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Tiling figures of the plane with two bars

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Abstract

Given two "bars", a horizontal one, and a vertical one (both of length at least two), we are interested in the following decision problem: is a finite figure drawn on a plane grid tilable with these bars. It turns out that if one of the bars has length at least three, the problem is NP-*complete*. If bars are dominoes, the problem is in P, and even *linear* (in the size of the figure) for certain classes of figures. Given a general pair of bars, we give two results: (1) a necessary condition to have a *unique* tiling for finite figures without holes, (2) a *linear algorithm* (in the size of the figure) deciding whether a unique tiling exists, and computing this one if it does exist. Finally, given a tiling of a figure (not necessarily finite), this tiling is the unique one for the figure if and only if there exists no subtiling covering a "*canonical*" *rectangle*.

Keywords: Tiling, bars, NP-complete

1. Introduction

Polyominoes and more generally pictures drawn on a plane grid are a subject which fascinated mathematicians for a long time [4, 9–11, 19]. They appeared of great interest for computer science in many research fields: computational geometry [2, 4], data organization in parallel computer architecture [17, 20], formal languages theory [7], decidability theory [1, 12] and so on. We have focused our attention on the problem of codicity and related questions.

We consider a number of problems relating to tilings of plane finite *figures* by non-overlapping *bars*. A figure consists of a number of unit squares and we are concerned with bars chosen from a pair of allowed types h_l (covering $l \geqslant 2$ horizontally

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contiguous squares) and v_m (covering $m \geqslant 2$ vertically contiguous squares). Given a finite figure and a particular pair of bar types, we are interested in deciding whether a tiling exists, whether it is unique and in actually finding a tiling if one does exist. Similar problems have been studied in the past [1, 10, 11, 14], concerning mainly more complex tile shapes than our bars but often simpler figures than our arbitrary ones (squares, quadrants or the whole plane). Results proved have included undecidability of tiling the plane with a given finite set of tiles [4], NP-completeness of tiling of finite figures [5, 8], and undecidability of tile codes, that is finite sets of tiles for which a tiling of any finite figure must be unique [1].

Since a non-connected figure is (uniquely) tilable if and only if each of its connected components is (uniquely) tilable, we consider only connected figures. A more significant distinction concerns the existence of *holes*, it turns out that absence of holes plays an important role.

The existence of a tiling can be decided in P for the simplest case h_2, v_2 (the *dominoes*), with or without holes. For all other pairs of bar types this problem is NP-complete for finite figures with holes and has recently been proved to be in P for those without holes [13].

For the simple case of two dominoes, we show efficient algorithms for certain problems. Finding a tiling, if one exists, can be performed efficiently in general and even in linear time in the size of the figure for certain classes of figures. There is also a linear time algorithm for deciding existence of a unique tiling for finite figures without holes.

For the case of a general pair of bar types h_l, v_m and finite figures without holes, we give two results on unique tilings. Namely, we give a necessary condition for a finite figure to have a unique tiling and also a necessary and sufficient condition that a given tiling is in fact unique.

2. Definitions

The plane is divided into *cells* which are unit squares with horizontal and vertical sides, so the set of cells is considered as $\mathbb{Z} \times \mathbb{Z}$. A *figure* F is a subset of cells; an *instance of* F is a translated image of F. Two different cells are *adjacent* if they have a common side. A figure F is *connected* if the following condition is satisfied: for two cells c and c' of f, there exists a sequence of cells of f, $c_0 = c, \ldots, c_n = c'$ such that c_i is adjacent to c_{i+1} for $i = 0, \ldots, n-1$. A connected figure is called a *piece*. A piece has no hole if its complement is also a piece (Fig. 1). A finite piece without holes is called a *polyomino*. A figure F is *horizontally convex* if the following condition holds:

$$\forall c, c' \in F \; (c = (x, y), c' = (x', y) \text{ and } x < x') \Rightarrow (\forall x < x'' < x' \; (c'' = (x'', y) \in F))$$

A vertically convex figure is defined in a similar way.

A *horizontal bar* (resp. *vertical bar*) h_l (resp. v_l) is the union of l horizontally (resp. vertically) contiguous cells ($l \geqslant 2$) (Fig. 2).

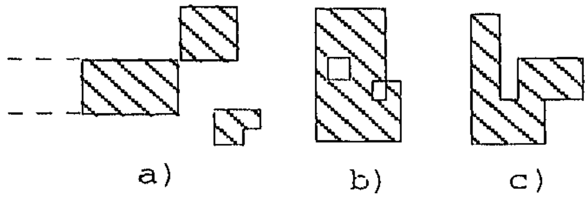

Fig. 1. a) a figure b) a finite piece c) a finite piece without holes (a polyomino).

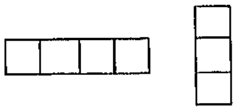

Fig. 2. A horizontal bar h_4 and a vertical bar v_3.

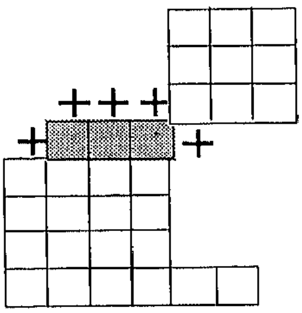

Fig. 3. A up vertical peak ($l = 5$, $k = 3$).

Let S be a set of a horizontal bar h_l and a vertical bar v_m. Let c be a cell. Two horizontal bars $h(c)$, $h'(c)$ and two vertical bars $v(c)$, $v'(c)$ are defined in the following way:

- $h(c)$ (resp. $h'(c)$) is the instance of h_l whose leftmost (resp. rightmost) cell is equal to c
- $v(c)$ (resp. $v'(c)$) is the instance of v_m whose lowest (resp. highest) cell is equal to c.

In the following the set S is fixed. A *tiling* T with S of a figure F is represented as two subsets T_h, $T_{v'}$ of F such that the sets $\{h(c) \mid c \in T_h\}$ and $\{v'(c) \mid c \in T_{v'}\}$ form a partition of F. A *subtiling* R of a tiling T is a pair of two subsets R_h and $R_{v'}$ of T_h and $T_{v'}$ which tile the subfigure F' of F equal to the union of the bars $h(c)$, $c \in R_h$ and $v'(c)$, $c \in R_{v'}$.

A *canonical rectangle* is a rectangle l by m. Clearly a canonical rectangle admits two tilings with S. It will play an important role later.

In a figure F, a *up horizontal peak* is a set p of k ($0 < k < l$) cells of F which is an instance of h_k, and such that the adjacent cells which are marked with a sign + (Fig. 3) do not belong to F. This definition depends on the set S. In the same way are defined *a down horizontal, left vertical, right vertical peak* (Fig. 5).

Let $c_1, \ldots, c_k$ be the cells of an up peak p, the *cover* $C(p)$ of the peak p is the rectangle $C(p) = v'(c_1) \cup \cdots \cup v'(c_k)$ (Fig. 4). In a similar way is defined the cover of other peaks (down, left or right).

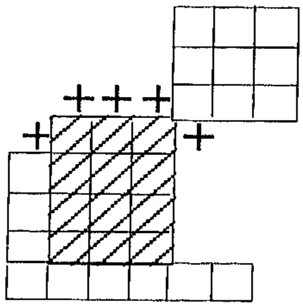

Fig. 4. The cover of the peak of Fig. 3, here $m = 4$.

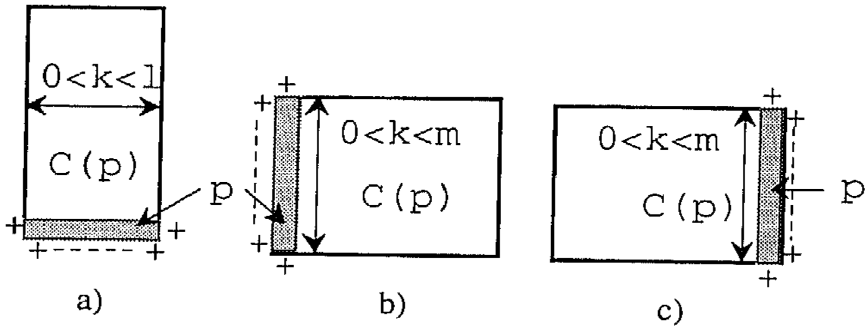

Fig. 5. a) a down peak, b) a left peak, c) a right peak, and their cover.

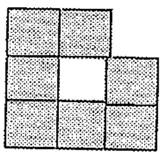

Fig. 6. A finite piece which is not a polyomino.

One can observe (it is important for the future) that in a piece without holes, it is impossible that two cells of the piece are just touching at a corner and that the two other cells at this corner do not belong to the piece (this situation is shown in Fig. 6), because in that case the connectivity implies that the piece has a hole.

This covering notion deals with the following property.

Lemma 2.1. *If a figure F admits a tiling T with S and has a up (resp. down, left, right) peak p, then $C(p) \subset F$ and $p \subset T_{v'}$ (resp. T_v, T_h, $T_{h'}$).*

Proof. Let T be a tiling of a figure F. If p is a up peak, it is clear that each cell c of the peak cannot be covered in T with an horizontal bar, but only by a vertical bar whose top cell is c, so the proof is done in that case, the other ones are symmetric cases. □

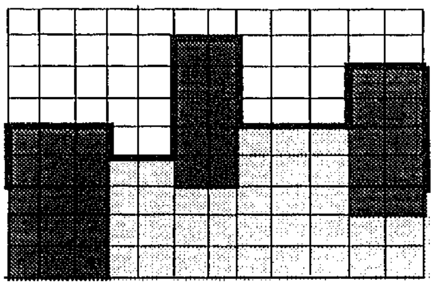

Fig. 7. A up chain of three peaks, $l = 4$, $m = 5$.

Let $A = \{X, X', Y, Y'\}$ a four letter alphabet. The outline of a figure F is orientated in such a way that the interior of the figure is on the right of the outline. So, if F is finite, every connected part of the outline is coded [7] with a word of A^*, X, X', Y, Y' standing for the following unit moves: X: $\rightarrow$, X': $\leftarrow$, Y: $\uparrow$, Y': $\downarrow$.

Let F be a figure. A up chain of k peaks ($k \geqslant 1$) (Fig. 7) is a partial outline of F which belongs to the set $Y^+(HV'HV)^{k-1}HY'^+$ with

$$H = \{X^h \mid 0 < h < l\}, \quad V' = \{Y'^h \mid 0 < h < m\}, \quad V = \{Y^h \mid 0 < h < m\}.$$

The number k is the *length* of the chain.

A down (resp. left, right) chain is defined in a similar way. The *set of peaks of a figure form a chain of k peaks* ($k > 0$) if there exists a partial outline of F which forms a chain of k peaks and covers all the peaks of F (if F has no peak, we will say that its set of peaks form a chain of 0 peaks).

3. Tiling finite pieces with $S_2 = \{h_2, v_2\}$

We first restrict our study to the particular case when the bars are dominoes. Classical results of graphs theory provide a polynomial algorithm to decide whether a finite figure admits a tiling with S_2. For particular classes of figures, we obtain linear algorithms.

Let F be a finite figure. The *graph G_F of the cells of F* is the symmetric graph whose vertices are the cells of F whose edges are the unordered pairs of cells with a common side. A *matching* (resp. *perfect matching*) of a unordered graph $G = (F, E)$ is a subset E' of E such that every vertex of F is incident to at most (resp. exactly) one edge of E.

Proposition 3.1. *There exists a tiling of a finite figure F with S_2 if and only if there exists a perfect matching of the graph G_F. Moreover, there exists a canonical one-to-one mapping between the tilings of F with S_2 and the perfect matchings of G_F.*

Corollary 3.2. *There exists a polynomial algorithm which:*

(1) *decides the existence of a tiling of a finite figure F with S_2, or not.*

(2) *provides a tiling of F with S_2, if such a tiling exists.*

This corollary comes from the fact that the problem of the maximal matching of a given graph G is solvable in polynomial time [12]. The algorithms to solve the problem are based on the alternating chains (i.e. chains with alternatively edges in and out of the matching) and the exchange operations on them. We give here a result, about those chains, that will be used later. This result is a direct corollary of a classical theorem of Berge [3] about maximal matchings.

Proposition 3.3. *Let F be a finite figure accepting a perfect matching. Let E be a matching on F and A_0 a vertex of F, unmatched by E. There exists a chain $[A_0, A_1, \ldots, A_p]$, alternating for E, such that A_p is unmatched by E.*

Tiling a horizontally convex finite figure with S_2

Usual algorithms in polynomial time do not use the fact that the graph is a subgraph of Z^2. Recently, Thurston [18] has proposed a linear algorithm for certain classes of figures (more precisely for holeless finite figures). His main result comes from an application of the theory of Cayley graphs. In this section, we propose a linear algorithm for a special kind of figures using directly the geometrical properties of the figures.

Given a figure F, there is a unique way to color each cell of F black or white if we impose that every black cell (x, y) has components whose sum $x + y$ is even, and a black cell cannot be adjacent to a black one (actually, F is a piece of a checkerboard).

The set of cells of F which have the same height is called *a row* of F. Let F be a *horizontally-convex finite figure* (i.e. a finite figure whose intersection with each horizontal line is a segment). For each row of F, we have exactly one cell at the leftmost position and another one at the rightmost position. So, the white cells at the leftmost extremity and the black cells at the rightmost extremity can be numbered, according to the decreasing heights, starting with number one. For the following, the white kth cell at the leftmost extremity will be called W_k, and the kth black cell at the rightmost extremity will be called B_k.

We define, by induction on k, a sequence C_k of *chains* of F by

• If W_k is lower than B_k then $C_k = [A_0, A_1, \ldots, A_p]$, with $A_0 = W_k$, and

(1) if A_i is a black cell, then $A_{i+1} = A_i + (1, 0)$,

(2) if A_i is a white cell whose height is the same as the height of B_k, then we have $A_{i+1} = A_i + (1, 0)$,

(3) if A_i is a white cell (strictly) lower than B_k, then

$$A_{i+1} = A_i + (0, 1) \text{ if } A_i + (0, 1) \text{ is in } F \text{ and } A_i + (0, 1) \text{ is not in the chain } C_{k-1},$$

$$A_{i+1} = A_i + (1, 0) \text{ otherwise.}$$

Informally, we obtain those chains by starting from W_k and climbing, if possible, until the height of B_k, according to the following rules:

(1) the chain is allowed to climb when we are on a white cell whose upper neighbor is in F and is not in the chain C_{k-1},

(2) when the clain is allowed to climb, it climbs,

(3) when the chain is not allowed to climb, it goes to the right.

In the other case, the definition is symmetric:

- If B_k is lower than W_k then $C_k = [A_0, A_1, \dots, A_p]$, with $A_0 = B_k$, and

(1) if A_i is a white cell, then $A_{i+1} = A_i + (-1, 0)$,

(2) if A_i is a black cell whose height is the same as the height of W_k, then we have $A_{i+1} = A_i + (-1, 0)$,

(3) if A_i is a black cell lower than W_k, then

$$A_{i+1} = A_i + (0, 1) \text{ if } A_i + (0, 1) \text{ is in } F \text{ and } A_i + (0, 1) \text{ is not in the chain } C_{k-1},$$

$$A_{i+1} = A_i + (-1, 0) \text{ otherwise}.$$

Theorem 3.4. *A horizontally convex finite figure F admits a tiling with S_2 if and only if there are as many cells W_k as cells B_k and, for each integer k, the extremities of C_k are W_k and B_k.*

Proof. Let E_0 be the matching such that each element of E_0 is a pair $\{(x, y), (x', y')\}$ where (x, y) is a black cell and $(x', y') = (x, y) + (1, 0)$. All the cells of F, except the cells W_k and B_k, are matched by E_0. Moreover, the chains C_k are alternating chains for E_0. The exchange operation on these chains gives a perfect matching of G_F.

Conversely, let $C_k = [A_0, A_1, \dots, A_p]$ be a chain such that $A_0 = W_k$, and, for each positive integer k', such that $k' < k$, let us call $W_{k'}$ and $B_{k'}$ the extremities of the chain $C_{k'}$. Let A_i be a white cell of C_k, lower than A_p. Let D_1 be the upper neighbor of A_i. We assume that $D_1 \neq A_{i+1}$

- If D_1 is not in F, then, for each non-negative integer n, the cell $D_1 + (-n, 0)$ is not in F, since F is horizontally convex.
- If D_1 is in F, let D_2 denote the left neighbor of D_1 and D_3 the right neighbor of D_1. The cell D_1 is in the chain C_{k-1}, due to condition 3 in the definition of the sequence C_k, case A_i lower than B_k. Then because the black cell D_1 cannot be the left end of a chain, the cells D_2 and D_3 are in the chain C_{k-1}, are lower than B_{k-1} and C_{k-1} starts at W_{k-1}.

Moreover, we now assume that k is the lowest integer such that the extremities of $C_k = [A_0, A_1, \dots, A_p]$ are not both W_k and B_k and A_i is the white cell of C_k, lower than A_p, such that $A_i + (0, 1)$ is in F and $A_{i+1} \neq A_i + (0, 1)$, with i being maximal (we will see later what happens when such a cell does not exist). From the previous remarks, we can get a sequence $[D_1, \dots, D_{2m+1}]$ of cells such that: (see Fig. 8)

- $D_1 = A_i$
- for each integer n such that $0 < n \leqslant m$, we have:

$$D_{2n+1} = D_{2n} + (0, 1) \quad \textit{and} \quad D_{2n+2} = D_{2n+1} + (-1, 0)$$

- for each integer n such that $0 \leqslant n < m$, D_{2n+1} and D_{2n+2} are in C_{k-n-1}
- the cell D_{2m+1} is not in F and, for each non-negative integer p, the cell $D_{2m+1} + (-p, 0)$ is not in F.

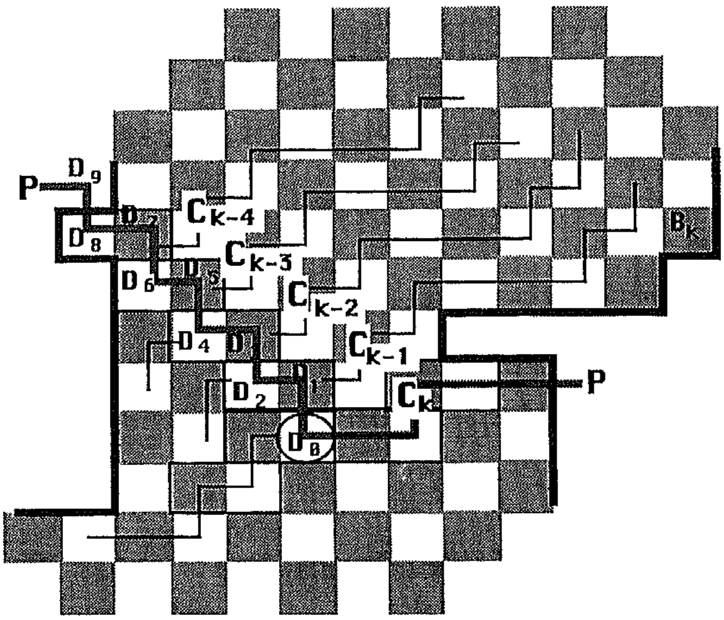

Fig. 8. Tiling of a horizontally convex figure with S_2.

Let P be the path $[D_{2m+1}, D_{2m}, \ldots, D_1, A_{i+1}, \ldots, A_p]$. We use the matching E of G_F, obtained in the following way: from the matching E_0, we do the exchange operation on the (alternating for E_0) chains $C_1, \ldots, C_{k-1}$. All the cells of F are matched by the matching E, except the cells $W_{k'}$ and $B_{k'}$, with $k' \geqslant k$. Thus, the only unmatched cell of F which is above path P is B_k. Assume that there exists an alternating chain C of matching E from B_k to a cell W_k' with $k' \geqslant k$. The chain C necessarily crosses the path P. Let U be the first cell of chain C which belongs to P. One can see that U cannot be a white cell. But the predecessor of any black cell B of C is the cell W such that V and B form a domino in the matching E. This yields that the predecessor of U belongs to P, which is a contradiction. Thus, there exists no alternating chain of E starting in B_k and finishing in an unmatched white cell (for more technical details, see [16]. Hence, because of the Proposition 3.3, G_F has no perfect matching.

When there exists no white cell of C_k, lower than A_p, such that $A_i + (0, 1)$ is in F and $A_{i+1} \neq A_i + (0, 1)$, we use the same arguments with $P = C_k$. The case where B_k is lower than W_k is treated in a symmetric way.

The algorithm given in this theorem spends a linear time since we can construct the chains C_k without passing more than a finite bounded number of times in each cell of F.

Uniqueness of the tiling of a finite figure with S_2

We give in this section a necessary condition for a finite figure to have a unique tiling with S_2.

Theorem 3.5. *If a finite piece has a single tiling with S_2, then it has at least two peaks.*

To give the proof, we need to define a new notion which fits dominoes.

A *strip* is a sequence $(c_0, \ldots, c_n)$ of *distinct* cells such that for $i = 0, \ldots, n-1$, c_i is adjacent to c_{i+1}. If c_n is adjacent to c_0 the strip is called a *ring* (see Fig. 9).

To a strip s we can associate a word $m(s)$ of $A^* = \{X, X', Y, Y'\}^*$ which is the concatenation of the letters corresponding to the translations from c_i to c_{i+1}, $0 \leqslant i < n$.

Lemma 3.6. *If s is a ring, then $m(s)$ has an odd length.*

Proof. The proof is simple. The curve coded by $m(s)$ is a simple closed curve, so it contains as many occurrences of X as occurrences of X', and the same for Y and Y'. □

Lemma 3.7. *Every ring has two tilings with S_2.*

Proof. Let $s = (c_0, \ldots, c_{2n+1})$ be a ring. The first tiling of s is the set of dominoes $\{\{c_i, c_{i+1}\}\ i \text{ even}\}$, the second one is the set of dominoes $\{\{c_i, c_{i+1}\} \mid i \text{ odd}\}$ (Fig. 10). □

Proof of Theorem 3.5. Let us consider a finite piece P_0 which admits a single tiling T with S_2. We will examine the cases: P_0 has no peak, one peak, and prove that each case leads to a contradiction, using previous lemmas.

Let us suppose P_0 has no peak. If T is a tiling of P_0, let $\{c_0, c_1\}$ be a set of adjacent cells which is a bar of T. Since P_0 has no peak there is a cell $c_2 \neq c_0$ in P_0 which is

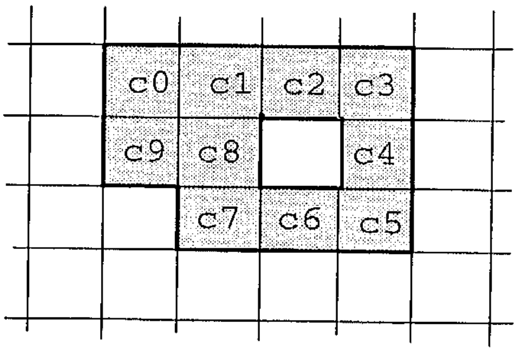

Fig. 9. A ring.

Fig. 10. Two tilings of a ring.

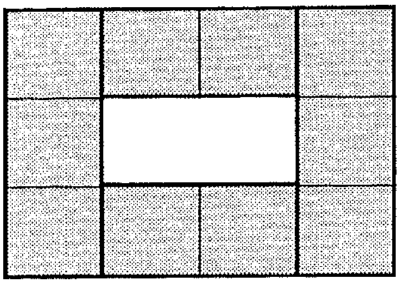

Fig. 11. $l = 2$, $m = 3$, a counter example.

adjacent to c_1. Let $\{c_2, c_3\}$ be the bar of T which covers c_2. Iterating the process, we build a sequence of bars $\{c_{2i}, c_{2i+1}\}$ of T. There exists a smallest integer j and an integer $i < 2j$ such that c_i and c_{2j+1} are adjacent. So, $(c_i, \ldots, c_{2j+1})$ is a ring. By Lemma 3.6, $2j + 2 - i$ is even, so i is even. Then, P_0 admits another tiling (Lemma 3.7).

Let us suppose that P_0 has one peak, the proof is the same as soon as we choose c_0 to be the peak. □

Theorem 3.5 does not hold if one of the bars has a length greater than two. Figure 11 provides a counter example: it is a finite piece which has a unique tiling and has a chain of 0 peaks, but it has a hole.

4. Tiling polyominoes with h_l and v_m

We give in this section two results. The first one is a linear algorithm to decide whether a given *trapeze* is tilable with dominoes. The second one is a necessary condition for a polyomino to be tiled in a unique way with h_l and v_m.

Tiling a trapeze with h_l, v_m

A *trapeze* T is a horizontally convex finite figure which has the following property: for each cell (x, y) of T, and each cell (x', y') in T such that $y' < y$, the cell $(x, y - 1)$ is also in T.

A *horizontal block* of a figure F is a rectangle $l \times j$, with $j < m$, included in F. A *vertical block* of a figure F is a rectangle $i \times m$, with $i < l$, included in F.

The *next cells* of a block B of a finite figure F are the cells (x, y) of F such that the cell $(x - 1, y)$ is in the rightmost column of B.

Clearly, each block has a unique tiling with h_l and v_m, and this tiling only contains one kind of bars.

Let us notice that a next cell of a block B is not in B.

Let T be a trapeze. The *upper fibre* of T is the longest sequence $(B_0, B_1, \ldots, B_p)$ of blocks of T, such that (see Fig. 12):

(1) the block B_0 contains the highest cell S of T such that the upper neighbor and the left neighbor of S are not in T. If the cell $S + (l - 1, 0)$ is in T, then B_0 is the bar

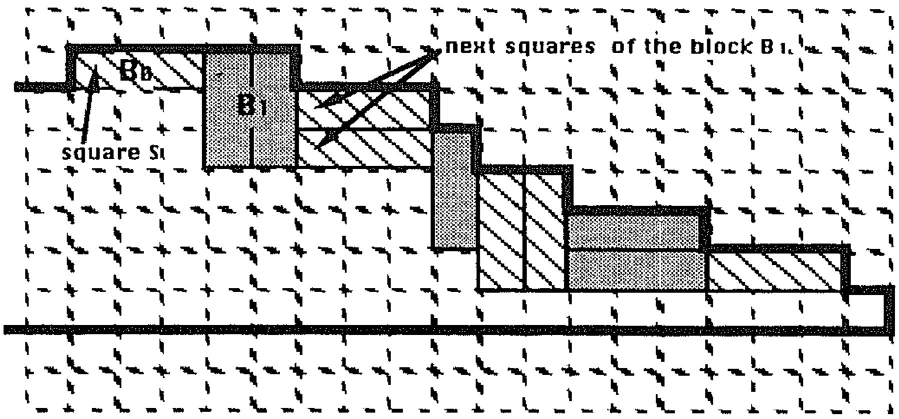

Fig. 12. Upper fibre of a trapeze.

h_l which has S in its leftmost column. Otherwise, B_0 is the largest vertical block of T which contains S.

(2) for each integer i such that $0 \leqslant i < p$, the block B_i of T has at least one next cell.

(3) If there exists a horizontal block of T whose upper-left corner is the highest next cell of B_i and whose lower-left corner is the right neighbor of the lower-right corner of B_i, then B_{i+1} is this block. Otherwise, B_{i+1} is the largest vertical block of T whose upper left corner is the highest next cell of B_i.

We say that the upper fibre $(B_0, B_1, \ldots, B_p)$ of T has a *good end* if the block B_p has no next cell.

Remark. If the upper fibre of T has a good end, then the figure $T' = T - (B_0 \cup B_1 \cup \cdots \cup B_p)$ is also a trapeze.

A trapeze T is *fibrous* if one of the following conditions holds:

(1) T is empty or the upper fibre of T has a good end,

(2) $T' = T - (B_0 \cup B_1 \cup \cdots \cup B_p)$ is fibrous.

In this case, the tiling of T that we obtain by the union of the tiling by fibres of T' and the tilings of the blocks B_i is called the tiling by fibres of T (if T is empty, its tiling by fibres is the empty tiling).

Theorem 4.1. *Let T be a trapeze. There exists a tiling of T with the bars h_l and v_m if and only if T is fibrous.*

Proof. By induction on the size of T. We have three cases.

In Figs. 13 and 14, blocks H_i are black, and the upper fibres of T and T'' are grey colored.

Case (A): *The upper fibre of T has not a good end.*

See Fig. 13. Let us denote $(B_0, B_1, \ldots, B_p) = (B_0, B_1, \ldots, B_k, V_{k+1}, \ldots, V_p)$, where B_k is the last horizontal block of the fibre. Let (x_0, y_0) be the highest next cell of V_p,

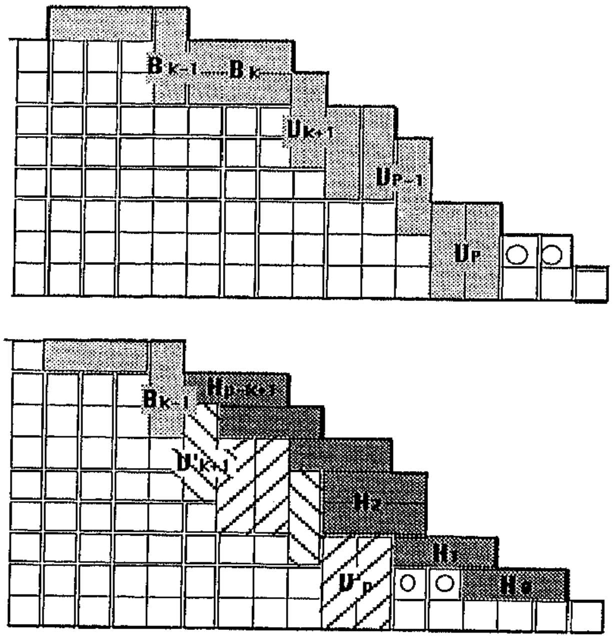

Fig. 13. A trapeze whose upper fibre has not a good end.

and let (x'_0, y_0) be the cell of the right side of T which is on the same line as (x_0, y_0).

We define the sequence $(H_0, H_1, \dots, H_{p-k+1})$ of horizontal blocks such that:

- H_0 is the tile h_l, included in F, which contains the cell (x'_0, y_0)
- H_i is the horizontal block whose right side is composed by the cells of B_{p-i+1} whose right neighbors are not in T.

We remark that each tiling of T uses the tiles used to tile the blocks H_i. So the trapeze $T'' = T - (H_0 \cup H_1 \cup \dots \cup H_{p-k+1})$ has a tiling with h_l and v_m, thus, by the induction hypothesis, T'' is fibrous. But the upper fibre of T'' is the sequence $(B_0, B_1, \dots, B_{k-1}, V'_{k+1}, \dots, V'_p)$, with $V'_{k+i} = V_{k+i} + (-l, 0)$. This fibre has not a good end, which is impossible.

Case (B): *The upper fibre of T has a good end, and its last block is a vertical block.* See Fig. 14. Let (x_1, y_1) be the upper-right corner of V_p, and V be the tile v_m whose highest cell is (x_1, y_1). If T admits a tiling containing V, then, by an obvious induction from the trapeze $T - V$, T is fibrous.

Now, assume that each tiling of T contains the tile h_l denoted by H, whose right side is the cell (x_1, y_1). Then the trapeze $T''' = T - (H \cup H_2 \cup \cdots \cup H_{p-k+1})$ has a tiling with h_l and v_m, thus, by induction hypothesis, T''' is tilable by fibres. The upper fibre of T''' is the sequence $(B_0, B_1, \dots, B_{k-1}, V'_{k+1}, \dots, V'_p, H_{p+1})$ where H_{p+1} denotes the rectangle $l \times m - 1$ whose right side is composed by the right side of V_p, except the cell (x_1, y_1). Let us consider the tiling of T that we obtain by the union of the tiling by fibres of T''' and the tilings of the blocks H_i : the rectangle $l \times m$ whose upper-right corner is (x_1, y_1) is tiled by m tiles h_l. We can put l tiles v_m instead of those tiles h_l. Thus, we obtain a tiling of T which contains the tile V, a contradiction.

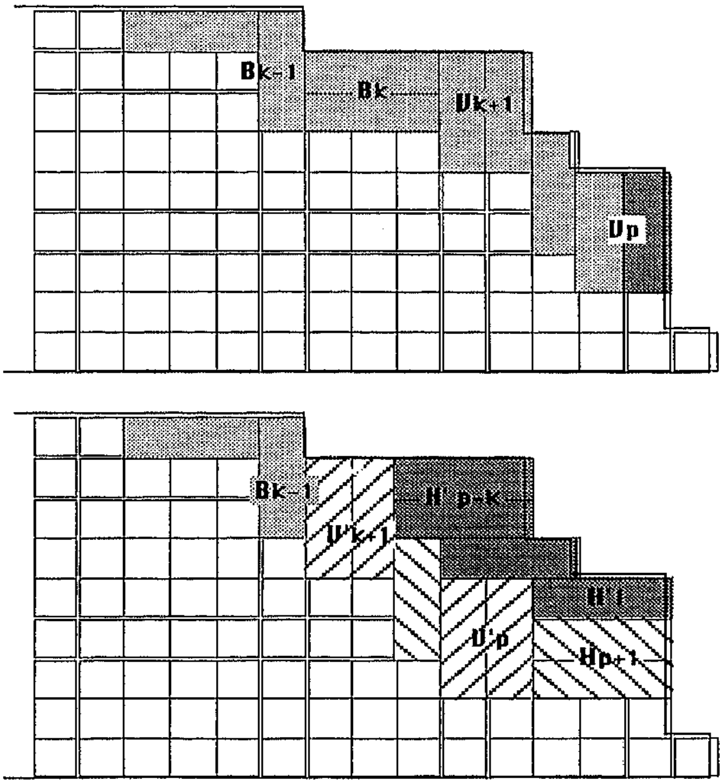

Fig. 14. A trapeze whose upper fibre has a good end, with a vertical block.

Case (C): *The upper fibre of T has a good end, and its last block is a horizontal block.* See Fig. 15. If T admits a tiling containing H, then, by an obvious induction using the trapeze $T-H$, T is fibrous. Now, let us assume that each tiling of T contains the tile V. Let P be a tiling of T. We define the sequence $(V_0, V_1, \ldots, V_k)$, where V_i denotes a tile v_m of P such that:

- $V_0 = V$,
- let us denote by (x_i, y_i) the highest cell of V_i. For $0 \leqslant i < k$, we have:
- $y_i - n < y_{i+1} < y_i$ and $x_i < x_{i+1}$
- If (x, y) is the highest cell of a tile v_m of P, and if (x, y) satisfies the conditions: $x_i < x$ and $y_i - m < y < y_i$, then we have: $x_{i+1} \leqslant x$
- If (x, y) is the highest cell of a tile v_m of P, then we cannot have simultaneously $x_k < x$ and $y_k - m < y < y_k$.

Let G be the set of the cells (x, y) such that (x, y) is in T and there exists an integer i such that: $x_i \leqslant x$ and (x_i, y) is a cell of V_i. The tiling P canonically gives a tiling P_G of the figure G. Moreover, the trapeze $T^{(4)} = T - G$ has a tiling with h_l and v_m, thus, by induction hypothesis, $T^{(4)}$ is fibrous. The upper fibre of $T^{(4)}$ is the sequence $(B_0, B_1, \ldots, B_{p-1}, V')$, where V' denotes the rectangle $(l-1) \times m$ whose upper side is the upper side of B_p, except the cell (x_1, y_1). Let us consider the tiling of T obtained by the union of the tiling by fibres of $T^{(4)}$ and the tiling P_G. Thus, the rectangle $l \times m$ whose upper-right corner is (x_1, y_1) is tiled by l tiles v_m. We can put

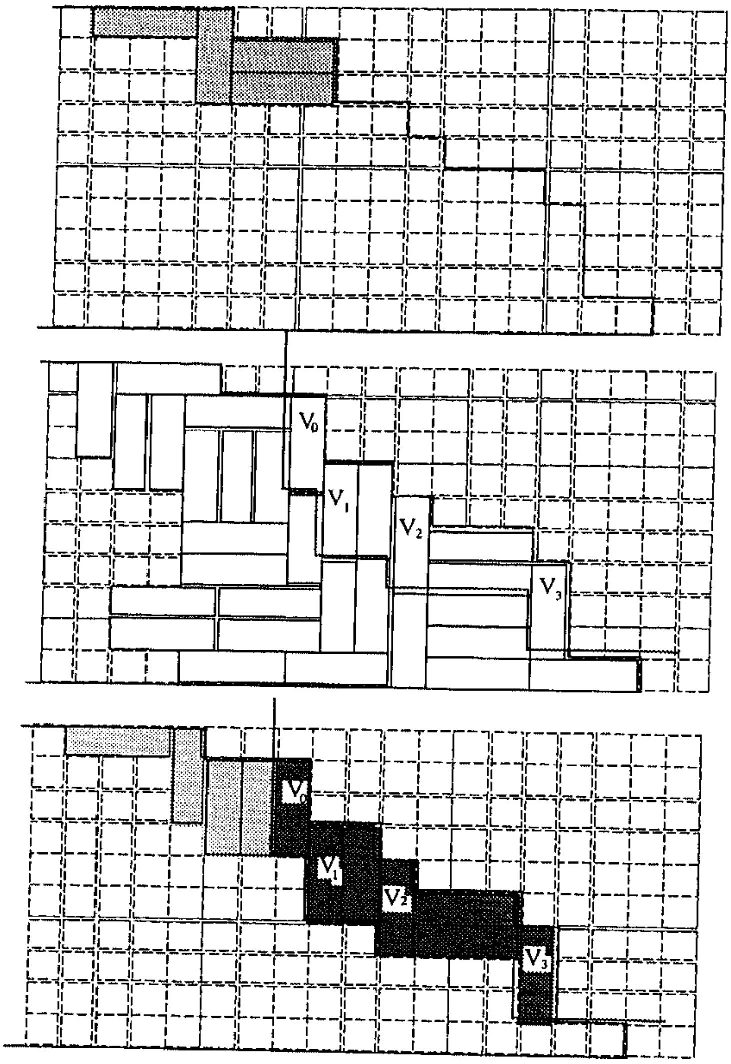

Fig. 15. A trapeze whose upper fibre has a good end, with a horizontal block.

m tiles h_l instead of those tiles v_m. Thus, we obtain a tiling of T which contains the tile H, a contradiction. □

The algorithm given by this theorem takes a linear time, since we can construct the fibres without passing more than a finite bounded number of times in each cell of T.

Remark. Theorems 3.4 and 4.1 easily give new proofs of a result from L. Bougé and M. Cosnard [6]. This result says that each coloured trapeze which has as much white cells as black cells admits a tiling with S_2.

Uniqueness of tilings of polyominoes with $S = \{h_l, v_m\}$, $l, m \geqslant 2$

We suppose $l, m \geqslant 2$ and $l + m > 4$.

The main result is the following theorem.

Theorem 4.2. *If a polyomino has a single tiling with S, then its set of peaks cannot form a chain.*

This theorem needs a proof by studying different cases, so it is based on three lemmas.

Lemma 4.3. *Let P_0 be a polyomino having a single tiling T_0 with S, and without peak. Let c_1 be the leftmost cell of P_0 on the top row. Then the connected components of the figure $P_1 = P_0 - p$, with $p = h(c_1)$ if $c_1 \in T_{0_h}$, $v'(c_1)$ otherwise, are polyominoes with at most one peak.*

Proof. Let P_0 be a polyomino having a single tiling T_0 with S. Let c_1 be the leftmost cell of P_0 on the top row. Two cases occur according to $c_1 \in T_{0_h}$ or $c_1 \in T_{0_v}$, (necessarily, c_1 belongs to one of these two sets).

First case: $c_1 \in T_{0_h}$.

Let $P_1 = P_0 - h(c_1)$ (P_1 is not empty since P_0 has no peak). We have to look at the connected components of P_1 and at their peaks. If P_1 has a peak p_1, this peak is adjacent to the bar $h(c_1)$, otherwise p_1 would be a peak of P_0 (adjacent means that at least one cell of p_1 is adjacent to a cell of $h(c_1)$), Figs. 17, 18, 19 give all the possible

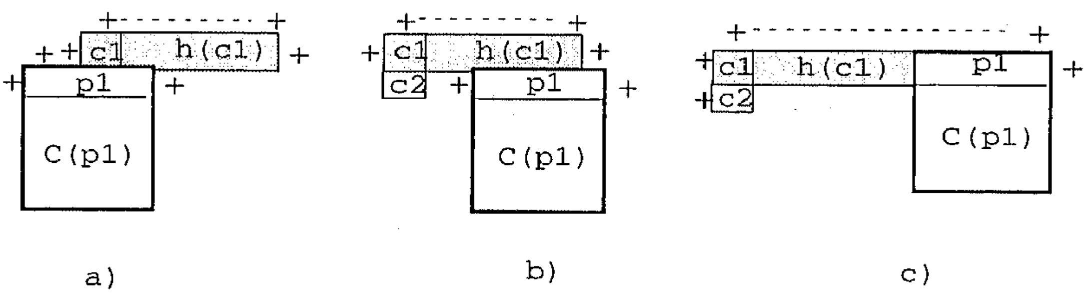

Fig. 16. The different positions of a up peak.

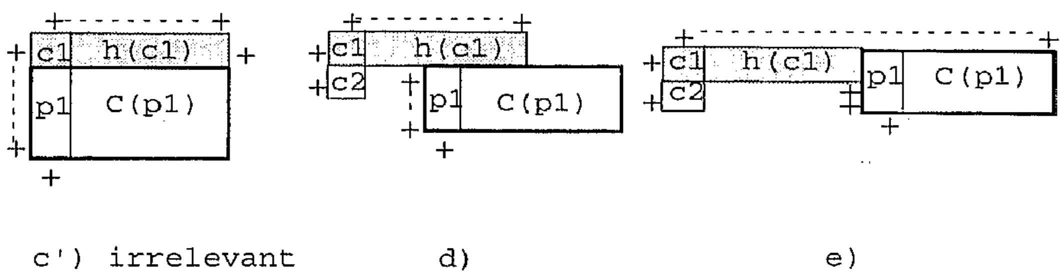

Fig. 17. The different positions of a left peak.

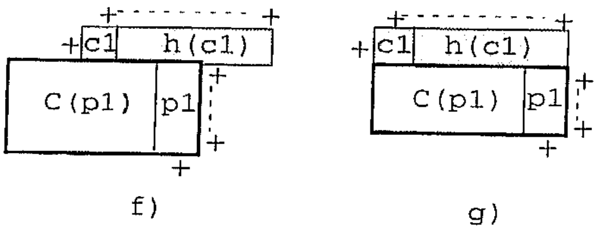

Fig. 18. The different positions of a right peak.

positions of p_1 related to $h(c_1)$. One can observe that the cell c_2 below c_1 belongs to P_0, otherwise P_0 would have a peak, and $C(p_1)$ is included in P_0.

- *p_1 is an up peak.*

In a), p_1 is adjacent to $h(c_1)$ on the down side and contains c_2 (p_1 can go beyond the left side of $h(c_1)$).

In b), p_1 is adjacent to $h(c_1)$ on the down side and does not contain c_2 (p_1 can go beyond the right side of $h(c_1)$).

In c), p_1 is adjacent to $h(c_1)$ on the right side.

The peak p_1 cannot be a down peak, because in that case, it cannot be adjacent to $h(c_1)$.

- *p_1 is a left peak*

In c′), p_1 is adjacent to $h(c_1)$ on the down side and contains c_1. So, $h(c_1) \cup C(p_1)$ is a canonical rectangle and then P_0 admits another tiling. It forces us to eliminate this case.

In d), p_1 is adjacent to $h(c_1)$ on the down side and does not contain c_1.

In e), p_1 is adjacent to $h(c_1)$ on the right side.

- *p_1 is a right peak*

Necessarily, p_1 is adjacent to $h(c_1)$ on the down side and contains c_1.

The case g) is a borderline case of f). In that case, $h(c_1) \cup C(p_1)$ is a rectangle but not necessarily a canonical one because the cell on the right of $h(c_1)$ possibly is a cell of P_0, so this case can happen.

Now, let us suppose that P_1 admits at least two peaks. Since these peaks are adjacent to $h(c_1)$, it is possible to order them along the outline of $h(c_1)$. Let p_1 and p'_1 two consecutive peaks of P_1 along $h(c_1)$.

Claim 1. *Necessarily, between p_1 and p'_1 there is a cell c which does not belong to P_0 and which has at least a corner which touches $h(c_1)$.*

Proof of Claim 1. We have just to scan all the possibilities for the positions of p_1 and p'_1: (a, b) (a, c), (a, d), (a, e) (b, c), (b, d), (b, e), (b, f), (c, f), (d, f), (e, f), and verify the claim. Some cases are impossible for small values of l, for example, if $l = 2$, the case b) is not possible; for $l < 6$ the pair (b, d) cannot occur. □

Claim 1 implies that in P_1 the connected components of two peaks are different, otherwise P_0 would have a hole precisely containing the cell c held up above. It turns out that every connected component of P_1 has at most one peak.

Second case: $c_1 \in T_{0_{v'}}$.

The cell c_3 adjacent to c_1 on the right belongs to P_0 otherwise P_0 would have a peak. Let $P_1 = P_0 - v'(c_1)$ (P_1 is not empty since P_0 has no peak). As in the first case we look at the different positions of a possible peak of P_1. We just give all the cases and we leave the reader to verify that the same scheme of reasoning leads to the end of the proof.

- p_1 *is a up peak* (see Fig. 19)

In case a), p_1 is adjacent to $v'(c_1)$ on the right side and does not contain c_3.

In case a′), p_1 is adjacent to $v'(c_1)$ on the right side and contains c_3. It leads to an irrelevant piece because the tiling has a subtiling which tiles exactly a canonical rectangle.

In case b), p_1 is adjacent to $v'(c_1)$ on the down side.

In case c), p_1 is adjacent to $v'(c_1)$ on the left side.

- p_1 *is a down peak* (see Fig. 20)

There is only one possibility: p_1 is adjacent to $v'(c_1)$ on the right side.

- p_1 *is a left peak* (see Fig. 21)

In case e), p_1 is adjacent to $v'(c_1)$ on the right side and contains c_3.

In case f), p_1 is adjacent to $v'(c_1)$ on the right side and does not contain c_3.

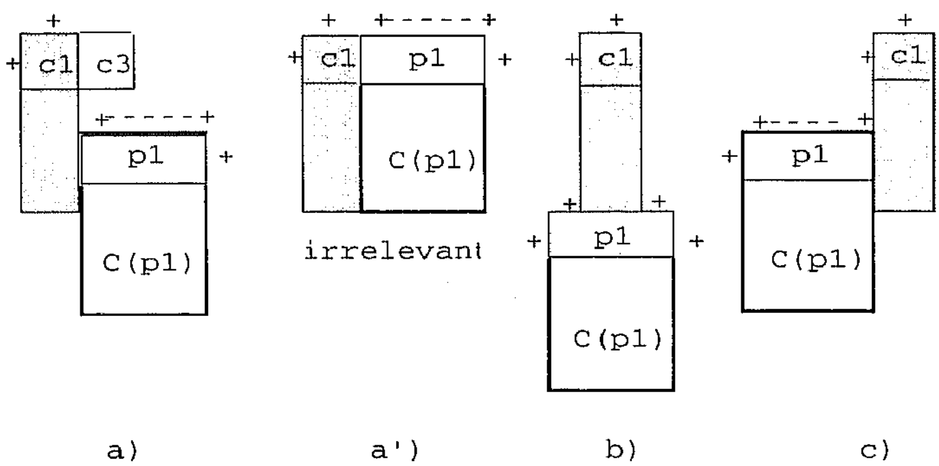

Fig. 19. The different positions of a up peak.

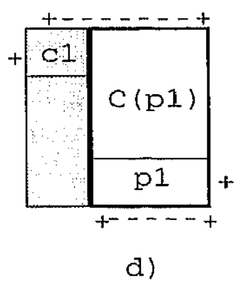

Fig. 20. The different positions of a down peak.

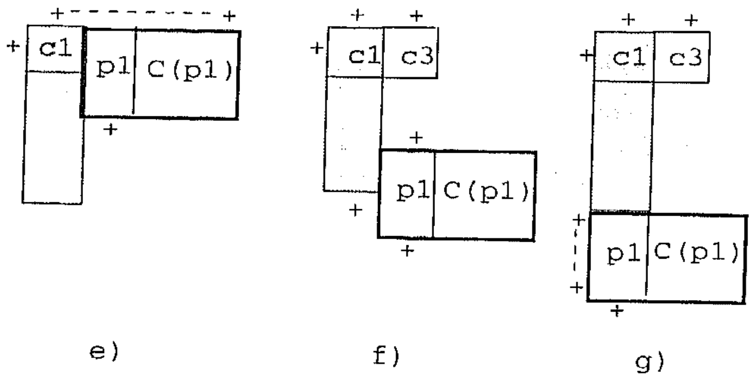

Fig. 21. The different positions of a left peak.

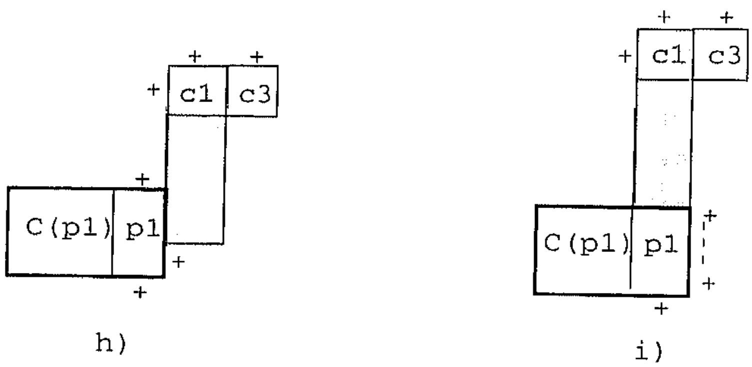

Fig. 22. The different positions of a right peak.

In case g), p_1 is adjacent to $v'(c_1)$ on the down side.
- p_1 *is a right peak* (see Fig. 22)

In case h), p_1 is adjacent to $v'(c_1)$ on the left side.
In case i), p_1 is adjacent to $v'(c_1)$ on the up side. □

Proofs of Lemma 4.4 and Lemma 4.5 work with the same kind of arguments.

Lemma 4.4. *Let P_0 be a polyomino having a single tiling T_0 with S, and having exactly one peak p_1. Then the connected components of the figure $P_1 = P_0 - C(p_1)$ are polyominoes and their set of peaks form a chain.*

Lemma 4.5. *Let P_0 be a polyomino having a single tiling T_0 with S, and whose set of peaks form a chain of length $k \geqslant 2$. Then the connected components of the figure $P_1 = P_0 - C(p_1) - \cdots - C(p_k)$, where $p_1, \cdots, p_k$ are the peaks of P_0, are pieces without holes and their set of peaks form a chain.*

Proof of Theorem 4.2. Let us suppose there exists a polyomino P_0 having a single tiling, and whose set of peaks form a chain. We consider a piece of *minimal size* having this property.

• If P_0 has no peak, we use Lemma 4.3. The figure P_1 (which is not empty, otherwise P_0 would have two peaks), has connected components without holes (otherwise P_0 would have a hole), with a smaller size than P_0, and having a single tiling. So, by Lemma 4.4, each of these connected components has at most one peak which form a chain, and we get a contradiction with the minimality of P_0.

• If P_0 has one peak, we use Lemma 4.5, and with the same kind of argument, we get a contradiction.

• At last, if P_0 has a set of peaks which form a chain of length $k \geqslant 2$, applying Lemma 4.5 gives in the same way a contradiction. So the proof of Theorem 4.2 is achieved. □

Theorem 4.2. provides an algorithm to decide whether a given figure without holes has a single tiling.

Theorem 4.6. *Let P be a finite figure without holes. There is a linear algorithm to decide whether there is a unique tiling of P and computes this tiling in the positive case.*

Proof. Figure P is given as the set of words representing the outlines of its connected components. The algorithm is the following.

(0) if P is empty, then P admits a single tiling with S, stop
(1) else look for peaks;
(2) if no peak, and P is not empty then P does not have a single tiling
(3) else remove from P the covers of all the peaks; go to (0).

This algorithm is linear in the size of P, because a first scanning of the outlines determines all the peaks; and after, the new peaks which appear are adjacent to the covers we have removed, so their research is a local one. If the algorithm leads to an empty figure, the single tiling is constituted of the bars which fill the peaks' covers we have removed. For the implementation, we can use a two-dimensional array to code the figure and a stack to store the peaks. □

5. Rigid and flexible tilings

In this section l and m satisfy $l, m \geqslant 2$.

The second result of Section 4 leads us to study uniqueness for a tiling of a figure (not necessarily finite) without holes.

A tiling T with S of a figure F is said to be *flexible* if there exists another tiling T' of F which is different from T only for a *finite* number of elements (that means that a finite subtiling can be changed, without changing the associated subfigure). A tiling which is not flexible is called a *rigid* tiling.

So we give here a characteristic property of rigidity for tilings with S, of pieces without holes. The main result is the following one.

Theorem 5.1. *Let F be a figure (finite or infinite) without holes. A tiling T_0 of F is rigid if and only if no subtiling of T_0 covers exactly a canonical rectangle.*

Proof. The condition is clearly necessary. Let us look at the sufficient condition. If F admits a flexible tiling T, there exists a finite *piece* without hole which is covered by a subtiling of T and admits also another tiling. Let P_0 be a minimal finite piece without hole satisfying this property. Let c_1 be the leftmost cell of P_0 on the highest row. We put $Q = h(c_1)$ if $c_1 \in T_h$ and $Q = v'(c_1)$ if $c_1 \in T_{v'}$. Let $P_1 = P_0 - Q$. Since P_0 has a minimal size, P_0 has no peak. Actually, if P_0 would have a peak p, then $P_0 - C(p)$ would also have several tilings (otherwise P_0 would have a single one), and so P_0 would not be of minimal size.

If we come back to the proof of Theorem 4.2, since P_1 has a single tiling, each connected component of P_1 has at least two peaks, but this is impossible (see the proof of Theorem 4.2) except if we are in the situation (c, c′) in the first case ($c_1 \in T_h$), or in the situation (a′, c) in the second case ($c_1 \in T_{v'}$). But in both these situations, T_0 admits a subtiling covering exactly a canonical rectangle. □

Corollary 5.2. *Every tiling of a finite piece without holes whose set of peaks forms a chain admits a subtiling covering exactly a canonical rectangle.*

Proof. It is a clear consequence of Theorem 4.2 and Theorem 5.1. □

Corollary 5.3. *A tiling T_0 of the plane with S is rigid if and only if no finite subtiling of T_0 covers exactly a canonical rectangle.*

6. Tiling arbitrary shaped finite figures with h_l, v_m

In contrast to the above results showing efficient algorithms for many tiling problems, we now show that tiling arbitrary shaped figures with any set of two bars more complex than h_2 and v_2 is hard (provided P ≠ NP).

Theorem 6.1. *It is NP-complete to decide whether a figure can be tiled with h_l and v_m bars for $l, m \geqslant 2$ unless $l = m = 2$.*

The proof will be given first for the simple case h_2, v_3 and then extended to the general case. Membership of the problem in NP is trivial; NP-hardness will be shown by a reduction from Planar 3-CNF Satisfiability [8].

The reduction will place gadgets corresponding to clauses and variables in a plane figure and connect them by 'cables'. We start by describing cables and their properties. In the figures illustrating cables and gadgets we classify squares as belonging to six sets a, b, c, d, e and f according to their x and y coordinates modulo 2 and 3 respectively as shown in Fig. 23 and we show possible tilings by using lower case

```
ababab...
cdcdcd...
efefef...
a.
c.
e.

.bababa         AbabaB         abababA         ababab
      C         C    D               C         C    D
      E         E    F               E         E    F
      Abababababa    bababa.         abababababA    Bababab
```

Fig. 23. The six types of square and a wire of Type a – b.

letters for squares covered by an h_2 domino and the corresponding upper case letters for squares covered by a v_3 bar.

Fig. 1 shows a *wire of type* a–b, that is a sequence of squares each connected orthogonally to its successor and none of them except possibly the first and last connected to another square of the figure, and such that the every change of direction (of which there must be at least one) is at a square of Type a or b. The wire has an implied orientation and has its first square of Type a and its last of Type b. As shown in the figure the wire can be tiled with its two end squares left uncovered and so added to the connecting gadgets (a *positive* signal or wire) or alternatively completely tiled (called a *negative* signal transmitted on the wire or simply a *negative wire*). The function of a wire in our constructions and the idea behind the terminology defined can be explained by remarking that the existence of a wire between two sections of a figure implies a dependency between the ways in which the two sections are paved in any possible paving of the whole figure, namely that either both sections are paved without a tile overlapping into the wire (negative) or each of them is paved with a tile overlapping only as far as the first square of the wire (positive because the two end squares are added to the section). A simpler way of describing this may be to say that in traversing a wire of Type a–b from its start to its end, a horizontal section composed of squares of Types a and b will be tiled by b–a tiles if the wire is positive but by a–b tiles if negative. We can similarly define wires of types b–a, c–d etc according to the type of squares at their beginning and end.

A *cable* consists of two wires of Types a–b and b–a with the a–b wire to the left of the b–a (with respect to their common orientation). If the two wires carry the same signal (which will always be forced in the cables in our constructed figures) then the cable is said to be *coherent* and to carry this same signal.

A central concept in our constructions is that of a *verifier*. A verifier is a figure included within a rectangle and having n pairs of squares designated in the row immediately above this rectangle; these squares being of Types a and b alternately reading from the left. We consider the question of whether the figure can be tiled as it is or with some or all of the pairs added. The answer is evidently a function f of n Boolean variables describing which pairs have been added, where we take variable X_i to be true iff the ith designated pair has been added. The verifier is called a *verifier* of f. If, moreover, the figure cannot be tiled with the addition of some non-empty subset of the designated squares other than a set of pairs, the figure is called a *strong*

```
                                A     B A     B A     B
Ċ     Ḋ Ċ     Ḋ Ċ     Ḋ         C     D C     D C     D
E     F E     F E     F         E     F E     F E     F
AB  AB  AB  AB  AB  AB          ab  ab  ab  ab  ab  ab
 DcdC    DcdC    DcdC            dcdc    dcdc    dcdc
EF  EfefeF  EfefeF  EF          ef  efefef  efefef  ef
A                    B          A                    B
C                    D          C                    D
efefefefefefefefefefef          EfefefefefefefefefefeF
```

Fig. 24. A strong verifier of $X_1 = X_2 = X_3$.

verifier. Fig. 24 illustrates a strong verifier of the function $X_1 = X_2 = X_3$. This verifier can be modified in an obvious way to handle a different number of variables.

By placing strong verifiers of functions in a larger figure with cables connecting their designated pairs, we obtain a figure which can be tiled if and only if the assignment of values to variables implicit in the paving of designated squares (true iff the cable is positive) satisfies the functions of all the verifiers. (This is because, firstly, by the construction of wires, each wire must carry a positive or negative signal, secondly, since the verifiers are strong, each cable must be coherent, and finally since the verifiers are verifiers, their functions must be satisfied by the implied assignment.)

In fact, to construct the figure which will be tilable iff a given planar 3-CNF expression is satisfiable, we will use a slightly weaker version of this scheme. Note that the argument of the previous paragraph is still valid if at least one of the verifiers at the ends of each cable is strong. Accordingly we will construct a figure with a verifier (not necessarily strong) for each clause and, for every variable, a strong verifier of $X_1 = X_2 = \cdots = X_c$, for c the number of clauses in which the variable occurs; now joining every designated pair of a clause verifier, by a cable, to the corresponding variable verifier will give the figure which can be tiled if and only if the expression is satisfiable and, because the expression is planar, this construction is possible without any need for cables to cross.

Figs. 25 to 28 show small components which are useful in constructing the required verifiers. Firstly Fig. 25 shows how an input wire of Type a–b (all inputs in these four figures are from the left) can *fork* into two wires of Types f–e and d–c, each carrying the same signal as the input.

Fig. 26 shows a component which can be regarded as a restricted OR- *gate*. The left part of the figure shows that if the lower input is negative, the upper input is passed unchanged to the output. The right part shows the unique tiling possible if the lower input is positive, namely with the upper input negative and the output positive. Thus this component can be tiled if and only if at most one input is positive and its output must be the OR of its two inputs.

Fig. 27 shows a *switch* which can be tiled in one of three ways (excluding those where the input cable is not coherent). The first shows input and output cables both positive; the second shows them both negative; but the third shows that, because of the 'short circuit' between the two wires, it is also possible to have a positive input and a negative output. In summary this component makes it possible to 'switch off' a positive signal on its input cable.

```
   fefe    Fef.
abab    .baB
   dcdc    Dcd.
```

Fig. 25. A fork.

```
.baba   abAba
  C       C
  E       E
abA     .ba
```

Fig. 26. A restricted OR-gate.

```
     bababa
     D
     F
.babaBababababab.
     C
     E
.ababA

      Babab.
      D
      F
abababAbabababa
      C
      E
 bababa

     Babab.
     D
     F
.babababababa.
     C
     E
.ababA
```

Fig. 27. A switch.

```
    Babab
    D
    F
    ba
dcdcdcdcdc
    fe
.abab

    baba.
    D
    F
    BA
dcdcDCdcdc
    FE
babaB
```

Fig. 28. A partial crossing.

Fig. 28 shows a *partial crossing* where a wire of Type a–b crosses one of Type c–d. The first part of the figure shows that, if the a–b wire is negative, the c–d may be negative or positive. The second part shows the only possible tiling with the a–b positive and illustrates that, in this case, the c–d must also be positive. In other words the two wires can cross each other and must carry the same signal at the output and the input but one combination of signals, namely a–b positive and c–d negative, is forbidden. A similar partial crossing can be constructed with any required combination of signals as the forbidden one.

Fig. 29 shows a verifier for the function of three variables which is true when exactly one of the three is true; the tiling is shown for the case that X_2 is true.

This verifier contains a number of the forks, OR-gates and partial crossings described above. The entering a–b wires are brought together by OR-gates to end at a wire which must be positive and the same thing happens to the c–d and e–f wires formed by forking the three b–a inputs. The fact that the verifier is not needed to be tiled if two inputs are positive means that the partial crossings and restricted OR-gates never have the combination of input signals which makes it impossible to tile them.

Note that this verifier of X_1 *or* X_2 *or* X_3 *and not* (*any two*) could be transformed into a verifier of $X_1 \neq X_2$ simply by making X_3 false, namely by omitting the designated pair of X_3 and not joining a cable there. If we change our point of view and regard the

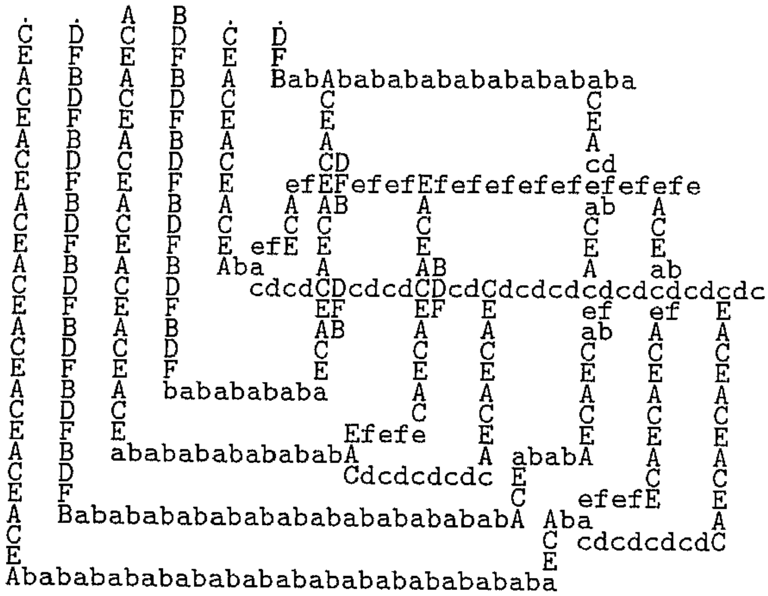

Fig. 29. A verifier of "one of X_1, X_2 and X_3".

two cables of this verifier as an 'input' and an 'output' cable, we obtain a 'negater' since the output must be the opposite of the input.

To transform Fig. 29 into a verifier for the simple clause X_1 *or* X_2 *or* X_3, we simply run each incoming cable into the input of a 'switch' like the one in Fig. 27 and the output of the switch into the input of the simple verifier of Fig. 29. The resulting figure can be tiled if one or more input cables are positive by switching off all but one of the positive cables (and not otherwise).

Finally we obtain a verifier for a general 3 literal clause by negating those inputs corresponding to negated variables. We do this by running their cables to the input of a negater constructed as noted above; then the output of that negater is run to the input of the X_1 *or* X_2 *or* X_3 verifier. This construction completes the reduction from Planar 3-CNF SAT to h_2, v_3 tiling, except for trivial details of the separation of verifiers needed to allow cables to run to their destinations without interference.

To extend the result to the general case of h_l and v_m with $l > 1$, $m > 1$ but not $l = m = 2$, we note that it is sufficient to prove it for $l > 1$, $m > 2$ since it will then follow for $l > 2$, $m > 1$ trivially. We will show a simple reduction from the h_2, v_3 case to this more general case.

Let A, B, C, D, E and F denote six non empty rectangles obtained by dividing a m by n rectangle by one vertical and two horizontal lines (e.g. ones whose respective dimensions are (horizontal first) : (1 by 1), ($l - 1$ by 1), (1 by 1), ($l - 1$ by 1), (1 by $m - 2$) and ($l - 1$ by $m - 2$). Given a plane figure P drawn on a plane with squares classified as of Types a, b, c, d, e and f as above, replace every square of the plane by a corresponding rectangle (e.g. a rectangle A for a square a) to obtain a figure P'. We claim that P' can be tiled by h_l and v_m if and only if P can be tiled by h_2 and v_3. In one direction this is clear : a bar covering part of P expands in a natural way into a number of the new bars covering the corresponding part of P', for instance in the example given a v_3 covering three squares dfb expands into $l-1$ bars v_m covering the three rectangles DFB. In the other direction, consider a tiling of P' and consider only those bars which include the top left corner square of at least one of the rectangles A, B, C, D, E and F. Any h_l (resp. v_m) which covers one of these corner squares must necessarily cover exactly two horizontally (resp. three vertically) adjacent such corners. Since these corner squares correspond exactly to the squares of P, the h_l and v_m which cover them give a natural covering of P by h_2 and v_3 completing the reduction.

If we consider a case where $l = m \geqslant 3$, which can be regarded as a question about one bar which can be rotated, then we have also proved the NP-completeness of the problem *perfect tile salvage* (Berman et al.).

It can be noted that the recent result of Laroche [14] shows that a similar reduction is possible from planar 1-in-3 Sat, avoiding the need for switches and negaters.

References

[1] D. Beauquier and M. Nivat, Codicity and simplicity in the plane, Publication interne LITP 88-66 (1988).
[2] D. Beauquier and M. Nivat, Tiling the plane with one tile, Topology and Category Theory in Computer Science (Oxford Univ. Press, Oxford, 1991) 291–334.
[3] C. Berge, Théorie des Graphes (Gauthier-Villars, Paris, 1971).
[4] R. Berger, The undecidability of the domino problem, Memoirs Amer. Math. Soc. 66 (1966).
[5] F. Berman, D. Johnson, T. Leighton, P.W. Shor and L. Snyder, Generalized Plana, J. Algorithms 11 (1990) 153–184.
[6] L. Bougé and M. Cosnard, Recouvrement d'une pièce trapézoïdale par des dominos, C R A S. à paraître.
[7] H. Freeman, On the encoding of arbitrary geometric configurations, IRE Trans. EC-10 (1961) 260–268.
[8] M.R. Garey, D.S. Johnson and C.H. Papadimitriou, unpublished manuscript.
[9] S.W. Golomb, Polyominoes (Georges Allen and Unwin, London, 1966).
[10] S.W. Golomb, Tilings with Sets of Polyominoes, J. Combin. Theory 9 (1970) 60–71.
[11] B. Grünbaum and G.C. Shephard, Tilings and patterns (Freeman, New York, 1986).
[12] J.D. Hopcroft and P. Karp SIAM J. Comput. 2 (1973) 225–231.
[13] C. Kenyon and R. Kenyon, Tiling polygons with rectangles, Preprint, 1992.
[14] P. Laroche, Planar 1-in-3 satisfiability is NP-complete, ASMICS Workshop on Tilings, Deuxièmes Journées Polyominos et pavages, Ecole Normale Supérieure de Lyon, 1992.
[15] D. Lichtenstein, Planar Satisfiability and its Uses, SIAM J. Comput. 11 (1982) 329–343.
[16] E. Remila, Pavage de figures par des barres et reconnaissance de graphes sous-jacents à des réseaux d'automates, Ph.D. Thesis Université Lyon I, 1992.
[17] H.D. Shapiro, Theoretical limitations on the efficient user of parallel memories, IEEE Trans. Comput. (1978).
[18] W.P. Thurston, Conway's tiling groups, AMM (1990) 757–773.
[19] H. Wang, Notes on a class of tiling problems, Fundam. Math. 82 (1975) 295–305.
[20] H.A.G. Wijshoff and J. van Leeuwen, The structure of periodic storage schemes for parallel memories, IEEE Trans. Comput. 34 (1985) 501–505.