Covering Rectangles by Disks: The Video

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Abstract

In this video, we motivate and visualize a fundamental result for covering a rectangle by a set of non-uniform circles: For any $\lambda \geq 1$, the critical covering area $A^*(\lambda)$ is the minimum value for which any set of disks with total area at least $A^*(\lambda)$ can cover a rectangle of dimensions $\lambda \times 1$. We show that there is a threshold value $\lambda_2 = \sqrt{\sqrt{7}/2 - 1/4} \approx 1.035797\ldots$, such that for $\lambda < \lambda_2$ the critical covering area $A^*(\lambda)$ is $A^*(\lambda) = 3\pi\left(\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}\right)$, and for $\lambda \geq \lambda_2$, the critical area is $A^*(\lambda) = \pi(\lambda^2 + 2)/4$; these values are tight. For the special case $\lambda = 1$, i.e., for covering a unit square, the critical covering area is $\frac{195\pi}{256} \approx 2.39301\ldots$. We describe the structure of the proof, and show animations of some of the main components.

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Related Version This contribution visualizes the main result of paper [1], which is part of SoCG 2020.

Supplement Material `https://github.com/phillip-keldenich/circlecover`

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1 Introduction

Given a collection of (not necessarily equal) disks, is it possible to arrange them so that they completely cover a given region, such as a square or a rectangle? Problems of this type have a variety of applications, but are notoriously difficult; see our related conference paper [1] for a more detailed overview.

In this contribution, we illustrate a fundamental result: If the total area of the disks is sufficiently large, they can always cover the region. More precisely, for any given λ, we identify the minimum value $A^*(\lambda)$ for which any collection of disks with total area at least $A^*(\lambda)$ can cover a rectangle of dimensions $\lambda \times 1$. We call $A^*(\lambda)$ the *critical covering area* for $\lambda \times 1$ rectangles and give a complete and tight characterization, along with a visual illustration of the involved proof techniques.

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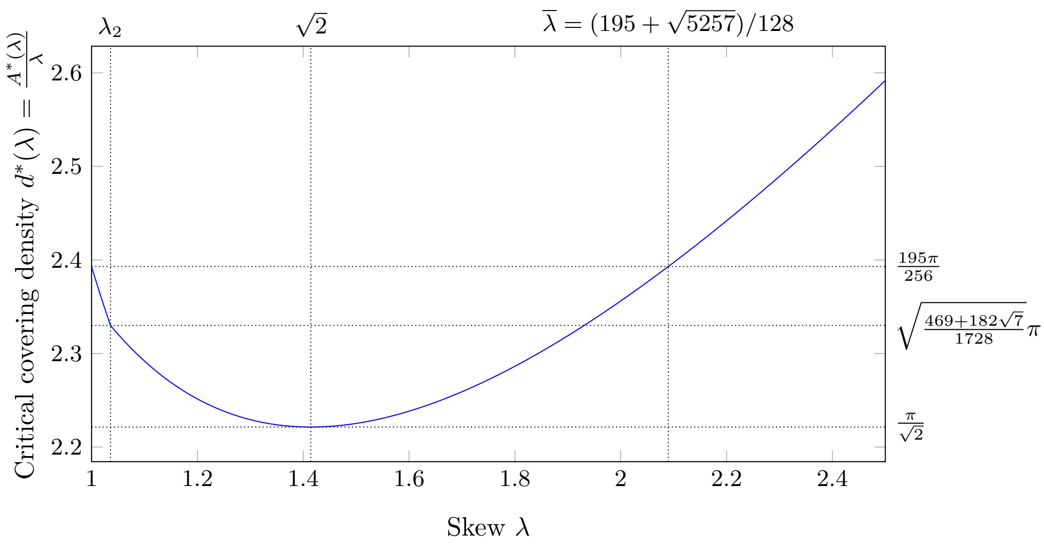

Figure 1 The critical covering density $d^*(\lambda)$ depending on λ and its values at the threshold value λ_2, the global minimum at $\sqrt{2}$ and the skew $\overline{\lambda}$ at which the density becomes as bad as for the square.

▶ **Theorem 1.** *Let $\lambda \geq 1$ and let $\mathcal{R}$ be a rectangle of dimensions $\lambda \times 1$. Let*

$$\lambda_2 = \sqrt{\frac{\sqrt{7}}{2} - \frac{1}{4}} \approx 1.035797\ldots, \text{ and } A^*(\lambda) = \begin{cases} 3\pi\left(\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}\right), & \text{if } \lambda < \lambda_2, \\ \pi\frac{\lambda^2+2}{4}, & \text{otherwise.} \end{cases}$$

(1) *For any $a < A^*(\lambda)$, there is a set D^- of disks with $A(D^-) = a$ that cannot cover $\mathcal{R}$.*

(2) *Let $D = \{r_1, \ldots, r_n\} \subset \mathbb{R}$, $r_1 \geq r_2 \geq \ldots \geq r_n > 0$ be any collection of disks identified by their radii. If $A(D) \geq A^*(\lambda)$, then D can cover $\mathcal{R}$.*

See Figure 1 for a graph showing the (normalized) critical covering area, called critical covering density $d^*(\lambda) = {A^*(\lambda)}/{\lambda}$, and Figure 2 for examples of worst-case configurations. The point $\lambda = \lambda_2$ is the unique real number greater than 1 for which the two bounds $3\pi\left(\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}\right)$ and $\pi\frac{\lambda^2+2}{4}$ coincide; see Figure 1. At this so-called *threshold value*, the worst case changes from three identical disks to two disks, which are the circumcircle $r_1^2 = \frac{\lambda^2+1}{4}$ and a disk $r_2^2 = \frac{1}{4}$; see Figure 2. The intuition behind the behavior of $d^*(\lambda)$ is as follows. The three-disk worst case is bad due to the fact that one of the three disks has to cover an entire edge of the rectangle. The efficiency of this placement improves when λ increases, because the size of the largest disk increases as well, while the length of the shorter edge remains constant. For the two-disk worst case, increasing λ initially improves the density, because the constant area contributed by the second disk becomes less significant. After this initial improvement, the quadratic growth of the largest disk compared to the linear growth of the rectangle dominates, leading to an overall linear increase in density.

2 High-level description

As shown in the video and illustrated in Figure 3, the proof consists of several components. In addition, there are a number of lemmas, which we describe first for easier reference.

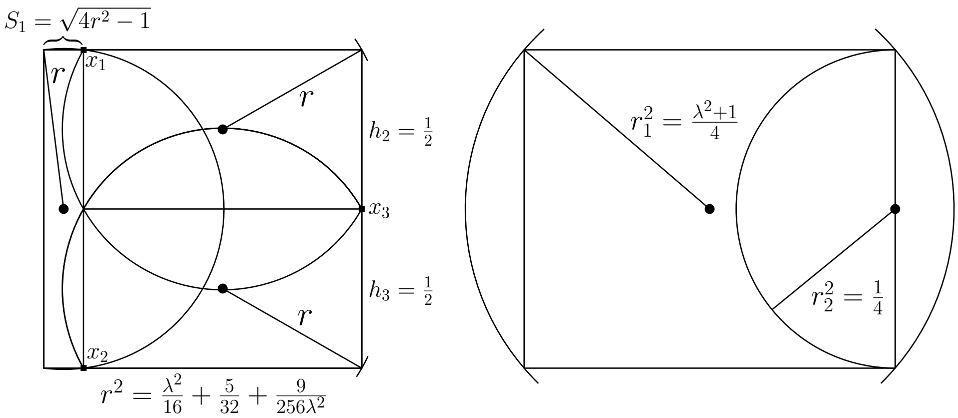

Figure 2 Worst-case configurations for small $\lambda \le \lambda_2$ (left) and for large skew $\lambda \ge \lambda_2$ (right).

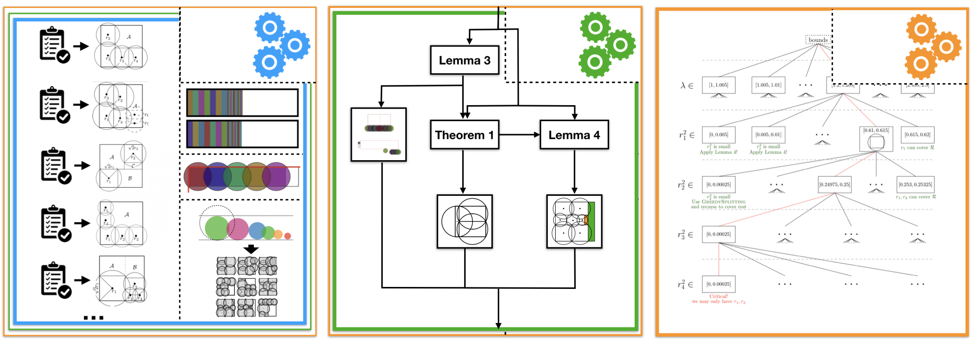

Figure 3 The different proof components. (Left) Individual covering routines. (Center) Recursive logic of the overall algorithmic approach. (Right) Case analysis for the computer-assisted proof.

2.1 Mathematical components

First is a lemma that describes the worst cases and shows tightness of our result.

▶ **Lemma 2.** *Let $\lambda \ge 1$ and let $\mathcal{R}$ be a rectangle of dimensions $\lambda \times 1$. (1) Two disks of radius $r_1 = \sqrt{\frac{\lambda^2+1}{4}}$ and $r_2 = \frac{1}{2}$ suffice to cover $\mathcal{R}$. (2) For any $\varepsilon > 0$, two disks of radius $r_1 - \varepsilon$ and r_2 do not suffice to cover $\mathcal{R}$. (3) Three identical disks of radius $r = \sqrt{\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}}$ suffice to cover $\mathcal{R}$. (4) For $\lambda \le \lambda_2$ and any $\varepsilon > 0$, three identical disks of radius $r_- := r - \varepsilon$ do not suffice to cover $\mathcal{R}$.*

For large λ, the *critical covering coefficient* $E^*(\lambda) := \frac{A^*(\lambda)}{\lambda\pi}$ of Theorem 1 becomes worse, as large disks cannot be used to cover the rectangle efficiently. If the *weight*, i.e., squared radius, of each disk is bounded by some $\sigma \ge r_1^2$, we provide the following lemma achieving a better covering coefficient $E(\sigma)$ for large λ.

▶ **Lemma 3.** *Let $\hat{\sigma} := \frac{195\sqrt{5257}}{16384} \approx 0.8629$. Let $\sigma \ge \hat{\sigma}$ and $E(\sigma) := \frac{1}{2}\sqrt{\sqrt{\sigma^2+1}+1}$. Let $\lambda \ge 1$ and $D = \{r_1, \ldots, r_n\}$ be any collection of disks with $\sigma \ge r_1^2 \ge \ldots \ge r_n^2$ and $W(D) := \sum_{i=1}^{n} r_i^2 \ge E(\sigma)\lambda$. Then D can cover a rectangle $\mathcal{R}$ of dimensions $\lambda \times 1$.*

The final component is the following Lemma 4, which also gives a better covering coefficient if the size of the largest disk is bounded. The bound required for Lemma 4 is smaller than for Lemma 3, yielding a better covering coefficient in return.

▶ **Lemma 4.** *Let $\lambda \geq 1$ and let $\mathcal{R}$ be a rectangle of dimensions $\lambda \times 1$. Let $D = \{r_1, \ldots, r_n\}$, $0.375 \geq r_1 \geq \ldots \geq r_n > 0$ be a collection of disks. If $W(D) \geq 0.61\lambda$, or equivalently $A(D) \geq 0.61\pi\lambda \approx 1.9164\lambda$, then D suffices to cover $\mathcal{R}$.*

2.2 Proof overview

The proofs of Theorem 1 and Lemmas 3 and 4 work by induction on the number of disks. For proving Lemma 3 for n disks, we use Theorem 1 for n disks. For proving Theorem 1 for n disks, we use Lemma 4 for n disks; Lemma 3 is only used for fewer than n disks. For proving Lemma 4 for n disks, we only use Theorem 1 and Lemma 3 for fewer than n disks. Therefore, there are no cyclic dependencies in our argument; however, we have to perform the induction for Theorem 1 and Lemmas 3 and 4 simultaneously.

The proofs of our result are constructive; they are based on an efficient recursive algorithm that uses a set of simple *routines*. These routines were derived by hand, in many cases based on problematic instances that were identified by the automatic prover and could not be handled by the routines that were already present. We go through the list of routines in some fixed order. For each routine, we check a sufficient criterion for the routine to work. We call these criteria *success criteria*. They only depend on the total available weight and a constant number of largest disks. If we cannot guarantee that a routine works by its success criterion, we simply disregard the routine; this means that our algorithm does not have to backtrack. We prove that, regardless of the distribution of the disks' weight, at least one success criterion is met, implying that we can always apply at least one routine. The number of routines and thus success criteria is large; this is where the need for automatic assistance comes from.

Typical routines are recursive; they consist of splitting the collection of disks into smaller parts, splitting the rectangle accordingly, and recursing, or recursing after fixing the position of a constant number of large disks. As a success criterion for recursion, we check whether Theorem 1 or Lemma 3 or 4 can be applied.

2.3 Interval arithmetic

We use interval arithmetic to prove that there always is a routine that works. In interval arithmetic, operations like addition, multiplication or taking a square root are performed on intervals $[a, b] \subset \mathbb{R}$ instead of numbers. After proving our result manually for large λ, this allows us to check a finite, discrete set of cases, instead of the continuum of all possible radii and λ. See our main paper [1] for details.

3 The video

The video starts with a motivation of the basic problem of covering a rectangle by disks, followed by a description of the main result. After an overview of the main three aspects of the proof (individual covering routines, recursive logic, case analysis), these are explained and illustrated in detail.

References

1 S. P. Fekete, U. Gupta, P. Keldenich, C. Scheffer, and S. Shah. Worst-Case Optimal Covering of Rectangles by Disks. In *Proceedings 36th International Symposium on Computational Geometry (SoCG 2020)*, pages 42:1–42:19, 2020.