# Worst-Case Optimal Covering of Rectangles by Disks*

**Sándor P. Fekete[1], Utkarsh Gupta[2], Phillip Keldenich[1], Christian Scheffer[1], and Sahil Shah[2]**

1 **Department of Computer Science, TU Braunschweig, Germany**
**{s.fekete,p.keldenich,c.scheffer}@tu-bs.de**

2 **Department of Computer Science & Engineering, IIT Bombay, India**
**{utkarshgupta149,sahilshah00199}@gmail.com**

## Abstract

We provide the solution for a fundamental problem of geometric optimization by giving a complete characterization of worst-case optimal disk coverings of rectangles: For any $\lambda \geq 1$, the critical covering area $A^*(\lambda)$ is the minimum value for which any set of disks with total area at least $A^*(\lambda)$ can cover a rectangle of dimensions $\lambda \times 1$. We show that there is a threshold value $\lambda_2 = \sqrt{\sqrt{7}/2 - 1/4} \approx 1.035797\ldots$, such that for $\lambda < \lambda_2$ the critical covering area $A^*(\lambda)$ is $A^*(\lambda) = 3\pi \left(\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}\right)$, and for $\lambda \geq \lambda_2$, the critical area is $A^*(\lambda) = \pi(\lambda^2 + 2)/4$; these values are tight. For the special case $\lambda = 1$, i.e., for covering a unit square, the critical covering area is $\frac{195\pi}{256} \approx 2.39301\ldots$. The proof uses a careful combination of manual and automatic analysis, demonstrating the power of the employed interval arithmetic technique.

## 1 Introduction

Given a collection of (not necessarily equal) disks, is it possible to arrange them so that they completely cover a given region, such as a square or a rectangle? Covering problems of this type are of fundamental theoretical interest, but also have a variety of different applications, most notably in sensor networks, communication networks and wireless communication [22], surveillance, robotics, and even gardening and sports facility management, as shown in Figure 1.

If the total area of the disks is small, it is clear that completely covering the region is impossible. On the other hand, if the total disk area is sufficiently large, finding a covering

* This is an extended abstract of our paper *Worst-Case Optimal Covering of Rectangles by Disks* [15]. A video presenting the main result can be found at `https://www.ibr.cs.tu-bs.de/users/fekete/Videos/Cover_full.mp4` .

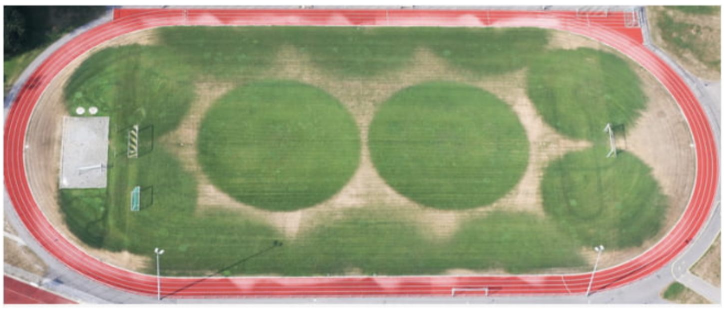

**Figure 1** An incomplete covering of a rectangle by disks: Sprinklers on a soccer field during a drought. (Source: dpa [13].)

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This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

seems easy; however, for rectangles with large aspect ratio, a major fraction of the covering disks may be useless, so a relatively large total disk area may be required. The same issue is of clear importance for applications: What fraction of the total cost of disks can be put to efficient use for covering? This motivates the question of characterizing a critical threshold: For any given $\lambda$, find the minimum value $A^*(\lambda)$ for which any collection of disks with total area at least $A^*(\lambda)$ can cover a rectangle of dimensions $\lambda \times 1$. What is the critical covering area of $\lambda \times 1$ rectangles? In this paper we establish a complete and tight characterization that generalizes to arbitrary rectangles by scaling and rotating.

## 1.1 Related Work

Like many other packing and covering problems, disk covering is typically quite difficult, compounded by the geometric complications of dealing with irrational coordinates that arise when arranging circular objects. This is reflected by the limitations of provably optimal results for the largest disk, square or triangle that can be covered by $n$ unit disks, and hence, the "thinnest" disk covering, i.e., a covering of optimal density. As early as 1915, Neville [27] computed the optimal arrangement for covering a disk by five unit disks, but reported a wrong optimal value; much later, Bezdek [6, 7] gave the correct value for $n = 5, 6$. As recently as 2005, Fejes Tóth [33] established optimal values for $n = 8, 9, 10$. Szalkai [32] gave an optimal solution for a small special case ($n = 3$) of a general problem posed by Connelly in 2008, who asked how one should place $n$ small disks of radius $r$ to cover the largest possible area of a disk of radius $R > r$. For covering arbitrary rectangles by $n$ unit disks, Heppes and Mellissen [20] gave optimal solutions for $n \leq 5$; Melissen and Schuur [24] extended this for $n = 6, 7$. See Friedman [19] for the best known solutions for $n \leq 12$. Covering equilateral triangles by $n$ unit disks has also been studied. Melissen [23] gave optimal results for $n \leq 10$, and conjectures for $n \leq 18$; the difficulty of these seemingly small problems is illustrated by the fact that Nurmela [28] gave conjectured optimal solutions for $n \leq 36$, improving the conjectured optimal covering for $n = 13$ of Melissen. Carmi, Katz and Lev-Tov [11] considered algorithms for covering point sets by unit disks at fixed locations. There are numerous other related problems and results; for relevant surveys, see Fejes Tóth [14] (Section 8), Fejes Tóth [34] (Chapter 2), Brass, Moser and Pach [10] (Chapter 2) and the book by Böröczky [9].

Even less is known for covering by non-uniform disks, with most previous research focusing on algorithmic aspects. Alt et al. [3] gave algorithmic results for minimum-cost covering of point sets by disks, where the cost function is $\sum_j r_j^\alpha$ for some $\alpha > 1$, which includes the case of total disk area for $\alpha = 2$. Agnetis et al. [2] discussed covering a line segment with variable radius disks. Abu-Affash et al. [1] studied covering a polygon minimizing the sum of areas; for recent improvements, see Bhowmick,Varadarajan and Xue [8]. Bánhelyi, Palatinus and Lévai [4] gave algorithmic results for the covering of polygons by variable disks with prescribed centers.

The dual question of *packing* unit disks into a square has also attracted attention. For $n = 13$, the optimal value for the densest square covering was only established in 2003 [18], while the optimal value for 14 unit disks is still unproven; densest packings of $n$ disks in equilateral triangles are subject to a long-standing conjecture by Erdős and Oler from 1961 [29] that is still open for $n = 15$. Many authors have considered heuristics for circle packing problems, see [31, 21] for overviews of numerous heuristics and optimization methods. The best known solutions for packing equal disks into squares, triangles and other shapes are published on Specht's website `http://packomania.com` [30]. Establishing the critical packing density, i.e., the disk area that can always be packed into a unit square, for (not

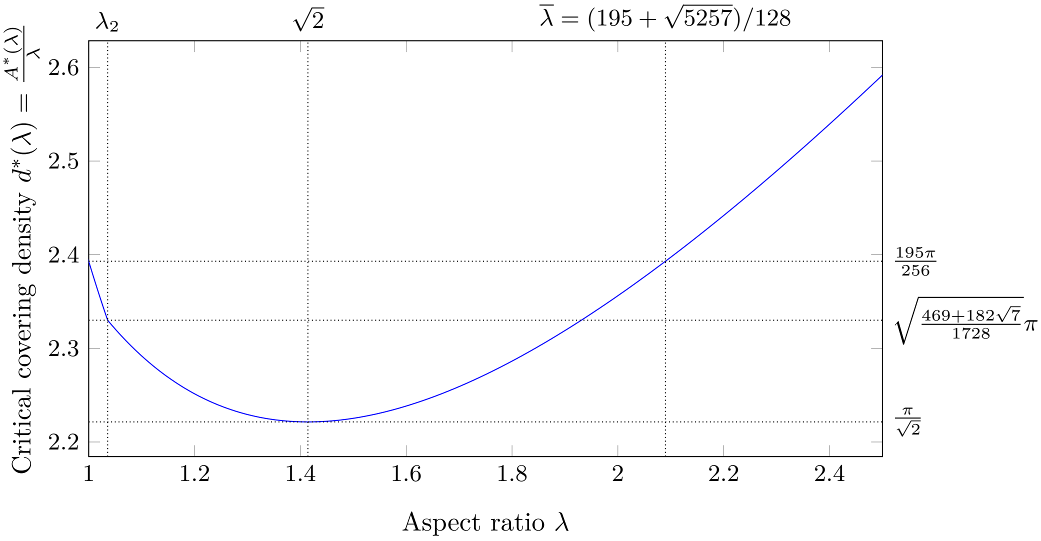


**Figure 2** The critical covering density $d^*(\lambda)$ depending on $\lambda$ and its values at the threshold value $\lambda_2$, the global minimum $\sqrt{2}$ and the aspect ratio $\overline{\lambda}$ at which the density becomes as bad as for the square.

necessarily equal) disks in a square was proposed by Demaine, Fekete, and Lang [12] and solved by Morr, Fekete and Scheffer [26, 17]. Using a recursive procedure for cutting the container into triangular pieces, they proved that the critical packing density of disks in a square is $\frac{\pi}{3+2\sqrt{2}} \approx 0.539$. The critical density for (not necessarily equal) disks in a disk was recently proven to be 1/2 by Fekete, Keldenich and Scheffer [16]; see the video [5] for an overview and various animations. The critical packing density of (not necessarily equal) squares was established in 1967 by Moon and Moser [25], who used a shelf-packing approach to establish the value of 1/2 for packing into a square.

For more related work, we refer the reader to the full version of our paper [15].

## 1.2 Our Contribution

We show that there is a threshold value $\lambda_2 = \sqrt{\sqrt{7}/2 - 1/4} \approx 1.035797\ldots$, such that for $\lambda < \lambda_2$ the critical covering area $A^*(\lambda)$ is $A^*(\lambda) = 3\pi \left(\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}\right)$, and for $\lambda \geq \lambda_2$, the critical area is $A^*(\lambda) = \pi(\lambda^2 + 2)/4$. These values are tight: For any $\lambda$, any collection of disks of total area $A^*(\lambda)$ can be arranged to cover a $\lambda \times 1$-rectangle, and for any $a(\lambda) < A^*(\lambda)$, there is a collection of disks of total area $a(\lambda)$ such that a $\lambda \times 1$-rectangle cannot be covered. (See Figure 2 for a graph showing the (normalized) critical covering density, and Figure 3 for examples of worst-case configurations.) The point $\lambda = \lambda_2$ is the unique real number greater than 1 for which the two bounds $3\pi \left(\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}\right)$ and $\pi\frac{\lambda^2+2}{4}$ coincide; see Figure 2. At this so-called *threshold value*, the worst case changes from three identical disks to two disks — the circumcircle $r_1^2 = \frac{\lambda^2+1}{4}$ and a disk $r_2^2 = \frac{1}{4}$; see Figure 3. For the special case $\lambda = 1$, i.e., for covering a unit square, the critical covering area is $\frac{195\pi}{256} \approx 2.39301\ldots$.

The proof uses a careful combination of manual and automatic analysis, demonstrating the power of the employed interval arithmetic technique.

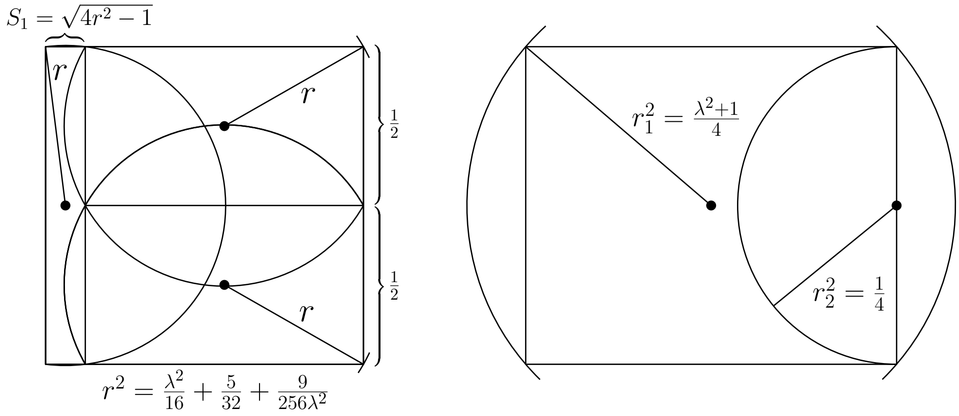


**Figure 3** Worst-case configurations for small $\lambda \le \lambda_2$ (left) and for large $\lambda \ge \lambda_2$ (right). Shrinking $r$ or $r_1$ by any $\varepsilon > 0$ in either configuration leads to an instance that cannot be covered.

## 2 High-Level Description

Our main theorem gives a closed-form solution for the *critical covering area* $A^*(\lambda)$ for any $\lambda \ge 1$, i.e., for any given rectangle $\mathcal{R}$, we determine the total disk area that is (1) sometimes necessary and (2) always sufficient to cover $\mathcal{R}$. Due to limited space, we only sketch the overall approach; details are contained in the full version [15] of the paper.

▶ **Theorem 2.1.** *Let $\lambda \ge 1$ and let $\mathcal{R}$ be a rectangle of dimensions $\lambda \times 1$. Let*

$$\lambda_2 = \sqrt{\frac{\sqrt{7}}{2} - \frac{1}{4}} \approx 1.035797\ldots, \text{ and } A^*(\lambda) = \begin{cases} 3\pi\left(\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}\right), & \text{if } \lambda < \lambda_2, \\ \pi\frac{\lambda^2+2}{4}, & \text{otherwise.} \end{cases}$$

(1) *For any $a < A^*(\lambda)$, there is a set $D^-$ of disks with $A(D^-) = a$ that cannot cover $\mathcal{R}$.*
(2) *Let $D = \{r_1, \ldots, r_n\} \subset \mathbb{R}$, $r_1 \ge r_2 \ge \ldots \ge r_n > 0$ be any collection of disks identified by their radii. If $A(D) \ge A^*(\lambda)$, then $D$ can cover $\mathcal{R}$.*

The critical covering area does not depend linearly on the area $\lambda$ of the rectangle; it also depends on the rectangle's aspect ratio. Figure 2 shows a plot of the dependency of the critical covering density $d^*(\lambda) := \frac{A^*(\lambda)}{\lambda}$, i.e., the amount of disk area required per rectangle area, on $\lambda$. In the following, to simplify notation, we factor out $\pi$ if possible; instead of working with the areas $A(D)$ or $A^*(\lambda)$ of the disks, we use their *weight* $W(D)$, i.e., their area divided by $\pi$. Similarly, we work with the covering coefficient $E^*(\lambda) := \frac{d^*(\lambda)}{\pi}$ instead of the density $d^*(\lambda)$; a lower covering coefficient corresponds to a more efficient covering.

As shown in Figure 2, the critical covering coefficient $E^*(\lambda)$ is monotonically decreasing from $\lambda = 1$ to $\sqrt{2}$ and monotonically increasing for $\lambda > \sqrt{2}$. For a square, $E^*(1) = \frac{195}{256}$; the point $\lambda > 1$ for which the covering coefficient becomes as bad as for the square is $\overline{\lambda} := \frac{195+\sqrt{5257}}{128} \approx 2.08988\ldots$; for all $\lambda \le \overline{\lambda}$, the covering coefficient is at most $\frac{195}{256}$.

### 2.1 Proof Components

The proof of Theorem 2.1 uses a number of components. First is a lemma that describes the worst-case configurations and shows tightness, i.e., claim (1), of Theorem 2.1 for all $\lambda$.

▶ **Lemma 2.2.** *Let $\lambda \geq 1$ and let $\mathcal{R}$ be a rectangle of dimensions $\lambda \times 1$. (1) Two disks of weight $r_1^2 = \frac{\lambda^2+1}{4}$ and $r_2^2 = \frac{1}{4}$ suffice to cover $\mathcal{R}$. (2) For any $\varepsilon > 0$, two disks of weight $r_1^2 - \varepsilon$ and $r_2^2$ do not suffice to cover $\mathcal{R}$. (3) Three identical disks of weight $r^2 = \frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}$ suffice to cover a rectangle $\mathcal{R}$ of dimensions $\lambda \times 1$. (4) For $\lambda \leq \lambda_2$ and any $\varepsilon > 0$, three identical disks of weight $r_-^2 := r^2 - \varepsilon$ do not suffice to cover $\mathcal{R}$.*

For large $\lambda$, the critical covering coefficient $E^*(\lambda)$ of Theorem 2.1 becomes worse, as large disks cannot be used to cover the rectangle efficiently. If the weight of each disk is bounded by some $\sigma \geq r_1^2$, we provide the following lemma achieving a better covering coefficient $E(\sigma)$ with $E^*(\overline{\lambda}) \leq E(\sigma) \leq E^*(\lambda)$. This coefficient is independent of the aspect ratio of $\mathcal{R}$.

▶ **Lemma 2.3.** *Let $\hat{\sigma} := \frac{195\sqrt{5257}}{16384} \approx 0.8629$. Let $\sigma \geq \hat{\sigma}$ and $E(\sigma) := \frac{1}{2}\sqrt{\sqrt{\sigma^2+1}+1}$. Let $\lambda \geq 1$ and $D = \{r_1, \dots, r_n\}$ be any collection of disks with $\sigma \geq r_1^2 \geq \dots \geq r_n^2$ and $W(D) = \sum_{i=1}^{n} r_i^2 \geq E(\sigma)\lambda$. Then $D$ can cover a rectangle $\mathcal{R}$ of dimensions $\lambda \times 1$.*

Note that $E(\hat{\sigma}) = \frac{195}{256}$ is the best covering coefficient established by Lemma 2.3, coinciding with the critical covering coefficient of the square established by Theorem 2.1. Thus, we can cover any rectangle with covering coefficient $\frac{195}{256}$ if the largest disk satisfies $r_1^2 \leq \hat{\sigma}$.

The final component is the following Lemma 2.4, which also gives a better covering coefficient if the size of the largest disk is bounded. The bound on the largest radius that is required for Lemma 2.4 is smaller than for Lemma 2.3; in return, the covering coefficient that Lemma 2.4 yields is better. We remark that the result of Lemma 2.4 is not tight.

▶ **Lemma 2.4.** *Let $\lambda \geq 1$ and let $\mathcal{R}$ be a rectangle of dimensions $\lambda \times 1$. Let $D = \{r_1, \dots, r_n\}$, $0.375 \geq r_1 \geq \dots \geq r_n > 0$ be a collection of disks. If $W(D) \geq 0.61\lambda$, or equivalently $A(D) \geq 0.61\pi\lambda \approx 1.9164\lambda$, then $D$ suffices to cover $\mathcal{R}$.*

## 2.2 Proof Overview

The proofs of Theorem 2.1 and Lemmas 2.3 and 2.4 work by induction on the number of disks. For proving Lemma 2.3 for $n$ disks, we use Theorem 2.1 for $n$ disks. For proving Theorem 2.1 for $n$ disks, we use Lemma 2.4 for $n$ disks; Lemma 2.3 is only used for fewer than $n$ disks; see Figure 4. For proving Lemma 2.4 for $n$ disks, we only use Theorem 2.1 and Lemma 2.3 for fewer than $n$ disks. Therefore, there are no cyclic dependencies in our argument; however, we have to perform the induction for Theorem 2.1 and Lemmas 2.3 and 2.4 simultaneously.

**Strategies.** The proofs of Theorem 2.1 and Lemma 2.4 are constructive; they are based on an efficient recursive algorithm that uses a set of simple *strategies*. We go through the list of strategies in some fixed order. For each strategy, we check a sufficient criterion for the strategy to work. We call these criteria *success criteria*. They only depend on the total available weight and a constant number of largest disks. If we cannot guarantee that a strategy works by its success criterion, we simply disregard the strategy; this means that our algorithm does not have to backtrack. We prove that, regardless of the distribution of the disks' weight, at least one success criterion is met, implying that we can always apply at least one strategy. The number of strategies and thus success criteria is large — more than 40 strategies considering over 500 combinatorially different placements of the largest disks, which would presumably need to be considered in a manual analysis. This is where the need for automatic assistance comes from.

**Recursion.** Typical strategies are recursive; they consist of splitting the collection of disks into smaller parts, splitting the rectangle accordingly, and recursing, or recursing after fixing the position of a constant number of large disks.

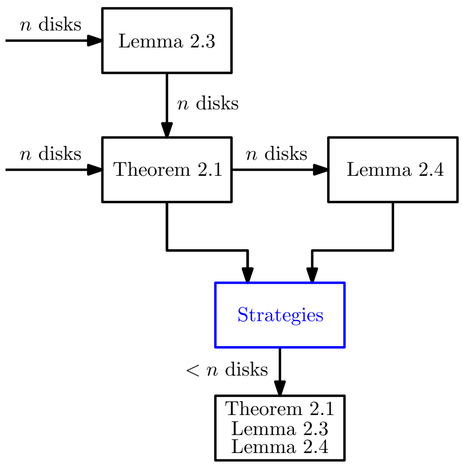


**Figure 4** The inductive structure of the proof; the blue parts are computer-aided.

In the entire remaining proof, the criterion we use to guarantee that recursion works is as follows. Given a collection $D' \subsetneq D$ and a rectangular region $\mathcal{R}' \subsetneq \mathcal{R}$, we check whether the preconditions of Theorem 2.1 or Lemma 2.3 or 2.4 are met after appropriately scaling and rotating $\mathcal{R}'$ and the disks. Note that, due to the scaling, the radius bounds of Lemmas 2.3 and 2.4 depend on the length of the shorter side of $\mathcal{R}'$. In some cases where we apply recursion, we have more weight than necessary to satisfy the weight requirement for recursion according to Lemma 2.3 or 2.4, but these lemmas cannot be applied due to the radius bound. In that case, we also check whether we can apply Lemma 2.3 or 2.4 after increasing the length of the shorter side of $\mathcal{R}'$ as far as the disk weight allows. This excludes the case that we cannot recurse on $\mathcal{R}'$ due to the radius bound, but there is some $\mathcal{R}'' \supset \mathcal{R}'$ on which we could recurse.

## 2.3 Interval Arithmetic

We use interval arithmetic to prove that there always is a strategy that works. In interval arithmetic, operations like addition, multiplication or taking a square root are performed on intervals $[a, b] \subset \mathbb{R}$ instead of numbers. Arithmetic operations on intervals are derived from their real counterparts as follows. The result of an operation $\circ$ in interval arithmetic is

$$[a_1, b_1] \circ [a_2, b_2] := \left[ \min_{x_1 \in [a_1,b_1], x_2 \in [a_2,b_2]} x_1 \circ x_2, \max_{x_1 \in [a_1,b_1], x_2 \in [a_2,b_2]} x_1 \circ x_2 \right].$$

Thus, the result of an operation is the smallest interval that contains all possible results of $x \circ y$ for $x \in [a_1, b_1], y \in [a_2, b_2]$. Unary operations are defined analogously.

# 3 Conclusion

Our worst-case values correspond to instances with only 2 or 3 relatively large disks; if we have an upper bound $R$ on the size of the largest disk, this gives rise to the critical covering area $A^*_R(\lambda)$ for $\lambda \times 1$-rectangles. Getting some tight bounds on $A^*_R(\lambda)$ would be interesting and useful. Establishing the critical covering density for disks and triangles is also open. We are optimistic that an approach similar to the one of this paper can be used for a solution.

*Computing* optimal coverings by disks is quite difficult. Deciding whether a given collection of disks can be packed into a unit square, is known to be NP-hard [12], the complexity of deciding whether a given set of disks can be used to cover a unit square is still open.

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