Online Circle Packing

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Abstract. We consider the online problem of packing circles into a square container. A sequence of circles has to be packed one at a time, without knowledge of the following incoming circles and without moving previously packed circles. We present an algorithm that packs any online sequence of circles with a combined area not larger than 0.350389 of the square's area, improving the previous best value of $\pi/10 \approx 0.31416$; even in an offline setting, there is an upper bound of $\pi/(3+2\sqrt{2}) \approx 0.5390$. If only circles with radii of at least 0.026622 are considered, our algorithm achieves the higher value 0.375898.

As a byproduct, we give an online algorithm for packing circles into a $1 \times b$ rectangle with $b \geq 1$. This algorithm is worst case-optimal for $b \geq 2.36$.

Keywords: Circle packing · Online algorithms · Packing density

1 Introduction

Packing a set of circles into a given container is a natural geometric optimization problem that has attracted considerable attention, both in theory and practice. Some of the many real-world applications are loading a shipping container with pipes of varying diameter [10], packing paper products like paper rolls into one or several containers [9], machine construction of electric wires [19], designing control panels [2], placing radio towers with a maximal coverage while minimizing interference [20], industrial cutting [20], and the study of macro-molecules or crystals [21]. See the survey paper of Castillo, Kampas, and Pintér [2] for an overview of other industrial problems. In many of these scenarios, the circles have to be packed *online*, i.e., one at a time, without the knowledge of further objects, e.g., when punching out a sequence of shapes from the raw material.

Even in an offline setting, deciding whether a given set of circles fits into a square container is known to be NP-hard [4], which is also known for packing squares into a square [11]. Furthermore, dealing with circles requires dealing with possibly complicated irrational numbers, incurring very serious additional geometric difficulties. This is underlined by the slow development of provably

A full version of this extended abstract is available at [8].

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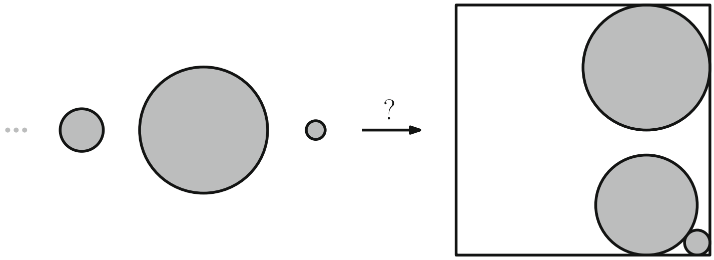

Fig. 1. Circles are arriving one at a time and have to be packed into the unit square. At this stage, the packed area is about 0.33. What is the largest $A \geq 0$ for which *any* sequence of total area A can be packed?

optimal packings of n identical circles into the smallest possible square. In 1965, Schaer [17] gave the optimal solution for $n = 7$ and $n = 8$ and Schaer and Meir [18] gave the optimal solution for $n = 9$. A quarter of a century later, Würtz et al. [3] provided optimal solutions for 10,11,12, and 13 equally sized circles. In 1998, Nurmela and Östergård [15] provided optimal solutions for $n \leq 27$ circles by making use of computer-aided optimality proofs. Markót and Csendes [13] gave optimal solutions for $n = 28, 29, 30$ also by using computer-assisted proofs within tight tolerance values. Finally, in 2002 optimal solutions were provided for $n \leq 35$ by Locatelli and Raber [12]; at this point, this is still the largest n for which optimal packings are known. The extraordinary challenges of finding densest circle packings are also underlined by a long-standing open conjecture by Erdős and Oler from 1961 [16] regarding optimal packings of n unit circles into an equilateral triangle, which has only been proven up to $n = 15$.

These difficulties make it desirable to develop relatively simple criteria for the packability of circles. A natural bound arises from considering the *packing density*, i.e., the total area of objects compared to the size of the container; the *critical packing density* δ is the largest value for which any set of objects of total area at most δ can be packed into a unit square; see Fig. 1.

In an offline setting, two equally sized circles that fit exactly into the unit square show that $\delta \leq \delta^* = \pi/(3 + 2\sqrt{2}) \approx 0.5390$. This is indeed tight: Fekete, Morr and Scheffer [7] gave a worst-case optimal algorithm that packs any instance with combined area at most δ^*; see Fig. 2 (left). More recently, Fekete, Keldenich and Scheffer [6] established 0.5 as the critical packing density of circles in a circular container.

The difficulties of offline circle packing are compounded in an online setting. This is highlighted by the situation for packing squares into a square, which does not encounter the mentioned issues with irrational coordinates. It was shown in 1967 by Moon and Moser [14] that the critical offline density is 0.5: Refining an approach by Fekete and Hoffmann [5], Brubach [1] established the currently best lower bound for online packing density of 0.4. This yields the previous best bound for the online packing density of circles into a square: Inscribing circles into bounding boxes yields a value of $\pi/10 \approx 0.3142$.

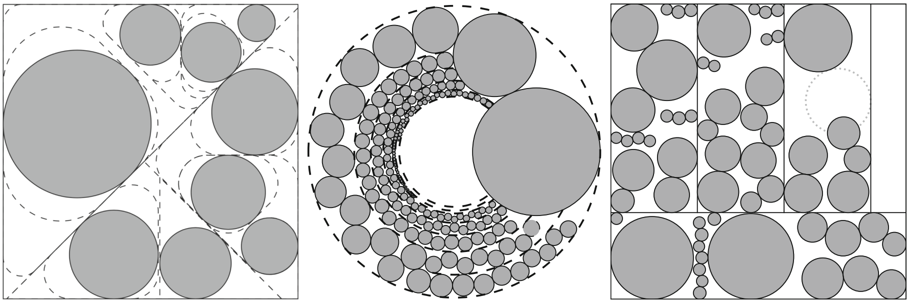

Fig. 2. Examples of algorithmic circle packings. (Left) The worst-case optimal offline algorithm of Fekete et al. [7] for packing circles into the unit square. (Center) The worst-case optimal offline algorithm of Fekete et al. [6] for packing circles into the unit circle. (Right) Our online algorithm for packing circles into the unit square.

1.1 Our Results

In this paper, we establish new lower bounds for the online packing density of circles into a square and into a rectangle. Note that in the online setting, a packing algorithm has to stop as soon as it cannot pack a circle. We provide three online circle packing results for which we provide constructive proofs, i.e., corresponding algorithms guaranteeing the claimed packing densities.

Theorem 1. *Let $b \geq 1$. Any online sequence of circles with a total area no larger than* $\min\left(0.528607 \cdot b - 0.457876, \frac{\pi}{4}\right)$ *can be packed into the $1 \times b$-rectangle R. This is worst-case optimal for $b \geq 2.36$.*

We use the approach of Theorem 1 as a subroutine and obtain the following:

Theorem 2. *Any online sequence of circles with a total area no larger than* 0.350389 *can be packed into the unit square.*

If the incoming circles' radii are lower bounded by 0.026623, the density guaranteed by the algorithm of Theorem 2 improves to 0.375898.

Theorem 3. *Any online sequence of circles with radii not smaller than* 0.026623 *and with a total area no larger than* 0.375898 *can be packed into the unit square.*

We describe the algorithm of Theorem 1 in Sect. 2 and the algorithm of the Theorems 2 and 3 in Sect. 3.

2 Packing into a Rectangle

In this section, we describe the algorithm, *Double-Sided Structured Lane Packing (DSLP)*, of Theorem 1. In particular, DSLP uses a packing strategy called *Structured Lane Packing (SLP)* and an extended version of SLP as subroutines.

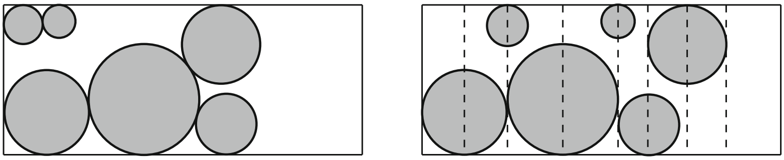

Fig. 3. Comparison of Tight Lane Packing (left) with Structured Lane Packing (right) for the same input. The former has a smaller packing length.

2.1 Preliminaries for the Algorithms

A *lane* $L \subset \mathbb{R}^2$ is an x- and y-axis-aligned rectangle. The *length* $\ell(L)$ and the *width* $w(L)$ of L are the dimensions of L such that $w(L) \leq \ell(L)$. L is *horizontal* if the length of L is given via the extension of L w.r.t. the x-axis. Otherwise, L is *vertical*. The *distance* between two circles packed into L is the distance between the orthogonal projections of the circles' midpoints onto the longer side of L. A lane is either *open* or *closed*. Initially, each lane is open.

Packing a circle C into a lane L means placing C inside L such that C does not intersect another circle that is already packed into L or into another lane. A *(packing) strategy* for a lane L is a set of rules that describe how a circle has to be packed into L. The *(packing) orientation* of a strategy for a horizontal lane is either *rightwards* or *leftwards* and the *(packing) orientation* of a strategy for a vertical lane is either *downwards* or *updwards*.

Let w be the width of L. Depending on the radius r of the current circle C, we say: C is *medium* (Class 1) if $w > r \geq \frac{w}{4}$, C is small if $\frac{w}{4} > r \geq 0.0841305w$ (Class 2), C is *tiny* (Class 3 or 4) if $0.0841305w > r \geq 0.023832125w$, and C is *very tiny* if $0.023832125w > r$ (Classes 5,6, ...). For a more refined classification of r, we refer to Sect. 2.3. The general idea is to reach a certain density within a lane by packing only relatively equally sized circles into a lane with SLP.

For the rest of Sect. 2, for $0 < w \leq b$, let L be a horizontal $w \times b$ lane.

2.2 Structured Lane Packing (SLP) – The Standard Version

Rightwards *Structured Lane Packing (SLP)* packs circles into L alternating touching the bottom and the top side of L from left to right, see Fig. 3 (right).

In particular, we pack a circle C into L as far as possible to the left while guaranteeing: (1) C does not overlap a vertical lane packed into L, see Sect. 2.3 for details[1]. (2) The distance between C and the circle C' packed last into L is at least $\min\{r, r'\}$ where r, r' are the radii of C, C', see Fig. 3 (right).

Leftwards *Structured Lane Packing* packs circles by alternatingly touching the bottom and the top side of L from right to left. Correspondingly, upwards and downwards *Structured Lane Packing* packs circles alternatingly touching the left and the right side of L from bottom to top and from top to bottom.

[1] Requiring that C does not overlap a vertical lane placed inside L is only important for the extension of SLP (see Sect. 2.3), because the standard version of SLP does not place vertical lanes inside L.

2.3 Extension of SLP – Filling Gaps by (Very) Tiny Circles

Now consider packing medium circles with SLP. We extend SLP for placing tiny and very tiny circles within the packing strategy, see Fig. 4. Note that small circles are not considered for the moment, such that (very) tiny circles are relatively small compared to the medium ones. In particular, if the current circle C is medium, we apply the standard version of SLP, as described in Sect. 2.2. If C is (very) tiny, we pack C into a vertical lane inside L, as described next.

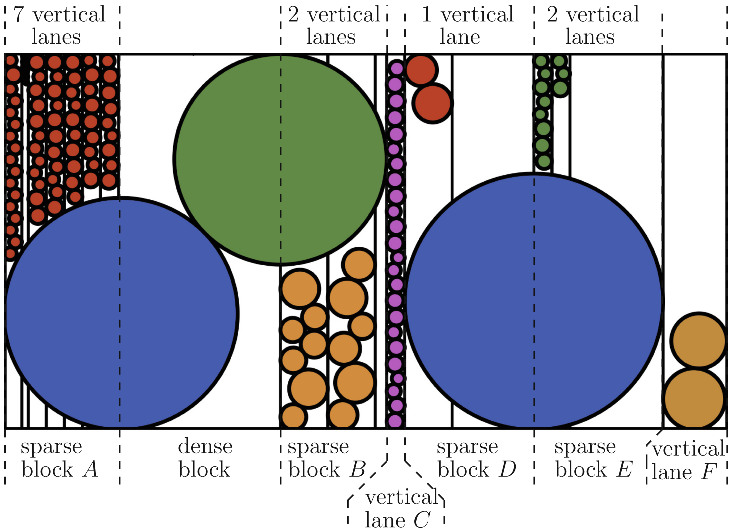

Fig. 4. A packing produced by extended SLP: 128 (3 medium, 17 tiny, and 108 very tiny) circles packed into 15 (1 medium, 4 tiny and 10 very tiny) lanes. A possible input order of the circles is 1 medium circle, 23 very tiny circles (filling the sparse block A), 1 medium circle, 13 tiny circles (filling the sparse block B), 24 very tiny circles (filling the vertical lane C), 1 medium circle, 2 tiny circles (filling the sparse block D), 11 very tiny circles (filling the sparse block E), and 2 tiny circles (filling the vertical lane F). (Color figure online)

We pack (very) tiny circles into vertical lanes inside L, see Fig. 4. The vertical lanes are placed inside *blocks* that are the rectangles induced by vertical lines touching medium circles already packed into L, see Fig. 3 (right).

Blocks that include two halves of medium circles are called *dense* blocks, while blocks that include one half of a medium circle are called *sparse* blocks. The area of L that is neither covered by a dense block or a sparse block is called a *free* block. *Packing* a vertical lane L' into a sparse block B means placing L' inside B as far as possible to the left, such that L' does not overlap another vertical lane already packed into B. Packing a vertical lane L' into L means placing L' inside L as far as possible to the left, such that L' does neither overlap another vertical lane packed into L, a dense block of L, or a sparse block of L.

We extend our classification of circles by defining classes i of lane widths w_i and *relative lower bounds* q_i for the circles' radii as described in Table 1. This means a circle with radius r belongs to class 1 if $0.5w \geq r > q_1 w_1$ and to class i if $q_{i-1}w_{i-1} \geq r > q_i w_i$, for $i \geq 2$. Only circles of class i are allowed to be packed into lanes of class i.

Table 1. Circles are classified into the listed classes. Note that the lower bounds to the circles' radii is relative to the lane width, e.g., the absolute lower bound for circles inside a small lane is $q_S w_2 = 0.168261 w_2$.

Class i	(Relative) lower bound q_i	Lane width w_i
1 (Medium)	$q_1 := q_M := 0.25$	$w_1 := w$
2 (Small)	$q_2 := q_S := 0.168261$	$w_2 := 2q_M w = 0.5 \cdot w$
3 (Tiny)	$q_3 := 0.371446$	$w_3 := 4q_M q_S w = 0.168261 \cdot w$
4 (Tiny)	$q_4 := 0.190657$	$w_4 := 8q_M q_S q_3 w \approx 0.125 \cdot w$
5 (Very tiny)	$q_5 := 0.175592$	$w_5 := 16q_M q_S q_3 q_4 w \approx 0.047664 \cdot w$
6 (Very tiny)	$q_6 := 0.170699$	$w_6 := 32q_M q_S q_3 q_4 q_5 w \approx 0.016739 \cdot w$
7 (Very tiny)	$q_7 := 0.169078$	$w_7 \approx 0.005715 \cdot w$
8 (Very tiny)	$q_8 := 0.168354$	$w_8 \approx 0.001932 \cdot w$
9 (Very tiny)	$q_9 := 0.168293$	$w_9 \approx 0.000651 \cdot w$
10 (Very tiny)	$q_{10} := 0.168272$	$w_{10} \approx 0.000219 \cdot w$
11 (Very tiny)	$q_{11} := 0.168265$	$w_{11} \approx 0.000074 \cdot w$
12 (Very tiny)	$q_{12} := 0.168263$	$w_{12} \approx 0.000025 \cdot w$
13 (Very tiny)	$q_{13} := 0.168262$	$w_{13} \approx 0.000008 \cdot w$
...	...	...
k (Very tiny)	$q_k := 0.168262$	$w_k := 2^{k-1} q_M q_S q_3 q_4 \cdot \ldots \cdot q_{k-1} w$

A sparse block is either *free*, *reserved* for class 3, *reserved* for class 4, *reserved* for all classes $i \geq 5$, or *closed*. Initially, each sparse block is free.

We use SLP in order to pack a circle C of class $i \geq 3$, into a vertical lane $L_i \subset L$ of class i and width w_i by applying the Steps 1–5 in increasing order as described below. When one of the five steps achieves that C is packed into a vertical lane L_i, the approach stops and returns successful.

- **Step (1):** If there is no open vertical lane $L_i \subset L$ of class i go to Step 2. Assume there is an open vertical lane L_i of class i. If C can be packed into L_i, we pack C into L_i. Else, we declare L_i to be closed.
- **Step (2):** We close all sparse blocks B that are free or reserved for class i in which a vertical lane of class i cannot be packed into B.
- **Step (3):** If there is an open sparse block B that is free or reserved for class i and a vertical lane of class i can be packed into B:

- **(3.1):** We pack a vertical lane $L_i \subset L$ of class i into B. If the circle half that is included in B touches the bottom of L, we apply downwards SLP to L_i. Otherwise, we apply upwards SLP to L_i.
- **(3.2):** If B is free and $i \in \{3, 4\}$, we reserve B for class i. If B is free and $i \geq 5$, we reserve B for all classes $i \geq 5$.
- **(3.3):** We pack C into L_i.

– **Step (4):** If a vertical lane of class i can be packed into L:
- **(4.1):** We pack a vertical lane L_i of class i into L and apply upwards SLP to L_i.
- **(4.2):** We pack C into L_i

– **Step (5):** We declare L to be closed and return failed.

2.4 Double-Sided Structured Lane Packing (DSLP)

We use SLP as a subroutine in order to define our packing strategy *Double-Sided Structured Lane Packing (DSLP)* of Theorem 1. In particular, additionally to L, we consider two small lanes L^1, L^2 that partition L, see Fig. 5.

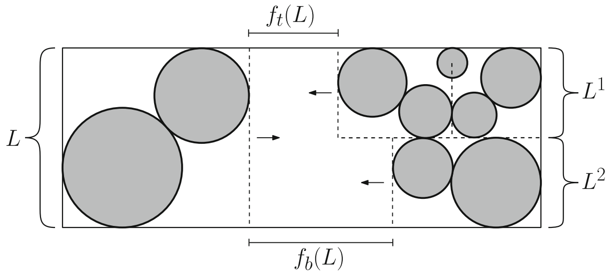

Fig. 5. A packing produced by DSLP: The medium lane is packed from left to right by medium circles. The two contained small lanes are packed simultaneously in parallel from right to left by small circles.

Rightwards *Double-Sided Structured Lane Packing (DSLP)* applies the extended version of rightwards SLP to L and leftwards SLP to L^1, L^2. If the current circle C is medium or (very) tiny, we pack C into L. If C is small, we pack C into that lane of L^1, L^2, resulting in a smaller packing length.

Leftwards DSLP is defined analogously, such that the extended version of leftwards SLP is applied to L and rightwards SLP to L^1, L^2. Correspondingly, upwards and downwards DSLP are defined for vertical lanes.

3 Packing into the Unit Square

We extend our circle classification by the class 0 of *large* circles and define a relative lower bound $q_0 := \frac{w}{2}$ and the lane width of corresponding *large* lanes as $w_0 := 1 - w$.

We set w to 0.288480 and 0.277927 for Theorem 2 respectively Theorem 3. In order to pack large circles, we use another packing strategy called *Tight Lane Packing (TLP)* defined as SLP, but without restrictions (1) and (2), see Fig. 3.

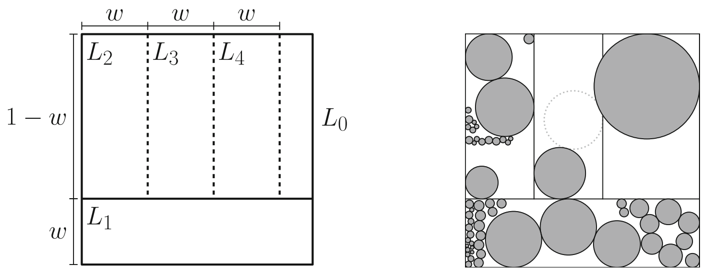

Fig. 6. Left: The unit square is divided into four lanes L_1, L_2, L_3, and L_4, into which medium, small, tiny, and very tiny circles are packed. Large circles are packed into a lane L_0 that overlaps L_1, L_2, and L_3. Right: An example packing. A medium circle (dotted) does not fit.

We cover the unit square by the union of one large lane L_0 and four medium lanes $L_1, \dots, L_k$ for $k = 4$, see Fig. 6. We apply TLP to L_0 and DSLP to $L_1, \dots, L_4$. The applied orientations for L_0, L_1, L_2 are leftwards, rightwards, and downwards. For $i = 3, 4$, the orientation for L_i is chosen such that the first circle packed into L_i is placed adjacent to the bottom side of L_i.

If the current circle to be packed is large, we pack C into the large lane L_0 and stop if C does not fit in L_0. Otherwise, in increasing order we try to pack C into $L_1, \dots, L_4$.

4 Analysis of the Algorithms

In this section we sketch the analysis of our approaches and refer to the appendix for full details. First, we analyze the packing density guaranteed by DSLP. Based on that, we prove our main results Theorems 1, 2, and 3.

4.1 Analysis of SLP

In this section, we provide a framework for analyzing the packing density guaranteed by DSLP for a horizontal lane L of width w. It is important to note that this framework and its analysis in this subsection deals with the packing of only one class into a lane.

We introduce some definitions. The *packing length* $p(L)$ is the maximal difference of x-coordinates of points from circles packed into L. The *circle-free length* $f(L)$ of L is defined as $\ell(L) - p(L)$. We denote the *total area* of a region $R \subset \mathbb{R}^2$ by $area(R)$ and the area of an $a \times b$-rectangle by $\mathcal{R}(a, b)$. Furthermore, we denote the area of a semicircle for a given radius r by $\mathcal{H}(r) := \frac{\pi}{2}r^2$. The total area of the circles packed into R is called *occupied area* denoted by $occ(R)$. Finally, the density $den(R)$ is defined by $occ(R)/area(R)$.

In order to apply our analysis for different classes of lanes, i.e., different lower bounds, we consider a general (relative) lower bound q for the radii of circles allowed to be packed into L with $0 < q \leq 1/2$. The following lemma deduces a lower bound for the density of dense blocks depending on q (Fig. 7).

Lemma 1. *Consider a dense block D containing two semicircles of radii r_1 and r_2 such that $0 < qw \leq r_1, r_2 \leq 1/2w$. Then $den(D)$ is lower-bounded by*

$$\delta : \left(0, \frac{1}{2}\right] \to \mathbb{R} \text{ with } q \mapsto \begin{cases} \pi q & 0 < q < \frac{1}{3\sqrt{3}} \\ \frac{\pi}{3\sqrt{3}} \approx 0.6046 & \frac{1}{3\sqrt{3}} \leq q \leq \frac{1}{3} \\ \frac{\pi q^2}{\sqrt{4q-1}} & \frac{1}{3} < q \leq \frac{1}{2}. \end{cases}$$

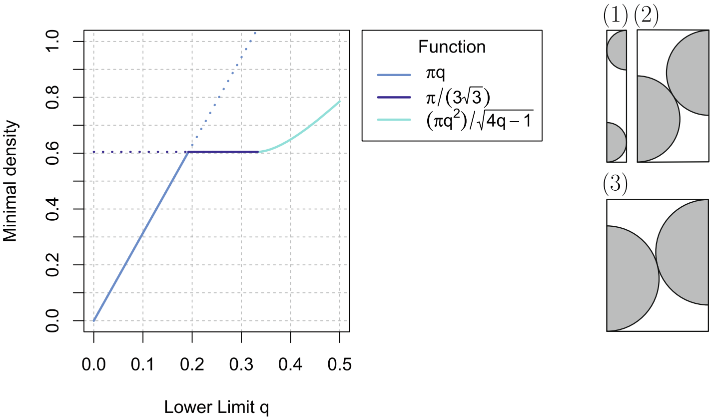

Fig. 7. (Left): A plot of $\delta(q)$ for its complete range. It provides the minimal density of a dense block whose two semicircles have a radius of at least $q \cdot w$. A lower bound of $q = 1/2$ leads to a minimal density of $\pi/4$ which is the ratio of a circle to its minimal bounding square. (Right): (1): $\delta(0.15) \approx 0.47123$, (2): $\delta(\frac{1}{3\sqrt{3}}) \approx 0.6046$, and (3): $\delta(0.4) \approx 0.6489$.

We continue with the analysis of sparse blocks. Sparse blocks have a minimum length qw. Lemma 2 states a lower bound for the occupied area of sparse blocks. This lower bound consists of a constant summand and a summand that is linear with respect to the actual length.

Lemma 2. *Given a density bound $\delta_{min} \leq \delta(q)$ for dense blocks. Let S be a sparse block and z be the lower bound for $\ell(S)$ with $\ell(S) \geq z \geq qw$. Then $occ(S) \geq \mathcal{R}(\ell(S) - z, w) \cdot \delta_{min} + \mathcal{H}(z)$.*

The occupied area of a sparse block is at least a semicircle of a smallest possible circle plus the remaining length multiplied by the lane width and by the minimal density δ_{min} of dense blocks. This composition is shown in Fig. 8.

Next, we combine the results of Lemmas 1 and 2. We define the term $min_{SLP}\big(p, w, z, \delta_{min}\big) := \mathcal{R}(p - 2z, w) \cdot \delta_{min} + 2 \cdot \mathcal{H}(z)$ for some $p, w, z, \delta_{min} > 0$ and state the following (see also Fig. 8 (4)).

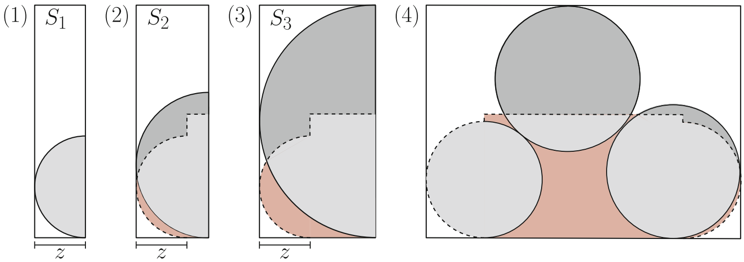

Fig. 8. (1)+(2)+(3) Three sparse blocks S_1, S_2, S_3 in order of ascending length. The grey coloured area (light and dark unified) represents the occupied area. The dashed area shows the lower bound of Lemma 2, which is composed of the smallest possible semicircle plus a linear part. The dark grey parts symbolize the area that exceeds the bound, whereas the red parts symbolize the area missing to the bound. Block S_1 has the minimal length p so that the occupied area and the bound are equal. For blocks of larger length, represented by S_2 and S_3, the dark grey area is larger than the red area. (4) A packing produced by SLP and the lower bound of Lemma 3.

Lemma 3. *Given a lane L packed by SLP, a lower bound q, and a density bound $\delta_{min} \leq \delta(q)$ for dense blocks. Let w be the width of L. The occupied area in L is lower-bounded by $min_{SLP}\big(p(L), w, qw, \delta_{min}\big)$.*

4.2 Analysis of DSLP

Let L be a horizontal lane packed by DSLP. We define $p_t(L)$ ($p_b(L)$) as the sum of the packing lengths of the packing inside L and the length of the packing inside the top (bottom) small lane inside L. Furthermore, we define $f_t(L) := \ell(L) - p_t(L)$ and $f_b(L) := \ell(L) - p_b(L)$, see Fig. 5.

By construction, vertical dense blocks packed into L have a density of at least $\delta(q_2)$. In fact, the definitions of circle sizes for all classes $i \geq 2$ ensure the common density bound $\hat{\delta} := \delta(q_2)$ for dense blocks.

We consider mixed dense blocks, that were defined as sparse blocks of L in which vertical lanes are packed, also as dense blocks and extend the lower bound $den(D) \geq \hat{\delta}$ to all kinds of dense blocks by the following Lemma.

Lemma 4. *Let D be a dense block of L. Assume all vertical lanes packed into D to be closed. Then $den(D) \geq \hat{\delta}$.*

As some vertical lanes may not be closed, we upper bound the error $\mathcal{O}(L)$ that we make by assuming that all vertical lanes $L_1, \ldots, L_n \subset L$ are closed.

Lemma 5. $\mathcal{O}(L_1 \cup \ldots \cup L_n) < 0.213297 \cdot w^2$.

We lower bound the occupied area inside L by using the following term:

$$min_{DSLP}(p_t(L), p_b(L), w, z, \delta_{min}) := \mathcal{R}\big(p_t(L) + p_b(L) - w - 4z, w_2\big) \cdot \delta_{min} + 2 \cdot \mathcal{H}\big(\frac{w}{4}\big) + 4\mathcal{H}(z),$$

where z denotes the minimal radius for the circles, i.e., $z := q_2 w_2$.

Applying Lemma 4 and Lemma 2 separately to L, L^1, and L^2, analogous to the combination of Lemmas 1 and 2 in the last subsection, and estimating the error $\mathcal{O}(L)$ with Lemma 5, yields lower bounds for the occupied areas of L, L^1, and L^2. Figure 9 separately illustrates the lower bounds for the occupied areas of L, L^1, and L^2 for two example packings and Lemma 6 states the result.

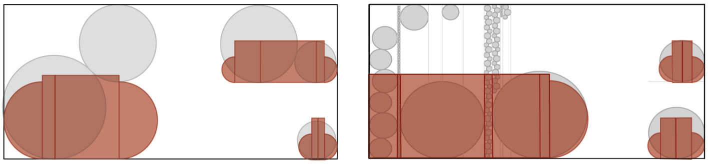

Fig. 9. Two example packings and the compositions of our lower bounds (red) for the occupied area implied by Lemma 6. Note that $\mathcal{O}(L)$ is not visualized. (Color figure online)

Lemma 6. $min_{DSLP}\big(p_t(L), p_b(L), w, z, \hat{\delta}\big) - \mathcal{O}(L) \geq occ(L)$.

4.3 Analysis of Packing Circles into a Rectangle

Given a $1 \times b$ rectangle R, we apply DSLP for packing the input circles into R.

Theorem 1. *Let $b \geq 1$. Any online sequence of circles with a total area no larger than* $\min\Big(0.528607 \cdot b - 0.457876, \frac{\pi}{4}\Big)$ *can be packed into the $1 \times b$-rectangle R. This is worst-case optimal for $b \geq 2.36$.*

The lower bound for the occupied area implied by Lemma 6 is equal to $\big(b - \frac{3}{4} - q_2\big) \cdot \hat{\delta} + \frac{\pi}{16} + \frac{\pi}{2}(q_2)^2 - 0.213297$. This is lower bounded by $\frac{\pi}{4}$ for $b \geq 2.36$. Hence, the online sequence consisting of one circle with a radius of $\frac{1}{2} + \epsilon$ and resulting total area of $\frac{\pi}{4} + \epsilon$ is a worst case online sequence for $b \geq 2.36$, see Fig. 10. This concludes the proof of Theorem 1.

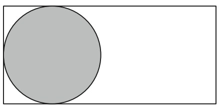

Fig. 10. A worst case for packing circles into an $1 \times b$-rectangle with $b \geq 2.36$ consists of one circle with radius $\frac{1}{2} + \epsilon$. The shown circle with radius $\frac{1}{2}$ just fits.

4.4 Analysis of Packing Circles into the Unit Square

In this section, we analyze the packing density of our overall approach for online packing circles into the unit square. In order to prove Theorem 3, i.e., a lower bound for the achieved packing density, we show that if there is an overlap or if there is no space in the last lane, then the occupied area must be at least this lower bound. Our analysis distinguishes six different higher-level cases of where and how the overlap can happen, see Fig. 11 left.

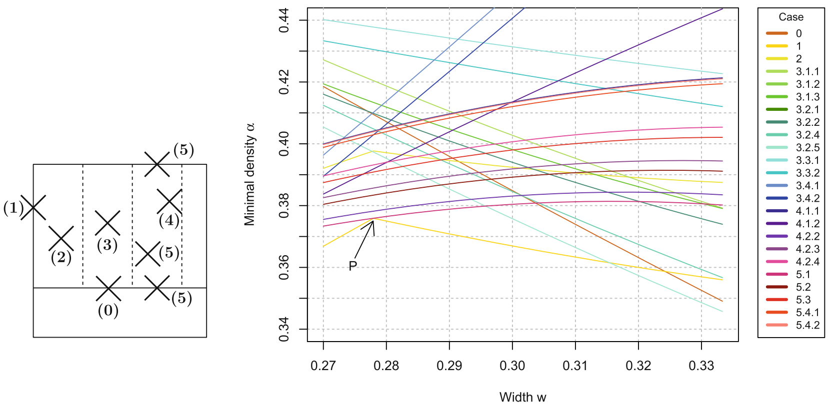

Fig. 11. (Left): Different cases for an overlap. Case 0: A single circle is too large for L_0. Case 1: L_0 exceeded. Case 2: Overlap in L_2. Case 3: Overlap in L_3. Case 4: Overlap in L_4 with large circle being involved. Case 5: Overlap in L_4 with no large circle being involved. (Right): A plot of 24 terms for corresponding 24 (sub-)cases for $q_S = 0.191578$. The point $P = (0.277927, 0.375898)$ is the highest point of the 0-level. Its y-value is the highest guaranteed packing density for circles with minimal radii of $0.191578 \cdot 0.277927/2 < 0.0266223$.

For each of the six cases and its subcases, we explicitly give a density function, providing the guaranteed packing density depending on the choice of w, see Fig. 11 right. The shown functions are constructed for the case of no (very) tiny circles with an alternative $q_2 = 0.191578$, which was chosen numerically in order to find a high provable density. The w with the highest guaranteed packing density of 0.375898 is $w = 0.277927$. This concludes the proof of Theorem 3.

Theorem 3. *Any online sequence of circles with radii not smaller than* 0.026623 *and with a total area no larger than* 0.375898 *can be packed into the unit square.*

With the same idea but with $w = 0.288480$ and all circles classes, especially with classes $i \geq 2$ as defined in Table 1, we prove Theorem 2.

Theorem 2. *Any online sequence of circles with a total area no larger than* 0.350389 *can be packed into the unit square.*

5 Conclusion

We provided online algorithms for packing circles into a square and a rectangle. For the case of a rectangular container, we guarantee a packing density which is worst-case optimal for rectangles with a skew of at least 2.36. For the case of a square container, we provide a packing density of 0.350389 which we improved to 0.375898 if the radii of incoming circles are lower-bounded by 0.026622.

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