

## Universal Guard Problems

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Received 18 February 2017

Revised 12 September 2017

Published 13 July 2018

Communicated by Seok-Hee Hong, Guest Editor

Given a set  $S$  of  $n$  points in the plane, how many *universal guards* are sometimes necessary and always sufficient to guard *any* simple polygon with vertex set  $S$ ? We call this problem a *Universal Guard Problem* and provide a spectrum of results. We give upper and lower bounds on the number of universal guards that are always sufficient to guard all polygons having a given set of  $n$  vertices, or to guard all polygons in a given set of  $k$  polygons on an  $n$ -point vertex set. Our upper bound proofs include algorithms to construct universal guard sets of the respective cardinalities.

*Keywords:* Combinatorial geometry; visibility; polygons; art gallery theorem; guarding; universal optimization.

### 1. Introduction

Problems of finding optimal covers are among the most fundamental algorithmic challenges that play an important role in many contexts. One of the best-studied prototypes in a geometric setting is the classic Art Gallery Problem (AGP), which asks for a small number of points (“guards”) required for covering (“seeing”) all of the points within a geometric domain. An enormous body of work on algorithmic aspects of visibility coverage and related problems (see, e.g., O’Rourke,<sup>22</sup> Keil,<sup>17</sup> and Shermer<sup>23</sup>) was spawned by Klee’s question for worst-case bounds more than

40 years ago: How many guards are always sufficient to guard all of the points in a simple polygon having  $n$  vertices? The answer, as shown originally by Chvátal,<sup>4</sup> and with a very simple and elegant proof by Fisk,<sup>10</sup> is that  $\lfloor n/3 \rfloor$  guards are always sufficient, and sometimes necessary, to guard a simple  $n$ -gon.

While Klee's question was posed about guarding an  $n$ -vertex *simple polygon*, a related question about *point sets* was posed at the 2014 NYU Goodman-Pollack Fest: Given a set  $S$  of  $n$  points in the plane, how many *universal* guards are sometimes necessary and always sufficient to guard any simple polygon with vertex set  $S$ ? This problem, and several related questions, are studied in this paper. We give the first set of results on universal guarding, including combinatorial bounds and efficient algorithms to compute universal guard sets that achieve the upper bounds we prove. We focus on the case in which guards must be placed at a subset of the input set  $S$  and thus will be vertex guards for any polygonalization of  $S$ .

A strong motivation for our study is the problem of computing guard sets in the face of uncertainty. In our model, we require that the guards are *robust* with respect to different possible polygonalizations consistent with a given set of points (e.g., obtained by scanning an environment). Our Universal Guard Problem is, in a sense, an extreme version of the problem of guarding a set of possible polygonalizations that are consistent with a given set of sample points that are the polygon vertices: In the universal setting, we require that the guards are a rich enough set to achieve visibility coverage for *all* possible polygonalizations. Another variant studied here is the *k-universal* guarding problem in which the guards must perform visibility coverage for a set of  $k$  different polygonalizations of the input points. Further, we study the case in which guards are required to be placed at non-convex hull points of  $S$ , or at points of a regular rectangular grid.

## ***Related work***

In addition to the worst-case results for the AGP, related work includes algorithmic results for computing a minimum-cardinality guard set. The problem of computing an optimal guard set is known to be NP-hard,<sup>22</sup> even in very basic settings such as guarding a 1.5D terrain.<sup>19</sup> Ghosh<sup>11,12</sup> observed that greedy set cover yields an  $O(\log n)$ -approximation for guarding with the fewest vertices. Using techniques of Clarkson<sup>5</sup> and Brönnimann–Goodrich,<sup>3</sup>  $O(\log OPT)$ -approximation algorithms were given, if guards are restricted to vertices or points of a discrete grid.<sup>7,8,13</sup> For the special case of *rectangle visibility* in orthogonal (rectilinear) polygons, an exact optimization algorithm is known.<sup>25</sup> Recently, for vertex guards (or discrete guards on the boundary) in a simple polygon  $P$ , King and Kirkpatrick<sup>18</sup> obtained an  $O(\log \log OPT)$ -approximation, by building  $\epsilon$ -nets of size  $O((1/\epsilon) \log \log(1/\epsilon))$  for the associated hitting set instances, and applying Ref. 3. For the special case of guarding 1.5D terrains, local search yields a PTAS.<sup>20</sup> Experiments based on heuristics for computing upper and lower bounds on guard numbers have been shown to perform very well in practice.<sup>1</sup> Methods of combinatorial optimization with insights

and algorithms from computational geometry have been successfully combined for the Art Gallery Problem, leading to provably optimal guard sets for instances of significant size.<sup>2,6,9,21,24</sup>

The notion of “universality” has been studied in other contexts in combinatorial optimization,<sup>14,16</sup> including the traveling salesman problem (TSP), Steiner trees, and set cover. For example, in the universal TSP, one desires a single “master” tour on all input points so that, for *any* subset  $S$  of the input points, the tour obtained by visiting  $S$  in the order specified by the master tour yields a tour that approximates an optimal tour on the subset.

### Our results

We introduce a family of universal coverage problems for the classic Art Gallery Problems. We provide a spectrum of lower and upper bounds for the required numbers of guards. See Tables 2 and 3 for a detailed overview, and the following Sec. 2 for relevant notation.

## 2. Preliminaries

For  $n \in \mathbb{N}$ , let  $\mathcal{S}(n)$  be the set of all discrete point sets in the plane that have cardinality  $n$ . A single *shell* (also known as a *convex layer*) of a point set  $S$  is the subset of points of  $S$  on the boundary of the convex hull of  $S$ . Recursively, for  $k \geq 2$ , a point set lies on  $k$  shells, if removing the points on its convex hull, leaves a set that lies on  $k - 1$  shells. We denote by  $\mathcal{S}_g(n) \subset \mathcal{S}(n)$  and  $\mathcal{S}(n, m) \subset \mathcal{S}(n)$  the set of all discrete point sets that form a rectangular  $a \times b$ -grid of  $n$  points for  $a, b, a \cdot b = n \in \mathbb{N}$ , and the set of all discrete point sets that lie on  $m$  shells for  $m \in \mathbb{N}$ , respectively.

By  $\mathcal{P}(S)$ , we consider a set of polygonizations. In particular, for  $S \in \mathcal{S}(n)$ , let  $\mathcal{P}(S)$  (resp.,  $\mathcal{H}(S)$ ) be the set of all simple polygons (resp., polygons with holes) whose vertex set equals  $S$ . Note that the number of polygonizations for a point set of  $n$  points may be exponential w.r.t.  $n$ .

Let  $P$  be a polygon. We say a point  $p \in P$  *sees* (w.r.t.  $P$ ) another point  $q \in P$  if  $pq \subset P$ ; we then write  $p \leftrightarrow_P q$ . The *visible region* (w.r.t.  $P$ ) of a point  $g \in P$  is  $V_P(g) = \{a \in P : g \leftrightarrow_P a\}$ . A point set  $G \subseteq S$  is a *guard set* for  $P$  if  $\bigcup_{g \in G} V_P(g) = P$ . Furthermore, we say that  $G$  is an *interior guard set* for  $P$  if  $G$  is a guard set for  $P$  and no  $g \in G$  is a vertex of the convex hull of  $P$ . For a guard set  $G$  of the point set  $S$ , we say that a point  $g \in G$  is *guarded* and a point  $u \in S \setminus G$  is *unguarded*.

For a set  $A$  of polygons we say that  $G \subseteq S$  is a(n) (interior) guard set of  $A$  if  $G$  is a(n) (interior) guard set for each  $P \in A$ . We denote by  $w(A)$  the minimum cardinality guard set for  $A$  and by  $i(A)$  the minimum cardinality interior guard set for  $A$ . Furthermore, for any given point set  $S$  we say that  $G \subseteq S$  is a *guard set* for  $S$  if  $G$  is a guard set for  $\mathcal{P}(S)$ .

Let  $k \geq 0$  be an arbitrarily chosen but fixed integer. For  $n \geq 0$ , the *k-universal guard number*  $\mathbf{u}_k(n)$  considers the maximal number of guards needed to guard

Table 1. The universal guard numbers considered in this paper.

<i>Universal guard numbers</i>	$\mathbf{u}(n)$	$\max_{S \in \mathcal{S}(n)} w(\mathcal{P}(S))$
<i>m-shelled universal guard numbers</i>	$\mathbf{s}(n, m)$	$\max_{S \in \mathcal{S}(n, m)} w(\mathcal{P}(S))$
<i>Interior universal guard numbers</i>	$\mathbf{i}(n)$	$\max_{S \in \mathcal{S}(n)} \mathbf{i}(\mathcal{P}(S))$
<i>k-universal guard numbers of simple polygons</i>	$\mathbf{u}_k(n)$	$\max_{S \in \mathcal{S}(n)} \max_{\substack{A \subseteq \mathcal{P}(S) \\ \text{s.t. }  A =k}} w(A)$
<i>k-universal guard numbers of polygons w. holes</i>	$\mathbf{h}_k(n)$	$\max_{S \in \mathcal{S}(n)} \max_{\substack{A \subseteq \mathcal{H}(S) \\ \text{s.t. }  A =k}} w(A)$
<i>Grid universal guard numbers</i>	$\mathbf{g}(n)$	$\max_{S \in \mathcal{S}_g(n)} w(\mathcal{P}(S))$

Table 2. Results for simple polygons. The approaches for the upper bounds for  $\mathbf{u}(n)$  and  $\mathbf{s}(n, m)$  also apply to polygons with holes, yielding the same upper bounds.

$m, n \in \mathbb{N}$	$\mathbf{u}(n)$	$\mathbf{s}(n, m)$	$\mathbf{g}(n)$	$\mathbf{i}(n)$
Lower bounds	$\left(1 - \Theta\left(\frac{1}{\sqrt{n}}\right)\right)n$	$\left(1 - \frac{1}{2(m-1)} - \frac{8m}{n(m-1)}\right)n$	$\lfloor \frac{n}{2} \rfloor$	$n - \mathcal{O}(1)$
Upper bounds	$\left(1 - \Theta\left(\frac{1}{n}\right)\right)n$	$\left(1 - \frac{1}{16n\left(1 - \frac{1}{2m}\right)}\right)n$	$\lfloor \frac{n}{2} \rfloor$	$n - \Omega(1)$

Table 3. Overview of our results for  $k$ -universal guard numbers of simple polygons and of polygons with holes. We give a new corresponding approach for the upper bounds of  $\mathbf{h}_1(n), \mathbf{h}_2(n), \dots$ . We also consider the lower bounds for  $\mathbf{u}_1(n), \mathbf{u}_2(n), \dots$  as lower bounds for  $\mathbf{h}_1(n), \mathbf{h}_2(n), \dots$ .

$n \in \mathbb{N}$	$\mathbf{u}_2(n)$	$\mathbf{u}_3(n)$	$\mathbf{u}_4(n)$	$\mathbf{u}_5(n)$	$\mathbf{u}_k(n)$ for $k \geq 6$	$\mathbf{h}_k(n)$ for $k \in \mathbb{N}$
Lower bounds	$\lfloor \frac{3n}{8} \rfloor$	$\frac{4n}{9}$	$\frac{n}{2} - \mathcal{O}(\sqrt{n})$	$\frac{n}{2} - \mathcal{O}(\sqrt{n})$	$\frac{5n}{9}$	$\frac{5n}{9}$
Upper Bounds	$\frac{5n}{9}$	$\frac{19n}{27}$	$\frac{65n}{81}$	$\frac{211n}{243}$	$(1 - (\frac{2}{3})^k)n$	$(1 - (\frac{5}{8})^k)n$

simultaneously  $k$  simple polygons that have the same set of  $n$  vertices, i.e. a maximal value of  $w(\{P_1, \dots, P_k\})$  for  $\{P_1, \dots, P_k\} \subseteq \mathcal{P}(\mathcal{S}(n))$ . Analogously, for  $n \geq 0$ , the  $k$ -universal guard number  $\mathbf{h}_k(n)$  considers the maximal number of guards needed to guard simultaneously  $k$  polygons that have the same set of  $n$  vertices, i.e., a maximal value of  $w(\{P_1, \dots, P_k\})$  for  $\{P_1, \dots, P_k\} \subseteq \mathcal{H}(\mathcal{S}(n))$ .

For  $k, m, n \in \mathbb{N}$ , the guard numbers are listed in Table 1.

### 3. Bounds for Universal Guard Numbers

In the following, we provide different lower and upper bounds for the universal guard numbers. In particular, the provided bounds can be classified by the number of shells on which the points of the considered point set are located.

#### 3.1. Lower bounds for universal guard numbers

In this section we give lower bounds for the universal guard numbers  $\mathbf{u}(n)$  and  $\mathbf{s}(n, m)$  for  $n \in \mathbb{N}$  and  $m \geq 2$ . In particular, we provide lower bound

constructions that can be described by the following approach: For any given  $n \in \mathbb{N}$  and  $m \geq 2$ , we construct a point set  $S_m \in \mathcal{S}(n)$  as follows.  $S_m$  is partitioned into pairwise disjoint subsets  $B_1, \dots, B_m$ , such that  $\bigcup_{i=1}^m B_i = S$ . For  $i \in \{1, \dots, m\}$ , each  $B_i$  lies on a circle  $C_i$  such that  $C_i$  is enclosed by  $C_{i+1}$  for  $i \in \{1, \dots, m-1\}$ . Furthermore,  $C_1, \dots, C_m$  are concentric and have “sufficiently large” radii; see Secs. 3.1.1, 3.1.2, and 3.1.3 for details. In particular, the radii depend on the approaches that are applied for the different cases  $m = 2$ ,  $m = 3$ , and  $m \geq 4$ . We place four equidistant points on  $C_m$ . The remaining points are placed on  $C_{m-1}, \dots, C_1$ .

Note that  $s(n, 1) = 1$  holds, because for every convex point set  $S \in \mathcal{S}(n)$ ,  $\mathcal{P}(S)$  consists of only the boundary of the convex hull of  $S$ . Thus we start with the case  $m = 2$ .

### 3.1.1. Lower bounds for $s(n, 2)$

We give an approach that provides a lower bound for  $s(n, 2)$ . In particular, for any  $n \in \mathbb{N}$ , we construct a point set  $S_2 \in \mathcal{S}(n)$  having  $n - 4$  equally spaced points that lie on circle  $C_1$  and 4 equally spaced points on a larger concentric circle  $C_2$ , such that these 4 points form a square containing  $C_1$ ; see Fig. 1. In order to assure that the constructed subsets of  $S_2$  and  $S_3, S_4, \dots$  (which are described later) are nonempty, we require  $n \geq 32$  for the remainder of Sec. 3.1.

Let  $v$  be a point from the square and let  $p, q$  be two consecutive points from the circle  $C_1$ , such that the segments  $vp$  and  $vq$  do not intersect the interior of the circle  $C_1$ ; see Fig. 1(a). We choose the side lengths of the square such that the cone  $c$  that is induced by  $p$  and  $q$  with apex at  $v$  contains at most  $\frac{n}{8}$  points from  $C_1$  for all choices of  $v, p$ , and  $q$ .

**Lemma 1.** *Let  $G$  be a guard set of  $S_2$ . Then we have  $|G| > \frac{n}{2} - 4$ .*

**Proof.** Suppose  $|G| \leq \lfloor \frac{n-4}{2} \rfloor - 1$ . This implies that there are two points  $p, q \in S_m \setminus G$  such that  $p$  and  $q$  lie adjacent on  $C_1$ ; see Fig. 1(b). Let  $w_1, w_2, w_3$ , and  $w_4$  be the four points from the square. At most two points  $v_1, v_2 \in \{w_1, w_2, w_3, w_4\}$  span a cone, such that the segments  $v_1p, v_1q, v_2p, v_2q$  do not intersect the interior of  $C_1$ .

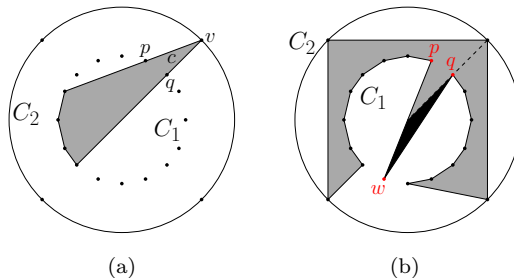


Fig. 1. Lower-bound construction for  $s(n, 2)$ .

Without loss of generality, we assume that these two different cones  $c_1$  and  $c_2$  exist.  $c_1$  and  $c_2$  contain at most  $\frac{n}{4}$  points from  $C$ . Thus, there is another point  $w \in S_2 \setminus G$  such that  $w \notin c_1 \cup c_2$ . This implies that there is a polygon in which  $w$  is not seen by a guard from  $G$ ; see Fig. 1(b). This is a contradiction to the assumption that  $G$  is a guard set.

Thus we have  $|G| > \lfloor \frac{n-4}{2} \rfloor - 1 \geq \frac{n-4}{2} - 2 = \frac{n}{2} - 4$ . This concludes the proof.  $\square$

**Corollary 2.**  $s(n, 2) \geq \lfloor \frac{n}{2} \rfloor - 4$ .

### 3.1.2. A first lower bound for $s(n, 3)$

The high-level idea is to guarantee in the construction of  $S_3$  that at most two points on  $C_1$  are unguarded; see Fig. 2 for the idea of the proof by contradiction. By constructing  $S_3 = B_1 \cup B_2 \cup B_3$  such that  $|B_1| = \lfloor \frac{n-4}{2} \rfloor$ ,  $|B_2| = \lceil \frac{n-4}{2} \rceil$ , and  $|B_3| = 4$ , we obtain  $|G| \geq \frac{n}{2} - 5$  for any guard set  $G$  of  $S_3$ .

We consider the lower-bound construction  $S_m$  for  $m - 1 = 2$  and  $n = (m - 1)2^l + 4 = 3 \cdot 2^l + 4$  for any  $l \geq 4$ , i.e., for all  $S_3 \in \mathcal{S}(2 \cdot 2^l + 4)$  for any  $l \geq 2$ . The argument can easily be extended to  $n \in \mathbb{N}$ .

The points of  $B_1$  and  $B_2$  are placed on  $C_1$  and  $C_2$ , such that they lie on  $2^{l-1}$  lines; see Fig. 2(a). Let  $v \in B_2$  be chosen arbitrarily and  $p, q \in B_1$  such that  $p$  and  $q$  are the neighbors of the point from  $B_1$  that corresponds to  $v \in B_2$ . We choose the radius of  $C_2$  such that the cone that is induced by  $p$  and  $q$  and with apex at  $v$  contains all points from  $B_1$ ; see the gray cone in Fig. 2(a). Furthermore, we choose the radius of  $C_1$  such that the square that is induced by the four points from  $B_3$  contains all points from  $B_1 \cup B_2$ .

The key construction that we apply in the proofs of our lower bounds are *chambers*.

**Definition 3.** Let  $S$  be a finite point set in the plane. Four points  $p_1, p_2, p_3, p_4 \in S$  form a *chamber* (illustrated in Fig. 2(b)), denoted  $\triangle(p_1, p_2, p_3, p_4)$ , if

- (1)  $p_1$  and  $p_2$  lie on different sides of the line  $p_3p_4$ ,
- (2)  $p_3$  and  $p_4$  lie on the same side of the line  $p_1p_2$ ,

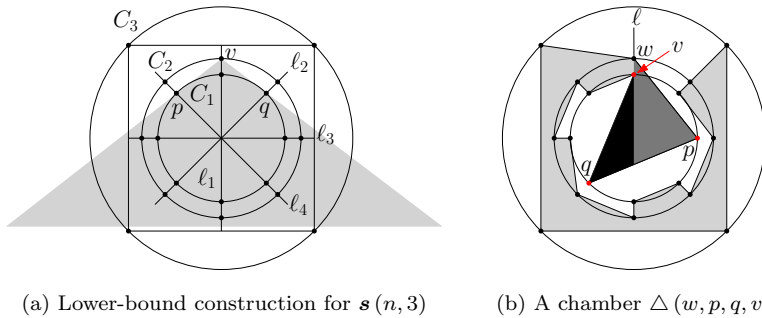


Fig. 2. The lower-bound construction for  $s(n, 3)$ .

- (3)  $p_1p_2p_3$  and  $p_2p_3p_4$  form a right turn or  $p_1p_2p_3$  and  $p_2p_3p_4$  form a left turn, and
- (4) there is no point from  $S$  that lies inside the polygon  $\langle p_1, p_2, p_3, p_4 \rangle$ .

Note that the only important property of a chamber applied in our lower bound constructions is that there is no guarded point that sees  $p_3$ .

Let  $G \subseteq S$ . We say that  $\triangle(p_1, p_2, p_3, p_4)$  is *empty* (with respect to  $G$ ) if  $p_2, p_3, p_4 \notin G$ . Let  $P \in \mathcal{P}(S)$ . We say that  $\triangle(p_1, p_2, p_3, p_4)$  is *part of*  $P$  if  $p_1p_2, p_2p_3, p_3p_4 \subset \partial P$ .

Our proofs are based on the following simple observation.

**Observation 4.** Let  $G$  be a guard set for a polygon  $P$ . There is no empty chamber that is part of  $P$ .

Based on Observation 4 we prove the following lemma, which we then apply to the construction above to obtain our lower bounds for  $s(n, m)$ . We recall that  $S_3$  is defined as the union of the point sets  $B_1$ ,  $B_2$ , and  $B_3$  constructed as point sets on the circles  $C_1$ ,  $C_2$ , and  $C_3$ , as illustrated in Fig. 2.

**Lemma 5.** Let  $G$  be a guard set for  $\mathcal{P}(S_3)$ . Then  $|B_1 \setminus G| \leq 2$ .

**Proof.** Suppose there are three points  $v, q, p \in B_1 \setminus G$ . Without loss of generality, we assume that  $q$  and  $p$  lie on different sides with respect to the line  $\ell$  that corresponds to the placement of  $v$ ; see Fig. 2(b). Furthermore, we denote the point from  $B_2$  that lies above  $v$  by  $w$ . By construction it follows that  $w, p, q$ , and  $v$  form an empty chamber  $\triangle(w, p, q, v)$ . Furthermore, we construct a polygon  $P \in \mathcal{P}(S_3)$  such that  $\triangle(w, p, q, v)$  is part of  $P$ ; see Fig. 2(b). By Observation 4 it follows that  $G$  is not a guard set for  $P$ , a contradiction. This concludes the proof.  $\square$

There is a corresponding construction for all other values  $n \in \mathbb{N}$ . In particular, we place four points equidistant on  $C_3$ ,  $\lceil \frac{n-4}{2} \rceil$  equidistant points on  $C_2$ , and  $\lfloor \frac{n-4}{2} \rfloor$  points on  $C_1$ , such that each point from  $C_1$  lies below a point from  $C_2$ . The same argument as above applies to the resulting construction of a point set. The constructions of  $S_m$  can be modified so that no three points lie on the same line, by a slight perturbation. Thus,  $S_3$  can be assumed to be in general position. We obtain the following corollary.

**Corollary 6.**  $s(n, 3) \geq \frac{n}{2} - 5$ .

**Proof.** Lemma 5 implies that at least  $\lfloor \frac{n-4}{2} \rfloor - 2$  points from  $B_1$  are guarded. Let  $G$  be an arbitrarily chosen guard set for  $\mathcal{P}(S_3)$ . Thus we establish  $|G| \geq \lfloor \frac{n-4}{2} \rfloor - 2 \geq \frac{n-4}{2} - 3 = \frac{n}{2} - 5$ .  $\square$

In the following section we generalize the above approach from the case of three shells to the case of  $m$  shells and combine that argument with the approach that

we applied in the case of  $m = 2$ . This also leads to the improved lower bound  $\mathbf{u}_3(n) \geq (\frac{3}{4} - \mathcal{O}(\frac{1}{n}))n$ .

### 3.1.3. (Improved) lower bounds for $\mathbf{u}(n)$ and $\mathbf{s}(n, m)$ for $m \geq 3$

In this section we give general constructions  $S_3, S_4, \dots$  of the point sets that yield our lower bounds for  $\mathbf{s}(n, m)$  for  $m \geq 3$ . The main difference in the construction of  $S_m$  for  $m \geq 3$ , compared to the previous section, is the choice of the radii of  $C_1, \dots, C_m$ . Similar as in the previous section, we guarantee that on each circle  $C_3, C_4, \dots$  at most  $\mathcal{O}(1)$  points are unguarded. The general idea is to choose five arbitrary points  $q_1, q_2, q_3, q_4, q_5$  on  $C_i$  for  $i \in \{3, 4, \dots\}$ . There are three points  $u_1, u_2, u_3 \in \{q_1, q_2, q_3, q_4, q_5\}$ , such that the triangle induced by  $u_1, u_2, u_3$  does not contain the common centerpoint of  $C_1, C_2, \dots$ . By choosing the radius of  $C_{i+1}$  sufficiently large, we obtain that there is a chamber  $\triangle(u_1, u_2, u_3, p)$ , where  $p$  is a point on  $C_{i+1}$ ; see Fig. 3. This implies that  $\triangle(u_1, u_2, u_3, p)$  is empty if  $q_1, q_2, q_3, q_4, q_5$  are unguarded. Thus, at most four points on  $C_i$  are allowed to be unguarded; see Corollary 9.

Finally, we show how the arguments for  $S_m$  yield lower bounds for  $\mathbf{s}(n, m)$  and  $\mathbf{u}(n)$ .

As in the previous section, the constructed point sets  $S_3, S_4, \dots$  can be modified to be in general position.

**The Construction of  $S_m$  for  $m \geq 3$ :** We construct  $S_m$  such that  $|B_1| = \dots = |B_{m-1}| = 2^l$ ,  $|B_m| = 4$ , and hence  $n = (m-1)2^l + 4$  for  $l \geq 4$ . In particular, similar to the construction of  $S_3$  from the previous section, we place the points of  $B_1, \dots, B_{m-1}$  equidistant on the circles  $C_1, \dots, C_{m-1}$ , such that the points lie on  $2^{l-1}$  lines  $\ell_1, \dots, \ell_{2^{l-1}}$ ; see Fig. 3(a).

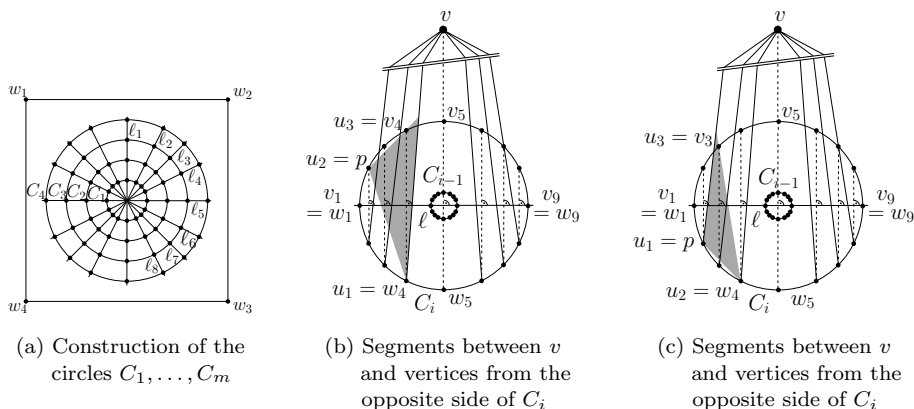


Fig. 3. Construction of  $S_m$  for  $n = 68$ . For a simplified illustration we changed the ratios of the circles' radii and we shortened the lines adjacent to  $v$ . In the configuration of Lemma 7, three points from  $C_i$  in the same half of  $C_i$  imply a chamber with a point  $v \in C_{i+1}$  that lies above  $\ell$ . Chambers with a point  $w \in C_{i+1}$  can be constructed symmetrically with respect to the line  $\ell$ .



In order to apply an argument that makes use of chambers, we need the following notation for points on a circle  $C_i$ . Let  $n' := 2^l$ . Let  $v_1, \dots, v_{1+n'/2}$  be the points on  $C_i$  to one side or on  $\ell \in \{\ell_1, \dots, \ell_{n'/2}\}$ . Let  $w_1, \dots, w_{1+n'/2}$  be their reflection across  $\ell$ ; see Figs. 3(b) and 3(c). Let  $v \in C_{i+1}$  be the point that lies above  $v_{1+n'/4}$ . As the construction of  $S_m$  is symmetric with respect to rotations the following discussion applies to each choice of  $\ell$  and  $v$  such that the segment between  $v$  and the centerpoint of the circles  $C_1, \dots, C_m$  lies orthogonal to  $\ell$ .

For  $i \in \{1, \dots, m-1\}$ , we choose the radius of  $C_{i+1}$  compared to the radius of  $C_i$  sufficiently large, such that  $v$ , two points  $v_j$  and  $w_j$  that lie orthogonal to  $\ell$ , and a fourth point  $p$  from  $C_i$  build a chamber  $\triangle(v, w_j, p, v_j)$ ; see Fig. 3(b). Simultaneously, we ensure that  $v$ ,  $p$ ,  $w_j$ , and  $v_{j-1}$  build another chamber  $\triangle(v, p, w_j, v_{j-1})$ ; see Fig. 3(c).

In particular, we have to choose the radius of  $C_{i+1}$  large enough such that the polygons bounded by the polygonal chains  $\langle v, w_j, p, v_j \rangle$  and  $\langle v, p, w_j, v_{j-1} \rangle$  do not contain any other points from  $S$ . In order to do this, we ensure that (1) the segment  $vw_j$  intersects  $C_i$  in the arc between  $v_j$  and  $v_{j+1}$ ; see Fig. 3(b), and (2) the segment  $vw_{j-1}$  intersects  $C_i$  in the arc between  $v_{j-1}$  and  $v_{j-2}$ ; see Fig. 3(c).

Finally, we place the four points  $w_1, w_2, w_3, w_4 \in B_m$  such that all circles lie in the convex hull of  $w_1, w_2, w_3$ , and  $w_4$ ; see Fig. 3(a).

**The Analysis of  $S_m$  for  $m \geq 3$ :** First we show that we can choose three points  $u_1, u_2, u_3$  from five arbitrarily chosen points from  $C_i$ , such that there is another point  $u \in C_{i+1}$  with  $\triangle(u, u_1, u_2, u_3)$  forming a chamber; see Lemma 7. Next, we construct a polygon  $P \in \mathcal{P}(S_m)$ , such that  $\triangle(u, u_1, u_2, u_3)$  is a part of  $P$ ; see Lemma 8. Finally, by combining Lemma 7 and Lemma 8 we establish that on each  $C_i$ , at most four points are allowed to be unguarded; see Corollary 9. This leads to several lower bounds for  $s(n, m)$  and  $u(n)$ .

**Lemma 7.** *Let  $q_1, q_2, q_3, q_4, q_5 \in C_i$  be chosen arbitrarily. There are three points  $u_1, u_2, u_3 \in \{q_1, q_2, q_3, q_4, q_5\}$  and a point  $u \in C_{i+1}$ , such that  $\triangle(u, u_1, u_2, u_3)$  is a chamber.*

**Proof.** We choose  $u_1, u_2, u_3$  from  $\{q_1, q_2, q_3, q_4, q_5\}$ , such that  $u_1, u_2, u_3$  lie in the same half of  $C_i$ , i.e., such that the centerpoint of  $C_i$  does not lie inside the triangle  $t$  that is induced by  $u_1, u_2, u_3$ , see Figs. 3(b) and 3(c). Without loss of generality, we assume that  $u_2$  lies between  $u_1$  and  $u_3$ . Otherwise, we rename the points appropriately.

We distinguish two cases. (C1) The number of points between  $u_1$  and  $u_3$  is odd and (C2) the number of points between  $u_1$  and  $u_3$  is even. For both cases (C1) and (C2) we can ensure the existence of a corresponding chamber for achieving the required contradiction; see Fig. 3(b) for even (C1) and Fig. 3(c) for odd (C2).  $\square$

Based on Lemma 7, we can construct the required polygon  $P$  such that the chamber constructed in Lemma 7 is part of  $P$ .

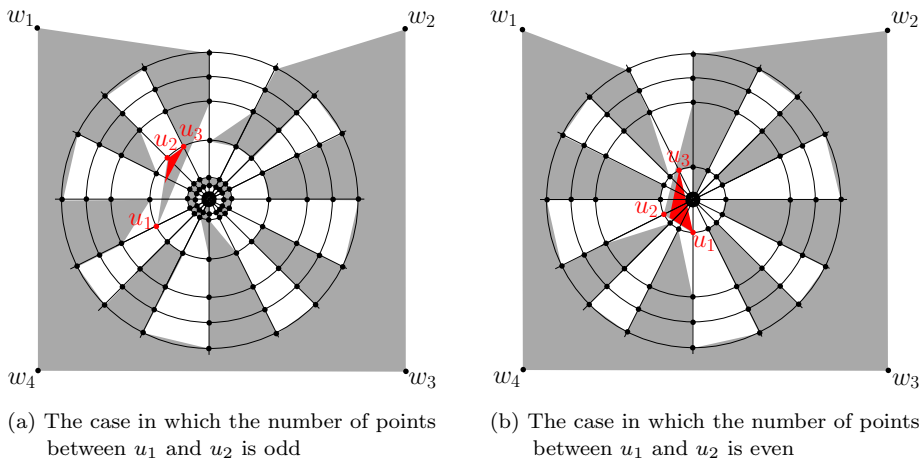


Fig. 4. Construction of  $\mathcal{P}$  for  $k = 6$  and  $n = 16$ . For a simplified illustration we changed the ratios of the circles' radii (otherwise the figure would become too large). The red regions illustrate areas that are not seen by any guard (color online).

**Lemma 8.** *There is a polygon  $P \in \mathcal{P}(S_m)$  such that  $\triangle(u, u_1, u_2, u_3)$  is part of  $P$ .*

**Proof.** We construct  $P$  for the cases (C1) and (C2) of Lemma 7 separately; see Fig. 4. In both cases we walk upwards on the line  $\ell \in \{\ell_1, \dots, \ell_{n'/2}\}$  until we reach  $C_1$ . Next we orbit  $C_i$  in a zig-zag approach and finally connect all points from  $C_{i-1}, \dots, C_1$  in a similar manner; see Fig. 4.  $\square$

The combination of Lemma 7 and Lemma 8 implies the following corollary.

**Corollary 9.** *Let  $G \subset S_m$  be a guard set of  $\mathcal{P}(S_m)$ . Then  $|B_i \setminus G| \leq 4$ , for  $i \in \{1, \dots, m-2\}$ .*

**Lower bounds for  $s(n, m)$  and  $u(n)$  that are implied by Corollary 9:** We combine the approach for  $s(n, 2)$  with Corollary 9, which yields the following lower bound for  $s(n, m)$  for  $m \geq 3$ .

**Corollary 10.** *Let  $m \geq 3$  and  $n' = 2^l$  with  $l \geq 4$ . Furthermore, let  $G \subseteq S_m$  be a guard set of  $S_m$ . Then we have  $|G| \geq (1 - \frac{1}{2(m-1)} + \frac{8m}{n(m-1)})|S_m|$ .*

**Proof.** By Corollary 9 it follows that  $(m-2)(n'-4)$  points from  $B_1 \cup \dots \cup B_{m-2}$  are guarded where  $n' = |B_1| = \dots = |B_{m-2}|$ . Furthermore, by applying the approach of Lemma 1 to  $B_{m-1}$  and  $B_m$  yields that at least  $\frac{n'}{2} - 8$  points from  $B_{m-1} \cup B_m$  are guarded because  $n' = |B_{m-1}|$ . In order to apply the approach of Lemma 1 to  $B_{m-1}$  and  $B_m$ , we again choose the radii of  $C_{m-1}$  and  $C_m$  large enough. W.l.o.g., we assume that all points from  $B_m$  are unguarded. By assuming that five other points from  $B_{m-1}$  are unguarded, we can construct three unguarded points  $w \in B_{m-1}$  and  $p, q \in B_{m-1}$ , such that  $p$  and  $q$  lie adjacent on  $C_{m-1}$ , see Fig. 1(b).

Following the approach of the proof of Lemma 8, we can guarantee that no point from  $B_{m-2}, \dots, B_1$  lies inside the cone with apex at  $w$  and induced by  $p$  and  $q$ , because we choose the radii of  $C_m$  and  $C_{m-1}$  large enough.

Thus we obtain  $|G| \geq (m-2)(n'-4) + \frac{n'}{2} - 8$ , which is lower-bounded by  $|S_m|(1 - \frac{1}{2(m-1)} - \frac{8m}{|S_m|(m-1)})$  because  $n' = \frac{|S_m|-4}{m-1}$ .  $\square$

**Theorem 11.**  $s(n, m) \geq n(1 - \frac{1}{2(m-1)} + \frac{8m}{n(m-1)})$  for  $m \geq 3$ .

By choosing  $m$  appropriately, we obtain the following lower bound:

**Lemma 12.** For any  $c < 1$  and any guard set  $G$  for  $S_m$  there is an  $m \in \mathbb{N}$  with  $|G| > c|S_m|$ .

**Proof.** The approach is to choose  $m := \lceil \frac{2n'}{n'-4-cn'} \rceil$ , which will imply  $|G| > c|S_m|$ .

Suppose  $|G| \leq c|S_m|$ . This leads to a contradiction as follows. We have  $|S_m| = 4 + (m-1)n'$ . Corollary 9 implies that on  $C_1, \dots, C_{m-2}$  there are at most four vertices that are unguarded. Thus,  $(m-2)(n'-4) \leq |G|$ . By assumption we know  $|G| \leq c(4 + (m-1)n')$ . Thus, we obtain  $(m-2)(n'-4) \leq c(4 + (m-1)n')$ , which implies that  $8 \leq 4$  holds because  $m = \lceil \frac{2n'}{n'-4-cn'} \rceil$ .  $\square$

By choosing  $c$  appropriately, Lemma 12 leads to our general upper bound for  $\mathbf{u}$  (n).

**Theorem 13.** There is an  $m \in \mathbb{N}$  such that  $|G| > (1 - \frac{10}{\sqrt{|S_m|}})|S_m|$  holds for any guard set  $G$  for  $\mathcal{P}(S_m)$ .

**Proof.** Lemma 12 implies that at least  $(1 - \frac{5}{n'})|S_m|$  points are guarded for  $c := (1 - \frac{5}{n'})$ . Note that we chose  $m := \lceil \frac{2n'}{n'-4-cn'} \rceil$  in the proof of Lemma 12. Furthermore, we have  $|S_m| = 4 + (m-1)n'$ . This implies  $m \leq \frac{2n'}{n'-4-(1-\frac{5}{n'})n'} + 1 = 2n'$ . Additionally, by combining  $m := \lceil \frac{2n'}{n'-4-cn'} \rceil$  and  $|S_m| = 4 + (m-1)n'$ , we obtain  $|S_m| \leq 4 + 2(n')^2$ , which implies that  $\sqrt{|S_m|/2} - \sqrt{2} \leq n'$ . As least  $(1 - \frac{5}{n'})|S_m|$  points are guarded, we get  $|G| \geq (1 - \frac{5\sqrt{2}}{\sqrt{|S_m|-2}})|S_m| > (1 - \frac{10}{\sqrt{|S_m|}})|S_m|$  as required.  $\square$

Theorem 13 establishes the lower bound for the universal guard number in Table 2.

**Theorem 14.**  $\mathbf{u}(n) \geq (1 - \frac{10}{\sqrt{n}})n$ .

Note that the lower bounds of this section are purely combinatorial, based on the order structure of the point sets. As a consequence, the constructed coordinates are of minor relevance, as we are not developing algorithms or performing reductions. In fact, the resulting coordinate numbers are exponential in value, but polynomial in encoding size. It may be of additional interest to consider point sets of coarser granularity, for which the coordinates have bounded size. An extreme case are full grids, which are discussed in Sec. 5.2.

### 3.2. Upper bounds for universal guard numbers

In the following we give an approach to computing a non-trivial guard set of a given point set. The number of the computed guards depends on the number  $m$  of shells of the point set  $S$ . This approach yields upper bounds for  $s(n, m)$  for  $m \geq 2$ .

For the case  $m = 1$ , a naïve approach is simply to choose one guard arbitrarily from  $S$ . In that case,  $\mathcal{P}(S)$  just consists of the polygon that corresponds to the boundary of the convex hull of  $S$  and an arbitrarily chosen point from  $S$  sees all points from all polygons of  $\mathcal{P}(S)$ .

In the following, we first give an approach for the case  $m = 2$ . Then, we generalize that approach to the case  $m \geq 3$ .

#### 3.2.1. Upper bound for $s(n, 2)$

First we describe the approach, followed by showing that the computed point set  $G$  is a guard set. This leads to an upper bound for  $|G|$ , which implies the required upper bound for  $s(n, 2)$ .

**The upper bound approach for two shells:** The high-level idea is to avoid areas that are unguarded by structures similar to chambers. In particular, in the case  $m = 2$ , a chamber, which we also call an *obstruction of type 1*, cannot be part of a simple polygon; otherwise, the boundary of  $P$  meets points at least twice, see Fig. 5(a). However, there is another structure, which we call an *obstruction of type 2*, that has an effect similar to that of chambers and that also may cause unguarded areas, see Fig. 5(b). In the example of Fig. 5(b), our approach guarantees that  $p_2$  or  $p_6$ ,  $p_2$  or  $p_4$ , and  $p_4$  or  $p_6$  is guarded.

More generally, for a point  $p$  on the outer shell, a point  $q$  on the inner shell is a *tangent point* of  $p$  if all points from the inner shell lie on the same side with respect to the line induced by  $p$  and  $q$ . Each point on the outer shell has two tangent points on the inner shell. In our approach we guarantee that two unguarded points on the inner shell are not separated by tangent points corresponding to a point from the outer shell, see Fig. 5(c).

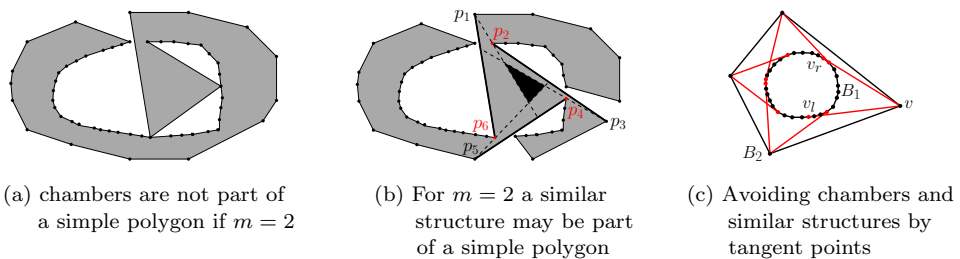


Fig. 5. Possible obstructions of type 2 in case of two shells and how we avoid them. In particular, the obstruction of type 2 illustrated in subfigure (b) corresponds to a chamber when  $p_3 = p_4$  and  $p_5 = p_6$  hold.

Our approach makes a case distinction as follows: Let  $B_1 \subset S$  be the points on the inner shell and  $B_2 \subset S$  be the points on the outer shell of the input point set  $S$ . If  $|B_2| \geq \sqrt{|B_1|}/2$  we take  $B_1$  as the guard set  $G$ . Otherwise, we compute for each  $v \in B_2$  the two corresponding tangent points  $v_l$  and  $v_r$  on  $B_1$ , see Fig. 5(c). Next, we compute a longest sequence  $\langle v_1, \dots, v_k \rangle$  of points from the inner shell such that  $\langle v_1, \dots, v_k \rangle$  does not contain any tangent points. Finally, we fix every second point from  $\langle v_1, \dots, v_k \rangle$  as unguarded and choose all other points from  $S$  as guarded.

**Analysis of the approach for two shells:** For the constructed point set  $G$ , we can guarantee that  $G$  is a guard set for  $\mathcal{P}(S)$  with  $|G| \leq (1 - \frac{1}{\sqrt{6|S|}})|S|$ :

**Theorem 15.** *For each point set  $S$  that lies on two convex hulls, we can compute in  $\mathcal{O}(|S| \log |S|)$  time a guard set  $G$  with  $|G| \leq (1 - \frac{1}{\sqrt{6|S|}})|S|$ .*

For the proof of Theorem 15, we first show  $|G| \leq (1 - \frac{1}{\sqrt{6|S|}})|S|$ , (Lemma 16) followed by showing that  $G$  is a guard set for  $\mathcal{P}(S)$ , see the partition of  $P$  described below and Lemma 17.

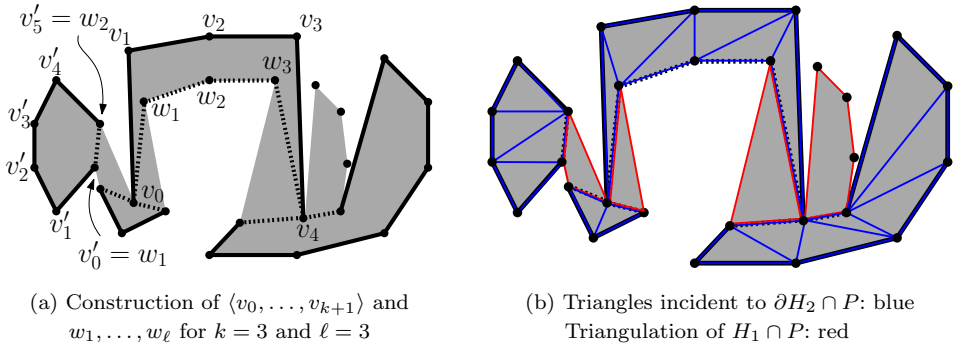
**Lemma 16.**  $|G| \leq (1 - \frac{1}{\sqrt{6|S|}})|S|$ .

**Proof.** For simplified presentation we denote  $n_1 := |B_1|$  and  $n_2 := |B_2|$ . We consider the two cases  $n_2 \geq \frac{\sqrt{n_1}}{2}$  and  $n_2 < \frac{\sqrt{n_1}}{2}$  separately:

- Assume that  $n_2 \geq \frac{\sqrt{n_1}}{2}$  holds. This is equivalent to  $4n_2^2 \geq n_1$ , which implies  $4n_2^2 + n_2 \geq n_1 + n_2 = |S|$ . This yields  $5n_2^2 \geq |S|$  and thus we obtain  $n_2 \geq \frac{\sqrt{|S|}}{\sqrt{5}}$ . Furthermore, we know that the number  $|G|$  of guarded points is equal to  $n_1$  because our approach sets  $G := B_1$ . Thus, we can upper-bound  $|G|$  by  $\frac{n_1}{|S|}|S| \leq \frac{n_1 + n_2 - n_2}{|S|}|S| \leq (1 - \frac{\sqrt{|S|}/\sqrt{5}}{|S|})|S| \leq (1 - \frac{1}{\sqrt{5|S|}})|S|$ .
- Assume that  $n_2 < \frac{\sqrt{n_1}}{2}$  holds. In that case we upper-bound  $|G|$  as follows:  $n_2 < \frac{\sqrt{n_1}}{2}$  implies that there are at most  $\sqrt{n_1}$  tangent points because for each point on the outer shell there are two tangent points on the inner shell. Thus, a longest sequence  $\langle v_1, \dots, v_k \rangle$  on the inner shell that does not contain any tangent points has a length of at least  $\sqrt{n_1} - 1$ . Thus, we obtain that at least  $\frac{\sqrt{n_1} - 1}{2}$  points are unguarded because we only choose every second point from  $\langle v_1, \dots, v_k \rangle$  as guarded.

Furthermore, by combining  $\frac{\sqrt{n_1}}{2} > n_2$  with  $|S| = n_1 + n_2$ , we get  $|S| \leq \frac{4}{3}n_1$ . This implies that the number of guarded points is upper-bounded by  $|S| - \frac{\sqrt{\frac{3}{4}|S|}}{2} + \frac{1}{2}$ , which is no larger than  $(1 - \frac{1}{\sqrt{6|S|}})|S|$ .  $\square$

In order to prove that  $G$  is a guard set for  $\mathcal{P}(S)$ , we consider an arbitrarily chosen but fixed polygon  $P \in \mathcal{P}(S)$  and construct a partition  $T$  of  $P$  into convex regions, such that each region  $t \in T$  is adjacent to a guarded point  $v \in G$ . This

Fig. 6. Stepwise construction of  $T$  (color online).

implies that  $G$  guards the polygon  $P$  because each point inside a convex region  $t$  can be seen by an arbitrarily chosen corner point from  $t$ .

**Partition of  $P$ :** We denote by  $H_1$  and  $H_2$  the convex hulls of  $B_1$  and  $B_2$ . Below, we first describe how to determine the regions (triangles) from  $P$  that are incident to points from the boundary of the convex hull of  $S$ , i.e. incident to  $\partial H_2 \cap P$ , see blue bounded regions in Fig. 6(b). After that we argue that the remaining parts of  $P$  are convex regions  $A \subseteq H_1$  that share at most a vertex, see red bounded regions in Fig. 6(b):

- (1) Triangles that are incident to  $\partial H_2 \cap P$ : Let  $\langle v_1, \dots, v_k \rangle$  be a maximal sequence of points from  $B_2$  that are connected by segments from  $\partial P$ , see Fig. 6(a). The predecessor  $v_0$  and successor  $v_{k+1}$  of  $v_1$  and  $v_k$  on  $\partial P$  do not lie on  $H_2$ , which implies that  $v_0$  and  $v_{k+1}$  lie on  $H_1$ . Otherwise,  $\langle v_1, \dots, v_k \rangle$  would not be maximal or another point  $p \in P$  would be isolated such that  $p$  cannot be part of  $P$ . Let  $\langle w_1, \dots, w_\ell \rangle$  be the sequence of points that lie on  $H_1$  between the segments  $v_0 v_1$  and  $v_k v_{k+1}$ , see Fig. 6(a). By walking simultaneously from  $v_1$  to  $v_k$  and from  $w_1$  to  $w_\ell$ , we triangulate the polygon that is bounded by  $\langle v_0, \dots, v_k \rangle$  and  $\langle w_1, \dots, w_\ell \rangle$ . We call the resulting triangles *type 2* regions.
- (2) Partition of the remaining parts: As no point from  $S$  lies in the interior of  $H_1$  it follows that the remaining areas of  $P$  that are not yet triangulated are convex polygons  $t \subseteq H_1$  that do not intersect each other, see Fig. 6(b). We call the resulting convex polygons *type 1*-regions.

**Lemma 17.** *Each region  $t \in T$  is adjacent to a point  $v \in G$ .*

**Proof.** We distinguish if the region  $t$  is of type 1 or of type 2:

- $t$  is of type 2:  $t$  is adjacent to a point  $v_1 \in H_1$  and adjacent to a point  $v_2 \in H_2$ . Because our approach ensures that all points from  $H_1$  or all points from  $H_2$  are guarded, it follows  $v_1 \in G$  or  $v_2 \in G$ .
- $t$  is of type 1: The region  $t$  is given via a sequence  $\langle w_1, \dots, w_\ell = w_1 \rangle$  of points from  $H_1$ , see Fig. 7. In the first case of our approach, we choose all points from

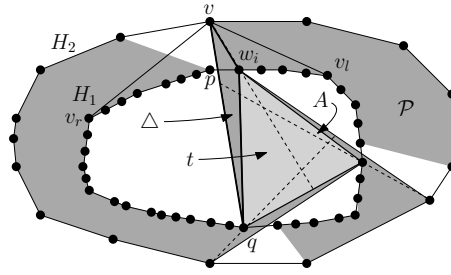


Fig. 7. A polygon  $P$  causing a region  $t \subset P$  of type 2 needed in the contradiction proof of Lemma 17. If the corners of  $t$  are not guarded, there is an area  $A \subseteq t$  that is not guarded. However, we prevent that all corners from  $t$  are unguarded by avoiding that unguarded points on  $H_1$  are separated by tangent points.

the inner shell  $B_1$  as guarded. Thus we obtain that  $w_1, \dots, w_\ell$  are guarded, which implies the lemma.

Next, we consider the situation achieved in the second case of our approach. In particular, we show that at least one point from  $w_1, \dots, w_\ell$  is guarded. For the sake of contradiction, we assume that  $w_1, \dots, w_\ell$  are unguarded. At least one edge from the boundary of  $t$  is not an edge of the boundary of  $P$  because otherwise the resulting circle of edges would imply that no point from  $S$  lies on the outer shell. Let  $w_i q$  be an edge from the boundary of  $t$  such that  $w_i q$  is not an edge of  $\partial P$ . This implies that the edge  $w_i q$  is shared by  $t$  and another type 2 triangle  $\Delta$ , see Fig. 7. Let  $v$  be the third corner of  $\Delta$ . As  $\Delta$  is of type 2, it follows that  $v$  lies on the outer shell of  $S$ . As type 2 triangles are constructed such that no point from  $S$  lies in the interior of  $\Delta$  it follows that even  $qv$  or  $w_i v$  intersects the boundary  $\partial H_1$  of the convex hull  $H_1$  of the inner shell in an edge  $w_i p$  or  $qp$ . In particular, the case in which both segments  $qv$  and  $w_i v$  are not intersecting cannot occur because  $w_i q$  is a segment that is shared by a type 1 region and a type 2 region. Without loss of generality, we assume that  $qv$  intersects  $\partial H_1$  in an edge  $w_i p \subset \partial H_1$ , see Fig. 7. This implies that the two unguarded points  $w_i$  and  $q$  are separated on  $H_1$  by the two tangent points  $v_l$  and  $v_r$  of  $v$ . Thus, our approach ensures that  $w_i$  or  $q$  is guarded, which is a contradiction to the assumption that  $w_1, \dots, w_\ell$  are unguarded. This concludes the proof.  $\square$

We obtain Theorem 15 by combining Lemma 16 and Lemma 17. Finally, Theorem 15 implies Corollary 18:

**Corollary 18.**  $s(n, 2) \leq (1 - \frac{1}{\sqrt{6n}})n$ .

### 3.2.2. Upper bounds for $s(n, m)$ for $m \geq 3$

In this section, we generalize the approach for two shells to the case  $m \geq 3$ .

Let  $B_1, \dots, B_m$  be the pairwise disjoint subsets of  $S$  that lie on the  $m$  shells of  $S$ . The high-level idea of the approach is a generalization of the approach for

$m = 2$  and described as follows. In particular, instead of one inner shell, we now consider  $m - 1$  inner shells  $B_1, \dots, B_{m-1}$  that may have tangent points from points of the outer shell  $B_m$ .

If  $|B_m|$  is “large enough” (larger than a value  $\lambda$ ), we set  $G = B_1 \cup \dots \cup B_{m-1}$ . Otherwise, we carefully choose one shell  $B_j$  for  $j \in \{1, \dots, m-1\}$  and select partially its points as unguarded. All the remaining points are selected as guarded.

In particular, we first compute the tangent points on  $B_j$  for all points from  $B_{j+1} \cup \dots \cup B_m$ . Next, we compute a longest sequence  $\langle v_1, \dots, v_k \rangle$  of points from  $B_j$  between to tangent points. Finally, we fix every second point from  $\langle v_1, \dots, v_k \rangle$  as unguarded and all remaining points from  $S$  as guarded.

It still remains to describe how to choose  $B_j$  in the second case of our approach. We choose  $B_j$  as the shell such that the number of unguarded points is maximized in the worst case for the above described approach. In particular, we choose  $j$  such that  $\frac{|B_j|}{2(|B_{j+1}| + \dots + |B_m|)} - 1$  is maximized. This maximises the number of unguarded points in the worst case because for each point from  $B_{j+1}, \dots, B_m$  there are at most two tangents on  $B_j$ . Furthermore, we decide if “ $|B_m|$  is large enough” by applying worst case balancing. In particular, we set  $\lambda$  to the lower bound for the number of unguarded points in the worst case, i.e.  $\lambda := \frac{|B_j|}{2(|B_{j+1}| + \dots + |B_m|)} - 1$ .

By applying a similar argument as for the case  $m = 2$ , we can show that the computed point set  $G \subseteq S$  is a guard set for  $\mathcal{P}(S)$ . The details are developed in the remainder of the subsection.

**Theorem 19.** *For any point set  $S$  that lies on  $m$  convex hulls we can compute in  $\mathcal{O}(n \log n)$  time a guard set  $G$  with  $|G| \leq (1 - \frac{1}{16|S|^{(1-\frac{1}{2m})}})|S|$ .*

This leads to our generalized upper bound for  $s(n, m)$  for  $m \geq 3$ :

**Corollary 20.**  $s(n, m) \leq (1 - \frac{1}{16n^{(1-\frac{1}{2m})}})n$ .

**Analysis.** In the following we establish an upper bound for  $|G|$  and show that  $G$  is a guard set for  $\mathcal{P}(S)$ . For a simplified presentation we define  $n_1 := |B_1|, \dots, n_m := |B_m|$ .

The following lemma is the key technical ingredient in our proof that the number of guarded points is bounded above by  $(1 - \frac{1}{16n^{(1-\frac{1}{2m})}})n$ .

**Lemma 21.** *The maximum of  $\frac{n_j}{2(n_{j+1} + \dots + n_m)} - 1$  and  $n_m$  is lower-bounded by  $\frac{1}{16}n^{\frac{1}{2m}}$ .*

**Proof.** For the sake of contradiction, assume that both values  $\frac{n_j}{2(n_{j+1} + \dots + n_m)} - 1$  and  $n_m$  are smaller than  $\frac{1}{16}n^{\frac{1}{2m}}$ . This implies that  $\frac{n_{m-\ell-1}}{2(n_{m-\ell} + \dots + n_m)} - 1 < \frac{1}{16}n^{\frac{1}{2m}}$  ( $\star$ ) holds for all  $\ell \in \{0, \dots, m-2\}$ . Based on that, we show that  $n_{m-\ell} < \frac{1}{16}n^{2^{\ell-m}}$  holds for all  $\ell \in \{0, \dots, m-1\}$ . Thus we can upper-bound  $n_1 + \dots + n_m$  as



follows:

$$n_1 + \cdots + n_m = n_{m-0} + \cdots + n_{m-(m-1)} \leq \frac{1}{16}n^{2^{-m}} + \cdots + \frac{1}{16}n^{2^{-1}} < n. \quad (1)$$

This is a contradiction because  $n = n_1 + \cdots + n_m$ , concluding the proof.

It still remains to prove that  $n_{m-\ell} < \frac{1}{16}n^{2^{\ell-m}}$  holds for all  $\ell \in \{0, \dots, m-1\}$ , which we do in the following. In particular, we show the stronger inequality  $n_{m-\ell} + \cdots + n_m < \frac{1}{16}n^{2^{\ell-m}}$  by induction over  $\ell$ , which implies  $n_{m-\ell} < \frac{1}{16}n^{2^{\ell-m}}$ , as required.

For  $\ell = 0$  we know by assumption that  $n_m < \frac{1}{16}n^{\frac{1}{2^m}}$  holds. Assume that  $n_{m-\ell} + \cdots + n_m < \frac{1}{16}n^{2^{\ell-m}}$  ( $\dagger$ ) holds. Based on that we show  $n_{m-\ell-1} + \cdots + n_m < \frac{1}{16}n^{2^{\ell+1-m}}$  as follows:

By the assumption ( $\star$ ), we know that  $\frac{n_{m-\ell-1}}{2(n_{m-\ell} + \cdots + n_m)} - 1 < \frac{1}{16}n^{\frac{1}{2^m}}$  holds. Combining this with the assumption  $n_{m-\ell} + \cdots + n_m < \frac{1}{16}n^{2^{\ell-m}}$  ( $\dagger$ ) of the induction yields  $\frac{n_{m-\ell-1}}{2n^{2^{\ell-m}}} - 1 < \frac{1}{16}n^{\frac{1}{2^m}}$ . This implies  $n_{m-\ell-1} < \frac{6}{256}n^{2^{\ell+1-m}}$ . A final application of the assumption ( $\star$ ) yields  $n_{m-\ell-1} + \cdots + n_m < \frac{6}{256}n^{2^{\ell+1-m}} + \frac{1}{16}n^{2^{\ell-m}}$ , which, in turn, is smaller than  $\frac{1}{16}n^{2^{\ell+1-m}}$ .  $\square$

By applying Lemma 21 we can upper-bound  $|G|$  as required:

**Corollary 22.**  $|G| \leq (1 - \frac{1}{16|S|^{\frac{2m-1}{2m}}})|S|$ .

**Proof.** Our approach guarantees that the number of unguarded points is lower-bounded by the maximum of  $\frac{n_j}{2(n_{j+1} + \cdots + n_m)} - 1$  and  $n_m$ . By Lemma 21, this is lower-bounded by  $\frac{1}{16}|S|^{\frac{1}{2^m}}$ . Thus, the number of guarded points can be upper-bounded by  $|S| - \frac{1}{16}|S|^{\frac{1}{2^m}} = (1 - \frac{1}{16|S|^{\frac{2m-1}{2m}}})|S|$ .  $\square$

Finally, we show that  $G$  is a guard set for  $\mathcal{P}(S)$ . In particular, we consider an arbitrarily chosen but fixed polygon  $P \in \mathcal{P}(S)$  and construct a partition  $T$  of  $P$  into convex regions, such that each region  $t \in T$  is adjacent to a vertex  $v \in G$ .

Roughly speaking, we extend the approach for determining a partition in the case of two shells to the case of  $m$  shells for  $m \geq 3$ . In particular, we repeatedly apply the first step of the above approach and remove the corresponding triangles from the polygon until the remaining points lie on one shell. Finally, we apply the second step of the approach for two shells to the area that is given by the remaining regions. In the following, we give the details of this approach.

**Partition of  $P$ :** For  $i \in \{1, \dots, m\}$ , let  $H_i$  be the convex hull of  $B_i$ . The basic idea for the construction of the partition of  $P$  is the following. Consecutively, for each  $i = m, \dots, 2$  we compute the triangles that are incident to  $\partial H_i \cap P$  just as we do for  $H_2$  in the case of two shells, see Figs. 8(b)–8(e). Finally, we argue that the remaining parts of  $P$  are convex regions  $t \subseteq H_1$  that do not intersect each other, see Fig. 8(f).

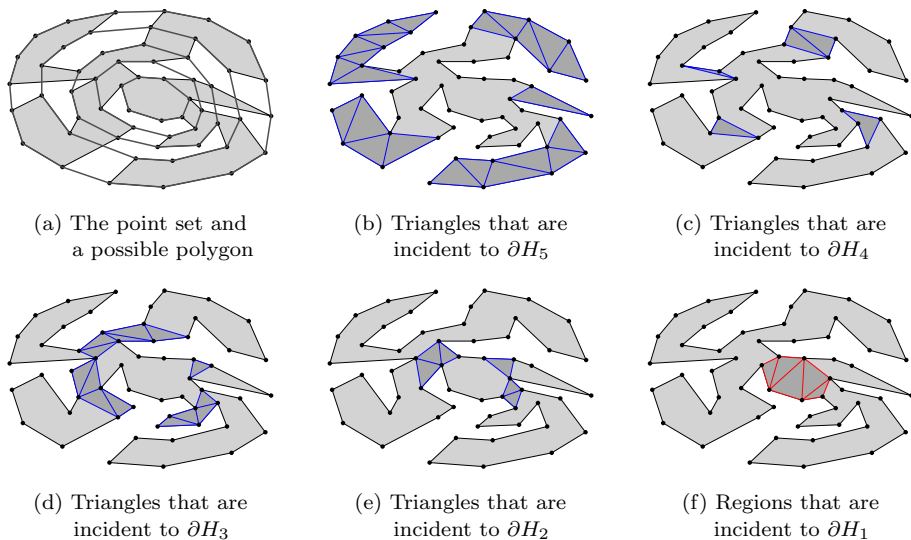


Fig. 8. Stepwise construction of the partition of  $P$  for the case of five shells.

- Triangles that are incident to outer shells: The construction of the triangles proceeds from  $H_m$  to  $H_2$ . In particular, we iterate the following construction for  $i = m, \dots, 2$ : Let  $\langle v_1, \dots, v_k \rangle$  be a maximal sequence of points on  $\partial H_i$  that are connected by segments from  $P$ , such that no segment  $v_j v_{j+1}$  intersects the interior of  $H_{i-1}$ . Let  $v_0$  and  $v_{k+1}$  be the points before and after  $v_1$  and  $v_k$  on the boundary of  $P$ . Let  $\langle w_1, \dots, w_\ell \rangle$  be the sequence of vertices on  $H_{i-1}$  that lies between the segments  $v_0 v_1$  and  $v_k v_{k+1}$ . By walking simultaneously from  $v_1$  to  $v_k$  and from  $w_1$  to  $w_\ell$ , we triangulate the polygon that is bounded by  $\langle v_1, \dots, v_k \rangle$  and  $\langle w_1, \dots, w_\ell \rangle$ . We call the resulting triangles *type  $i$  regions*.

We remove all type  $i$  regions from  $P$  and repeat the above construction for  $i := i - 1$  until  $i = 1$ .

- Partition of the remaining parts: By the same argument as in the case of two shells we know that the remaining parts of  $P$  are convex polygons  $t \subseteq H_1$  that do not intersect each other. We call the resulting convex polygons *type 1 regions*.

**Lemma 23.** *Each region  $t \in T$  is adjacent to a point  $v \in P$  such that  $v \in G$ .*

**Proof.** All triangles that are not of type  $j$  are adjacent to a point  $v \in G$ . Thus we assume, without loss of generality, that  $t$  is of type  $j$ . By the same argument we are allowed to assume that all points of  $t$  lie on  $\partial H_j$ ; by the same argument as applied for type 1 regions in the case of two shells, it follows that at least one vertex of  $t$  is guarded. This concludes the proof.  $\square$

**Theorem 19.** *For each point set  $S$  that lies on  $m$  convex hulls we can compute in  $\mathcal{O}(n \log n)$  time a guard set  $G$  with  $|G| \leq (1 - \frac{1}{16|S|^{\frac{2m-1}{2m}}})|S|$ .*

#### 4. Bounds for the $k$ -Universal Guard Numbers

One of the difficulties of the general Universal Art Gallery Problem is the huge number of possible polygonalizations of a given set of  $n$  points; on the other hand, the classic Art Gallery Problem considers only *one* polygon. This makes it natural to investigate the dependency of the guard number on the number  $k$  of polygons that are to be covered.

In the following we state several lower and upper bounds for various  $k$ -universal guard numbers; recall that  $k$ -universal guard numbers are defined as follows: For a set  $A$  of  $k \geq 0$  given simple polygons, we denote by  $u_k(n)$  the minimum cardinality guard set for  $A$ . Analogously, for a set  $A$  of  $k \geq 0$  given polygons, we denote by  $h_k(n)$  the minimum cardinality guard set for  $A$ .

##### 4.1. Lower bounds for $u_k(n)$

**Theorem 24.**  $u_2(n) \geq \lfloor \frac{3n}{8} \rfloor$ .

In the proof of Theorem 24, we make use of a construction establishing a lower bound for the traditional Art Gallery Problem for one polygon  $P_{\text{holes}}$  with holes. In particular, we construct two simple polygons whose union covers  $P_{\text{holes}}$ .

**Proof.** For each  $n \in \mathbb{N}$  we give a pair of simple polygons that have a common set of vertices of size  $n$ , such that each guard set for  $\{P_1, P_2\}$  has a size of at least  $\lfloor \frac{3n}{8} \rfloor$ . This implies  $u_2(n) \geq \lfloor \frac{3n}{8} \rfloor$ .

Consider the polygon  $P$  that is illustrated in Fig. 9(a). Each guard set for  $P$  has size at least  $\lfloor \frac{3n}{8} \rfloor$ , where  $n$  is the number of vertices of  $P$ . We construct two

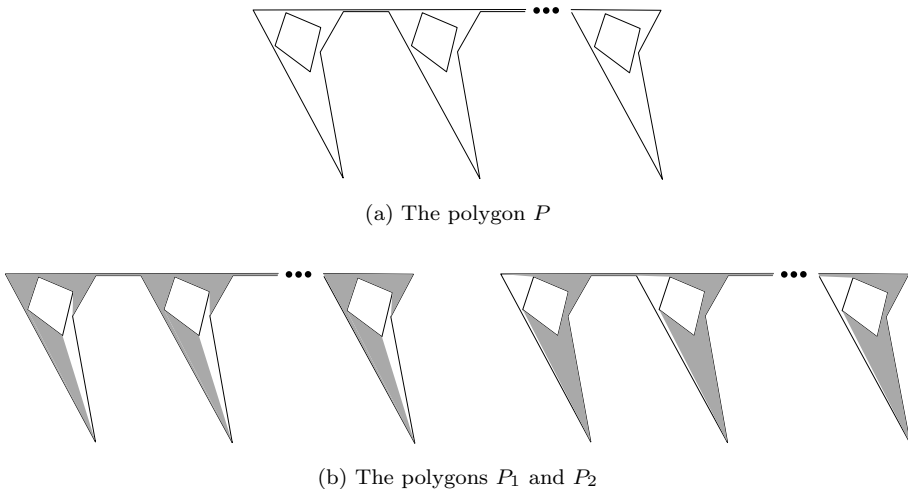


Fig. 9. A  $\frac{3n}{8}$  lower-bound construction for  $u_2(n)$ : Covering a  $\frac{3n}{8}$  lower-bound construction for  $h_1(n)$ .

polygons  $P_1$  and  $P_2$ , as illustrated in Fig. 9(b). We have  $P_1 \cup P_2 = P$  at which  $P_1$ ,  $P_2$ , and  $P$  have the same vertices. Furthermore, we have  $a \leftrightarrow_P b$  if  $a \leftrightarrow_{P_1} b$  and  $a \leftrightarrow_{P_2} b$ . Thus, a guard set for  $\{P_1, P_2\}$  is at least as large as a guard set for  $P$ . This concludes the proof.  $\square$

**Theorem 25.**  $u_3(n) \geq \lfloor \frac{4n}{9} \rfloor$ .

**Proof.** For each  $n \in \mathbb{N}$  we give a set of three simple polygons that have a common set of vertices of size  $n$ , such that each guard set for  $\{P_1, P_2, P_3\}$  has a size of at least  $\lfloor \frac{4n}{9} \rfloor$ . This implies  $u_3(n) \geq \lfloor \frac{4n}{9} \rfloor$ .

First, consider an example (see Fig. 10), with three simple polygons on a set of  $n = 9$  points. By a brute-force check of all  $\binom{9}{3}$  possible triples of points, we see

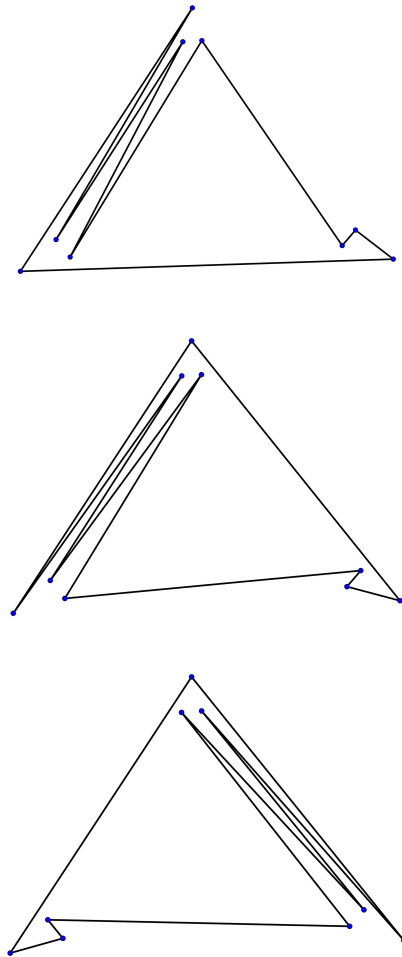


Fig. 10. The polygons  $P_1$ ,  $P_2$ ,  $P_3$  require four 3-universal guards for  $u_3(n)$  when  $n = 9$ .

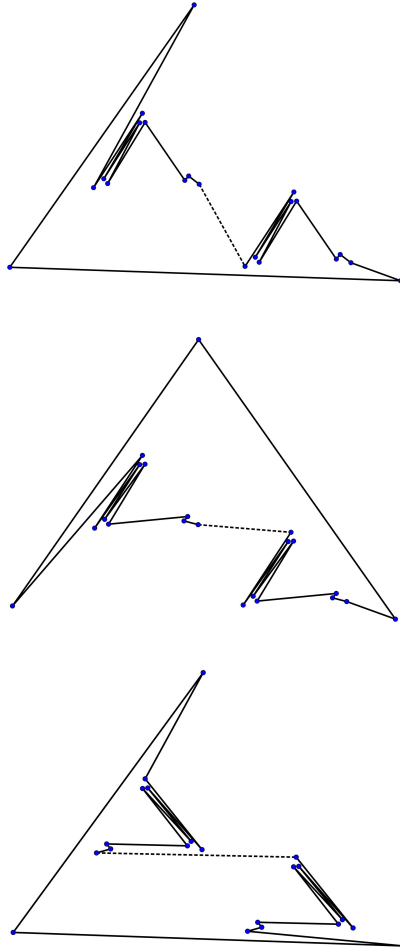


Fig. 11. The general polygons  $P_1, P_2, P_3$  require  $\lfloor \frac{4n}{9} \rfloor$  3-universal guards for  $\mathbf{u}_3(n)$ .

that three guards do not suffice to guard all three polygons; however, four guards easily do.

We extend the example (Fig. 11), by connecting copies of the polygons in Fig. 10 with the vertices of a much larger bounding triangle. In this way, for each point set of size nine, we need at least four guards; for large enough  $n$ , we can ignore the three vertices of the outer big triangle. This concludes the proof.  $\square$

**Theorem 26.**  $\mathbf{u}_5(n) \geq \mathbf{u}_4(n) \geq \frac{n}{2} - 8\sqrt{n} - 23$ .

**Proof.** For each  $n \in \mathbb{N}$  we give a set of four of simple polygons  $P_1, P_2, P_3$ , and  $P_4$  that have a common set of  $n$  vertices. Let  $G$  be a guard set for  $\{P_1, P_2, P_3, P_4\}$ . We show  $|G| \geq \frac{1}{2}n - 8\sqrt{n} - 23$ .

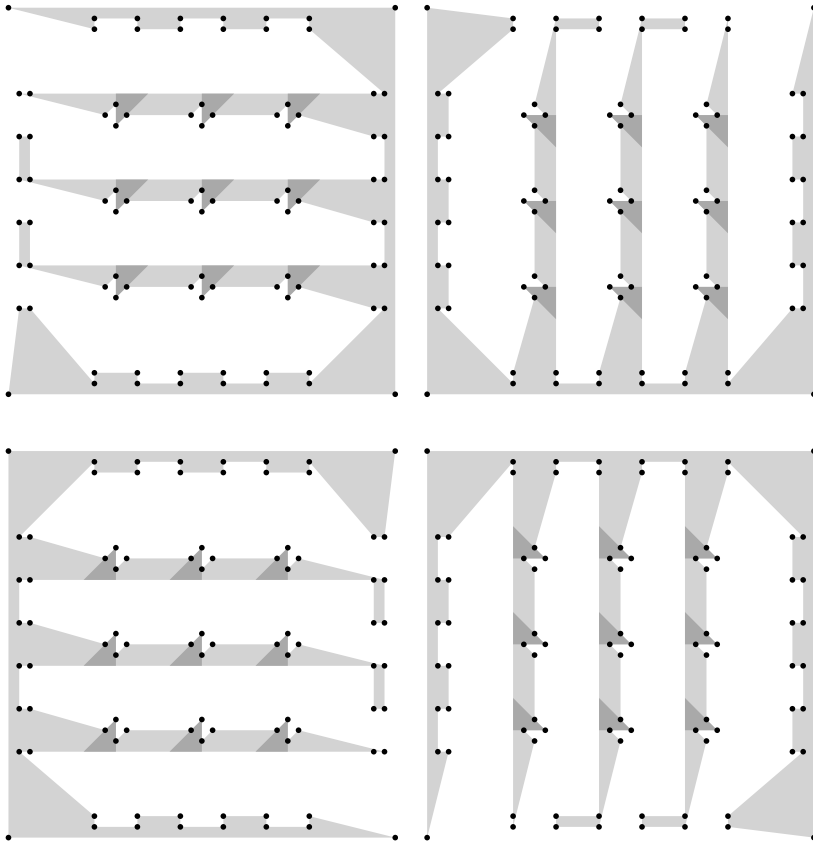


Fig. 12. Lower-bound construction of  $\frac{1}{2}n - 8\sqrt{n} - 4$  for  $k$ -universal guard numbers.

First, we give the required construction of  $P_1$ ,  $P_2$ ,  $P_3$ , and  $P_4$ , such that  $n = (4\ell)^2 + 16\ell + 4$  for  $\ell \in \mathbb{N}$  and show that a corresponding guard set needs at least  $\frac{n}{2} - 16\sqrt{n} - 4$  guards. Next we show how to extend the construction and the corresponding argument appropriately to an arbitrary  $n \in \mathbb{N}$ .

We construct  $P_1$ ,  $P_2$ ,  $P_3$ , and  $P_4$ , as illustrated in Fig. 12. The vertices in the middle block are structured in groups of size four. Assume that one of these groups has only one guarded point. This implies that the other points are unguarded and thus build an unguarded area in  $P_1$ ,  $P_2$ ,  $P_3$ , or  $P_4$ , as illustrated by the dark gray cones. Hence, each of these groups has two guarded points. This implies  $|G| \geq \frac{1}{2}(4\ell)^2 = \frac{n}{2} - 8\ell - 2 \geq \frac{n}{2} - 8\sqrt{n} + 4$ , because  $16\ell^2 + 16\ell + 4 = n$ .

Finally, we give an extension of the above approach to an arbitrary  $n \in \mathbb{N}$ . Let  $\ell$  be the largest integer such that  $16\ell^2 + 16\ell + 4 \leq n$ . We apply the above construction for  $16\ell^2 + 16\ell + 4$  points. All the remaining points are added in a new row and column of the above construction. The worst case for that approach is

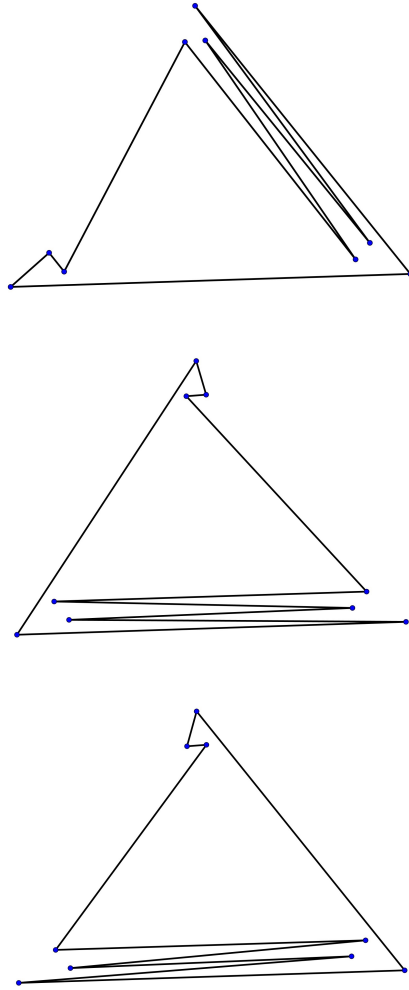


Fig. 13. The polygons  $P_4, P_5, P_6$ , together with  $P_1, P_2, P_3$ , require five 6-universal guards for  $u_6(n)$  when  $n = 9$ .

$n = 16\ell^2 + 16\ell + 4 + 19$ , as 19 additional points are needed until the first new guard is enforced. This concludes the proof.  $\square$

**Theorem 27.**  $u_k(n) \geq \lfloor \frac{5n}{9} \rfloor$  for  $k \geq 6$ .

**Proof.** This proof is similar to the proof of Theorem 25. In addition to  $P_1, P_2, P_3$  in Fig. 10, we add three more polygons  $P_4, P_5, P_6$ ; refer to Fig. 13. Then, by the same argument, the polygons  $P_4, P_5, P_6$ , together with  $P_1, P_2, P_3$ , require 5 6-universal guards. The extensions are also similar since they are essentially symmetric to the three polygons in Theorem 25. This concludes the proof.  $\square$

## 4.2. Upper bounds for $k$ -universal guard numbers

We give non-trivial upper bounds for  $\mathbf{u}_k(n)$  and  $\mathbf{h}_k(n)$ , for all values  $n, k \in \mathbb{N}$ . In particular, we provide algorithms that efficiently compute guard sets for  $\mathcal{P}(S)$  and  $\mathcal{H}(S)$  for any given  $S \in \mathcal{S}(n)$  and analyze the computed guard sets.

Broadly speaking, our algorithms implement the following iterative approach. Let  $S \in \mathcal{S}(n)$  and  $\{P_1, \dots, P_k\}$  be a set of  $k$  polygons on which the considered Art Gallery Problem is to be solved. We iteratively process each polygon  $P_i$  in the sequence  $\langle P_1, \dots, P_k \rangle$ ; after processing  $P_i$ , we guarantee that the current set  $G$  is a guard set for  $\{P_1, \dots, P_i\}$ . In processing  $P_i$ , let  $A$  be the union of the portions of  $P_i$  that are not yet guarded by the current set  $G \subseteq S$  of guards obtained already, while processing  $\langle P_1, \dots, P_{i-1} \rangle$ . We simply solve the corresponding Art Gallery Problem for coverage of  $A$  and add the appropriately selected guards to the set  $G$ .

By realizing the above approach for simple polygons, we show the following.

**Theorem 28.**  $\mathbf{u}_k(n) \leq (1 - (\frac{2}{3})^k)n$ .

**Proof.** Let  $\{P_1, \dots, P_k\} \subset \mathcal{P}(S)$ . We compute a guard set for  $\{P_1, \dots, P_k\}$  by the above approach by iteratively processing each polygon  $P_i$  as follows. We compute a triangulation  $T$  of  $P_i$  and then a 3-coloring of  $T$ . Let “red” be a corresponding smallest color w.r.t.  $S \setminus G$ . We add all points from  $S \setminus G$  to  $G$  that are colored red.

By induction it follows that  $G$  is a guard set for  $\{P_1, \dots, P_k\}$ , because the red points guard all triangles of  $T$ . Furthermore, all guards that were previously selected to guard  $\{P_1, \dots, P_{i-1}\}$  are still elements of  $G$  after  $P_i$  is processed.

By induction it follows that  $|G| \leq (1 - (\frac{2}{3})^k)n$ , because the number of the unselected points after  $i$  iterations is lower bounded by  $(\frac{2}{3})^i n$ .  $\square$

Hoffmann *et al.*<sup>15</sup> showed  $\mathbf{h}_1(n) \leq \lfloor \frac{3n}{8} \rfloor$ . Our approach implies for the traditional guard number  $\mathbf{h}_1(n) \leq \lfloor \frac{n}{2} \rfloor$ .

The following theorem shows that we can combine our approach with the method from Ref. 15.

**Theorem 29.**  $\mathbf{h}_k(n) \leq (1 - (\frac{5}{8})^k)n$ .

**Proof.** Let  $\{P_1, \dots, P_k\} \subset \mathcal{P}(S)$ . We compute a guard set for  $\{P_1, \dots, P_k\}$  as described above, in which we iteratively process each polygon  $P_i$  as follows. We compute a triangulation  $T$  of  $P_i$ . Let  $A$  be the union of all triangles that are not adjacent to a point from  $G$ . This means that  $A$  forms a union of polygons, which may have holes. Two of these polygons share at most a discrete set of points, but not edges. The approach of Ref. 15 computes a guard set for one polygon  $P$  with holes by determining a covering of  $P$  by convex subpolygons, called *rooms*. Two rooms  $r_1, r_2 \subset P$  are connected by a *door* that corresponds to the area in which  $r_1$  and  $r_2$  overlap. Hoffmann *et al.*<sup>15</sup> compute a guard set by placing guards at vertices that lie inside doors. They consider three different classes of doors. One type of doors corresponds to so-called *V-shapes*. A *V-shape* is defined by a concave vertex



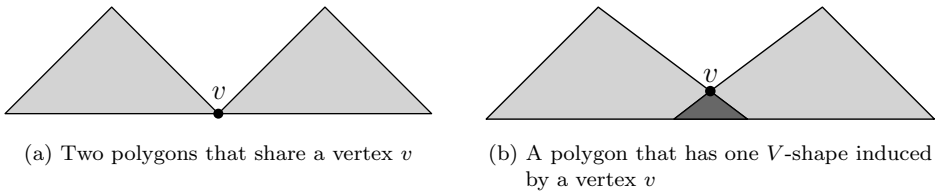


Fig. 14. Transformation of adjacencies between polygons by building V-shapes.

of  $P$ , such that the extension of its adjacent edges of the boundary of  $P$  intersects a common third edge from  $\partial P$ , see Fig. 14(b).

We choose a maximal set  $\{P'_1, \dots, P'_\ell\}$  of polygons from  $A$ , such that there is no other polygon  $P' \in A$  that is adjacent to a polygon from  $\{P'_1, \dots, P'_\ell\}$ . Now we apply a modified version of the approach from<sup>15</sup> to  $P'_1 \cup \dots \cup P'_\ell$ . In particular, we also identify vertices, in which two polygons from  $P'_1, \dots, P'_\ell$  are adjacent, as V-shapes.

It follows by induction, by the correctness of the approach from Ref. 15, that the final  $G$  is a guard set for  $\{P_1, \dots, P_k\}$ .

The approach from Ref. 15 computes a guard set of size  $\lfloor \frac{3n'}{8} \rfloor$  for one polygon, where  $n'$  is the complexity of the considered polygon. Thus, it follows by induction that  $|G| \leq (1 - (\frac{5}{8})^k)n$  holds, because the number of the unselected points after  $i$  iterations is lower bounded by  $(\frac{5}{8})^i n$ .  $\square$

## 5. Other Variants

In this section, we consider two variants of the Universal Art Gallery Problem: the case in which guards are allowed to be placed only at input points  $S$  that are interior to the convex hull of  $S$ , and the case in which the input set  $S$  is a regular grid of points. In both cases we obtain tight bounds on the universal guard number.

### 5.1. Interior guards

In the Interior Universal Guards Problem (IUGP) we allow guards to be placed only at points of  $S$  that are not convex hull vertices of  $S$ . Note that placing guards at *all* interior points is sufficient to guard any polygonalization of  $S$ , since the  $CH(S)$  vertices are convex vertices in any polygonalization of  $S$ ; it is a simple fact is that the reflex vertices of any simple polygon see all of the polygon. Our main result in this section is a proof that it is sometimes necessary to place guards at all interior points, in order to have a universal guard set.

**Theorem 30.** *The interior universal guard number satisfies  $i(n) = n - \Theta(1)$ . In particular, there exist configurations of  $n$  points  $S$ , for arbitrarily large  $n$ , for which  $CH(S)$  is a triangle, and the only universal guard set using only interior guards is the set of all  $n - 3$  interior points.*

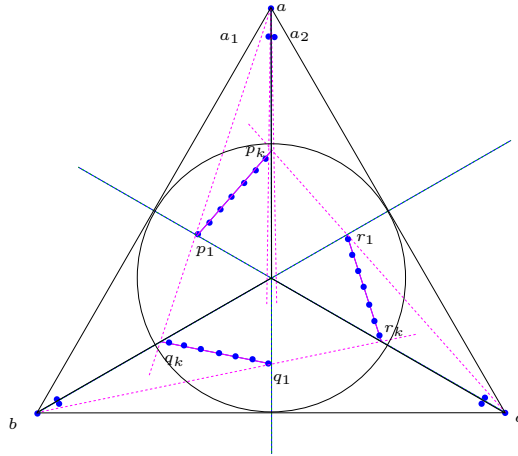


Fig. 15. The construction of the instance showing that for some input sets  $S$  of  $n = 9 + 3k$  points, if guards are not allowed to be placed at convex hull vertices, then all interior points of  $S$  may be required to be in a universal guard set.

**Proof.** Figure 15 shows the structure of the construction. The set  $S$  consists of the following  $n = 9 + 3k$  points:

- $a, b, c$ , which are the vertices of the convex hull of  $S$ ; in the example in the figure, the triangle  $\Delta abc$  is equilateral;
- three pairs of points, with  $a_1, a_2$  very close to  $a$ ,  $b_1, b_2$  very close to  $b$ , and  $c_1, c_2$  very close to  $c$ ; these 6 points are in convex position;
- three sets of  $k$  points, with each set of points collinear, and the set of  $3k$  points in convex position; denote the points by  $p_1, \dots, p_k$ ,  $q_1, \dots, q_k$ , and  $r_1, \dots, r_k$ , with the points indexed in order along the segments  $p_1p_k$ ,  $q_1q_k$ , and  $r_1r_k$ .

(Our construction is for values of  $n$  of the form  $n = 9 + 3k$ , for positive integers  $k$ ; our construction can likely be modified for values of  $n \neq 9 + 3k$ .)

In more details, the properties of the point configuration are as follows.

- (1) All of the points  $p_i$  lie to the right of the oriented line through  $aa_1$ ; similarly, points  $q_i$  are to the right of  $bb_1$  and points  $r_i$  are to the right of  $cc_1$ .
- (2) The line  $ap_i$  passes between points  $q_{k-i+1}$  and  $q_{k-i+2}$ . (A similar statement holds for lines  $bq_i$  and  $cr_i$ .) The line  $aq_i$  passes between points  $p_{k-i+1}$  and  $p_{k-i+2}$ . (A similar statement holds for lines  $br_i$  and  $cp_i$ .) We call this the “interleaving rays property”. See Fig. 15(right).

In order to argue that such a configuration exists, for arbitrarily large  $k$  (and thus for arbitrarily large  $n = 9 + 3k$ ), we give a procedure for placing the points  $p_i, q_i, r_i$  along their respective segments. We begin with a placement of points with  $k = 2$ , as shown, zoomed in, in Fig. 16. (The point  $q_1$  is shown collinear with  $a$  and

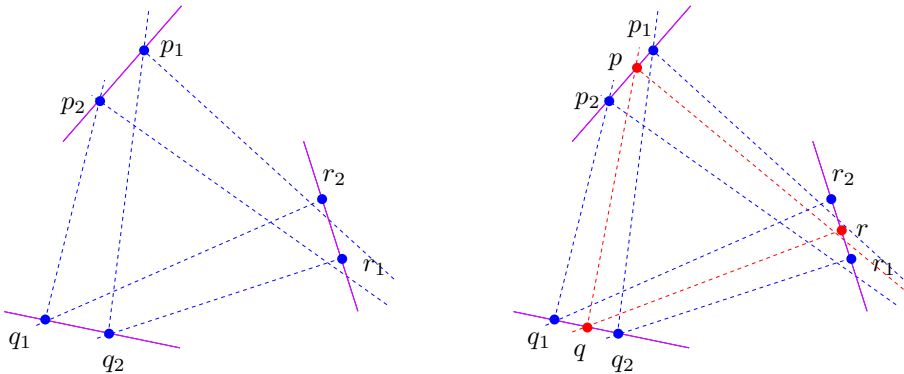


Fig. 16. A zoomed-in view of the construction in Fig. 15. Left: Placement of  $k = 2$  points on each of the three segments, in order that the interleaving rays property holds for this small instance. Right: Addition of the intermediate points  $p, q, r$  on the three segments, while preserving the interleaving rays property.

$p_2$ , and  $r_2$  is shown collinear with  $b$  and  $q_1$ ; however, the point  $q_1$  is just to the left (by an arbitrarily small amount) of the oriented line  $ap_2$ , and  $r_2$  is just to the left of oriented line  $bq_1$ . Similarly,  $q_2$  is just left of  $ap_1$  and  $r_1$  is just left of  $bq_2$ , etc.) Then, we claim that we can place new points  $p$  between  $p_1$  and  $p_2$ ,  $q$  between  $q_1$  and  $q_2$ , and  $r$  between  $r_1$  and  $r_2$ , while preserving the interleaving rays property. See Fig. 16(right). (The existence of such a point  $p$  along the segment  $p_1p_2$  follows from the intermediate value theorem: as  $p$  varies from  $p_1$  to  $p_2$ , the corresponding position of  $r$  (on  $r_1r_2$ , just to the left of where line  $bq$  intersects  $r_1r_2$ , where  $q$  is the point on  $q_1q_2$ , just to the left of where line  $ap$  intersects  $q_1q_2$ ) varies from  $r_1$  (which is below  $cp_1$ ) to the point  $r_2$  (which is above  $cp_2$ )). We then reindex the points to be  $p_1, p_2, p_3, q_1, q_2, q_3$ , and  $r_1, r_2, r_3$ . We then apply this argument recursively to place 2 new points in the 2 gaps (along segments  $p_1p_2, p_2p_3, q_1q_2, q_2q_3, r_1r_2$ , and  $r_2r_3$ ), and repeat, placing 4 new points in 4 gaps, then 8 new points, etc. Doing so allows the instance size to grow (exponentially) with each iteration, showing that the construction yields arbitrarily large instances.

We claim that every point of  $S$  interior to the convex hull of  $S$  must have a guard in any universal guarding that is not allowed to place guards at the convex hull vertices ( $a, b, c$ ). To see this, we show polygonalizations that would have some portion of the polygon unguarded if not all interior points of  $S$  were guarded. In Fig. 17(left) we give a polygonalization of  $S$  showing that if  $a_1$  is not guarded, then, even if all other interior points are guarded, a portion of the polygon (shown in gray) is not seen. In Fig. 17(right), we give a polygonalization of  $S$  showing that if  $p_i$  is not guarded, then, even if all other interior points are guarded, a portion of the polygon (shown in gray) is not seen. Finally, we remark that it was not necessary for the point configuration to be degenerate, with groups of collinear points; the construction can be done with points in general position.  $\square$

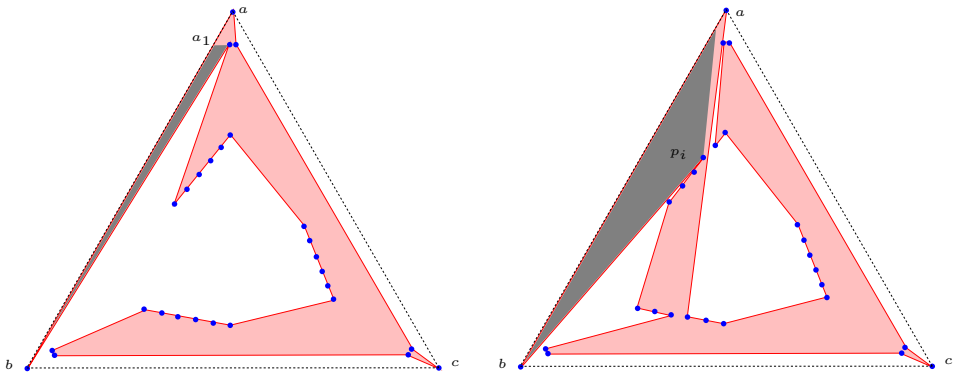


Fig. 17. Left: A polygonalization of  $S$  showing that if  $a_1$  is not guarded, then, even if all other interior points are guarded, a portion (in gray) of the polygon is not seen. Right: A polygonalization of  $S$  showing that if  $p_i$  is not guarded, then, even if all other interior points are guarded, a portion (in gray) of the polygon is not seen.

We remark that the configuration of points  $S$  given in the proof above can be universally guarded with approximately  $n/2$  guards, if we permit guards at the three convex hull vertices: With guards at  $a, b, c$ , and at the 6 points  $a_1, a_2, b_1, b_2, c_1, c_2$ , we need only to place guards at every other point in the collinear sequences  $p_1, p_2, \dots, p_k$ ,  $q_1, q_2, \dots, q_k$ , and  $r_1, r_2, \dots, r_k$ , in order to guard  $S$  universally, as one can readily check. Thus, the reason so many guards ( $|S| - 3$ ) were needed was because of the requirement to avoid guarding the convex hull vertices.

## 5.2. Full grid sets

A natural special case arises when considering universal guards for a set  $S$  of points that are the  $n = n_x \times n_y$  set of grid points (within a rectangle) on an integer lattice. For this case we achieve a tight worst-case bound.

**Theorem 31.**  $g(n) = \lfloor \frac{n}{2} \rfloor$ , for rectangular grids of  $n = n_x \times n_y$  grid points, with each of  $n_x, n_y$  above a constant.

**Proof.** There are two parts to the proof: First, we must show that  $\lfloor \frac{n}{2} \rfloor$  guards suffice to guard a set  $S$  of grid points (that is sufficiently large). Second, we must show necessity of  $\lfloor \frac{n}{2} \rfloor$  guards, arguing that fewer guards than this will result in the lack of full coverage for some polygonalizations of  $S$ .

The proof of sufficiency (that  $\lfloor \frac{n}{2} \rfloor$  guards suffice for universal guarding) is based on either of two different patterns of guard selection: (1) place guards at the odd positions on odd-numbered rows and at even positions on even-numbered rows of the grid (i.e., place guards in the grid according to white squares on a checkboard); or (2) place guards at all positions on the even-numbered rows. Both (1) and (2) place  $\lfloor \frac{n}{2} \rfloor$  guards. The two methods to place guards are shown in Fig. 18. In order to show that these placements yield universal guard sets (i.e., guard every possible

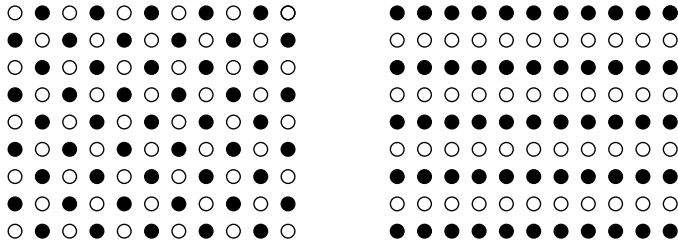


Fig. 18. Placement of guards at grid points according to pattern (1), left, and pattern (2), right. Hollow dots denote guarded points, and solid dots denote unguarded points.

polygonalization of the input points), we argue that, for either of the two placement strategies, any *empty grid triangle* (i.e., a triangle whose vertices are grid points, with no other grid points interior to the triangle or on its boundary) must have at least one of its three vertices guarded. This then implies that any polygonalization  $P$  of the grid points  $S$  is guarded, since any such simple polygon  $P$  has a triangulation, whose triangles are empty grid triangles, every one of which has a guard on at least one corner. Since  $P$  has a triangulation with nondegenerate triangles (having nonempty interiors), we restrict ourselves to nondegenerate empty grid triangles.

We give the argument for placement method (1); the argument is very similar for placement method (2). Consider a (nondegenerate) empty grid triangle,  $\Delta abc$ , and assume, for contradiction, that all three of its vertices  $\{a, b, c\}$  are unguarded according to the placement scheme (1). Then, since  $\Delta abc$  is an empty grid triangle, the parallelogram defined by the pair of (integral) vectors  $b - a$  and  $c - a$  has no grid points on its interior or on its boundary segments, other than at the vertices  $a$ ,  $b$ ,  $c$ , and  $b + (c - a)$ , all of which are unguarded. Since these parallelograms tile the plane, this implies that all grid points are unguarded, a contradiction.

The proof of necessity is based on examining local configurations of unguarded grid points that force certain grid points to be guarded, in order that every polygonalization is fully guarded. In particular, we observe that if a grid point  $a \in S$  (that is not one of the 4 corners of the bounding rectangle of  $S$ ) has both an unguarded horizontal neighbor and an unguarded vertical neighbor, then a polygonalization of  $S$  that connects these three unguarded points in order can result in a chamber (causing an unseen triangular region), even if all other points of  $S$  are guarded. See Fig. 19 for an example, showing locally a portion (three edges) of the polygonalization that leaves an unguarded portion (shaded gray). The full polygonalization of each local configuration is shown by checking each of the cases, to see that each can be fully polygonalized within a large enough rectangular grid: a 4-by-5 point grid is sufficient to contain each local configuration as part of a full polygonalization of the 4-by-5 grid. Then, if the grid set  $S$  is sufficiently large to contain a 4-by-5 subgrid, we claim that the grid points  $S$  have a polygonalization that would leave a chamber if three such grid points (point  $a$  and one of its horizontal and one of its vertical neighbors) are unguarded. Refer to Fig. 20 for an illustration of the

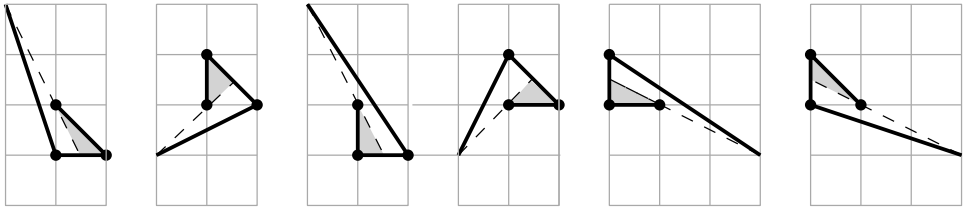


Fig. 19. An unguarded grid point and its unguarded neighbors above and to the right of it can result in a chamber (causing an unseen region (shaded)) in a polygonalization. Here, only 3 edges of the polygonalization are shown.

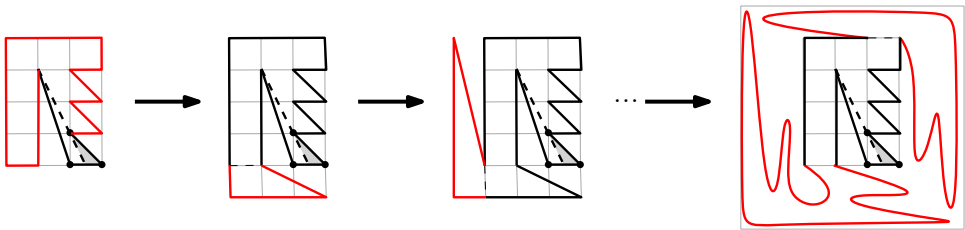


Fig. 20. Demonstrating that a local configuration of three unguarded points,  $a$  and a horizontal and a vertical neighbor of  $a$ , has a polygonalization that leaves a chamber (causing an unseen region (shown shaded)), even if all other grid points are guarded. First, we show a polygonalization within a 4-by-5 subgrid, then we illustrate how a larger containing grid  $S$  admits a polygonalization.

polygonalization of a 4-by-5 subgrid, and its extension to a polygonalization to the full grid  $S$ . Thus, for a sufficiently large grid  $S$ , any 2-by-2 subgrid of  $S$  (not in one of the 4 corners of the bounding rectangle of  $S$ ) must have at least two of its four grid points guarded.  $\square$

## 6. Conclusion

There are many open problems that are interesting challenges for future work. In particular, can the upper bound approaches for  $\mathbf{u}_k(n)$  and  $\mathbf{h}_k(n)$  be improved by making use of the number of shells? Can the general approach of Theorem 28 be improved? What about lower bounds for  $k$ -UGP for  $k \geq 7$ ?

The combinatorial lower-bound constructions provided in Sec. 3.1 make use of coordinates of exponential value (but polynomial encoding size). Do the bounds change if the range of coordinate values is polynomially bounded? This is related to the grid problems studied in Sec. 5.2. An additional question arises from grid sets in higher dimensions; given their simplicity, they seem amenable to extensions to 3D, and even to higher dimensions.

Is the bound of  $\frac{1}{2}n$  for the intersection-free  $k$ -UGP tight? Further questions consider the setting in which each vertex  $v$  has a bounded candidate set of vertices that may be adjacent to  $v$ . Other variants arise when the ratio of the lengths of the edges of the considered polygons is upper- and lower-bounded by given constants. It may also be interesting to explore possible relations between universal guard

problems and universal graphs. Finally, there are many interesting algorithmic questions about computing universal guard sets that are left for future work.

## Acknowledgments

We thank Joseph O'Rourke for comments and suggestions that improved the paper. We thank members of the Computational Geometry Group at Stony Brook for early discussions on the topic of this paper. Q. Li and J. Mitchell are partially supported by the National Science Foundation (CCF-1526406).

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