionally, since all variables are non-zero, we introduce a triangle forcing $x$ to be different from $p_0$. Note that in general, we can ensure that two points $x_1, x_2$ are distinct, by introducing a new point $q$ and adding a triangle $(x_1, x_2, q)$ with $\mathcal{A}'(x_1, x_2, q) := 1$. Now the absolute value of $X$ is defined by $\|p_0 x\|$. If $x$ is on the same side of $p_0$ as $p_1$, then the value of $X$ is positive, otherwise it is negative. We accept currently both positive and negative values, but later will force all original variables to be positive.

Let us describe the gadgets for addition and multiplication, see Figure 4. We start with the addition constraint $X + Y = Z$. Let $x, y, z$ be the points encoding the values of $X, Y, Z$, respectively, recall that they lie on $\ell$ and they do not coincide with $p_0$.

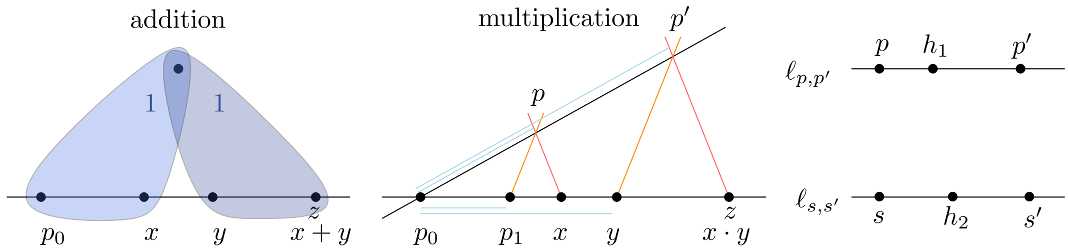



Figure 4: Gadgets for addition, multiplication, and construction of a parallel line.

We introduce a point $q_1$ and prescribe the areas $\mathcal{A}'(p_0, x, q_1) = (y, z, q_1) = 1$, see on the left of Figure 4. Since the two triangles have the same height, the distance between $y$ and $z$ agrees with the distance between $p_0$ and $x$. Thus, the value of $Z$ is either $X + Y$ or $X - Y$. Analogously, we introduce a point $q_2$ and define $\mathcal{A}'(p_0, y, q_2) = \mathcal{A}'(x, z, q_2) = 1$, implying that $Z$ is either $Y + X$ or $Y - X$. This is satisfied either if $Z = X + Y$ (the intended solution) or if $Z = X - Y = Y - X$. The second solution implies that $X = Y$ and thus $Z = 0$. This contradicts the fact that $z \neq p_0$ and hence shows the correctness of the addition gadget.

As a preparation to the multiplication gadget, we show how to enforce on the four pairwise different points $p, p', s, s'$ that $\ell_{p,p'}$ is parallel to $\ell_{s,s'}$, without adding additional constraints on any of the four points, see on the right of Figure 4. We construct two new points $h_1$ and $h_2$ such that $h_1$ is on line $\ell_{p,p'}$ and $h_2$ is on line $\ell_{s,s'}$ by defining $\mathcal{A}'(p, p', h_1) = \mathcal{A}'(s, s', h_2) = 0$. We want to construct a trapezoid with corner points $p, h_1, s, h_2$ such that the side $ph_1$ is parallel to $sh_2$. For this, we prescribe the areas $\mathcal{A}'(p, h_1, s) = \mathcal{A}'(p, h_1, h_2) = 1$ and the areas $\mathcal{A}'(s, h_2, p) = \mathcal{A}'(s, h_2, h_1) = 2$ , see left of Figure 5.

We want to show that $s$ and $h_2$ lie on the same side of the line $\ell_{p,h_1}$. Assume by contradiction that $\ell_{p,h_1}$ separates $s$ and $h_2$. If $p, h_1$ are on the same side of $\ell_{s,h_2}$ then the triangle $(s, h_2, p)$ is contained in or contains the triangle $(s, h_2, h_1)$, see middle of Figure 5. This contradicts that both triangles have the same area; recall that $p \neq h_1$.

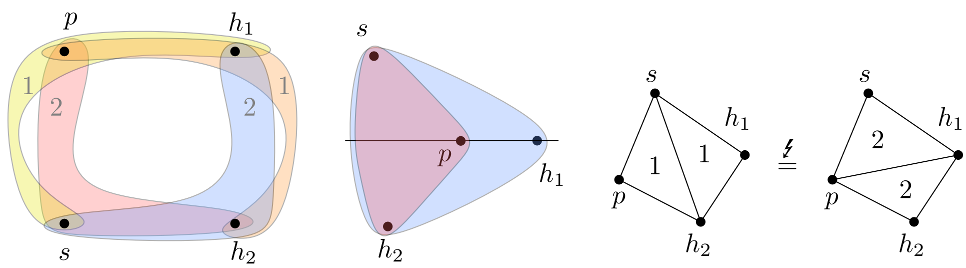



Figure 5: Forcing a trapezoid.

Consequently, $\ell_{s,h_2}$ separates $p$ and $h_1$ and the quadrangle can be partitioned by each of the two diagonals $sh_2$ and $ph_1$. See right of Figure 5 to observe the following contradiction: $2 = \mathcal{A}(p, h_1, s) + \mathcal{A}(p, h_1, h_2) = \mathcal{A}(s, h_2, h_1) + \mathcal{A}(s, h_2, p) = 4$. Thus $s, h_2$ lie on the same side of $\ell_{p,h_1}$ and $s$ and $h_1$ have the same distance to $\ell_{p,h_1}$ by the prescribed area. Therefore, the segments $ph_1$ and $sh_2$ and the lines $\ell_{p,p'}$ and $\ell_{s,s'}$ are parallel. Moreover, no other constraints are imposed on the points $p, p', s, s'$.

To construct a multiplication gadget for the constraint $X \cdot Y = Z$, let $x, y, z$ ($\neq p_0$) be the points encoding the values of $X, Y, Z$, respectively. First we introduce two points $p, p'$ which do not lie on $\ell$, but the three points $p_0, p, p'$ are collinear by setting $\mathcal{A}'(p_0, p, p') = 0$. We enforce the following pairs of lines to be parallel by the parallel-line construction: $\ell_{p_1,p}$ with $\ell_{y,p'}$ and $\ell_{x,p}$ with $\ell_{z,p'}$, see Figure 4. By the intercept theorem, the following ratios coincide (also for negative variables): $|p_0p|/|p_0p'| = |p_0p_1|/|p_0y| = |p_0x|/|p_0z|$. By the definition of $x, y, z$, we obtain $1/Y = X/Z$, which implies that $X \cdot Y = Z$.

Next, we introduce universally quantified variables $Y_1, Y_2, \ldots, Y_m$. For every $Y_i$, let $y_i$ be the point encoding its value, we know that $y_i$ is on $\ell$ and $y_i \neq p_0$. We introduce a triple $f_i = (p_0, r, y_i)$, whose area is universally quantified. Recall that $r$ is a point with $\mathcal{A}'(p_0, p_1, r) = 1$.

Now we show how to enforce each original variable $X$ to be positive. We add an existentially quantified variable $S_X$ and the constraint $X = S_X \cdot S_X$. The variable $S_X$ may not be positive, but this is not required. This finishes the description of the reduction. It is easy to see that it runs in polynomial time.

It remains to argue that $\Psi$ is true if and only if for every area assignment of the non-prescribed triples together with the area assignment of the prescribed triples there exists a placement of the points such that every triple has the correct area. Let $V(Y_1), \ldots, V(Y_m) \in \mathbb{R}^+$ be some values of the universally quantified variables. Then for every $i \in \{1, \ldots, m\}$ the area assignment of the triple $(p_0, r, y_i)$ enforces the positions of $y_i$ to correspond to $V(Y_i)$. Consequently, the position of the point encoding the variable $S_{Y_i}$ is either $\sqrt{(V(Y_i)}$, or $-\sqrt{(V(Y_i))}$ (both alternatives are plausible). Fixing a value of $X_i$ fixes $S_{X_i}$ to be either $\sqrt{V(X_i)}$ or $-\sqrt{V(X_i)}$ (both plausible), so it only remains to show that if there exists values $V(X_1), \ldots, V(X_n)$ such that $\Phi(Y, X)$ holds, then there is an $\mathcal{A}$-realizing placement of the points $x_1, \ldots, x_n$ and vice versa. This follows immediately from the correctness of the gadgets and the fact that auxiliary points for $S_{X_i}$ and $S_{Y_j}$ can be set arbitrarily and are independent for each gadget. ☐

# 4 Hardness of Fixed Prescribed Area

In this section we prove Theorem 3 by presenting a reduction from Planar-ETRINV.

**Theorem 3.** Fixed Prescribed Area *is* $\exists\mathbb{R}$*-complete.*

*Proof.* Let $\Psi = \exists X_1 \ldots X_n : \Phi(X_1, \ldots, X_n)$ be an instance of Planar-ETRINV. Recall that we can assume that if $\Psi$ is a yes-instance, then it has a solution, in which the values of variables are in the interval $(0, \lambda)$ with $\lambda = 5$. We construct a plane graph $G_\Psi = (V, E)$, a face area assignment $\mathcal{A}$ of inner faces of $G_\Psi$, and fixed positions of a subset of vertices, such that $G_\Psi$ has a realizing drawing respecting pre-drawn vertices if and only if $\Phi$ is satisfiable by real values from the interval $(0, \lambda)$.

Consider the incidence graph of $\Phi$ and fix an orthogonal plane drawing on an integer grid, see Figure 6 for an example. We design several types of gadgets: *variable gadgets* representing variables, as well as *inversion* and *addition gadgets*, realizing corresponding constraints. Moreover, we construct *wires* and *splitters* in order to copy and transport information. Some vertices in our gadgets will have prescribed positions. We call such vertices *fixed* and all other vertices are *flexible*.

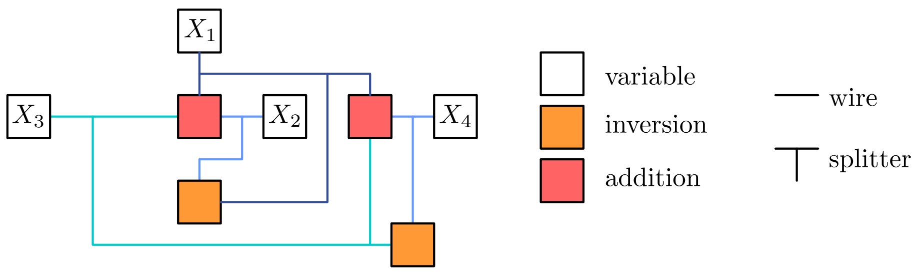



Figure 6: Incidence graph of $\Psi = (X_1 + X_2 = X_3) \wedge (X_1 X_2 = 1) \wedge (X_1 + X_4 = X_3) \wedge (X_4 \cdot X_3 = 1)$.

The *variable gadget* for a variable $X$ consists of eight fixed vertices $a, b, c, d, e, f, g, h$ and one flexible $v_x$, see left image of Figure 7. The fixed vertices $a, b, e, h$ form a rectangle of area 6. The vertices $c, d, v_x$ form a triangle of area $1/2$. The area of the face $aefv_xcdv_xghb$ is 5.5. Observe that the prescribed area of the triangle $cdv_x$ forces $v_x$ to lie on the line containing $fg$ and by the planarity constraint, it must lie in the interior of the segment $fg$. Denoting the Euclidean distance of two points $p, q$ by $\|pq\|$, $\lambda\|fv_x\|$ specifies the value of $X$. We symbolized $\|fv_x\|$ by a bold gray line in Figure 7. Now, we introduce a constant 1 with a constraint $X = 1$ by using a variable gadget. Instead of making vertex $v_x$ free, we

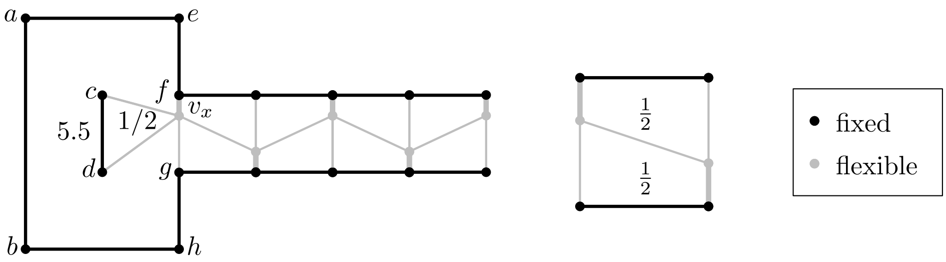



Figure 7: The variable gadget together with an attached wire (left). A wire gadget (right).

make it fixed and place it in such a way that $\|fv_x\| = 1/\lambda$. In the following, we sometimes assume that a flexible vertex is forced to lie on a specified segment. Note that this property can be induced by a variable gadget.

The *wire gadget* consists of several box-like fragments: four fixed vertices positioned as the corners of an axis-parallel unit square and two opposite fixed edges, see Figure 7. Each of the other two sides of the square is subdivided by a flexible vertex, these two vertices are joined by an edge. Each of the two quadrangular faces has a prescribed area of $\frac{1}{2}$.

Note that if one of the flexible vertices is collinear with its fixed neighbors, so is the other one. The planarity constraint ensures that each flexible vertex lies between the corresponding fixed vertices. Moreover, the segment representing the value of the variable changes from top to bottom or vice versa. Hence, if necessary, we may use an odd number of fragments in order to invert the side where the value is represented. Wires are used to connect other gadgets.

The *splitter gadget* contains a central fixed square of area 1, where each side is adjacent to a triangle of area $\frac{1}{2}$, see Figure 8. Each triangle fixes a flexible vertex on a line. These flexible vertices and their neighbors on the boundary of the splitter gadget are identified with the appropriate vertices in a variable gadget or a wire; this is how we connect the splitter with other gadgets. Observe that the value of one variable fixes the values of all variables by pushing the area circularly. Note that each face area 1 has exactly two flexible vertices which are forced to lie on specific segments. Thus if one of them is determined, clearly the other is also uniquely determined in a realizing drawing. Note that we may use the splitter gadget not only for splitting wires, but also for realizing turns.

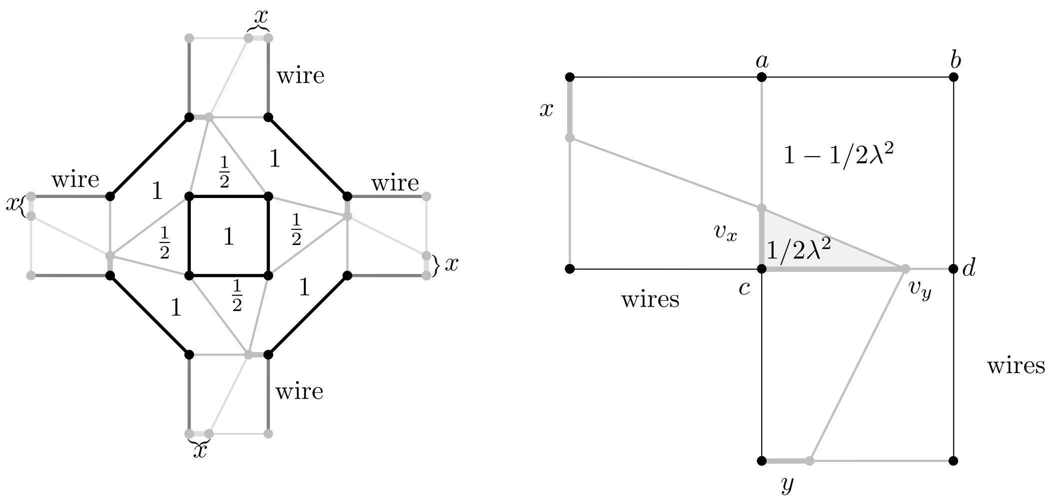



Figure 8: The splitter gadget, also enables left and right turns (left). The inversion gadget (right).

Now let us discuss gadgets for inversion and addition constraints. First consider an inversion constraint $X \cdot Y = 1$. The *inversion gadget* consists of four fixed vertices $a, b, c, d$, two flexible vertices $v_x$ and $v_y$, and the edge $v_x v_y$, see Figure 8. The fixed vertices belong to a unit square. By linking the inversion gadget with variable gadgets or wires, we can ensure that $v_x$ belongs to the segment $ac$ and $v_y$ belongs to the segment $cd$. Let $x$ be the distance from $c$ to $v_x$, and $y$ be the distance of $c$ to $v_y$. Suppose that these distances represent, respectively, the values of variables $X$ and $Y$ in the way described in the paragraph about variable gadgets; this is obtained by identifying them with appropriate vertices of such a gadget or a wire. Namely, the value of $X$ is $\lambda \cdot x$, and the value of $Y$ is $\lambda \cdot y$. To realize the area $1/(2\lambda^2)$ of the triangle $v_x v_y c$, the lengths $x$ and $y$ are forced to satisfy $xy/2 = 1/(2\lambda^2)$. This implies that the value of $X \cdot Y$ is $\lambda x \cdot \lambda y = 1$. Consequently, the inversion constraint is satisfied.

Finally, we define an *addition gadget* for an addition constraint $X + Y = Z$. We introduce fixed vertices $a, b, c, d, e, f, g, h, i, j, k, l$ placed on an integer grid, as depicted in Figure 9. Moreover, flexible vertices $v_x, v_y, v_z$ encode the values of variables $X, Y, Z$, respectively; they are identified by appropriate flexible vertices in variable or wire gadgets. This ensures that $v_x$,$v_y$, $v_z$ are forced to lie on the segments $cg$, $fj$, and $kl$, respectively. Finally, we introduce the flexible vertex $q$, joined to $deih$. The areas of faces are set as follows:

$f_1 = v_x, c, a, b, f, v_y, e, q, d \quad \mathcal{A}(f_1) = 3.5$
$f_2 = v_x, d, q, h, v_z, k, g \quad \mathcal{A}(f_2) = 2$
$f_3 = e, v_y, j, i, q \quad \mathcal{A}(f_3) = 1$
$f_4 = v_z, l, i, q, h \quad \mathcal{A}(f_4) = 1$

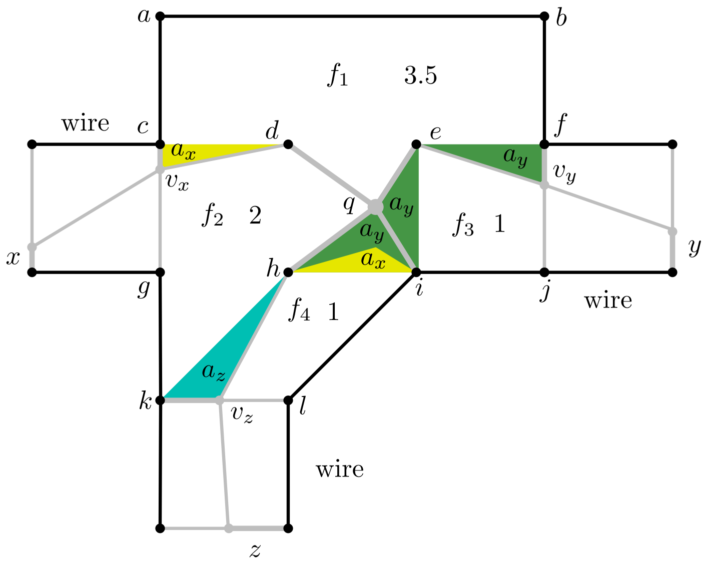



Figure 9: The addition gadget.

Let $x := \|cv_x\|$ and $y := \|fv_y\|$. Since $v_x, v_y, v_z$ encode the values of $X$, $Y$, and $Z$, respectively, we know that $X = \lambda x$, $Y = \lambda y$, and $Z = \lambda z$. Moreover, the area of the triangle $c, v_x, d$ is $a_x := x/2$, the area of the triangle $e, f, v_y$ is $a_y := y/2$ and the area of $k, h, v_z$ is $a_z := z/2$. The area of the triangle $d, e, q$ is forced to be $1/2 - a_x - a_y$, because of the constraint of on the area of $f_1$. Note if $\Psi$ is a positive instance, we may assume that $Z = X + Y < \lambda$, thus the area of the triangle $d, e, q$ is positive. On the other hand, if $X + Y > \lambda$, it is easy to see that there is no way to put flexible points in a way that conforms prescribed areas. Since the triangles $d, e, q$ and $h, i, q$ have parallel bases and share the third vertex, we observe that the area of the triangle $h, i, q$ is forced to be $a_x + a_y$. It is easy to see that the area of $f_4$ equals $1 = A(h, k, l, i) + A(h, q, i) - A(h, k, v_z) = 1 + a_x + a_y - a_z$. Thus we enforce $a_x + a_y = a_z$. It remains to show that all faces have the correct area given $a_x + a_y = a_z$. Observe that the area of the triangle $e, q, i$ is forced to equal $a_y$, because of the area of $f_3$. Consequently, the triangle $d, q, h$ has area $1/2 - a_y$. It holds that $\mathcal{A}(f_2) = A(c, d, h, k) - A(c, v_x, d) + A(d, h, q) + A(h, k, v_z) = 1.5 - a_x + (1/2 - a_y) + a_z = 2$.

Finally, observe that by connecting the gadgets we might have introduced some inner faces, which do not have specified areas. However, the area of each gadget is fixed, so we exactly know what the areas of these newly introduced faces are. We finish the construction by fixing the area for each remaining face, according to the way it is drawn in the plane.

Hence we created a planar graph $G_\Psi$ where the positions of some vertices are fixed, and an area assignment $\mathcal{A}$, such that $G_\Psi$ has a redrawing conforming $\mathcal{A}$ and respecting fixed positions of vertices if and only if $\Psi$ is true. The correctness of the construction follows easily from the correctness of individual gadgets. The size of $G$ is clearly polynomial in $|\Psi|$, so the proof is complete. □

## 5 Containment of Area and Volume Universality in $\forall\exists\mathbb{R}$

In this section, we show that Area Universality and Volume Universality* are contained in $\forall\exists\mathbb{R}$. For a triangle with vertices $v_1, v_2, v_3$ in counter-clockwise order, we can compute its area $A(v_1, v_2, v_3)$ by

$$2 \cdot A(v_1, v_2, v_3) = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} =: \mathrm{Det}(v_1, v_2, v_3).$$

This formula generalizes to simple polygons. Let $v_1, \ldots, v_n$ be the vertices of a simple polygon $\mathcal{P}$ in counter-clockwise order then the area $A$ of $\mathcal{P}$ can be computed by

$$2 \cdot A(\mathcal{P}) = \sum_{i=2}^{n-1} \mathrm{Det}(v_1, v_i, v_{i+1}). \tag{1}$$

The formula above is called the *shoelace formula* and was first described by Meister [16] (see also the related Wikipedia page [28]). Given a face area assignment $\mathcal{A}$ of a plane graph $G$, we want that each face fulfills an equation as in Equation 1. We call the system of equations for $G$ *area equations*. Clearly the existence of a realizing drawing implies a real solution of the area equations. The next proposition shows that we can also ensure by polynomial equations that the vertex placement yield a crossing-free equivalent drawing.

**Proposition 1.** Area Universality *is in* $\forall\exists\mathbb{R}$.

*Proof.* Let $G = (V, E, F)$ be a plane graph where $V$, $E$, and $F$ denote the set of vertices, edges and faces, respectively. We assume that $G$ has $n$ vertices, i.e., $V = \{v_1, \ldots, v_n\}$. We describe a polynomial-sized formula $\Psi \in$ UETR such that $G$ is area universal if and only if $\Psi$ is true. The idea is to use the universal quantifiers for the areas and the existential quantifiers for the coordinates of the vertices.

Recall that $G$ is area-universal if for all positive real numbers $A_1, \ldots, A_k$ there exist vertex positions $(X_1, Y_1, \ldots, X_n, Y_n)$ encoding a crossing-free, straight-line, and equivalent drawing of $G$ where the area of face $f_i$ is $A_i$. By definition, two drawings of $G$ are equivalent if they have the same outer face and rotation system. Hence, we ensure these properties by the following three conditions:

(i) All pairs of independent edges are crossing-free (noCrossing).
(ii) The cyclic order around each vertex is correct (Ordering).
(iii) Face areas are given by the universally quantified variables (Area).
(iv) The outer face has an area equal to the sum of all inner faces.

By noCrossing, Ordering, Area, Outer, we denote the predicates ensuring the four properties. Note that these properties imply that the outer face is preserved since only the outer face has as area equal to the sum of all the other areas.

The formula $\Psi$ is of the following form:

$$\forall(A_1, \ldots, A_k)\ \exists(X_1, Y_1, \ldots, X_n, Y_n)\colon \big((A_1 \geq 0) \wedge \ldots \wedge (A_f \geq 0)\big) \Rightarrow$$

$$\Bigg(\bigwedge_{\substack{e,e' \in E \\ e \cap e' = \emptyset}} \textsc{noCrossing}(e, e')\Bigg) \wedge \Bigg(\bigwedge_{v \in V} \textsc{Ordering}(v, N(v))\Bigg) \wedge \Bigg(\bigwedge_{f_i \in F} \textsc{Area}(A_i, f_i)\Bigg) \wedge \textsc{Outer}((f)_{f \in F, o}).$$

Here, the variables $(X_i, Y_i)$ represent the coordinates of $v_i$, $N(v)$ denotes the neighborhood of $v$. The outer face is denoted as $o$. For clarity of presentation, we have not replaced the vertices, edges and faces by its corresponding variables.

The predicate Area is already defined by the Equation 1. Note that Area requires that each face is drawn without crossings in order to be correct; this condition is enforced by noCrossing. It remains to describe noCrossing and Ordering.

Let us start with the noCrossing condition. For notational convenience, we denote the edges by $e = uv$ and $e' = rs$ and the coordinates of a point/vertex $v$ by $(v_x, v_y)$. Note that the edges $e$ and $e'$ are assumed to be independent, thus the points $u, v, r, s$ are pairwise different. Recall that a point $p$ is strictly left of an oriented line through the points $a, b$ if and only if $\mathrm{Det}(p, a, b) > 0$. It is easy to see that $e = uv$ and $e' = rs$ non-crossing is equivalent to at least one of the following conditions, see also left of Figure 10:

(i) Points $r$ and $s$ are on the same side of $\ell(u,v) \iff \mathrm{Det}(r,u,v) \geq 0 \wedge \mathrm{Det}(s,u,v) \geq 0$.

(ii) Points $u$ and $v$ are on the same side of $\ell(r,s) \iff \mathrm{Det}(u,r,s) \geq 0 \wedge \mathrm{Det}(v,r,s) \geq 0$.

Here $\ell(x,y)$ denotes the line through the points $x, y$. Note that we allow degenerate drawings as we allow that the area assignment is zero.

As indicated, each of theses conditions can easily be described as a product of determinants and thus each noCrossing condition is a conjunction of these two products.

Next, we describe the Ordering condition. For this it is important to distinguish whether or not $v$ is a reflex vertex of one of its adjacent faces. First assume $v$ is not a reflex vertex. Denote by $(v_1, \dots, v_d)$ the vertices around $v$ in counter-clockwise order. We first enforce the condition that $v_{i+1}$ is left of the line $\ell(v, v_i)$, for $i = 1, \dots, (d-1)$ and $v_1$ left of $\ell(v, v_d)$. See the middle of Figure 10. In case that $v$ is a reflex vertex of one of its faces, exactly one of the conditions is negated (the one corresponding to the reflex angle at $v$). We make a disjunction of all $d$ variants.

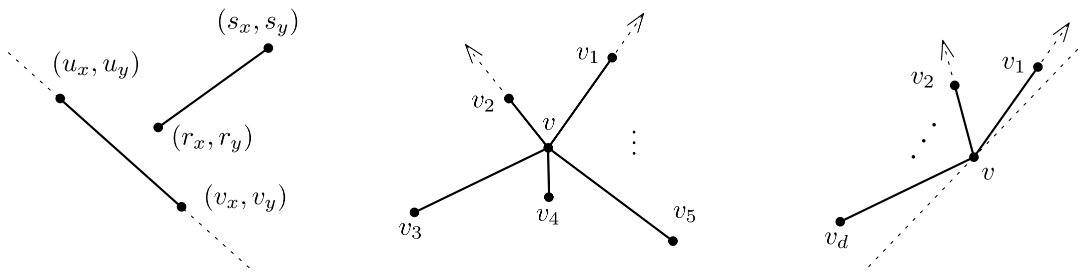



Figure 10: left: two non-crossing segments. middle: $v$ surrounded by its neighbors. right: $v$ is a reflex vertex to one of its faces

We also need to force the winding number around $v$ to be 1. That is, that the sum of the cyclic sequence of angles between consecutive neighbors of $v$ is $2\pi$. For this, we use the disjunction over each $s, t \in \{1, \dots, d\}$ not necessarily distinct such that $s \neq t + 1 \mod d$ of the following conditions. The first coordinate of each of the vectors $(v_s - v), (v_{s+1} - v), \dots, (v_t - v)$ is strictly positive, and that the first coordinate of each the vectors $(v_{t+1} - v), \dots, (v_{s-1} - v)$ is non-positive, where all indices are in the integers mod $d$. This divides the neighbors of $v$ into two contiguous intervals, one that is strictly to the right of $v$ and another that is to the left or vertically aligned with $v$.

The Outer condition is described by $\text{Area}(o) = \sum_{f \in F} \text{Area}(f)$.

In this way, we have described the formula $\Psi$ completely and correctness of the reduction follows from correctness of the predicates Ordering, Area, and noCrossing. Note that the size of $\Psi$ is polynomial in $n$. ☐

We conclude this section by showing that $\exists\mathbb{R}$ and $\forall\exists\mathbb{R}$ containment of Prescribed Volume and Volume Universality*, respectively.

**Proposition 10.** Prescribed Volume *is in* $\exists\mathbb{R}$.

*Proof.* Let $\mathcal{S}$ be a simplicial complex on the ground set $V$ with maximal sets $T$ together with a volume assignment $\mathcal{V}$. We describe an ETR-formula $\Psi$ that is true if and only if $\mathcal{S}$ has a crossing-free $\mathcal{V}$-realization. $\Psi$ is of the following form:

$$\underset{i \in V}{\exists} (X_i, Y_i, Z_i) \underset{\tau, \tau' \in T}{\exists} (X_{\tau,\tau'}, Y_{\tau,\tau'}, Z_{\tau,\tau'}, B_{\tau,\tau'}) :$$
$$\left( \bigwedge_{\tau,\tau' \in T} \text{noCrossing}(\tau, \tau') \right) \wedge \left( \bigwedge_{\tau \in T} \text{Volume}(\mathcal{V}_\tau, \tau) \right)$$

Here $\text{noCrossing}(\tau, \tau')$ is given by the conjunction for each $i \in \tau$,

$$X_{\tau,\tau'} X_i + Y_{\tau,\tau'} Y_i + Z_{\tau,\tau'} Z_i + B_{\tau,\tau'} \geq 0$$

and for each $j \in \tau'$,

$$X_{\tau,\tau'} X_j + Y_{\tau,\tau'} Y_j + Z_{\tau,\tau'} Z_j + B_{\tau,\tau'} \leq 0.$$

The intuition is that $X_{\tau,\tau'}, Y_{\tau,\tau'}, Z_{\tau,\tau'}$, and $B_{\tau,\tau'}$ represent the coefficients of a hyperplane separating the two tetrahedra $\tau, \tau'$. For $\tau = \{h, i, j, k\}$, the condition $\textsc{Volume}(A_\tau, \tau)$ is given by

$$\det \begin{pmatrix} X_h & X_i & X_j & X_k \\ Y_h & Y_i & Y_j & Y_k \\ Z_h & Z_i & Z_j & Z_k \\ 1 & 1 & 1 & 1 \end{pmatrix} = 6A_\tau.$$

This finishes the proof. □

**Proposition 4.** Volume Universality* *is in* $\forall\exists\mathbb{R}$.

*Proof.* We will show that Volume Universality* is in $\forall\exists\mathbb{R}$ by describing an arbitrary instance as an UETR-formula. We are given as input a pure abstract simplicial complex realizable in $\mathbb{R}^3$ over a ground set $V$ with maximal sets $T$ and a partial assignment of the volumes $\mathcal{V} : T' \to \mathbb{R}_0^+$. The corresponding formula is defined as follows:

$$\underset{\tau \in T}{\forall}(A_\tau) \underset{i \in V}{\exists} (X_i, Y_i, Z_i) \underset{\tau,\tau' \in T}{\exists}(X_{\tau,\tau'}, Y_{\tau,\tau'}, Z_{\tau,\tau'}, B_{\tau,\tau'}) :$$
$$\left( \bigwedge_{\tau \in T} A_\tau \geq 0 \right) \wedge \left( \bigwedge_{\tau \in T'} \mathcal{V}(\tau) = A_\tau \right) \Rightarrow \left( \bigwedge_{\tau,\tau' \in T} \textsc{noCrossing}(\tau, \tau') \right) \wedge \left( \bigwedge_{\tau \in T} \textsc{Volume}(A_\tau, \tau) \right)$$

Predicates noCrossing and Volume are defined as in Proposition 10. □

# 6 Volume-realization of simplicial complexes in 3D is hard

We start with some basic definitions and observations. Recall that a *simplex* $s \subseteq \mathbb{R}^d$ is the convex hull of at most $d + 1$ affinely independent points in general position, the *vertices* of $s$ are denoted by $\text{vert}(s)$. By affinely independent, we mean that there are no $r_i \in \mathbb{R}$ such that $r_1 + \cdots + r_k = 1$ and $v_0 = r_1 v_1 + \cdots + r_k v_k$ where $\{v_0, \ldots, v_k\} = \text{vert}(s)$.

For a simplex $s$, the convex hull of a subset of $\text{vert}(s)$ is a *face* of $s$. A *simplicial complex* $\mathcal{S}$ is a set of simplices in $\mathbb{R}^d$ that satisfies the following conditions:

- Any face of a simplex from $\mathcal{S}$ is also a simplex in $\mathcal{S}$.
- The intersection of two simplices $s, t \in \mathcal{S}$ is either $\emptyset$ or a face of both simplices $s$ and $t$.

An *abstract simplicial complex* is a collection of finite sets $\Sigma$ that is closed under inclusion. That is, if $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ then $\tau \in \Sigma$. Note that to define an abstract simplicial complex $\Sigma$ it suffices to only specify the maximal sets of $\Sigma$. We say $\Sigma$ is a *pure d*-dimensional abstract simplicial complex when all of the maximal sets in $\Sigma$ have exactly $d + 1$ elements.

We say an abstract simplicial complex $\Sigma$ is *realizable* in $\mathbb{R}^d$ when there is a simplicial complex $\mathcal{S} \subseteq \mathbb{R}^d$ that has a simplex $\tilde{\sigma}$ for each set $\sigma \in \Sigma$ and $\tilde{\tau} \in \mathcal{S}$ is a face of a simplex $\tilde{\sigma} \in \mathcal{S}$ if and only if $\tau \subseteq \sigma$. We say $\mathcal{S}$ *realizes* $\Sigma$. Note that every simplicial complex $\mathcal{S}$ realizes the abstract simplicial complex consisting of the set of vertices of each simplex of $\mathcal{S}$. Note also that a crossing-free drawing of an abstract simplicial complex is not necessarily a simplicial complex, since some simplices may be degenerate, or may intersect along their boundary where they do not share a common face.

**Basic properties** Let $\mathcal{S} = (V, F)$ be a simplicial 3-complex (i.e., a simplicial complex in $\mathbb{R}^3$). Lets first recall that the volume of a tetrahedron $t = \{a, b, c, d\}$ can be expressed in terms of the area of one triangle $\triangle := \{a, b, c\}$ and its respective height $h$:

$$\text{vol}(t) = {}^1\!/\!_3 \cdot \text{area}(\triangle) \cdot h.$$

As suggested by abstract simplicial complexes, we think of a simplex $s$ mostly as the set of vertices $\text{vert}(s)$. However, talking about area or volume, we refer to the volume of the convex hull of $\text{vert}(s)$. Given two simplices $s, t$ we define $s \oplus t$ as the simplex consisting of the convex hull of $vert(s) \cup vert(t)$. For a simplicial complex $\mathcal{S}$ and a vertex $v$, we define the *cone of* $v$ *over* $\mathcal{S}$ by

$$\mathcal{S} \oplus v := \{s \oplus v \mid s \in \mathcal{S}\}.$$

In particular, we will be interested in cones $G \oplus v$ over plane triangulations $G$ for a vertex $v$. We also say $v$ is an *apex* for $G$ in $G \oplus v$.

Let $\mathcal{V}$ be a volume-assignment of a simplicial complex $\mathcal{S}$ containing $G \oplus v$ for some plane triangulation $G$ and apex $v$. We define *the induced area-assignment* $\mathcal{A}_v$ of $v$ on $G$ as follows: For each triangle $abc$ of $G$, we set $\mathcal{A}_v(\{a,b,c\}) := \mathcal{V}(\{a,b,c,v\})$.

**Lemma 11.** *Let $G$ be a plane triangulation and $\mathcal{S}$ be a simplicial complex containing $G \oplus v$ for some vertex $v$. Let $\mathcal{V}$ be a volume assignment of $\mathcal{S}$. If $G$ is coplanar in a $\mathcal{V}$-realizations of $\mathcal{S}$, then the induced area-assignment $\mathcal{A}_v$ is realizable for $G$.*

*Proof.* For each triangle $abc$ of $G$ and its corresponding tetrahedron $abcv$ with height $h$ it holds that: $\text{vol}(abcv) = {}^1\!/_3 \cdot h \cdot \text{area}(abc)$. If $G$ is coplanar, then the height $h$ of all tetrahedra with respect to this plane coincides. Hence, the volume of the tetrahedra translates immediately to the area of the triangles (scaled by ${}^1\!/_3h$). If the volume assignment $\mathcal{V}$ is realized for $\mathcal{S}$, then the induced area assignment $\mathcal{A}_v$ scaled by $\lambda := {}^1\!/_3h$ is realized for $G$. By scaling the plane accordingly, we find an $\mathcal{A}_v$-realizing drawing of $G$. □

Let $\mathcal{S}$ be a simplicial complex with volume assignment $\mathcal{V}$, $T$ a plane triangulation and $v$ a vertex of $\mathcal{S}$ but not of $T$. Let $T \oplus v$ be contained in $\mathcal{S}$. Consider the (weak) dual graph $D$ of $T$: The vertex set of $D$ consists of the inner faces of $T$ and the edge set represents adjacent faces. We call an edge $e$ in $D$ *good*, if in any $\mathcal{V}$-realization of $\mathcal{S}$, the two adjacent triangles of $e$ are coplanar. The *coplanar graph* of $T$ (with respect to $\mathcal{V}$) is the subgraph of $D$ consisting of good edges.

**Lemma 12.** *Let $T$ be plane triangulation, $v$ be a vertex, and $\mathcal{S}$ be a simplicial complex containing $v \oplus T$. $T$ is coplanar in all $\mathcal{V}$-realization of $\mathcal{S}$ if and only if the coplanar graph of $T$ is connected.*

*Proof.* Suppose $T$ is coplanar in all $\mathcal{V}$-realization of $\mathcal{S}$. Then clearly all edges of $D$ are good. Consequently, the coplanar graph of $T$ is connected.

Suppose the coplanar graph of $T$ is connected. Then, by transitivity of coplanarity, $T$ is coplanar in all $\mathcal{V}$-realization of $\mathcal{S}$. □

## 6.1 Reduction from prescribed area for triangulations

We show that the realizabitity of an area-assignment of any triangulation can be reduced to an instance of Prescribed Volume.

**Proposition 5.** *There is a polynomial time algorithm that takes as input any plane triangulation $G$ with positive area-assignment $\mathcal{A}$ and outputs a simplicial complex $\mathcal{S}$ with volume assignment $\mathcal{V}$ such that $\mathcal{A}$ is realizable for $G$ if and only if $\mathcal{V}$ is realizable for $\mathcal{S}$.*

*Proof.* For a plane triangulation $G$, we define a simplicial complex $\mathcal{S}_G$ in the following way, see Figure 11a: Take two copies of $G$ and glue them along an outer edge resulting in the graph $F$. We introduce two new vertices $x$ and $y$ and define $S_G := (x \oplus F) \cup (y \oplus F)$. Note that $x$ and $y$ are apices for their neighborhoods.

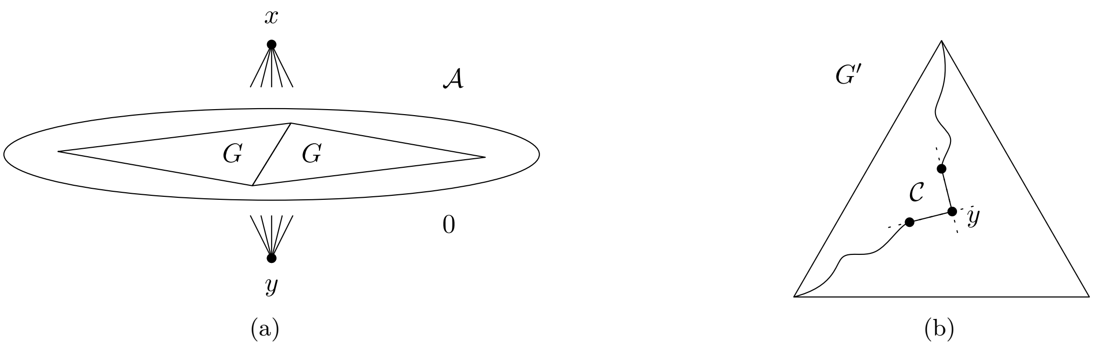



Figure 11: (a) Construction of $\mathcal{S}_G$ and volume assignment $\mathcal{V}_{\mathcal{A},0}$.
(b) If a copy $G'$ of $G$ is not coplanar, then $y$ coincides with an interior vertex of $G'$.

For a given area-assignment $\mathcal{A}$ of $G$, we consider the following volume assignment of $\mathcal{S}_G$:

$$\mathcal{V}_{\mathcal{A},0}(s) := \begin{cases} \mathcal{A}(t) & \text{for } s = t \oplus x \\ 0 & \text{for } s = t \oplus y \end{cases}$$

We show the following: $\mathcal{A}$ is realizable for $G$ if and only if $\mathcal{V}_{\mathcal{A},0}$ is realizable for $\mathcal{S}_G$.

If $\mathcal{A}$ is realizable for $G$, then we consider a realizing drawing of $G$ and its copy in the plane. Placing $x$ at height 3 above the plane and $y$ at height 0 below the plane yields a $\mathcal{V}$-realization for $\mathcal{S}_G$.

Now assume that $\mathcal{V}_{\mathcal{A},0}$ is realizable for $\mathcal{S}_G$. We will show that the vertices of at least one copy of $G$ in any $\mathcal{V}_{\mathcal{A},0}$-realization of $\mathcal{S}_G$ are coplanar. To this end, let us assume that the vertices of one of the copies $G'$ of $G$ in $F$ is not coplanar. We will call an interior edge $e$ of $G'$ a *crease edge* when the two triangles adjacent to $e$ are not coplanar. We claim that $y$ must be at an interior vertex of $G'$.

Observe that the area of each triangle of $G'$ must be positive, since each of the tetrahedra formed by a triangle of $G'$ with the vertex $x$ has positive volume. Hence, for every triangle $t$ of $G'$, we must have that $y$ is in the plane spanned by $t$.

Let $\mathcal{C}$ be the partition of the the triangles of $G'$ consisting of the maximal subsets of triangles of $G'$ that are coplanar. Since the vertices of $G'$ are not coplanar, $C$ must have multiple parts, and the boundaries of these parts must include at least one interior vertex. Hence, there is a closed walk $W$ in $G'$ around the boundary of a part of $\mathcal{C}$ that includes at least one interior vertex, see Figure 11b. Since, the walk $W$ is around the boundary of a part of $\mathcal{C}$, all the edges of $W$ that are interior edges of $G'$ are in fact crease edges.

For any crease edge $e = t_1 \cap t_2$ adjacent to triangles $t_1$, $t_2$, the planes spanned by $t_1$ and $t_2$ intersect in a line $l$, and since $y$ is in both of these planes, $y$ must be on the line $l$. Observe that $l$ is also the line spanned by the edge $e$.

Since all triangles have positive area, the walk $W$ must include some interior vertex $w$ where it bends. That is, the edges, $e_1$, $e_2$ of $W$ that are adjacent to $w$ are not collinear. Since $y$ is on both the line spanned by $e_1$ and the line spanned by $e_2$, we must have that $y = w$ at the point where these lines meet. Thus, the claim holds, $y$ is at an interior vertex of $G'$.

Since each of the tetrahedra formed by a triangle of $F$ with the vertex $x$ has positive volume and these tetrahedra are non-crossing, none of the interior vertices of $F$ can coincide. Therefore, $y$ can be an interior vertex of at most one of the copies $G$ that constitute $F$, which means that the vertices of a copy $G''$ of $G$ in $F$ are coplanar.

Since the vertices of $G''$ are contained in a plane $P$, each triangle $t$ of $G''$ has area $(\frac{3}{h})\mathcal{V}_{\mathcal{A},0}(t \oplus x)$ where $h$ is the distance between $x$ and $P$. Thus, if $\mathcal{V}_{\mathcal{A},0}$ is realizable, then scaling $G''$ by $\frac{h}{3}$ as above provides a realization of $\mathcal{A}$. $\square$

## 6.2 Prescribed Volume is hard

In this section we prove:

**Theorem 6.** Prescribed Volume *is* $\exists\mathbb{R}$*-complete.*

*Proof.* Containment of Prescribed Volume follows from Proposition 10. It remains to show $\exists\mathbb{R}$-hardness. We reduce from ETRINV. Let $\Psi = \exists X_1 \ldots X_n : \Phi(X_1, \ldots, X_n)$ be an instance of ETRINV, where $\Psi$ is a conjuction of constraints expressing additions, inversions or introducing a constant. We will, however, extend this proof in order to also prove Theorem 7. This is why we handle the inversion constraints as general multiplications constraints (in order to be able to reduce from Constrained-UETR).

We construct a simplicial complex $\mathcal{S} = (V, F)$ and a volume assignment $\mathcal{V}$ for the tetrahedra in $F$, such that $\mathcal{S}$ has a $\mathcal{V}$-realization if and only if $\Phi$ is satisfiable. We start with a description of our essential building block.

**Coplanar gadget** The coplanar gadget forces several triangles of *equal area* to lie in a common plane, see Figure 12a. These triangles will be free to one half space and hence, accessible for our further construction. Indeed, all but one vertex will lie in the same plane. We call this plane the *(base) plane* of the coplanar gadget.

For the construction, consider Figure 12b. The coplanar gadget consists of 3 layers of tetrahedra. The

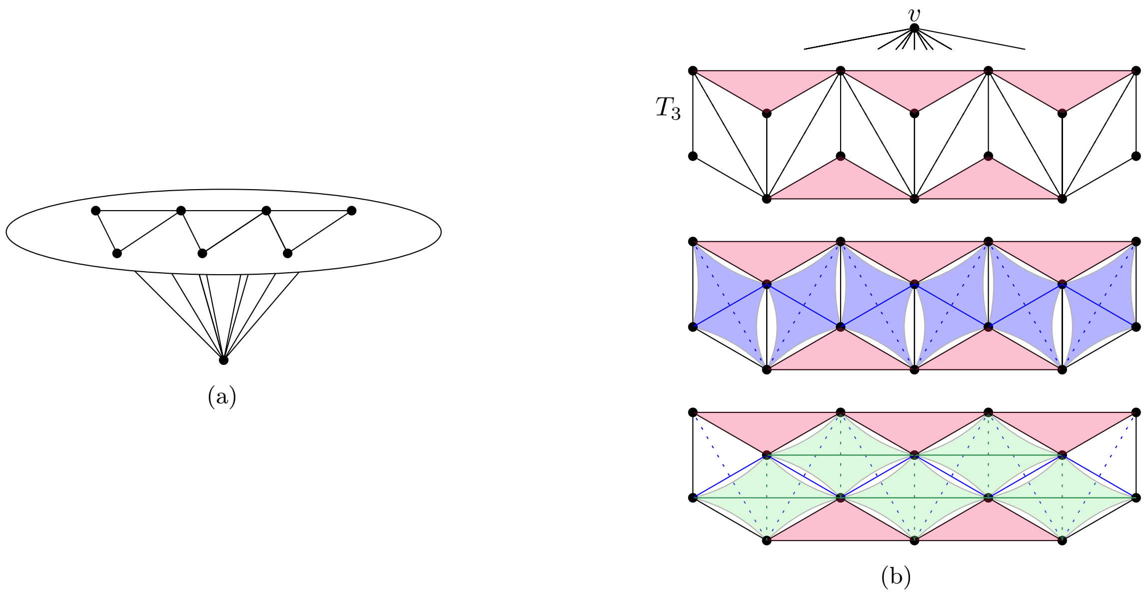



Figure 12: The coplanar gadget in (a) and its construction in (b).

first layer forces the triangles to be of equal size and the two other layers force coplanarity. We start with the planar graph $T_k$ depicted in the top of Figure 12b. (The number $k$ denotes the number of red triangles in the top row). We introduce a new vertex $v$ and a tetrahedron $v \oplus t$ for each of the triangles $t \in T_k$. We assign a volume of 1 to each of the tetrahedra. In other words, $v$ is an apex for $T_k$ and $\mathcal{V}(t \oplus v) = 1$.

Now, we come to the two layers enforcing co-planarity. Observe that the dual graph of the white triangles is an even path. Consider a maximum matching $M$ in this path. For each matching edge, we introduce a tetrahedron of volume 0 consisting of the two triangles in the matching, see the blue tetrahedra in Figure 12b. In any realization, these quadruples of points are now coplanar (due to the volume-assignment). Now we consider the path without $M$. This is also a matching and for each matching edge, we introduce a tetrahedron of volume 0 consisting of the two triangles in the matching, see the green tetrahedra in Figure 12b. Hence, the coplanar graph of $T_k$ is connected and hence, by Lemma 12, the vertices of $T_k$ are coplanar in every volume-realization. Lemma 11 in turn implies that the areas of all triangles in $T_k$ are of equal size due to their tetrahedra with $v$. Note that the actual area depends on the distance of the plane to $v$. When we refer to a *triangle* of the base plane, we always mean one of the red triangles. Note that the coplanar gadget helps to force a set of points to be coplanar, by inserting a tetrahedron for each point with its private triangle from the base plane. This set of points is then *accessible* from almost all sides. We will make use of this fact to force points on a line.

**Forcing points on a line** We use the coplanar gadget to force a set of points representing the values of the variables to lie on a common line $\ell$. In order to do so, we take two coplanar gadgets and enforce that their base planes $E, E_\ell$ are not parallel. This can be achieved by choosing two triangles $\Delta_1, \Delta_2$ (of equal area) on $E$ and two distinct points $v_1, v_2$ from $E_\ell$ and inserting two tetrahedra $v_1 \oplus \Delta_1$ and $v_2 \oplus \Delta_2$ of different volume (1 and 2).

Now consider a set of points $\{x_1, \ldots, x_n\}$. For each $x_i$, we introduce two tetrahedra: one with a triangle on $E$ and the other with a triangle on $E_\ell$, see Figure 13.

We prescribe the volumes to be 1. Since the planes $E, E_\ell$ are not parallel, but the triangles on each plane have the same area, all $x_i$ must lie in the intersection of the two planes (parallel to $E$ and $E_\ell$, respectively.) We call this line $\ell$.

These points will represent the values of our variables $X_i$ in the following way. We introduce two special points $p_0$ and $p_1$ on the line $\ell$ and two other free points $q, q'$ not necessarily on $\ell$. The tetrahedron $p_0, p_1, q, q'$ of volume 1 ensures that $p_0$ and $p_1$ are distinct. Without loss of generality we assume that $\|p_0 p_1\| = 1$. The value of variable $X_i$ is represented by $\|p_0 x_i\|$.

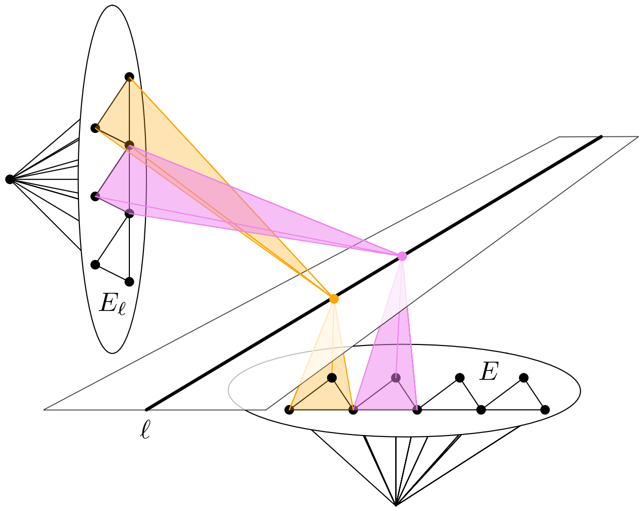



Figure 13: Forcing points to lie on a line $\ell$.

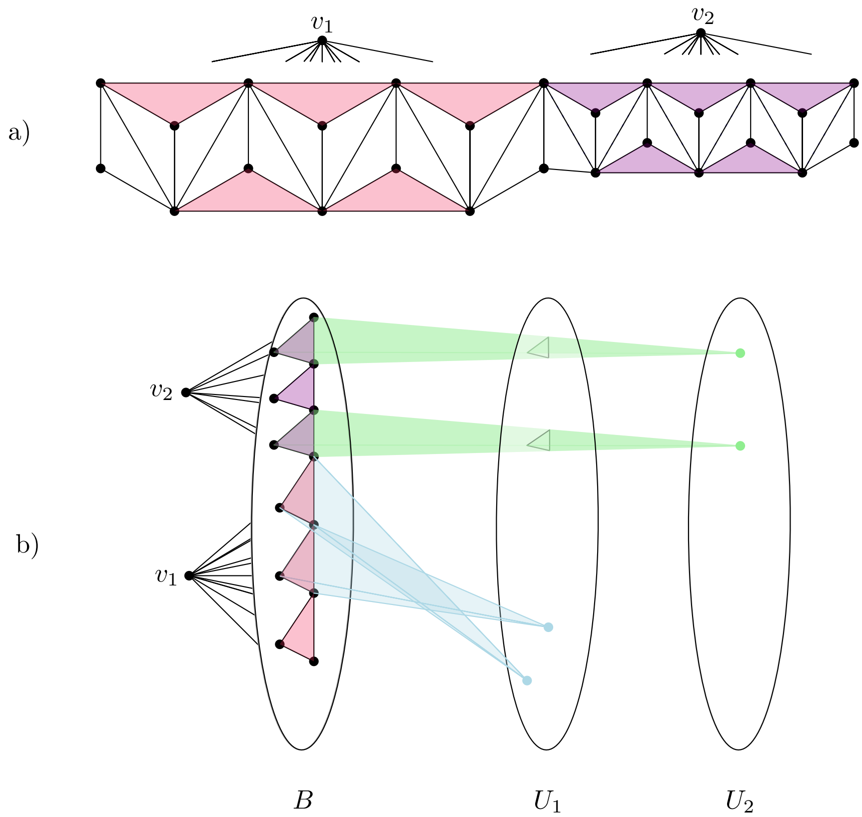



Figure 14: The parallel gadget. a) The triangulation $T_{2k}$ consists of part $T^1$ with red triangles of equal area and $T^2$ with violet triangles of equal area. For $i \in \{1, 2\}$, each $T^i$ has its private apex $v_i$.
b) The points of $U_1$ and $U_2$ are both in a common plane parallel to $B$. However, these two planes are independent.

**Parallel gadget** The parallel gadget is an extension of the coplanar gadget in order to force two sets of points to lie in two parallel planes that can vary independently of each other, see also Figure 14. Let $U_1$ and $U_2$ be two vertex sets. Consider a coplanar gadget with base plane $B$.

The construction has a triangulation $T_{2k}$ and two layers of degenerate tetrahedra forcing the triangulation to be coplanar in the same way as the degenerate tetrahedra of the coplanar gadget.

The triangulation $T_{2k}$ is broken into two parts, each isomorphic to some $T_k$. We denote these parts by $T^1$ and $T^2$, respectively. For each part $T^i$, there is an apex vertex $v_i$ and a simplex $v_i \oplus t$ for each $t \in T^i$, and all of these simplices are assigned volume 1. Consequently, the triangles of $T_{2k}$ are coplanar and the triangles from the same part $T^i$ have the same area. Note that the areas of the triangles of $T^1$ (red triangles in Figure 14) are independent from the areas of the triangles of $T^2$ (violet triangles in Figure 14). We then add vertices $u \in U_i$ and for each triangle $t_i$ of $T^i$ a simplex $u \oplus t_i$ that is assigned volume 1. As in the coplanar gadget the vertices $u \in U_i$ from the same set are coplanar and the planes of coplanarity are each parallel to the plane that contains the triangulation $T$.

Clearly, we can generalize this construction to force $h$ sets of points to lie in $h$ parallel planes for any constant $h$.

**Addition and Multiplication gadget** For both, the addition and multiplication gadgets, we mimic the ideas of the von Staude constructions.

An important step is to construct a line parallel to a given line through a specified point. Let $\ell_1$ be a line forced by coplanar gadgets with the already mentioned plane $E$ and a new plane $E_1$. Let $p$ be some point. We construct a line $\ell_2$ parallel to $\ell_1$ through $p$ by introducing a parallel gadget extending the base plane $E_1$. We force points to be on $\ell_2$ by tetrahedra with triangles on $E_1$ (belonging to the parallel gadget) and $E$ (if they do not already exist), see Figure 15.

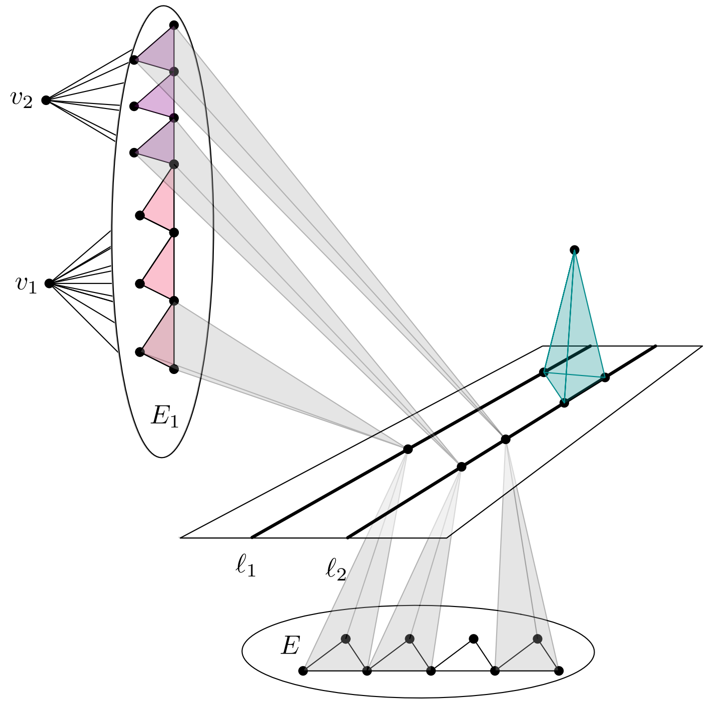



Figure 15: Constructing a parallel line. The cyan tetrahedron enforces $\ell_1$ and $\ell_2$ to be distinct.

In case we need to enforce that $\ell_1$ and $\ell_2$ do not coincide, we insert a tetrahedron $t$ with volume 1 consisting of two new points on $\ell_1$, a new point on $\ell_2$ and an arbitrary new point. This tetrahedron $t$ is displayed in cyan in Figure 15. Due to the positive volume of $t$, all four vertices $\mathrm{vert}(t)$ are distinct and affinely independent. Consequently, in any $\mathcal{V}$-realization, $\ell_1$ and $\ell_2$ are two parallel but distinct lines.

**Addition gadget** For the *addition gadget* consider Figure 16a. Suppose we want to enforce the constraint $X + Y = Z$. Thus the value of two variables $x, y$ lying on the line $\ell$ should add up to the value of the point $z$ also on $\ell$. We introduce a line $\ell'$ parallel to $\ell$ through a new point $u_1$ and ensure that the two lines are distinct as described above (analogous to the cyan tetrahedron in Figure 15). Using the parallel gadget we can enforce later more points to lie on $\ell'$.

Then, we introduce a line $\ell_g$ through $p_0$ and $u_1$ and a parallel line $\ell'_g$ through $y$. Again, we use the parallel gadget. These two lines are green in Figure 16a. We introduce the intersection point $u_2$ of $\ell'$ and $\ell'_g$. Then we construct the line $\ell_r$ through $u_1$ and $x$ and introduce the parallel line $\ell'_r$ through $u_2$. These two lines are red in Figure 16a. The intersection point of $\ell$ and $\ell'_r$ is identified with $z$. By parallelity of the lines, it follows that $\|u_1u_2\| = \|p_0y\| = \|xz\|$ and hence, $z$ is representing the value $\|p_0z\| = \|p_0x\| + \|p_0y\|$.

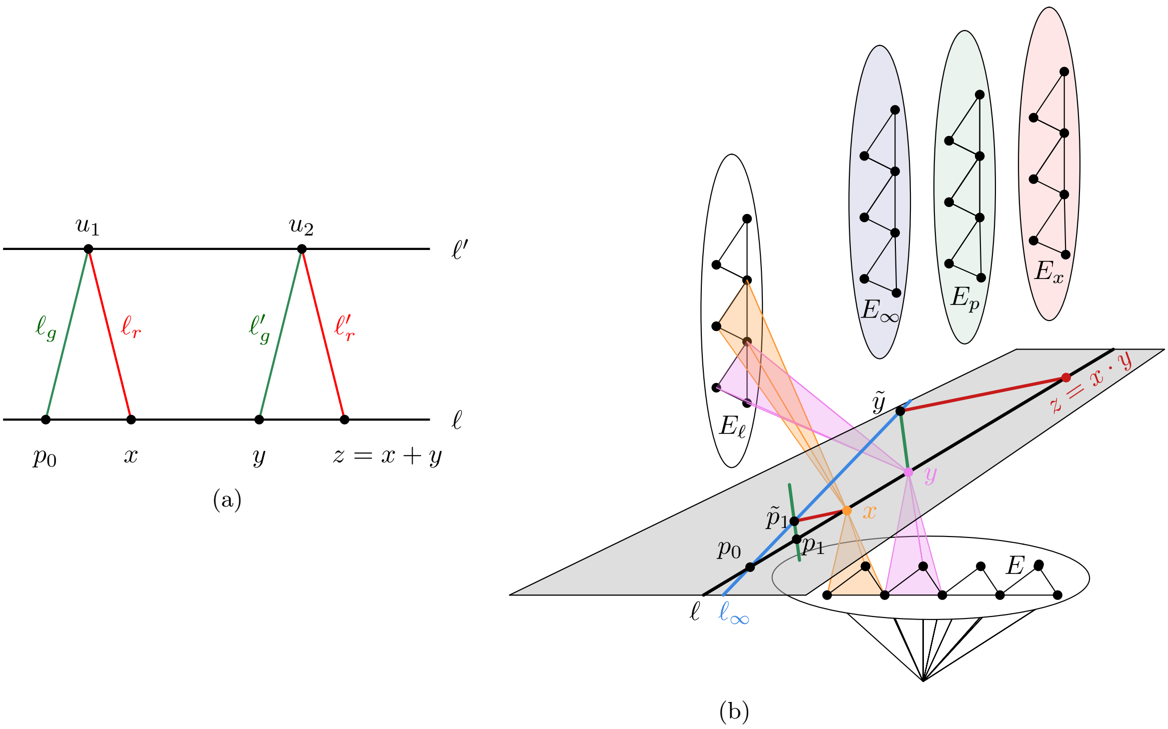



Figure 16: The addition gadget in (a) and the multiplication gadget in (b).

**Multiplication gadget** Now, we come to the *multiplication gadget*. (Note that an inversion gadget would suffice. However, we will extend this construction to prove ∀∃ℝ-hardness for Volume Universality*. For this reduction we need the multiplication gadget, since we are going to reduce from Constrained-UETR.)

We will mimic the multiplication gadget from von Staude, which we previously used to prove Theorem 2, see Figure 4. We start by introducing a new coplanar gadget with base plane $E_\infty$ to create a line $\ell_\infty$ intersecting $\ell$ in $p_0$ and spanning a plane with $\ell$ parallel to $E$. The points on $\ell_\infty$ are forced by the planes $E$ and $E_\infty$. We ensure that $\ell_\infty$ and $\ell$ do not coincide by introducing a tetrahedron of positive volume consisting of $p_0$, a new point on $\ell$, a new point on $\ell_\infty$, and an arbitrary new point.

For the construction of the multiplication gadget, we choose a special point $\tilde{p}_1$ on $\ell_\infty$ and introduce the line $p_1\tilde{p}_1$. A parallel gadget with base plane $E_p$ (together with $E$) forces a line parallel to $p_1\tilde{p}_1$ through $y$. These lines are depicted as green lines in Figure 16b.

For each multiplication constraint including a fixed variable $x$, we need to introduce one private plane $E_x$. It forces lines parallel to $x\tilde{p}_1$ (through $y\tilde{y}$). These lines are depicted as red lines in Figure 16b. We have already argued in the proof of Theorem 2, that the intercept theorem implies the correctness of this multiplication gadget. Recall that the intercept theorem implies

$$\frac{\|\tilde{p}_1p_0\|}{\|\tilde{y}p_0\|} = \frac{\|p_1p_0\|}{\|yp_0\|} = \frac{\|xp_0\|}{\|zp_0\|}.$$

Thus $\|xp_0\| \cdot \|yp_0\| = \|zp_0\|$. Note that this gadget works for both cases $x \le y$ and $x > y$.

This finishes the construction of $\mathcal{S}$ and its volume assignment $\mathcal{V}$. Next we will show that this yields a reduction.

**Proposition 13.** *There exists a $\mathcal{V}$-realization of $\mathcal{S}$ if and only if $\Psi$ is a* YES*-instance.*

If there exists a $\mathcal{V}$-realization of $\mathcal{S}$ then by construction of the coplanar gadget, all $x_i$ are on a common line $\ell$. By construction of the addition and multiplication gadgets all constraints of $\Psi$ are satisfied. Hence setting $X_i := \|p_0 x_i\|$ is a satisfying assignment, and it follows that $\Psi$ is a YES-instance.

Now, suppose that $\Psi$ is a YES-instance. We will describe an explicit placement of all the previously described objects. We will give mostly explicit coordinates for the objects that are responsible for the encoding of variables and the actual calculations. For most of the coplanar and parallel gadgets, we will place them very far away in "generic" directions. In this way most tetrahedra will become long and skinny, but it also ensures that they do not intersect, as those directions are sufficiently different. The following thought experiment might help to give an intuition: Consider a set $Q$ of points in $\mathbb{R}^3$ and a set $R$ of rays emitting from $Q$. When we choose the directions of the rays in a generic direction, we will not expect any two to intersect. In fact we will expect that they have some positive distance. Those long and skinny tetrahedra can be thought of as thickened rays.

As a start, we assume there exist values for $X_1, \ldots, X_k$ fulfilling the addition and multiplication constraints. By slight abuse of notation, we will also denote the values of $X_1, \ldots, X_k$ by the same symbol. We have to show that there exists a $\mathcal{V}$-realization of $\mathcal{S}$. In order to do so, we place the point $x_i = (X_i, 0, 0)$. For the line $\ell$ of our construction, we choose the $x$-axis. We can then build a coplanar gadget with triangulation $T$ in the plane $E = \{(X, Y, -1)\}$ and a second coplanar gadget with triangulation $T_\ell$ in the plane $E_\ell = \{(X, -1, Z)\}$. We can then add all the tetrahedra forcing the points $x_i$ to be on the line $\ell$.

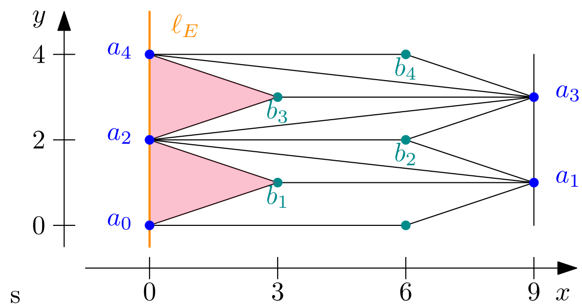



Figure 17: Placing the triangulation of a coplanar gadget.

We position the triangulation $T$ in the plane $E$ so that triangles $t_i$ where we attach tetrahedra with a vertex at the points $x_i$ are positive in the second coordinate and so that successive triangles have larger second coordinate everywhere. Specifically, we can choose points $a_0, \ldots, a_{2k}$ and $b_0, \ldots, b_{2k-1}$, as shown in Figure 17, given explicitly by

$$a_i = \begin{cases} (0, i, -1) & \text{for } i \text{ even} \\ (9, i, -1) & \text{for } i \text{ odd} \end{cases}$$
$$b_i = (a_{i-1} + a_i + a_{i+1})/3.$$

Let $t_i$ be the triangle with vertices $\{a_{2i}, b_{2i+1}, a_{2i+2}\}$ and note that it has area 3. We can then put $v = (0, 0, -2)$. Let us denote the line through all $a_2, a_4, a_6, \ldots$ as the *skeleton-line* $\ell_E$ of the triangles in $E$. As depicted in Figure 13, $\ell_E$, is orthogonal to $\ell$. Therefore we may add the simplices given in the definition of the coplanar gadget. Observe these tetrahedra all have pair-wise disjoint interior for all positions of $x_i$ on $\ell$. To be more specific, let $T_i = t_i \oplus x_i$ and $T_j = t_j \oplus x_j$ be two such tetrahedra with $i < j$. We define a plane $P$ that separates the interior of the two tetrahedra as follows. Let $k_i$ be the line $(X, 2i, -1)$ parallel to $\ell$. The unique plane that contains $\ell$ and $k_i$ defines $P$, see Figure 18. Similarly, we position the triangulation $T_\ell$ so that triangles $t'_i$ where we attach tetrahedra connecting to the points $x_i$ are positive in the third coordinate and so that successive triangles have larger third coordinate everywhere. In other words, the skeleton line $\ell_{E_\ell}$ of $T_\ell$ is orthogonal to both $\ell_E$ and $\ell$. This

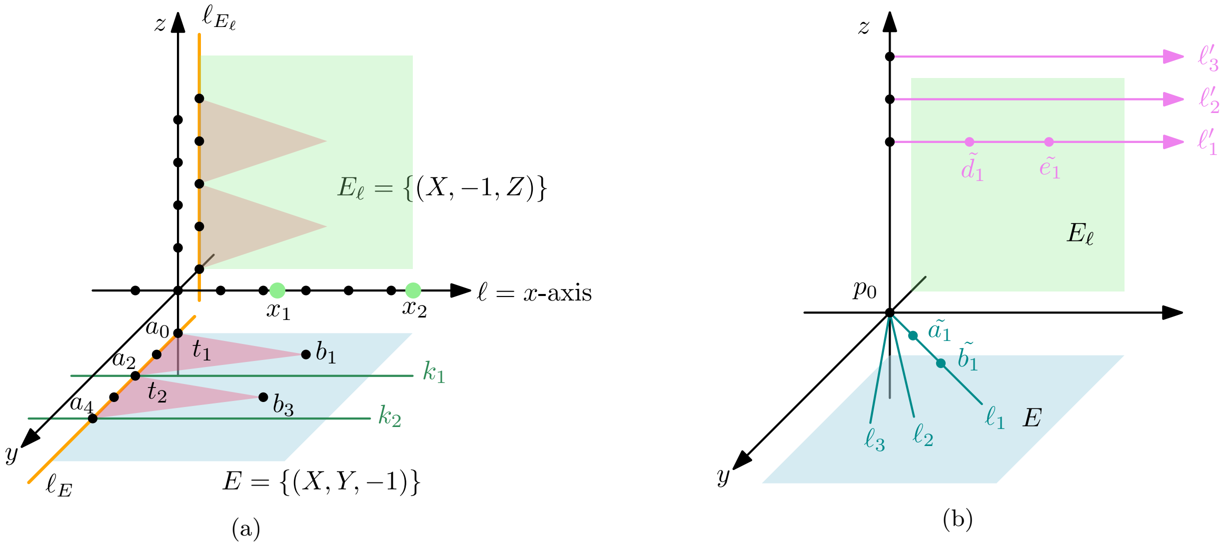



Figure 18: Placement of the base planes $E$ and $E_\ell$ and other related objects.

may be done in the same way as for $T$ by swapping the second and third coordinates. We may extend the triangulation $T_\ell$ to include vertices that are negative in the third coordinate, and attach a tetrahedron to two of these, $t'_0$ and $t'_{-1}$ as given above, with a vertex at $a_0$ and $a_1$ respectively, which then have volumes 1 and 2. Observe these tetrahedra all have pair-wise disjoint interior.

For the addition and multiplication gadget we assume that the formula $\Phi$ is "long enough", i.e., its length $n := |\Phi|$ is at least $C$, for some sufficiently large constant $C$. Otherwise we may replace $n$ with a sufficiently large constant where needed in the construction.

For each multiplication constraint $C_i \equiv [X \cdot Y = Z]$ in $\Phi$, we will attach a copy of the multiplication gadget. Each multiplication gadget introduces 2 new points, $\tilde{a}_i$ and $\tilde{b}_i$ that respectively correspond to the points $\tilde{p}_1$ and $\tilde{y}$ in the definition of the multiplication gadget, and introduces one new line $\ell_i$ that corresponds to the line $\ell_\infty$ in the definition of the multiplication gadget. We choose these lines $\ell_i$ so that successive lines have greater slope. Specifically, we choose $\ell_i = \{X, iX, 0\}$ to have slope $i$. We extend the triangulation $E_\ell$ and add tetrahedra connected triangles of $T_\ell$ to the points $\tilde{a}_i, \tilde{b}_i$ to force these points to be in the plane parallel to $E_\ell$ through the line $\ell$, in this case the second coordinate plane. Note that we are free to choose $\tilde{a}_i$ on the line $\ell_i$, and this determines $\tilde{b}_i$, provided that both points have positive first coordinate.

For each addition constraint $C \equiv [X + Y = Z]$ in $\Phi$, we will attach a copy of the addition gadget. Each addition gadget introduces two new points $\tilde{d}_i$ and $\tilde{e}_i$ that respectively correspond to $u_1$ and $u_2$ in the definition of the addition gadget. We put the points $\tilde{d}_i$ and $\tilde{e}_i$ on the ray $\ell'_i = \{(X, 0, n+i) : X > 0\}$. Note that these rays do not intersect the lines $\ell_i$ from the multiplication gadgets. Again we extend $T_\ell$ and add tetrahedra connecting a triangle of $T_\ell$ to each of these points. Note that we are free to choose $\tilde{d}_i$ on the ray $\{(X, 0, n+i) : X > 0\}$, and this determines $\tilde{e}_i$.

For each coplanar or parallel relation in an addition gadget or a multiplication gadget we choose a generic direction to add a coplanar or parallel gadget. To do so, let $v_i = (i, n \cdot i, 1)$, then we add a triangulation $T_i$ starting with the point $n^3 v_i$. We may choose all planes of the triangulations $T_i$ to have slope at most 1 with respect to the second coordinate plane. This means that the tetrahedra connecting points in the second coordinate plane to those triangles that are within a constant radius of $n^3 v_i$ have height that is of order $n^4$. Therefore, the triangles will have area that is of order $n^{-4}$. Since the triangulation has only finitely many triangles, the entire triangulation can be placed within a distance of order $n^{-4}$ from the point $n^3 v_i$. Thus, for $n$ large enough, for each point $x$ among $x_i, \tilde{a}_i, \tilde{b}_i, \tilde{d}_i, \tilde{e}_i$, the tetrahedra connecting $x$ to the coplanar and parallel gadgets will only intersect at $x$.

Consider a pair of distinct vertices $x, x'$ among the vertices $x_i, \tilde{a}_i, \tilde{b}_i, \tilde{d}_i, \tilde{e}_i$ and two tetrahedra $t, t'$ connecting those vertices to a parallel or coplanar gadget. If both $x$ and $x'$ are chosen to be on the line $\ell$, then $t$ and $t'$ can never intersect, so assume that $x'$ is not on $\ell$. The tetrahedra are contained in cylinders $C, C'$ of radius of order at most $n^{-4}$. The projection of the cylinder $C$ to the transverse plane of $C'$

is a strip of width of order $n^{-4}$ that has slope of order at least $n^{-2}$ with respect to the line on which the point $x'$ is chosen. Thus, if we fix the position of $x$, there is an interval of length of order at most $n^{-2}$ where we cannot place $x'$ without $t$ and $t'$ intersecting. Therefore, if we choose the points $\tilde{a}_i$ and $\tilde{d}_i$ successively, each time we place a point, if we consider positions that are spaced $n^{-2}$ apart, at most a factor of $n$ of these will cause a forbidden intersection with a tetrahedron already placed. Thus, we only have consider a factor of $n$ such positions to be guaranteed a position that does not cause a forbidden intersection. This completes the proof of Proposition 13.

Finally, we argue that the underlying abstract simplicial complex of $\mathcal{S}$ can be realized by a (non-degenerate) simplicial complex.

**Proposition 14.** *$\mathcal{S}$ is realizable in $\mathbb{R}^3$ as a simplicial complex without volume constraints.*

To see this, we just follow the same construction as above, but we start with points $x_i$ placed arbitrarily along the first coordinate axis. We add the coplanar gadgets that force the points to be on $\ell$, but we perturb the vertices of the triangulation slightly to realize the abstract simplicial complex defined by the coplanar gadget as a simplicial complex. In the case of the triangulation $T$ this is done by moving each point $b_i$ to the point $b_i + (0, 0, 1/2)$, and similarly for $T_\ell$. In general this may be done by moving the inner points of the triangulation away from the apex $v$ of the coplanar or parallel gadget. Next, we add the points $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i, \tilde{d}_i, \tilde{e}_i$ in the upper half of the second coordinate plane without regard to collinearities or parallel lines. Then, we add the coplanar and parallel gadgets used in the addition and multiplication gadgets as above, but we modify the coplanar and parallel gadgets to be simplicial complexes by relaxing the coplanarity of the triangulation in the same way as with $T$.

This completes the proof of Theorem 6. □

## 6.3 Volume Universality* is hard

In this section we prove the following theorem.

**Theorem 7.** Volume Universality* *is* $\forall\exists\mathbb{R}$*-complete.*

*Proof.* By Proposition 4, Volume Universality* is contained in UETR. In order to prove hardness, we reduce from Constrained-UETR. Let $\Psi$ be a formula of the form:

$$\Psi = (\forall Y_1, \ldots, Y_m \in \mathbb{R}^+)(\exists X_1, \ldots, X_n \in \mathbb{R}^+)\colon \Phi(Y_1, \ldots, Y_m, X_1, \ldots, X_n).$$

Recall that $\Phi$ is a conjunction of constraints of the form $X = 1$, $X + Y = Z$ and $X \cdot Y = Z$. We extend the construction from the proof of Theorem 6. Since we already introduced gadgets for the constraints, it only remains to introduce the universally quantified variables $Y_i$. For every $Y_i$, let $y_i$ be the point encoding its value. We force $y_i$ to lie on $\ell$ with the planes $E$ and $E_\ell$. As before $\|p_0 y_i\|$ represents the

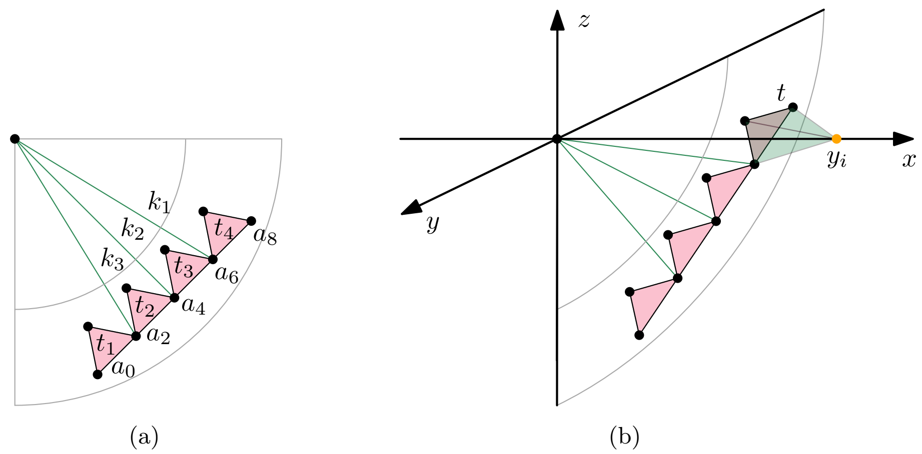



Figure 19: (a) Depiction of base plane $E_Y$ to insert the universally quantified variables $Y_i$ and (b) its placement in space.

value of $Y_i$, see also Figure 19 b). Moreover, we introduce a coplanar gadget with base plane $E_Y$. We ensure that $\ell$ intersects $E_Y$ by identifying the point $p_0$, which is contained in $E_Y$ and in $\ell$. Consequently, each plane parallel to $E_Y$ intersects $\ell$ in exactly one point.

We introduce a tetrahedron of volume 1, connecting one of the triangles on $E_Y$ to the point $p_1$ representing the value 1. For each universally quantified variable, we introduce a tetrahedron of *flexible* volume $Y_i$ consisting of a triangle of $E_Y$ and the point $y_i$. For each value of $Y_i$, the point $y_i$ is uniquely determined (since it lies on $\ell$ and the height of the tetrahedron is prescribed). In particular, the value of $Y_i$ is $\|p_0 y_i\|$.

**Proposition 15.** *$\mathcal{S}$ is volume-universal if and only if $\Psi$ is a* YES*-instance.*

Suppose $\Psi$ is a NO-instance. Then there exist the values $Y_1, \ldots, Y_m \in \mathbb{R}^+$ such that for all values for $X_1, \ldots, X_n \in \mathbb{R}^+$ the formula $\Phi$ is false. Suppose by contradiction that $\mathcal{S}$ is volume-universal. Then there exists a realization in particular for the chosen values for $Y_1, \ldots, Y_m$. By the correctness of the addition and multiplication gadgets, it follows that setting $X_i$ to $\|p_0 x_i\|$ is a satisfying assignment. This contradicts the fact that our choice for values $Y_1, \ldots, Y_m$ is certifying $\Psi$ to be a NO-instance.

Now, suppose $\Psi$ is a yes-instance. Consider fixed but arbitrary $(Y_1, \ldots, Y_m \in \mathbb{R}^+)$. Hence, we consider our constructed simplicial simplex where all tetrahedra have fixed volume. We construct a realization exactly as is Theorem 6. It only remains to add our new points $y_i$ and their tetrahedra to the planes $E_Y$ and $E$ and $E_\ell$. We set the plane $E_Y = \{(0, Y, Z)\}$ and its triangulation is completely contained inside $\{(0, y, z) : y, z \leq 0\}$. The triangulation $T_Y$ of $E_Y$ is placed within a quarter circle as depicted in Figure 19 a). We choose the triangulation small enough such that the lines $k_1, k_2, \ldots$ in $E_Y$ through $a_{2i}$ and the origin are separating adjacent triangles.

First we observe that no two tetrahedra of points $y_i$ intersect in their interior. For this let $t, t'$ be two such tetrahedra. We note that their underlying triangles are separated in the $E_Y$ plane by at least one of the $k_i$'s. Define $P$ as a plane containing such a $k_i$ and the $x$-axis. The plane $P$ separates $t$ and $t'$.

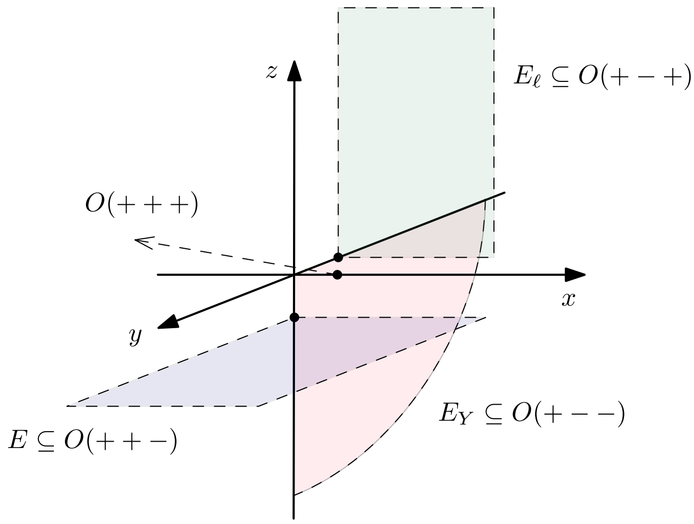



Figure 20: The construction is simultaneously realizable, because the tetrahedra of $E_\ell$ are all contained in the octant $O(+-+)$, the tetrahedra of $E$ are all contained in the octant $O(++-)$, the tetrahedra of $E_Y$ are all contained in the octant $O(+--)$, and the tetrahedra of the multiplication and addition gadgets are all contained in the octant $O(+++)$.

It remains to show that non of the newly introduced tetrahedra intersect one of the other tetrahedra, see Figure 20. For this we define the octants

$$O(+++) = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\},$$
$$O(+-+) = \{(x, y, z) : x \geq 0, y \leq 0, z \geq 0\},$$
$$O(++-) = \{(x, y, z) : x \geq 0, y \geq 0, z \leq 0\},$$
$$O(+--) = \{(x, y, z) : x \geq 0, y \leq 0, z \leq 0\}.$$

It is easy to observe that all important tetrahedra of $E_Y$ are in the octant $O(+--)$. The important tetrahedra of $E$ are in the octant $O(++-)$. The important tetrahedra of $E_\ell$ are in the octant $O(+-+)$. All the tetrahedra for the addition and the multiplication gadgets are in the octant $O(+++)$. This implies that none of the new tetrahedra intersect any of the previous ones. □

# 7 Potentially hard problems

The purpose of this section is to present interesting problems which are potentially contained in the introduced complexity classes. Therefor we discuss several candidates for ∀∃ℝ-complete and ∃∀ℝ-complete problems. We also present some problems, which may seem to be ∀∃ℝ-complete, but in fact are not. We draw connections to existing concepts in computer science, like robustness and imprecision. Although ∀∃ℝ and ∃∀ℝ are different complexity classes, it is worth mentioning that the complement of every language in ∃∀ℝ belongs to ∀∃ℝ. Thus from an algorithmic point of view they are equally difficult.

We want to emphasize again that this section is highly speculative. Its contribution is pointing at potentially interesting future problems rather than giving technical insights. We start with the problem we find most natural, besides Area Universality. Afterwards we come to the notions of universal extension problem, imprecision, and robustness. We conclude with a problem that has a similar flavor as area-universality but turns out to be polynomial time solvable.

## 7.1 Candidates for ∀∃ℝ- and ∃∀ℝ-complete problems

A very natural metric for point sets is the so-called Hausdorff distance. For two sets $A, B \subseteq \mathbb{R}^d$, the Hausdorff distance $d_H(A, B)$ is defined as

$$d_H(A,B) = \max\{\sup_{a\in A} \inf_{b\in B} \|ab\|, \sup_{b\in B} \inf_{a\in A} \|ab\|\}$$

where $\|ab\|$ denotes the Euclidean distance between the two points $a$ and $b$. We define the corresponding algorithmic problem as follows:

Hausdorff Distance
**Input:** Quantifier-free formulas $\Phi$ and $\Psi$ in the first-order theory of the reals, $t \in \mathbb{Q}$.
**Question:** Is $d_H(S_\Phi, S_\Psi) \leq t$?

Recall that $S_\Phi$ is defined as $\{x \in \mathbb{R}^n : \Phi(x)\}$. Marcus Schaefer pointed out the following interesting question:

**Open Problem:** Is computing the Hausdorff distance of semi-algebraic sets ∀∃ℝ-complete?

In [26], Schaefer and Štefankovič studied the following very common notion of distance, which we call here *d-distance* for clarity. The $d$-distance between two sets $A$ and $B$ in $\mathbb{R}^d$ is defined as $d(A, B) = \inf\{\|a, b\| : a \in A, b \in B\}$. They show the following lemma.

**Lemma 16** ([26]). *Deciding if two semi-algebraic sets have d-distance zero is ∃ℝ-complete.*

Note that containment in ∃ℝ is not clear for the Hausdorff-distance. However, it is contained in ∀∃ℝ. An encoding of Hausdorff-distance as a formula of the first order theory of the reals looks as follows:

$$(\forall a \in A, \varepsilon > 0\ \exists b \in B : \|ab\| < \varepsilon + t)\ \wedge\ (\forall b' \in B, \varepsilon > 0\ \exists a' \in A : \|a'b'\| < \varepsilon + t).$$

The formula holds true if and only if $d_H(A, B) \leq t$. This can be easily reformulated as a formula in prenex form with two blocks of quantifiers. By using the following logical equivalences:

$$\big(\forall x \exists\, y\colon \Phi(x,y)\big) \wedge \big(\forall x'\ \exists y'\colon \Phi'(x',y')\big) \equiv \forall x, x'\ \exists y, y'\colon \Phi(x,y) \wedge \Phi'(x',y') \text{ and}$$
$$\forall x \in X\ \exists y \in X : \Phi(x,y) \equiv \forall x\ \exists y : x \in X \Rightarrow \big(y \in Y \wedge \Phi(x,y)\big),$$

where $\Phi$ and $\Phi'$ are two quantifier free formulas, and $\equiv$ indicates that the two formulas are logically equivalent.

By definition, a semi-algebraic set is a subset $S$ of $\mathbb{R}^n$ defined by a finite sequence of polynomial equations and strict inequalities or any finite union of such sets.

As a step towards answering the question of Schaefer, we prove hardness of a variant where quantifier-free formulas are part of the input. Given a quantifier-free formula $\Gamma$ of the first-order theory of the reals with $n$ free variables, we denote by $S_\Gamma := \{x \in \mathbb{R}^n : \Gamma(x)\}$ the semi-algebraic set defined by $\Gamma$. By $\pi_k : \mathbb{R}^n \to \mathbb{R}^k$ we denote the projection onto the first $k$ coordinates. Note that the description complexity of a quantifier-free formula $\Gamma'$ defining $\pi_k(S_\Gamma)$ may exceed the description complexity of $\Gamma$.

> HAUSDORFF DISTANCE OF PROJECTIONS
> **Input:** Quantifier-free formulas $\Phi$ and $\Psi$ in the first-order theory of the reals, $k \in \mathbb{N}$.
> **Question:** Is $d_H(\pi_k(S_\Phi), \pi_k(S_\Psi)) = 0$?

**Lemma 17.** HAUSDORFF DISTANCE OF PROJECTIONS *is* $\forall\exists\mathbb{R}$*-complete.*

*Sketch of proof.* It is easy to see that HAUSDORFF DISTANCE OF PROJECTIONS is in $\forall\exists\mathbb{R}$. This can be done by explicitly writing out the definition as a decision problem, as explained above.

To see hardness we reduce from VOLUME UNIVERSALITY*. An instance of VOLUME UNIVERSALITY* consists of a pure abstract simplicial complex $\Sigma$ realizable in $\mathbb{R}^3$, and a partial volume assignment $\mathcal{V}'$ for a subset of 4-simplices $T'$. We construct $\Phi$ and $\Psi$ and a number $k$ such that their projection has Hausdorff distance 0 if and only if $\Sigma$ is realizes all volume-assignments respecting $\mathcal{V}'$.

The idea is that $\Phi$ describes the set of all volume assignments and $\Psi$ describes the subset of realizable volume assignments.

To describe the placement of the vertices and to formulate all the constraints given in the problem description of VOLUME UNIVERSALITY*, we need some additional variables to describe the placement of the vertices and to formulate all the constraints given in the problem description. We leave this as an exercise to the reader. The projection takes care of removing those variables again from the set $S_\Psi$.

Note that the Hausdorff distance of any two sets is zero if and only if they have the same closure. Therefore the Hausdorff distance of $S_\Psi$ and $S_\Phi$ is 0 if and only if all volume-assignments are realizable. □

We could easily scale the range of those variables that we removed by the projection. This would make them less important, but it is not clear if this suffices prove the hardness of the basic variant of the problem.

**Universal Extension Problems.** Before we explain the general concept of an Universal Extension problem, we take a look at a specific example: the ART GALLERY PROBLEM. In this problem we are given a simple polygon $P$ and we say that a point $p$ *sees* another point $q$ if the line segment $pq$ is fully contained in $P$. The ART GALLERY PROBLEM asks for a smallest set of points, which are called guards, such that every point inside the polygon is seen by at least one guard. It was recently shown that ART GALLERY PROBLEM is $\exists\mathbb{R}$-complete [1]. In an extension version of ART GALLERY PROBLEM, we are given a polygon and a partial set of guards $G_1$. The task is to find a set of guards $G_2$ such that every point inside the polygon is seen by at least one guard from the set $G_1 \cup G_2$. So the spirit of an extension problem is to give a partial solution as an additional input and ask if it is extendable to a full solution. Now we define the universal extension variant of the ART GALLERY PROBLEM as follows. The input consists of a simple polygon $P$ and a set of regions $R_1, \ldots, R_t$ inside $P$, such that for every guard placement of $G_1 = \{g_1, \ldots, g_t\}$ such that $g_i \in R_i$ it holds that there exists a second set of guards $G_2$ of some given size such that $G_1 \cup G_2$ guard the entire polygon. We denote this as the UNIVERSAL GUARD EXTENSION problem. For an example consider Figure 21.

> UNIVERSAL GUARD EXTENSION
> **Input:** A polygon $P$, regions $R_1, \ldots, R_t$ inside $P$, and a number $k \in \mathbb{N}$.
> **Question:** Is it true that for every placement of guards $G_1 = \{g_1, \ldots, g_t\}$ with $g_i \in R_i$, there exists a set $G_2$ of $k$ guards, such that $G_1 \cup G_2$ guard $P$ completely?

**Open Problem:** Is UNIVERSAL GUARD EXTENSION $\forall\exists\mathbb{R}$-complete?

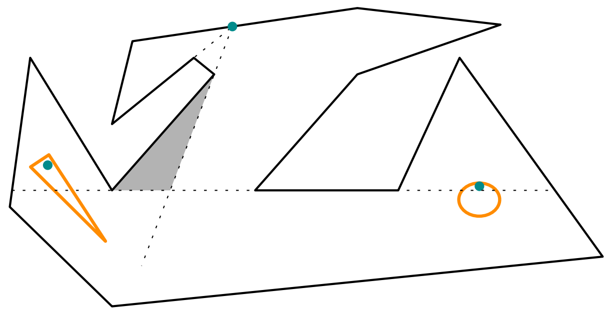


Figure 21: If two guards in the orange regions are badly placed it is impossible to guard the remaining polygon with just one guard.

The spirit is that we do not want to extend one given partial solution (as in the extension variant), but all possible partial solutions. In this example we require the guards to lie in specific regions. This is necessary, as otherwise we had to consider such extreme cases as a partial solution with all the guards in a single point, which would not be meaningful. As we will later see it seems to be a common theme that we have to impose some extra conditions on partial solutions and there is usually a choice which conditions we want to impose.

Many problems in combinatorics, computational geometry, and computer science ask for the realization of a certain object, for instance: a triangulation, an independent set, a plane drawing of a given graph, a plane drawing of a graph with prescribed areas, or a set of points in the plane realizing a certain combinatorial structure. These questions become extension questions as soon as a part of the solution is already created and the question is, whether there is a way to finish it. For instance: can a given plane matching be extended to a perfect matching, can a given plane graph be extended to a 3-regular graph. When going to an extension question the difficulty might increase. This is the case for matching extensions: Given a set with an even number of points in the plane there always exists a perfect, straight-line, crossing-free matching on that set of points. However, it is NP-hard to decide if a given partial matching can be extended to a perfect one [17]. This is a good reason why our main results must be considered with care as Theorems 2 and 7 are by its nature extension results.

We take this one step further and ask whether there is a way to complete *every* “reasonable” partial realization. As explained before, we might need to impose some problem-specific conditions, for the question to become meaningful.

Without giving explicit definitions, we mention here a few more problems that are worth to consider:

- Order-Type Extension (see [15]),
- Extension variant of the Steinitz problem (see [22])
- Graph Metric Extension (see [19]),
- Simultaneous Graph Embedding Extension ([7]).

**Imprecision.** Again, we use the Art Gallery Problem as an example. We introduce the Imprecise Guarding and Universal Guarding problems. The underlying idea is to guard a polygon, but we know the polygon only in an imprecise way. One might think of two different scenarios. In the first scenario, we want to know whether it is always possible to guard the polygon with $k$ guards, no matter how the actual polygon behaves. In the second scenario, we want to find a set of *universal guards* that will guard every possible polygon. Figure 22 depicts an example of a universal guard set.

> Guarding under Imprecision.
> **Input:** A set of unit disks $d_1, \ldots, d_n$ and a number $k \in \mathbb{N}$.
> **Question:** Is it true that for every set of points $p_1 \in d_1, \ldots, p_n \in d_n$ the polygon described by $p_1, \ldots, p_n$ can be guarded by $k$ guards?

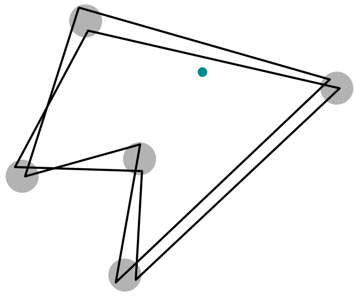


Figure 22: The cyan guard sees every point of the imprecisely given polygon.

> Universal Guard Set.
> **Input:** A set of unit disks $d_1, \ldots, d_n$ and a number $k \in \mathbb{N}$.
> **Question:** Is there a set $G$ of $k$ guards such that for every set of points $p_1 \in d_1, \ldots, p_n \in d_n$ the polygon described by $p_1, \ldots, p_n$ is guarded by $G$?

Both solution concepts seem sensible as they are able to deal with the situation that the input is not known precisely, nevertheless, they guarantee a precise solution. Algorithms solving this problem must be more fault-tolerant in the sense that they are forgiving towards small errors in the input. It is easy to see that Guarding under Imprecision is contained in $\forall\exists\mathbb{R}$ and that Universal Guard Set is contained in $\exists\forall\mathbb{R}$. Therefore, we wonder:

**Open Problem:** Is Guarding under Imprecision $\forall\exists\mathbb{R}$-complete?

**Open Problem:** Is Universal Guard Set $\exists\forall\mathbb{R}$-complete?

For any geometric problem, we can ask whether there is a universal solution that still works under any small perturbation. However, most problems with geometric input are contained in NP. On the other hand, problems that are known to be $\exists\mathbb{R}$-hard have usually combinatorial input. The Art Gallery Problem has this unique selling point, which makes it distinct from many other geometric problems. This means there might not be too many natural problems, which become $\forall\exists\mathbb{R}$-complete if the algorithm is required to deal with imprecision. Another exception to this rule of thumb are problems related to linkages (see e.g. [8])

Let us give in this context another example to clarify these observations. Consider a unit-disk intersection graph $G$ given explicitly by a set of disks in the plane. We can ask for a dominating set. Now the problem becomes more challenging, if we ask if the graph can be dominated even after some small perturbation of the disks. The perturbations of the input can also be understood as the inherent imprecision of the input.

> Domination of Imprecise Unit Disks.
> **Input:** A set of unit disks $d_1, \ldots, d_n$, a number $k \in \mathbb{N}$, and a number $\delta \in \mathbb{Q}$.
> **Question:** Is it true that for every translation of each disk by a vector of length at most $\delta$, it is possible to dominate the resulting disk intersection graph by $k$ disks?

Note that the problem of finding a dominating set in a graph is contained in NP. However, it is unclear if this remains the case even with the perturbation. Additionally, the perturbation might be captured best with real-valued variables. However, as a dominating set can be described in a discrete way, it looks unlikely to be $\forall\exists\mathbb{R}$-complete. However the following weaker question might be true.

**Open Problem:** Is Imprecise Domination of Unit Disks $\Sigma_2^p$ and $\exists\mathbb{R}$-complete?

Indeed, we think that it might be true that there is a plausible class between $\Sigma_2^p$ and $\forall\exists\mathbb{R}$ capturing imprecision issues.

**Robustness.** As above, we use the Art Gallery Problem as an illustrative example and we define the Robust Guarding problem as follows.

> Robust Guarding.
> **Input:** A simple polygon $P$ and a number $k \in \mathbb{N}$.
> **Question:** Does there exist a set of unit disks $D = \{d_1, \ldots, d_k\}$, each fully contained inside the polygon, such that for every placement of guards $G = \{g_1, \ldots, g_k\}$ with $g_i \in d_i$ guards the whole polygon $P$?

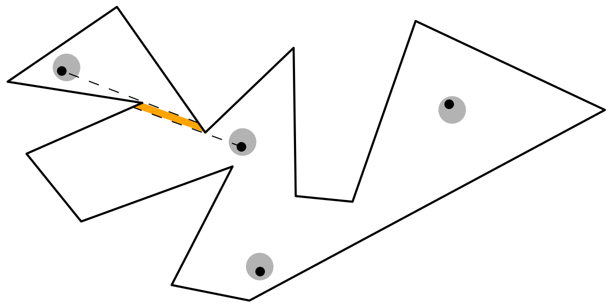


Figure 23: This guarding is not robust. The orange region might not be guarded.

It can be easily seen that this problem is contained in $\exists\forall\mathbb{R}$. The positions of the disks are existentially quantified, the positions of the guards are universally quantified, and the remaining formula enforces the guards to be inside the disks, the disks inside the polygon, and the condition that the guards are indeed guarding the whole polygon.

**Open Problem:** Is Robust Guarding $\exists\forall\mathbb{R}$-complete?

In more general terms we employ the notion that a solution is robust, if it remains a valid solution also if it gets slightly perturbed. Another example is Robust Order Types Realizability. Given a set of points $P$ in the plane, each ordered triple is either in clockwise or counter clockwise direction. In Order Type Realizability, we are given an orientation for each triple of some abstract set and we are asked to find a set of points in the plane with that given order type. Robustness asks if the solution remains correct, even after some small perturbation. As order types stay fixed under uniform scaling it makes sense to restrict the points to lie in a unit square.

> Robust Order Type Realizability.
> **Input:** An abstract order type of $n$ points and a rational number $w > 0$.
> **Question:** Does there exists a set of disks $d_1, \ldots, d_n$ with radius $w$ in the unit square $[0, 1]^2$ such that every set of points $p_1 \in d_1, \ldots, p_n \in d_n$ has the given order type?

**Open Problem:** Is Robust Order Type Realizability $\exists\forall\mathbb{R}$-complete?

We assume here that $w$ is encoded in binary. Furthermore, if an order type is realizable for some $w_1$ then it is also robustly realizable for $w_2 < w_1$. Thus this robustness problem becomes an optimization problem with respect to the parameter $w$. It is conceivable that at least an approximate solution to the problem can be found by restricting the centers of the disks to a fine grid. This would imply that we can find a polynomial-sized witness for an approximate solution and thus the robust problem might become easier than the basic version for all these problems. Recall that for most of the mentioned recognition problems are $\exists\mathbb{R}$-complete.

In a similar spirit, we can define the robust variants for all kinds of recognition problems of intersections graphs, such as intersection graph of unit disks, segments, rays, unit segments, or of your favorite geometric object.

**Open Problem:** Is Robust Recognition of Intersection graphs $\exists\forall\mathbb{R}$-complete?

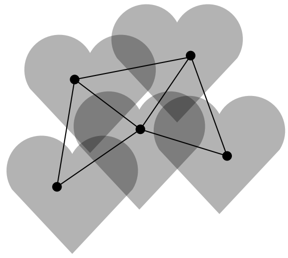


Figure 24: An intersection graph of hearts.

**Environmental Nash-Equilibria.** Finding Nash equilibria (see [20]) is one of the most important problems in game theory and it is known to be PPAD-complete [9]. As every game has a Nash equilibrium, there is no point in formulating a corresponding decision problem. However, if we restrict our attention to a small region of the strategy space, the problem becomes ∃ℝ-complete already for two players [26]. Restricting the strategy spaces makes sense, as one might be given an approximate solution and wants to find an exact solution "close" to the approximate one. A natural question we could ask is whether in a three-player game the first two players can find a Nash equilibrium for any fixed behavior of the third player. The underlying idea is that the third player simulates a potentially changing environment.

## 7.2 A false candidate – Universal Graph Metric

Given a graph $G = (V, E)$ together with some edge weights $w : E \to \mathbb{R}^+$, we can ask for an embedding $\varphi$ of $G$ in the plane such that for each edge $u, v$ the distance $\text{dist}(u, v)$ in the plane equals the edge weight $w(uv)$. In this case we say $\varphi$ realizes the edge-weight $w$. Interestingly, it is ∃ℝ-complete to decide if an edge weight with all weights equal 1 can be realized, see [25].

Many edge weights, are trivially not attainable, as they might not satisfy the triangle inequality. For the purpose of concreteness let us say that a metric is *reasonable* if there exists a Euclidean Space of some dimension, into which the graph is embeddable. It is easy to see that the dimension can be upper bounded by the number of vertices.

We are ready to define the Universal Graph Metric problem.

> Universal Graph Metric
> **Input:** A graph $G$.
> **Question:** Is it true that for every reasonable edge weight function, there exists a realizing embedding in the Euclidean plane?

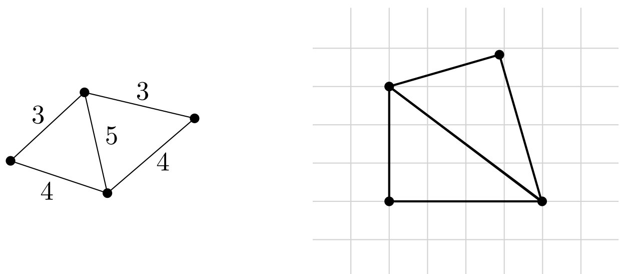



Figure 25: A graph with given edge weights (on the left) and an embedding in the Euclidean plane realizing the edge weights (on the right).

Universal Graph Metric clearly has a similar spirit as area-universality. The edge lengths are universally quantified and the drawing is existentially quantified. However, a simple argument shows the following (surprising), but known, fact:

**Proposition 18** (Belk & Connelly [4]). Universal Graph Metric *is in P.*

*Proof.* The argument builds on the powerful theory of minor closed graph classes. Recall that a graph $F$ is a *minor* of a graph $G$ ($F \prec G$), if we can attain $F$ from $G$ by deleting edges, vertices, or contracting edges. A graph class $\mathcal{G}$ is minor-closed, if $G \in \mathcal{G}$ and $F \prec G$ imply $F \in \mathcal{G}$. A celebrated and deep theorem of Robertson and Seymour [24] implies that every minor-closed graph class can be characterized by a finite number of forbidden minors. Since testing if a graph $G$ contains a fixed graph $F$ as a minor can be tested in polynomial time, the membership of a graph in any minor-closed graph class is checkable in polynomial time.

It is easy to observe that the implicitly defined graph class of Yes-instances of the Universal Graph Metric problem is indeed minor-closed: Let $\mathcal{G}$ denote the class of Yes-instances for all *reasonable* weight function. For the following argument refer to Figure 26.

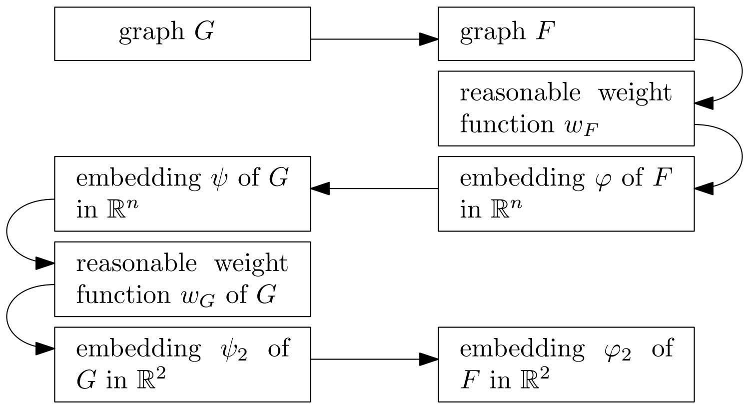



Figure 26: The line of argument, why Universal Graph Metric is a minor closed property.

Let $G \in \mathcal{G}$ be a graph with a minor $F$. Let $w_F$ be a reasonable weight function of $F$. We need to show that $F$ has an embedding respecting $w_F$ into $\mathbb{R}^n$. (Here $n$ denote the number of vertices of $G$, we can assume that the embedding is wlog into $\mathbb{R}^n$ as we have at most $n$ vertices.) For that purpose consider an embedding $\varphi$ of $F$ into $\mathbb{R}^n$ realizing $w_F$. This exists by definition since $w_F$ is reasonable. We can extend $\varphi$ to an embedding $\psi$ of $G$. For contracted edges, we use weight 0, deleted vertices are placed arbitrarily, and weights of deleted edge are inferred by the embedding. This defines a weight function $w_G$ of $G$, which is, as we have just shown, reasonable. Thus there is an embedding $\psi_2$ of $G$ into $\mathbb{R}^2$ as $G \in \mathcal{G}$. This embedding restricted to the vertices of $F$ gives an embedding of $w_F$ into $\mathbb{R}^2$. Thus $F \in \mathcal{G}$. □

It is easy to see that $K_4$ is a forbidden minor and is known to be the only one [3, 4]. If we choose *reasonable* to be all weight functions, then due to the triangle inequality, the yes-instances are exactly the set of trees and the forbidden minor is the triangle. In a recent Master Thesis by Muller [19] other meanings of *reasonable* are discussed.

# A Appendix – Proof of Theorem 9

In this section we prove Theorem 9:

**Theorem 9.** Constrained-UETR *is* $\forall\exists\mathbb{R}$*-complete.*

Our main tool is the following result by Schaefer and Štefankovič. We state it in a sligthly different way, which is more suitable to our notation.

**Lemma 19** (Schaefer, Štefankovič [26])**.** *Let* $\Phi(x)$ *be a quantifier-free formula of the first order theory of the reals, with the vector of variables* $X = (X_1, X_2, \ldots, X_n)$*. We can construct in polynomial time a polynomial* $F\colon \mathbb{R}^{n+k} \to \mathbb{R}$ *of degree 4, for some* $k = O(|\Phi|)$*, so that*

$$\{x \in \mathbb{R}^n \,:\, \Phi(x) \text{ is true}\} = \{x \in \mathbb{R}^n \,:\, (\exists u \in \mathbb{R}^k)\; F(x,u) = 0\}.$$

*The coefficients of* $F$ *have bitlength* $O(|\Phi|)$*.*

**Theorem 9.** Constrained-UETR *is* $\forall\exists\mathbb{R}$*-complete.*

*Proof.* We observe that Constrained-UETR is in $\forall\exists\mathbb{R}$, since the formula

$$(\forall Y_1, Y_2, \ldots, Y_m \in \mathbb{R}^+)(\exists X_1, X_2, \ldots, X_n \in \mathbb{R}^+)\colon \Phi(X,Y)$$

is equivalent to the following instance of UETR:

$$(\forall Y_1, \ldots, Y_m \in \mathbb{R})(\exists X_1, \ldots, X_n \in \mathbb{R})\colon \left(\bigwedge_{i=1}^{m} (Y_i > 0) \Rightarrow \bigwedge_{i=1}^{n} (X_i > 0) \wedge \Phi(X,Y)\right).$$

To prove $\forall\exists\mathbb{R}$-hardness, we consider an instance of UETR

$$\Psi = (\forall Y_1, Y_2, \ldots, Y_m \in \mathbb{R})(\exists X_1, X_2, \ldots, X_n \in \mathbb{R})\colon \Phi(X,Y).$$

We will create an equivalent Constrained-UETR formula

$$\overline{\Psi} = (\forall Y_1, Y_2, \ldots, Y_{\overline{m}} \in \mathbb{R}^+)(\exists X_1, X_2, \ldots, X_{\overline{n}} \in R^+)\colon \overline{\Phi}(X,Y),$$

such that the size of the new formula is polynomial in the size of the original one. The reduction consists of several steps.

### Transforming $\Phi$ into a single polynomial.

First, we want to simplify $\Phi$. To do so, we apply Lemma 19 to $\Phi$, and let $F\colon \mathbb{R}^{n+m+k} \to \mathbb{R}$ be the obtained polynomial. We define a UETR formula $\Psi'$ as follows:

$$\Psi' = (\forall Y_1, \ldots, Y_m \in \mathbb{R})(\exists X_1, \ldots, X_n \in \mathbb{R})(\exists U_1, \ldots, U_k \in \mathbb{R})\; F(Y,X,U) = 0.$$

First note that $\Psi'$ is equivalent to $\Psi$. To see this, we define

$$S = \{(y,x) \in \mathbb{R}^{m+n} : \Phi(x,y)\} \quad \text{and} \quad S' = \{(y,x) \in \mathbb{R}^{m+n} : \exists u : F(y,x,u) = 0\},$$

as in Lemma 19. We know that $S' = S$ and $\Psi$ is true if and only if the orthogonal projection of $S$ onto the first $m$ coordinates equals $\mathbb{R}^m$. Similarly, $\Psi'$ is true if and only if the orthogonal projection of $S = S'$ onto the first $m$ coordinates equals $\mathbb{R}^m$.

### Transforming $F$ into a conjuction of constraints.

We transform $F$ into an equivalent conjuction of constraints. First note that we can rewrite $F$ as a sum of monomials $F = f_1 + \ldots + f_\ell$ in polynomial time, because $F$ has degree at most 4. We show how to replace each monomial $f$ by an equivalent set of constraints, each of which is either a multiplication, an addition, or an introduction of a constant 1. Consider a mononomial

$$f = s \cdot c \cdot Z_1 \cdot Z_2 \cdots Z_t$$

of $F$, where $s \in \{-1, 1\}$ is the sign, $c$ is a positive integer of bitlength $O(|\Phi'|) = O(|\Phi|)$, and $Z$'s are non necessarily distinct variables. For convenience, we introduce the existential variables $I_0, I_1, I_{-1}, I_2$, whose values will be forced to be 0, 1, $-1$, and 2 respectively. We obtain this with the following constraints:

$$\begin{aligned} I_0 =& I_0 + I_0 \\ I_1 =& 1 \\ I_0 =& I_{-1} + I_1 \\ I_2 =& I_1 + I_1. \end{aligned}$$

We start by describing a variable holding the value $sc$, using only the above constraints. Let $c_\ell c_{\ell-1} \dots c_0$ be binary expansion of $c$, i.e., $c = \sum_{i=0}^{\ell} c_i \cdot 2^i$ (note that each $c_i$ is either 0 or 1). We introduce new existential variables and the constraints described below(the index of each variable shows its intended value). The idea is to use the recursion

$$c_l \dots c_t = 2 \cdot c_l \dots c_{t+1} + c_t.$$

Those are the correct constraints:

$$\begin{aligned} V_{2c_\ell} =& I_2 \cdot I_{c_\ell} \\ V_{2c_\ell + c_{\ell-1}} =& V_{2c_\ell} + I_{c_{\ell-1}} \\ V_{4c_\ell + 2c_{\ell-1}} =& I_2 \cdot V_{2c_\ell + c_{\ell-1}} \\ &\vdots \\ V_c =& V_{\sum_{i=1}^{\ell} c_i \cdot 2^i} + I_{c_0} \\ V_{sc} =& I_s \cdot V_c \end{aligned}$$

Observe that $V_{sc} = sc$. Next, we introduce new existential variables $V_f, V_{Z_1}, V_{Z_1 Z_2}, \dots, V_{Z_1 Z_2 \dots Z_t}$, and add the following constraints:

$$\begin{aligned} V_{Z_1 Z_2} =& Z_1 \cdot Z_2 \\ V_{Z_1 Z_2 Z_3} =& V_{Z_1 Z_2} \cdot Z_3 \\ &\vdots \\ V_{Z_1 Z_2 \dots Z_t} =& V_{Z_1 Z_2 \dots Z_{t-1}} \cdot Z_t \end{aligned}$$

Finally, we add a constraint $V_f = V_{sc} \cdot V_{Z_1 Z_2 \dots Z_t}$, which ensures that the value of $V_f$ is equal to the value of $f$. We repeat the same for every monomial $f$.

Recall $F = f_1 + f_2 + \dots + f_\ell$, where $f_1, f_2, \dots, f_\ell$ are monomials. We introduce new existential variables and the following constraints:

$$\begin{aligned} V_{f_1 + f_2} =& V_{f_1} + V_{f_2} \\ V_{f_1 + f_2 + f_3} =& V_{f_1 + f_2} + V_{f_3} \\ &\vdots \\ I_0 =& V_{f_1 + f_2 + \dots + f_{\ell-1}} + V_{f_\ell} \end{aligned}$$

This way we constructed the formula

$$\Psi'' = (\forall Y'_1, Y'_2, \dots, Y'_{m'} \in \mathbb{R})(\exists X'_1, X'_2, \dots, X'_{n'} \in R) \colon \Phi''(X', Y'),$$

which is equivalent to $\Psi$, its size is polynomial in $|\Psi|$, and $\Phi''$ is a conjuction of introducing constant 1, addition and multiplication constraints.

**Changing ranges of quantifiers.**

Next, we want to exchange quantifiers ranging over all reals with quantifiers ranging over positive reals. For each variable $Z$ of $\Psi''$, we introduce two positive variables $Z^+$ and $Z^-$. If $Z$ is universally quantified, then so are both $Z^+$ and $Z^-$; analogously in the case if $Z$ is existentially quantified. Every appearance of $Z$ in $\Phi''$ is substituted by $(Z^+ - Z^-)$; let $\Phi'''$ be the formula obtained in such a way. It is easy to observe that the constructed formula is equivalent to $\Psi'$. However, the structure of constraints is destroyed.

**Restoring the form of constraints.**

Now we want to restore the constraints. We re-introduce the existential variable $I_1$ with constraint $I_1 = 1$. Then, every constraint in $\Phi'''$ we transform it in the following way:

**Introducing a constant.** A constraint $(Z^+ - Z^-) = 1$ is transformed into $Z^+ = Z^- + I_1$.

**Addition.** A constraint $(X^+ - X^-) + (Y^+ - Y^-) = (Z^+ - Z^-)$ is equivalent to the following expression: $X^+ + Y^+ + Z^- = X^- + Y^- + Z^+$. We introduce new positive, existentially quantified variables and the constraints:

$$
\begin{aligned}
V_{X^+ + Y^+} &= X^+ + Y^+ \\
V_{X^- + Y^-} &= X^- + Y^- \\
V_{X^- + Y^- + Z^+} &= V_{X^- + Y^-} + Z^+ \\
V_{X^+ + Y^+} + Z^- &= V_{X^- + Y^- + Z^+}.
\end{aligned}
$$

**Multiplication.** A constraint $(X^+ - X^-) \cdot (Y^+ - Y^-) = (Z^+ - Z^-)$ is equivalent to the following expression: $X^+Y^+ + X^-Y^- + Z^- = X^+Y^- + X^-Y^+ + Z^+$. We introduce new positive, existentially quantified variables and the constraints:

$$
\begin{aligned}
V_{X^+Y^+} &= X^+ \cdot Y^+ \\
V_{X^-Y^-} &= X^- \cdot Y^- \\
V_{X^+Y^-} &= X^+ \cdot Y^- \\
V_{X^-Y^+} &= X^- \cdot Y^+ \\
V_{X^+Y^+ + X^-Y^-} &= V_{X^+Y^+} + V_{X^-Y^-} \\
V_{X^+Y^- + X^-Y^+} &= V_{X^+Y^-} + V_{X^-Y^+} \\
V_{X^+Y^- + X^-Y^+ + Z^+} &= V_{X^+Y^- + X^-Y^+} + Z^+ \\
V_{X^+Y^- + X^-Y^+} + Z^- &= V_{X^+Y^- + X^-Y^+ + Z^+}.
\end{aligned}
$$

Note that now all constraints are of desired forms and all variables are strictly positive. This way we produced a formula

$$\overline{\Psi} = (\forall Y_1, Y_2, \ldots, Y_{\overline{m}} \in \mathbb{R}^+)(\exists X_1, X_2, \ldots, X_{\overline{n}} \in \mathbb{R}^+)\ \overline{\Phi}(X, Y),$$

in which $\overline{\Phi}$ is a conjuction of equality checks, additions, and multiplications. Moreover, $\overline{\Psi}$ is equivalent to $\Psi$. Observe that the number of introduced variables and the length of $\overline{\Phi}$ are polynomial in $|\Phi|$. This completes the proof. ☐

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