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CONFLICT-FREE COLORING OF GRAPHS[*]

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Abstract. A *conflict-free k-coloring* of a graph assigns one of k different colors to some of the vertices such that, for every vertex v, there is a color that is assigned to exactly one vertex among v and v's neighbors. Such colorings have applications in wireless networking, robotics, and geometry and are well studied in graph theory. Here we study the natural problem of the *conflict-free chromatic number* $\chi_{CF}(G)$ (the smallest k for which conflict-free k-colorings exist). We provide results both for *closed* neighborhoods $N[v]$, for which a vertex v is a member of its neighborhood, and for *open* neighborhoods $N(v)$, for which vertex v is not a member of its neighborhood. For closed neighborhoods, we prove the conflict-free variant of the famous Hadwiger Conjecture: If an arbitrary graph G does not contain K_{k+1} as a minor, then $\chi_{CF}(G) \leq k$. For planar graphs, we obtain a tight worst-case bound: three colors are sometimes necessary and always sufficient. In addition, we give a complete characterization of the algorithmic/computational complexity of conflict-free coloring. It is NP-complete to decide whether a planar graph has a conflict-free coloring with *one* color, while for outerplanar graphs, this can be decided in polynomial time. Furthermore, it is NP-complete to decide whether a planar graph has a conflict-free coloring with *two* colors, while for outerplanar graphs, two colors always suffice. For the *bicriteria* problem of minimizing the number of colored vertices subject to a given bound k on the number of colors, we give a full algorithmic characterization in terms of complexity and approximation for outerplanar and planar graphs. For open neighborhoods, we show that every planar *bipartite* graph has a conflict-free coloring with at most four colors; on the other hand, we prove that for $k \in \{1, 2, 3\}$, it is NP-complete to decide whether a planar bipartite graph has a conflict-free k-coloring. Moreover, we establish that any *general* planar graph has a conflict-free coloring with at most eight colors.

Key words. conflict-free coloring, planar graphs, complexity, worst-case bounds

AMS subject classifications. 05C10, 05C15, 05C69, 05C83

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1. Introduction.

Coloring the vertices of a graph is one of the fundamental problems in graph theory, both scientifically and historically. Proving that four colors always suffice to color a planar graph [6, 7, 27] was a tantalizing open problem for more than 100 years; the quest for solving this challenge contributed to the development of graph theory, but also to computers in theorem proving [29]. A generalization that is still unsolved is the *Hadwiger Conjecture* [19]: A graph is k-colorable if it has no K_{k+1} minor.

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Over the years, there have been many variations on coloring, often motivated by particular applications. One such context is wireless communication, where "colors" correspond to different frequencies. This also plays a role in robot navigation, where different beacons are used for providing direction. To this end, it is vital that in any given location, a robot is adjacent to a beacon with a frequency that is unique among the ones that can be received. This notion has been introduced as *conflict-free coloring*, formalized as follows. For any vertex $v \in V$ of a simple graph $G = (V, E)$, the *closed neighborhood* $N[v]$ consists of all vertices adjacent to v and v itself. A *conflict-free k-coloring* of G assigns one of k different colors to a (possibly proper) subset $S \subseteq V$ of vertices, such that for every vertex $v \in V$, there is a vertex $y \in N[v]$, called the *conflict-free neighbor* of v, such that the color of y is unique in the closed neighborhood of v. The *conflict-free chromatic number* $\chi_{CF}(G)$ of G is the smallest k for which a conflict-free coloring exists. Observe that $\chi_{CF}(G)$ is bounded from above by the proper chromatic number $\chi(G)$ because in a proper coloring, every vertex is its own conflict-free neighbor. Similar questions can be considered for *open neighborhoods* $N(v) = N[v] \setminus \{v\}$.

Conflict-free coloring has received an increasing amount of attention. Because of the relationship to classic coloring, it is natural to investigate the conflict-free coloring of planar graphs. In addition, previous work has considered either general graphs and hypergraphs (e.g., see [26]) or geometric scenarios (e.g., see [21]); we give a more detailed overview in what follows. This adds to the relevance of conflict-free coloring of planar graphs, which constitute the intersection of general graphs and geometry. In addition, the subclass of outerplanar graphs is of interest, as it corresponds to subdividing simple polygons by chords.

There is a spectrum of different scientific challenges when studying conflict-free coloring. What are worst-case bounds on the necessary number of colors? When is it NP-hard to determine the existence of a conflict-free k-coloring? When is it polynomially solvable? What can be said about approximation? Are there sufficient conditions for more general graphs? And what can be said about the bicriteria problem, in which also the number of colored vertices is considered? We provide extensive answers for all of these aspects, basically providing a complete characterization for planar and outerplanar graphs.

1.1. Our contribution.

We present the following results; items 1–7 are for closed neighborhoods, while items 8–10 are for open neighborhoods.

1. For general graphs, we provide the conflict-free variant of the Hadwiger Conjecture: If G does not contain K_{k+1} as a minor, then $\chi_{CF}(G) \leq k$.
2. It is NP-complete to decide whether a planar graph has a conflict-free coloring with *one* color. For outerplanar graphs, this question can be decided in polynomial time.
3. It is NP-complete to decide whether a planar graph has a conflict-free coloring with *two* colors. For outerplanar graphs, two colors always suffice.
4. Three colors are sometimes necessary and always sufficient for conflict-free coloring of a planar graph.
5. For the bicriteria problem of minimizing the number of colored vertices subject to a given bound $\chi_{CF}(G) \leq k$ with $k \in \{1, 2\}$, we prove that the problem is NP-hard for planar and polynomially solvable in outerplanar graphs.
6. For planar graphs and $k = 3$ colors, minimizing the number of colored vertices does not have a constant-factor approximation unless P = NP.

7. For planar graphs and $k \geq 4$ colors, it is NP-complete to minimize the number of colored vertices. The problem is fixed-parameter tractable (FPT) and allows a polynomial-time approximation scheme (PTAS).
8. Four colors are sometimes necessary and always sufficient for conflict-free coloring with open neighborhoods of planar bipartite graphs.
9. It is NP-complete to decide whether a planar bipartite graph has a conflict-free coloring with open neighborhoods with k colors for $k \in \{1, 2, 3\}$.
10. Eight colors always suffice for conflict-free coloring with open neighborhoods of planar graphs.

1.2. Related work.

In a geometric context, the study of conflict-free coloring was started by Even et al. [16] and Smorodinsky [28], who motivate the problem by frequency assignment in cellular networks: There, a set of n base stations is given, each covering some geometric region in the plane. The base stations service mobile clients that can be at any point in the total covered area. To avoid interference, there must be at least one base station in range using a unique frequency for every point in the entire covered area. The task is to assign a frequency to each base station minimizing the number of frequencies. On an abstract level, this induces a coloring problem on a hypergraph where the base stations correspond to the vertices and there is a hyperedge between some vertices if the range of the corresponding base stations has a nonempty common intersection.

If the hypergraph is induced by disks, Even et al. [16] prove that $\mathcal{O}(\log n)$ colors are always sufficient. Alon and Smorodinsky [5] extend this by showing that each family of disks, where each disk intersects at most k others, can be colored using $\mathcal{O}(\log^3 k)$ colors. Furthermore, for unit disks, Lev-Tov and Peleg [24] present an $\mathcal{O}(1)$-approximation algorithm for the number of colors. Horev, Krakovski, and Smorodinsky [22] extend this by showing that any set of n disks can be colored with $\mathcal{O}(k \log n)$ colors, even if every point must see k distinct unique colors. Abam, de Berg, and Poon [1] discuss the problem in the context of cellular networks where the network has to be reliable even if some number of base stations fail, giving worst-case bounds for the number of colors required.

For the dual problem of coloring a set of points such that each region from some family of regions contains at least one uniquely colored point, Har-Peled and Smorodinsky [20] prove that with respect to every family of pseudo-disks, every set of points can be colored using $\mathcal{O}(\log n)$ colors. For rectangle ranges, Elbassioni and Mustafa [15] show that it is possible to add a sublinear number of points such that a conflict-free coloring with $\mathcal{O}(n^{3/8\cdot(1+\varepsilon)})$ colors becomes possible. Ajwani et al. [3] complement this by showing that coloring a set of points with respect to rectangle ranges is always possible using $\mathcal{O}(n^{0.382})$ colors. For coloring points on a line with respect to intervals, Cheilaris et al. [11] present a 2-approximation algorithm, and a $\left(5 - \frac{2}{k}\right)$-approximation algorithm when every interval must see k uniquely colored vertices. Hoffman et al. [21] give tight bounds for the conflict-free chromatic art gallery problem under rectangular visibility in orthogonal polygons: $\Theta(\log \log n)$ are sometimes necessary and always sufficient. Chen et al. [14] consider the online version of the conflict-free coloring of a set of points on the line, where each newly inserted point must be assigned a color upon insertion, and at all times the coloring has to be conflict-free. Also in the online scenario, Bar-Noy et al. [10] consider a certain class of k-degenerate hypergraphs which sometimes arise as intersection graphs of geometric objects, presenting an online algorithm using $\mathcal{O}(k \log n)$ colors.

On the combinatorial side, some authors consider the variant in which all vertices need to be colored; note that this does not change asymptotic results for general graphs and hypergraphs: it suffices to introduce one additional color for vertices that are left uncolored in our constructions. Regarding general hypergraphs, Ashok, Dudeja, and Kolay [8] prove that maximizing the number of conflict-freely colored edges in a hypergraph is FPT when parameterized by the number of conflict-free edges in the solution. Cheilaris, Smorodinsky, and Sulovsky [12] consider the case of hypergraphs induced by a set of planar Jordan regions and prove an asymptotically tight upper bound of $\mathcal{O}(\log n)$ for the conflict-free list chromatic number of such hypergraphs. They also consider hypergraphs induced by the simple paths of a planar graph and prove an upper bound of $\mathcal{O}(\sqrt{n})$ for the conflict-free list chromatic number. For hypergraphs induced by the paths of a simple graph G, Cheilaris and Tóth [13] prove that it is coNP-complete to decide whether a given coloring is conflict-free if the input is G. Regarding the case in which the hypergraph is induced by the neighborhoods of a simple graph G, which resembles our scenario, Pach and Tárdos [26] prove that the conflict-free chromatic number of an n-vertex graph is in $\mathcal{O}(\log^2 n)$. Glebov, Szabó, and Tardos [18] extend this from an extremal and probabilistic point of view by proving that almost all $G(n,p)$-graphs have conflict-free chromatic number $\mathcal{O}(\log n)$ for $p \in \omega(1/n)$, and by giving a randomized construction for graphs having conflict-free chromatic number $\Theta(\log^2 n)$. In more recent work, Gargano and Rescigno [17] show that finding the conflict-free chromatic number for general graphs is NP-complete, and prove that the problem is FPT with respect to vertex cover or neighborhood diversity number.

2. Preliminaries.

For every vertex $v \in V$, the *open neighborhood* of v in G is denoted by $N_G(v) := \{w \in V(G) \mid vw \in E(G)\}$, and the *closed neighborhood* is denoted by $N_G[v] := N_G(v) \cup \{v\}$. We sometimes write $N(v)$ instead of $N_G(v)$ when G is clear from the context.

A partial k-coloring of G is an assignment $\chi : V' \to \{1, \dots, k\}$ of colors to a subset $V' \subseteq V(G)$ of the vertices. χ is called *closed-neighborhood conflict-free k-coloring* of G iff, for each vertex $v \in V$, there is a vertex $w \in N_G[v] \cap V'$ such that $\chi(w)$ is unique in $N_G[v]$, i.e., for all other $w' \in N_G[v] \cap V'$, $\chi(w') \neq \chi(w)$. We call w the conflict-free neighbor of v. Analogously, χ is called *open-neighborhood conflict-free k-coloring* of G iff, for each vertex $v \in V$, there is a conflict-free neighbor $w \in N_G(v)$.

In order to avoid confusion with *proper k-colorings*, i.e., colorings that color all vertices such that no adjacent vertices receive the same color, we use the term *proper coloring* when referring to this kind of coloring. The minimum number of colors needed for a proper coloring of G, also known as the chromatic number of G, is denoted by $\chi_P(G)$, whereas the minimum number of colors required for a closed-neighborhood conflict-free coloring of G (G's *closed-neighborhood conflict-free chromatic number*) is written as $\chi_{CF}(G)$. The *open-neighborhood conflict-free chromatic number* of G is $\chi_O(G)$. To improve readability we sometimes omit the type of neighborhood if it is clear from the context.

Note that, because every vertex satisfies $v \in N[v]$, every proper coloring of G is also a closed-neighborhood conflict-free coloring of G, and thus $\chi_{CF}(G) \leq \chi_P(G)$. The same does not hold for open neighborhoods. There is no constant factor $c_1 > 0$ such that either $c_1 \cdot \chi_O(G) \leq \chi_P(G)$ or $c_1 \cdot \chi_P(G) \leq \chi_O(G)$ holds for all graphs G.

For closed neighborhoods, we define the *conflict-free domination number* $\gamma^k_{CF}(G)$ of G to be the minimum number of vertices that have to be colored in a conflict-free k-coloring of G. We set $\gamma^k_{CF}(G) = \infty$ if G is not conflict-free k-colorable. Because the set of colored vertices is a dominating set, the conflict-free domination number satisfies $\gamma^k_{CF}(G) \geq \gamma(G)$ for all k, where $\gamma(G)$, the domination number of G, is the size of a minimum dominating set of G. Moreover, for any graph, there is a $k \leq \gamma(G)$ such that $\gamma^k_{CF}(G) = \gamma(G)$.

We denote the complete graph on n vertices by $K_n := (\{1,\dots,n\}, \{\{u,v\} \mid u,v \in \{1,\dots,n\},\ u \neq v\})$, and the complete bipartite graph on n and m vertices as $K_{n,m}$. We define the graph $K_n^{-3} := (V(K_n), E(K_n) \setminus E(K_3))$, which is obtained by removing any three edges forming a single triangle from a K_n.

We also provide a number of results for outerplanar graphs. An outerplanar graph is a graph that has a planar embedding for which all vertices belong to the outer face of the embedding. An outerplanar graph is called *maximal* iff no edges can be added to the graph without losing outerplanarity [9]. Maximal outerplanar graphs can also be characterized as the graphs having an embedding corresponding to a polygon triangulation, which illustrates their particular relevance in a geometric context. In addition, maximal outerplanar graphs exhibit a number of interesting graph-theoretic properties. Every maximal outerplanar graph is chordal, a 2-tree, and a series-parallel graph. Also, every maximal outerplanar graph is the visibility graph of a simple polygon.

For some of our NP-hardness proofs, we use a variant of the planar 3-SAT problem, called Positive Planar 1-in-3-SAT. This problem was introduced and shown to be NP-complete by Mulzer and Rote [25] and consists of deciding whether a given positive planar formula in conjunctive normal form with at most 3 literals per clause (3-CNF) allows a truth assignment such that in each clause, exactly one literal is true.

Definition 2.1 (positive planar formulas). *A formula ϕ in 3-CNF is called* positive planar *iff it is both* positive *and* backbone planar*. A formula ϕ is called* positive *iff it does not contain any negation, i.e., iff all occurring literals are positive. A formula ϕ, with clause set $C = \{c_1,\dots,c_l\}$ and variable set $X = \{x_1,\dots,x_n\}$, is called* backbone planar *iff its associated graph $G(\phi) := (X \cup C, E(\phi))$ is planar, where $E(\phi)$ is defined as follows:*

- *$x_i c_j \in E(\phi)$ for a clause $c_j \in C$ and a variable $x_i \in X$ iff x_i occurs in c_j,*
- *$x_i x_{i+1} \in E(\phi)$ for all $1 \leq i < n$.*

The path formed by the latter edges is also called the backbone *of the formula graph $G(\phi)$.*

3. Closed neighborhoods: Conflict-free coloring of general graphs.

In this section we consider the Conflict-Free k-Coloring problem on general simple graphs with respect to closed neighborhoods. In section 3.1, we prove that this problem is NP-complete for any $k \geq 1$. In section 3.2, we provide a sufficient criterion that guarantees conflict-free k-colorability. In section 3.3, we consider the conflict-free domination number and prove that, for any $k \geq 3$, there is no constant-factor approximation algorithm for γ^k_{CF}.

3.1. Complexity.

Theorem 3.1. Conflict-Free k-Coloring *is NP-complete for any fixed $k \geq 1$.*

Membership in NP is clear. For $k \geq 3$, we prove NP-hardness using a reduction from Proper k-Coloring. For $k \in \{1,2\}$, refer to section 4, where we prove Conflict-Free k-Coloring of planar graphs to be NP-complete for $k \in \{1,2\}$.

Central to the proof is the following lemma, which enables us to enforce certain vertices to be colored, and both ends of an edge to be colored using distinct colors.

LEMMA 3.2. *Let G be any graph, $u, v \in V(G)$, and $vu = e \in E(G)$. If $N(v)$ contains two disjoint and independent copies of a graph H with $\chi_{CF}(H) = k$, not adjacent to any other vertex $w \in G$, every conflict-free k-coloring of G colors v. If the same holds for u and, in addition, $N_G(u) \cap N_G(v)$ contains two disjoint and independent copies of a graph J with $\chi_{CF}(J) = k - 1$, not adjacent to any other vertex $w \in G$, every conflict-free k-coloring of G colors u and v with different colors.*

Proof. Assume towards a contradiction that there was a conflict-free k-coloring χ that avoids coloring v. Then, due to the copies of H being independent, disjoint, and not connected to any other vertex, the restriction of χ to the vertices of each of the two copies must induce a conflict-free coloring on H. As $\chi_{CF}(H) = k$, this implies that χ uses k colors on each copy. Therefore, in the open neighborhood of v, there are at least two vertices colored with each color. This leads to a contradiction, because v cannot have a conflict-free neighbor.

For the second proposition, suppose there was a conflict-free coloring assigning the same color to u and v. Without loss of generality, let this color be 1. As every vertex of the two copies of J now sees two occurrences of color 1, color 1 cannot be the color of the unique neighbor of any vertex of J, and any occurrence of color 1 on the vertices of J can be removed. Therefore, we can assume each of the two copies of J to be colored in a conflict-free manner using the colors $\{2, \ldots, k\}$. Observe that, due to $\chi_{CF}(J) = k - 1$, each of these colors must be used at least once in each copy. This implies that both u and v see each color at least twice: The two copies of J enforce two occurrences of the colors $\{2, \ldots, k\}$, and color 1 is assigned to both u and v, which are connected by an edge. This is a contradiction, and therefore, both u and v must be colored with distinct colors. $\square$

Next, we give an inductive construction of graphs, G_k, with $\chi_{CF}(G_k) = k$. The proof of NP-hardness relies on this hierarchy.

1. The first graph G_1 of the hierarchy consists of a single isolated vertex. G_2 is a $K_{1,3}$ with one edge subdivided by another vertex, or, equivalently, a path of length 3 with a leaf vertex attached to one of the inner vertices.
2. Given G_k and G_{k-1}, G_{k+1} is constructed as follows for $k \geq 2$:
   - Take a complete graph $G = K_{k+1}$ on $k + 1$ vertices.
   - To each vertex $v \in V(K_{k+1})$, attach two disjoint and independent copies of G_k, adding an edge from v to every vertex of both copies of G_k.
   - For each edge $e = vw \in E(K_{k+1})$, add two disjoint and independent copies of G_{k-1}, adding an edge from v and w to every vertex of both copies.

The number of vertices of the graphs G_k obtained by the above construction satisfies the recursive formula

$$|G_1| = 1, |G_2| = 5, |G_{k+1}| = (k + 1) \cdot (2|G_k| + k|G_{k-1}| + 1),$$

which is in $\Omega\left(2^k\right)$ and $\mathcal{O}\left(2^{k \log k}\right)$. Figure 1 depicts the graph G_3, which in addition to being planar is a series-parallel graph.

LEMMA 3.3. *For G_k constructed in this manner, $\chi_{CF}(G_k) = k$.*

Proof. The proof uses induction over k. Application of Lemma 3.2 implies that all vertices of the K_{k+1} underlying G_{k+1} have to be colored using different colors. Therefore, $\chi_{CF}(G_{k+1}) \geq k + 1$. By coloring all $k + 1$ vertices of the underlying K_{k+1} with a different color, we obtain a conflict-free $(k + 1)$-coloring of G_{k+1}, implying $\chi_{CF}(G_{k+1}) \leq k + 1$. $\square$

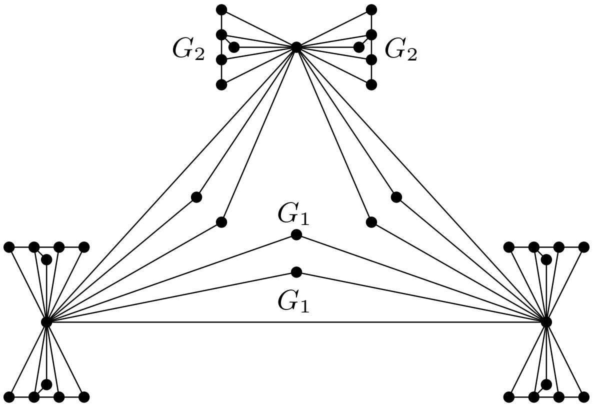


FIG. 1. *The graph G_3.*

LEMMA 3.4. *For $k \geq 2$, k-*COLORING $\preccurlyeq$ CONFLICT-FREE *k-*COLORING*. Therefore, for $k \geq 3$,* CONFLICT-FREE *k-*COLORING *is NP-complete.*

Proof. Given a graph G for which to decide proper k-colorability for a fixed k, we construct a graph G' that is conflict-free k-colorable iff G is k-colorable. G' is constructed from G by attaching two copies of G_k to each vertex $v \in V(G)$, by adding an edge from v to each vertex of the copies of G_k. For each edge $uv \in E(G)$, we attach two copies of G_{k-1} to both endpoints of uv by adding an edge from u and v to all vertices of both copies. As k is fixed, $|G_k|$ and $|G_{k-1}|$ are constant, implying that G' can be constructed in polynomial time.

A proper k-coloring of G induces a conflict-free k-coloring of G' by leaving all other vertices uncolored. On the other hand, by Lemma 3.2, a conflict-free k-coloring χ of G' colors all vertices $v \in V(G)$ and for every edge, the colors of both endpoints are distinct. Therefore, the restriction of χ to $V(G)$ is a proper k-coloring of G. □

3.2. A sufficient criterion for k-colorability. In this section we present a sufficient criterion for conflict-free k-colorability together with an efficient heuristic that can be used to color graphs satisfying this criterion with k colors in a conflict-free manner. This heuristic is called *iterated elimination of distance-*3*-sets* and is detailed in Algorithm 3.1. The main idea of this heuristic is to iteratively compute maximal sets of vertices at pairwise (edge) distance at least 3, coloring all vertices in one of these sets using one color, and then removing these vertices and their neighbors until all that remains is a collection of disconnected paths, which can then be colored using one color.

THEOREM 3.5. *Let G be a graph and $k \geq 1$. If G has neither K_{k+2} nor K_{k+3}^{-3} as a minor, G admits a conflict-free k-coloring that can be found in polynomial time using iterated elimination of distance-*3*-sets.*

Proof. For $k = 1$, a graph G with neither a K_3 nor a $K_4^{-3} = K_{1,3}$ minor consists of a collection of isolated paths. A path on $3n$ vertices can be colored with one color by coloring the middle vertex of every three vertices. This does not color the vertices at either end, so up to two vertices can be removed from the path to get colorings for paths on $3n - 1$ and $3n - 2$ vertices.

For $k \geq 2$, we use induction as follows: First, we color an inclusionwise maximal subset $D \subseteq V$ of vertices at pairwise distance at least 3 to each other using color 1.

Algorithm 3.1. Iterated elimination of distance-3-sets.

```
$i \leftarrow 1, \chi \leftarrow \emptyset$
Remove all isolated paths from $G$
while $G$ is not empty do
    $D \leftarrow \emptyset$
    For each component of $G$, select some vertex $v$ and add it to $D$
    while there is a vertex $w$ at distance $\geq 3$ from all vertices in $D$ do
        Choose $w$ at distance exactly 3 from some vertex in $D$
        $D \leftarrow D \cup \{w\}$
    $\forall u \in D : \chi(u) \leftarrow i$
    $i \leftarrow i + 1$
    Remove $N[D]$ from $G$
    Remove all isolated paths from $G$
Color all removed isolated paths using color $i$
```

This set D is chosen such that each vertex $v \in D$ is at distance exactly 3 from some $v' \in D$. Coloring D provides a conflict-free neighbor of color 1 to every vertex in $N[D]$. Therefore, the vertices in $N[D]$ are covered and can be removed from the graph. The remaining graph consists of vertices at distance 2 to some vertex in D; we call these vertices *unseen* in the remainder of the proof. We show that the remaining graph has no K_{k+1} and no K_{k+2}^{-3} as a minor. By induction, iterated elimination of distance-3-sets requires $k-1$ colors to color the remaining graph, and thus k colors suffice for G.

If the graph is disconnected, iterated elimination of distance-3-sets works on all components separately, so we can assume G to be connected. We claim that there is no set U of unseen vertices that is a cutset of G. Suppose there were such a cutset U, and let H be any component of $G \setminus U$ not containing v, the first selected vertex during the construction of D. At least one vertex of H is colored: every vertex in U is at distance at least two from every colored vertex not in H; therefore, every vertex in H is at distance at least 3 from every colored vertex not in H. Consider the iteration where the first vertex w of H is added to the set of colored vertices D. At this point, w is at distance exactly 3 from some colored vertex not in H. However, this implies w is adjacent to some vertex from U, contradicting the fact that all vertices in U are unseen.

Now, suppose for the sake of contradiction that the set W of unseen vertices contains a K_{k+1} or K_{k+2}^{-3} minor. W is not the whole graph, because at least one vertex is colored, so there must be a vertex v not in the K_{k+1} or K_{k+2}^{-3} minor. For every vertex $w \in W$, there is a path from v to w that intersects W only at w. Otherwise, $W \setminus \{w\}$ would be a cutset separating v from w. So, if the graph induced by W had a K_{k+1} or K_{k+2}^{-3} minor, we could contract $G \setminus W$ to a single vertex, which would be adjacent to all vertices in W, yielding a K_{k+2} or K_{k+3}^{-3} minor of G, which does not exist. □

Observe that G_{k+1} contains a K_{k+3}^{-3} as a minor, but not a K_{k+2}, proving that just excluding K_{k+2} as a minor does not suffice to guarantee k-colorability. Moreover, note that K_{k+1} is a minor of K_{k+2} and K_{k+3}^{-3}.

This yields the following corollary, which is the conflict-free variant of the Hadwiger Conjecture.

COROLLARY 3.6. *All graphs that do not have K_{k+1} as a minor are conflict-free k-colorable.*

3.3. Conflict-free domination number.

In this section we consider the problem of minimizing the number of colored vertices in a conflict-free k-coloring for a fixed k, which is equivalent to computing γ_{CF}^k. We call the corresponding decision problem k-CONFLICT-FREE DOMINATING SET. We show that approximating the conflict-free domination number in general graphs is hard for any fixed k. In section 5 we discuss the k-CONFLICT-FREE DOMINATING SET problem for planar graphs.

THEOREM 3.7. *Unless P = NP, for any $k \geq 3$, there is no polynomial-time approximation algorithm for $\gamma_{CF}^k(G)$ with constant approximation factor.*

Proof. We use a reduction from proper k-COLORING for the proof. Assume towards a contradiction that there was a polynomial-time approximation algorithm for $\gamma_{CF}^k(G)$ with approximation factor $c \geq 1$. Let G be a graph on n vertices for which we want to decide k-colorability. For each vertex v of G, add $M := (n+1)(c+1)$ vertices u_v to G and connect them to v. For each edge vw of G, add M vertices u_{vw} to G and connect them to both v and w. Let G' be the resulting graph. Clearly, the size of G' is polynomial in the size of G. Additionally, G' is planar if G is, and G' has a conflict-free k-coloring of size n iff G is properly k-colorable: Any proper k-coloring of G is a conflict-free k-coloring of G', as every vertex added to G is either adjacent to two distinctly colored vertices of G, or adjacent to just one vertex of G. Conversely, let χ be a conflict-free coloring of G', coloring just n vertices. If χ did not assign a color to some vertex v of G, it would have to color all $M \geq n+1$ neighbors of v. If χ assigned the same color to any pair v, w of vertices adjacent in G, it would have to color all M vertices adjacent only to v and w. Therefore, χ is a proper coloring of G. Running a c-approximation algorithm $\mathcal{A}$ for γ_{CF}^k on G' results in an approximate value $\mathcal{A}(G') \leq c \cdot \gamma_{CF}^k(G')$. We have $\mathcal{A}(G') \leq c \cdot n < M$ if G is k-colorable, and $\mathcal{A}(G') \geq M$ if G is not; thus we could decide proper k-colorability in polynomial time. □

4. Closed neighborhoods: Planar conflict-free coloring.

This section deals with the PLANAR CONFLICT-FREE k-COLORING problem, which consists of deciding conflict-free k-colorability for fixed k on planar graphs. Due to the 4-color theorem, we immediately know that every planar graph is conflict-free 4-colorable. This naturally leads to the question of whether there are planar graphs requiring 4 colors or whether fewer colors might already suffice for a conflict-free coloring, which we address in the following two sections.

4.1. Complexity.

For $k \in \{1, 2\}$ colors, we show that the problem of deciding conflict-free k-colorability on planar graphs is NP-complete. This implies that 2 colors are not sufficient.

THEOREM 4.1. *Deciding planar conflict-free 1-colorability is NP-complete.*

Proof. Membership in NP is obvious. The proof of NP-hardness is done by reduction from the problem POSITIVE PLANAR 1-IN-3-SAT. From a positive planar 3-CNF formula ϕ with clauses $C = \{c_1, \dots, c_l\}$ and variables $X = \{x_1, \dots, x_n\}$ we construct in polynomial time a graph $G_1(\phi)$ such that ϕ is 1-in-3-satisfiable iff $G_1(\phi)$ admits a conflict-free 1-coloring.

First, find and fix a planar embedding d of $G(\phi)$. $G_1(\phi)$ is constructed from $G(\phi)$ and d as follows: For every variable x_i, there is a cycle $Z_i = (z_{i,1}, \dots, z_{i,12})$ of length 12. The vertices $z_{i,1}, z_{i,4}, z_{i,7}, z_{i,10}$ are referred to as *true* vertices of Z_i; all other vertices are *false* vertices. Moreover, vertices $z_{i,1}, z_{i,2}, z_{i,3}$ are called *upper* vertices of Z_i, and vertices $z_{i,7}, z_{i,8}, z_{i,9}$ are called *lower* vertices of Z_i. Additionally, vertices $z_{i,4}, z_{i,5}, z_{i,6}$ are called *right* vertices of Z_i, and $z_{i,10}, z_{i,11}, z_{i,12}$ are called *left* vertices of Z_i.

For each clause c_j, there is a cycle $(c_{j,1}, \ldots, c_{j,4})$ of length 4 in $G_1(\phi)$. To each variable x_i for $i \in \{2, \ldots, n-1\}$, we associate two disjoint sequences $U_i = \left(u_j\right)_{j=1}^{|U_i|}$ and $L_i = \left(l_j\right)_{j=1}^{|L_i|}$ of clauses x_i appears in. The sequences are constructed using a clockwise (with respect to d) enumeration of the edges of x_i in $G(\phi)$, starting with $x_{i-1}x_i$. Let $(x_{i-1}x_i, x_ic_{j_1}, \ldots, x_ic_{j_\lambda}, x_ix_{i+1}, x_ic_{j_{\lambda+1}}, \ldots, x_ic_{j_\mu})$ be the sequence of edges encountered in this manner, and set $U_i := (c_{j_1}, \ldots, c_{j_\lambda})$ and $L_i := (c_{j_{\lambda+1}}, \ldots, c_{j_\mu})$. For $i \in \{1, n\}$, L_i is empty and U_i contains all clauses x_i appears in, again in clockwise order. In $G_1(\phi)$, the clauses and variables are connected such that for each clause c_j that x_i occurs in, either the upper or the lower *true* vertex of x_i is adjacent to $c_{j,1}$. More precisely, for variable x_i, if $c_j = u_m$, we add the edge $c_{j,1}z_{i,1}$ to connect the upper true vertex to the clause. If $c_j = l_m$, we add $c_{j,1}z_{i,7}$ to connect the lower true vertex to the clause. Because the order of edges around each vertex is preserved by the construction, the graph $G_1(\phi)$ obtained in this way can be embedded in the plane by a suitable adaptation of d. See Figure 2 for an example of the construction.

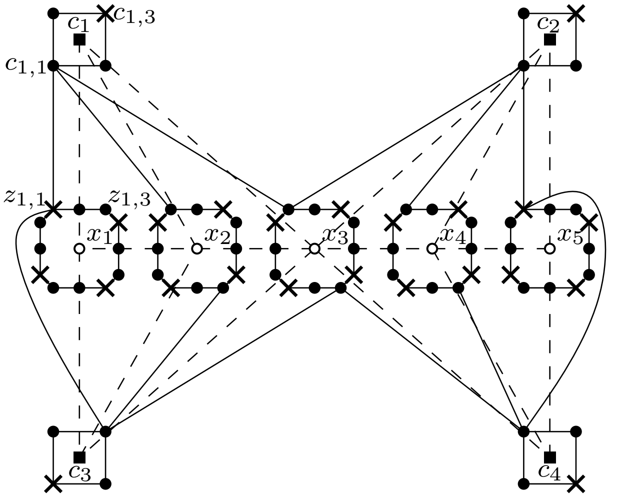


FIG. 2. *A formula graph $G(\phi)$ (dashed) and the corresponding $G_1(\phi)$ (solid).*

Now we prove that $G_1(\phi)$ is conflict-free 1-colorable iff ϕ is 1-in-3-satisfiable. Regarding necessity, a valid truth assignment $b : X \to \mathbb{B}$ yields a valid conflict-free coloring by coloring the vertex $c_{j,3}$ of every clause, coloring all true vertices of variables with $b(x_i) = 1$, and coloring the false vertices $z_{i,3}, z_{i,6}, z_{i,9}, z_{i,12}$ of all other variables. Thus, in every cycle Z_i, every third vertex is colored, providing a conflict-free neighbor to every vertex of Z_i. Moreover, in each clause, by virtue of $c_{j,3}$ being colored, vertices $c_{j,2}, c_{j,3}, c_{j,4}$ have a conflict-free neighbor. Because b is a valid truth assignment, for each clause, the vertex $c_{j,1}$ is adjacent to exactly one colored true vertex. Therefore, the coloring constructed in this way is conflict-free.

Regarding sufficiency, we first argue that the vertices $c_{j,1}, c_{j,2}, c_{j,4}$ can never be colored: If $c_{j,1}$ receives a color, then $c_{j,3}$ still enforces that one of $c_{j,2}, c_{j,3}, c_{j,4}$ is colored, leading to a contradiction in either case. If $c_{j,2}$ receives a color, then $c_{j,4}$ cannot have a conflict-free neighbor and vice versa. Therefore, no clause vertex can be the conflict-free neighbor of any vertex of Z_i. Thus, the conflict-free neighbor of *every* vertex of Z_i must itself be a vertex of Z_i. Moreover, the conflict-free neighbor of every vertex $c_{j,1}$ must be a true vertex. Thus, there are exactly three ways to color each cycle Z_i: either by coloring the true vertices (one possibility), or by coloring every other false vertex (two possibilities). A valid conflict-free 1-coloring of $G_1(\phi)$ satisfies the property that for each clause c_j, exactly one of the true vertices adjacent to $c_{j,1}$ is colored. Hence, a valid conflict-free 1-coloring of $G_1(\phi)$ induces a valid truth assignment b by setting $b(x_i) = 1$ iff all true vertices of x_i are colored. $\square$

THEOREM 4.2. *It is NP-complete to decide whether a planar graph admits a conflict-free* 2*-coloring.*

The proof requires the gadget $G_{\leq 1}$ depicted in Figure 3. $G_{\leq 1}$ consists of three vertices v, w_1, w_2 forming a triangle. Each edge ux of the triangle has two corresponding vertices y^1_{ux}, y^2_{ux}, each connected to u and x. Furthermore, both w_1 and w_2 are attached to two copies of a cycle on four vertices, where every vertex of both cycles is adjacent to the corresponding w_i. $G_{\leq 1}$ can be used to enforce that the vertices connected to its central vertex v are colored using at most one distinct color.

LEMMA 4.3. *Let $G = (V, E)$ be any graph, let $v \in V$, and let G' be the graph resulting from adding a copy of $G_{\leq 1}$ to G by identifying v in G with v in $G_{\leq 1}$. Then* (1) *G' is planar if G is, and* (2) *every conflict-free* 2*-coloring of G' leaves v uncolored and uses at most one color on $N_G[v]$.*

Proof. The planarity of G' follows from the planarity of G by the observation that $G_{\leq 1}$ is planar and can be embedded in any face incident to v in a planar embedding of G. Now consider a conflict-free 2-coloring χ of G'. χ must color both w_1 and w_2. Otherwise, χ restricted to each of the two 4-cycles adjacent to w_i must be a valid conflict-free 2-coloring. However, as C_4 requires at least two different colors, w_i then sees two occurrences of both colors, and thus cannot have a conflict-free neighbor anymore. Furthermore, $\chi(w_1) \neq \chi(w_2)$, as otherwise $y^1_{w_1w_2}$ and $y^2_{w_1w_2}$ must both be colored with the other color; but then, w_1 and w_2 again see two occurrences of both colors. By an analogous argument, χ must not color v. Moreover, χ cannot use more than one color on $N_G[v]$, because v already sees one occurrence of each color, so adding another occurrence of both colors would yield a conflict at v. $\square$

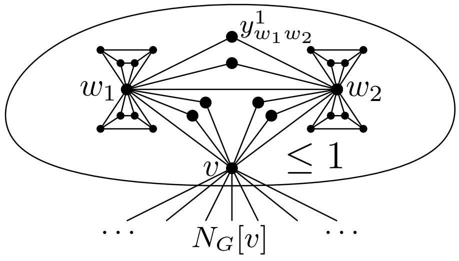


FIG. 3. *Gadget $G_{\leq 1}$.*

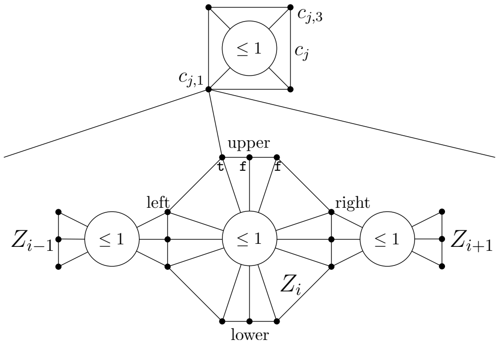


FIG. 4. *Clause and variable gadget for $k = 2$.*

Proof of Theorem 4.2. NP-hardness is proven by constructing, in polynomial time, a planar graph $G_2(\phi)$ from the graph $G_1(\phi)$ used in the hardness proof for $k = 1$, such that $G_2(\phi)$ is conflict-free 2-colorable iff $G_1(\phi)$ is conflict-free 1-colorable.

The construction is carried out by adding a gadget $G_{\leq 1}$ to every variable cycle Z_i of $G_1(\phi)$, to every clause cycle, and between the right and left vertices of two adjacent variable cycles Z_i and Z_{i+1}. This is depicted in Figure 4. More precisely, for every cycle Z_i, we add one copy of gadget $G_{\leq 1}$ and connect its central vertex v to all vertices of the cycle. In a planar embedding of $G_2(\phi)$, these gadgets can be embedded within the face defined by the cycles Z_i and thus do not harm planarity. By Lemma 4.3, this enforces that on every cycle, only one color can be used. Moreover, for every edge $x_i x_{i+1}$ in $G(\phi)$, we add one copy of $G_{\leq 1}$ that we connect to the right vertices of x_i and the left vertices of x_{i+1}. This preserves planarity because these gadgets and the added edges can be embedded in the face crossed by $x_i x_{i+1}$ in some fixed embedding d of $G(\phi)$. As one of the right vertices of x_i and one of the left vertices of x_{i+1} must be colored, this enforces that the same single color must be used to color all cycles Z_i. Finally, we add a copy of $G_{\leq 1}$ to every clause c_j and connect it to $c_{j,1}, \dots, c_{j,4}$. Again, this preserves planarity because the gadget may be embedded in the face defined by $(c_{j,1}, \dots, c_{j,4})$.

We now argue that $G_2(\phi)$ is conflict-free 2-colorable iff $G_1(\phi)$ is conflict-free 1-colorable. A 1-coloring of $G_1(\phi)$ induces a 2-coloring of $G_2(\phi)$ by copying the color assignment and coloring the internal vertices of the added gadgets as described in the proof of Lemma 4.3. Now, let $G_2(\phi)$ be conflict-free 2-colorable and fix a valid 2-coloring χ. In each clause, χ must color $c_{j,3}$ and neither $c_{j,1}$, nor $c_{j,2}$, nor $c_{j,4}$ can be colored. Therefore, no clause vertex can be the conflict-free neighbor of any vertex of Z_i. Thus, the conflict-free neighbor of *every* vertex of Z_i must itself be a vertex of Z_i. Moreover, the conflict-free neighbor of every vertex $c_{j,1}$ must be a true vertex. As there is only one color available to color all cycle vertices of all variables, the restriction of χ to the vertices of $G_1(\phi)$ yields a valid 1-coloring except for the fact that some $c_{j,3}$ might use a different color than the one used for the variables. However, this can be fixed by simply replacing all occurring colors with one single color. Hence, $G_2(\phi)$ is conflict-free 2-colorable iff $G_1(\phi)$ is conflict-free 1-colorable. ◻

4.2. Sufficient number of colors.

As shown above, it is NP-complete to decide whether a planar graph has a conflict-free k-coloring for $k \in \{1, 2\}$. On the positive side, we can establish the following result, which follows from the more general results discussed in section 3.2.

COROLLARY 4.4 (of Theorem 3.5). *Every outerplanar graph is conflict-free* 2-*colorable, and every planar graph is conflict-free* 3-*colorable. Moreover, such colorings can be computed in polynomial time.*

Outerplanar graphs are not the only interesting graph class for which one might suspect two colors to be sufficient. Two other interesting subclasses of planar graphs are series-parallel graphs and pseudomaximal planar graphs. However, each of these classes contains graphs that do not admit a conflict-free 2-coloring: The graph G_3 as defined in section 3 is an example of a series-parallel graph requiring three colors. Figure 5 depicts a maximal outerplanar graph O_9 satisfying $\chi_{CF}(O_9) = 2$. This graph can be used to obtain a pseudomaximal planar graph M with $\chi_{CF}(M) = 3$ by adding two copies of O_9 to the neighborhood of every vertex of a triangle, similar to the construction of G_3, and adding gadgets on the inside of the triangle as depicted in Figure 6.

Furthermore, observe that Theorem 4.4 does not hold if every vertex must be colored. In this case, there are outerplanar graphs requiring 3 colors for a conflict-free

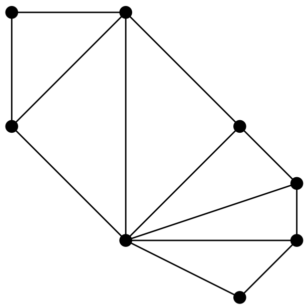


FIG. 5. *The maximal outerplanar graph O_9.*

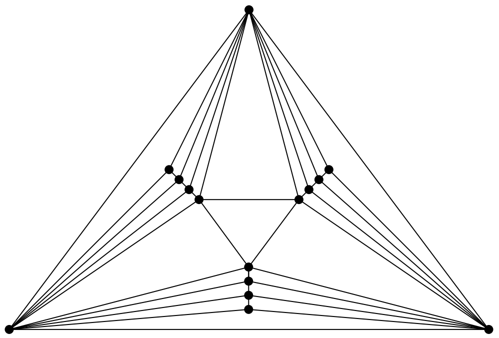


FIG. 6. *The pseudomaximal planar graph M, without the O_9 gadgets.*

coloring. One can obtain an example of such a graph by adding a chord to a cycle of length 5.

5. Closed neighborhoods: Planar conflict-free domination.

In this section we consider the decision problem k-CONFLICT-FREE DOMINATING SET for planar graphs. In section 5.1, we deal with the cases when $k \in \{1,2\}$ for planar and outerplanar graphs, and we give a polynomial-time algorithm to compute an optimal conflict-free coloring of outerplanar graphs with $k \in \{1,2\}$ colors. Section 5.2 discusses the problem for $k \geq 3$.

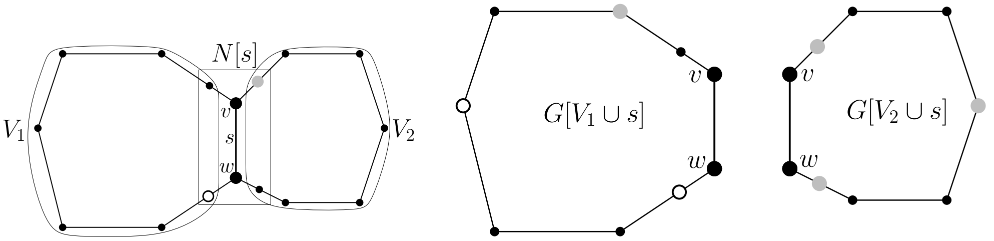
5.1. At most two colors.

We start by pointing out that, for every conflict-free 1-colorable graph G, $\gamma^1_{CF}(G) = \gamma(G)$ holds. Moreover, Corollary 5.1 discusses the complexity of k-CONFLICT-FREE DOMINATING SET, and Theorem 5.2 states positive results for outerplanar graphs.

COROLLARY 5.1 (of Theorems 4.1 and 4.2). *k-*CONFLICT-FREE DOMINATING SET *is NP-complete for $k \in \{1,2\}$ for planar graphs.*

THEOREM 5.2. *Let $k \in \{1,2\}$, and let G be an outerplanar graph. We can decide in polynomial time whether $\chi_{CF}(G) \leq k$. Moreover, we can compute a conflict-free k-coloring of G that minimizes the number of colored vertices in $\mathcal{O}(n^{4k+1})$ time.*

The proof of Theorem 5.2 relies on a polynomial-time algorithm that computes a k-coloring of the input outerplanar graph G if and only if such a coloring exists. Intuitively speaking, our algorithm works as follows. For each vertex $v \in G$ and each edge $vw \in G$, we consider all possible assignments of conflict-free neighbors to v and w and colors to these conflict-free neighbors. Each such assignment is called a *neighborhood configuration*. Because the number of colors is constant and there is at most one conflict-free neighbor per color for each vertex, there are only polynomially many neighborhood configurations for each vertex or edge.



(a) An outerplanar graph G and an edge separator $s = \{v, w\}$ splitting G into components V_1 and V_2 with a neighborhood configuration of s (white and gray vertices).

(b) Colorings extending the neighborhood configuration of s. These colorings are conflict-free on V_1 and V_2 and can be combined to a conflict-free coloring of G. Note that $v \notin V_1$ does not need to have a conflict-free neighbor in the coloring of V_1.

FIG. 7. *If we fix a neighborhood configuration of a separator of G and find conflict-free colorings of the separated components that extend this neighborhood configuration, we can combine these colorings to a conflict-free coloring of G.*

We decompose the outerplanar input graph at vertex separators (articulation points) and edge separators (edges shared by faces); removing the vertices of a separator splits the graph into several components. The following key property of this decomposition is the basis for our dynamic programming algorithm; see Figure 7. Let s be a separator in our graph, and let $V_1, \dots, V_k$ be the vertex sets of the components of G after removing s. If we fix a neighborhood configuration $\mathcal{C}$ of s and find, for each component $G[V_i \cup s]$, a coloring extending $\mathcal{C}$ that is conflict-free on V_i, then we can combine these colorings to a coloring of G.

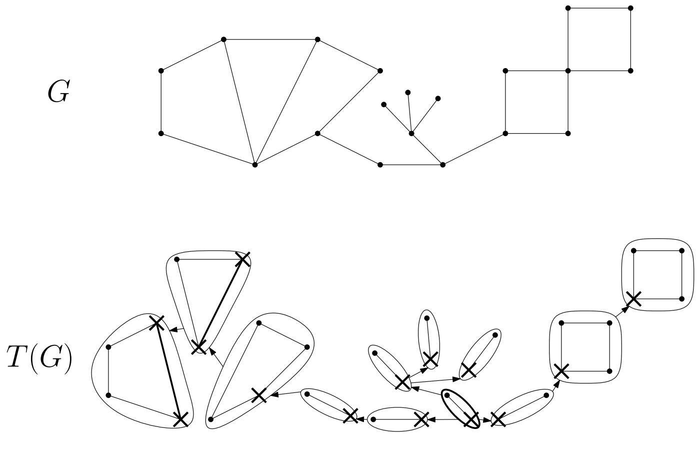
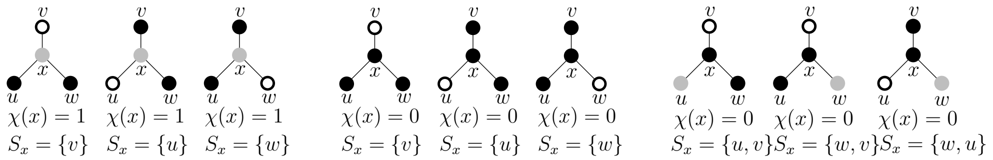


FIG. 8. *The arborescence $T(G)$ (bottom) for an outerplanar graph G (top). The vertices of the incoming separator of each atom are marked ×; incoming edge separators are drawn in bold. The root atom is drawn with bold outline and has an arbitrarily chosen vertex as incoming separator.*

Our decomposition yields an arborescence of components as depicted in Figure 8, with edges between components that share a separator, using an arbitrary component as root and directing all edges accordingly. In this arborescence, each component except the root has a unique incoming edge corresponding to a separator, called the *incoming separator* of the component marked × in Figure 8. Starting at the leaves, we use dynamic programming on this arborescence as follows. For each component and each possible neighborhood configuration of the incoming separator, we compute a conflict-free k-coloring that extends the neighborhood configuration and minimizes the number of colored vertices, or find that this neighborhood configuration does not allow a conflict-free k-coloring. At the root, this allows us to determine whether the graph is conflict-free k-colorable. Moreover, if the graph is colorable, we can retrieve a coloring that minimizes the number of colored vertices. In the following, we give a detailed formal description of this algorithm, prove its correctness, and analyze its runtime.

5.1.1. Preliminaries. Let $G = (V, E)$ be an outerplanar graph. W.l.o.g., we assume that G is connected and has at least two vertices. Let $\chi : V' \subseteq V(G) \rightarrow \{0, 1, \ldots, k\}$ be a partial coloring of the vertices of G, and let $v \in V$. Observe that χ defined like this modifies the definition given in the introduction by assigning color 0 to uncolored vertices. We begin by defining *vertex neighborhood configurations.* Intuitively speaking, a vertex neighborhood configuration assigns a color to v and lists all conflict-free neighbors of v together with their color; see Figure 9. In Figures 9, 10, and 11, black disks correspond to uncolored vertices, gray disks correspond to vertices colored by 1, and empty disks correspond to vertices colored by 2.



(a) x is colored by 1 and has one conflict-free neighbor.

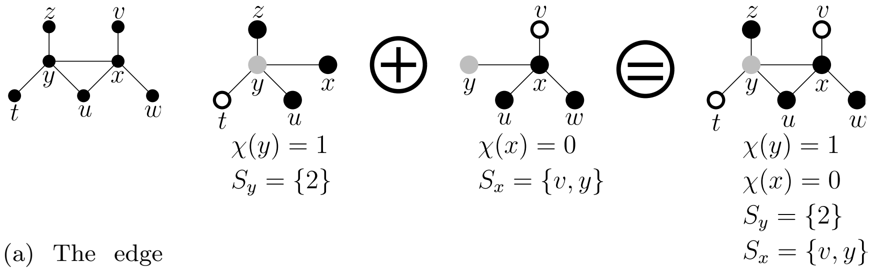
(b) x is uncolored and has one conflict-free neighbor.

(c) x is uncolored and has two conflict-free neighbors.

FIG. 9. *A vertex x with three neighbors and the possible neighborhood configurations of s, modulo switching labels of the colors.*

DEFINITION 5.3 (vertex neighborhood configuration). *A* vertex neighborhood configuration *is a tuple $\mathcal{C}_v = [\chi(v), S_v, \rho_v]$, where $\chi(v) \in \{0, 1, \ldots, k\}$ denotes the color of v; if $\chi(v) = 0$, we regard v as uncolored; see Figure 9. The set $\emptyset \neq S_v \subseteq N[v]$ contains all conflict-free neighbors of v. Because there is at most one conflict-free neighbor for each color, S_v contains at most k elements. Finally, $\rho_v : S_v \rightarrow \{1, \ldots, k\}$ is an injective assignment of colors to the conflict-free neighbors of v such that $v \in S_v$ implies $\chi(v) = \rho_v(v)$.*

We call two vertex neighborhood configurations $\mathcal{C}_u, \mathcal{C}_v$ for adjacent vertices u and v *compatible* if they do not contradict each other in the following sense; see Figure 10. First, they must not assign different colors to the same vertex. Second, after combining the partial colorings induced by $\mathcal{C}_u$ and $\mathcal{C}_v$, all conflict-free neighbors specified in the neighborhood configurations must remain conflict-free. An edge neighborhood configuration consists of two compatible vertex neighborhood configurations for its endpoints. Formally, we define this as follows.



(a) The edge xy and the neighbors of x and y.

(b) The combination of two neighborhood configurations of x and y.

FIG. 10. *The combination of two neighborhood configurations of two adjacent vertices x and y results in a neighborhood configuration of the edge xy.*

DEFINITION 5.4 (edge neighborhood configuration). *For an edge uv, we say that $\mathcal{C}_u = [\chi(u), S_u, \rho_u]$ and $\mathcal{C}_v = [\chi(v), S_v, \rho_v]$ are* compatible, *denoted by $\mathcal{C}_u \leftrightarrow \mathcal{C}_v$, if the following conditions hold; see Figure* 10.

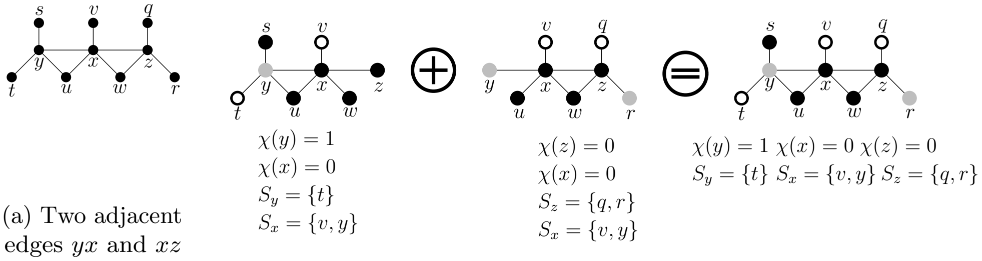
1. *For every $w \in S_v \cap S_u$, $\rho_u(w) = \rho_v(w)$. If u is in S_v, then $\chi(u)$ must be $\rho_v(u)$, and vice versa.*
2. *The combined coloring*

$$\rho_{uv} : S_u \cup S_v \cup \{u, v\} \to \{0, \dots, k\}, w \mapsto \begin{cases} \chi(w) & \textit{if } w \in \{u, v\}, \\ \rho_u(w) & \textit{if } w \in S_u, \\ \rho_v(w) & \textit{otherwise} \end{cases}$$

   must be injective on $N[v]$ and $N[u]$, with the exception that both u and v may receive color 0.

An edge neighborhood configuration *of $e = uv$ is a pair $\mathcal{C}_e = [\mathcal{C}_u, \mathcal{C}_v]$ of compatible vertex neighborhood configurations. For $w \in \{u, v\}$, $\mathcal{C}_e^w$ shall denote the neighborhood configuration of w contained in $\mathcal{C}_e$.*

Observe that we can check in $\mathcal{O}(k)$ time whether a pair of vertex neighborhood configurations is compatible. For a pair of incident edges, we call a pair of edge neighborhood configurations *compatible* if the neighborhood configuration of v is the same in both neighborhood configurations; see Figure 11.



(a) Two adjacent edges yx and xz and the neighbors of y, x, and z.

(b) Two compatible neighborhood configurations of the two adjacent edges yx and xz.

FIG. 11. *Compatible neighborhood configurations of adjacent edges.*

Definition 5.5 (compatibility). *If $\mathcal{C}_e^v = \mathcal{C}_{e'}^v$ for a pair $e = uv, e' = vw$ of incident edges, then we say $\mathcal{C}_{e'}$* is compatible with *$\mathcal{C}_e$; see Figure* 11.

We observe that if we have a neighborhood configuration for each edge and all these neighborhood configurations are pairwise compatible, the colors of all vertices are fixed in a consistent manner, and we can thus derive a conflict-free k-coloring from the neighborhood configurations.

Observation 5.6. Let $\mathcal{C}$ be a set of edge neighborhood configurations containing one neighborhood configuration $\mathcal{C}_e$ for each edge e. If $\mathcal{C}_e$ and $\mathcal{C}_{e'}$ are compatible for *every* pair $e = uv$, $e' = vw$ of incident edges, a conflict-free k-coloring can be obtained from $\mathcal{C}$.

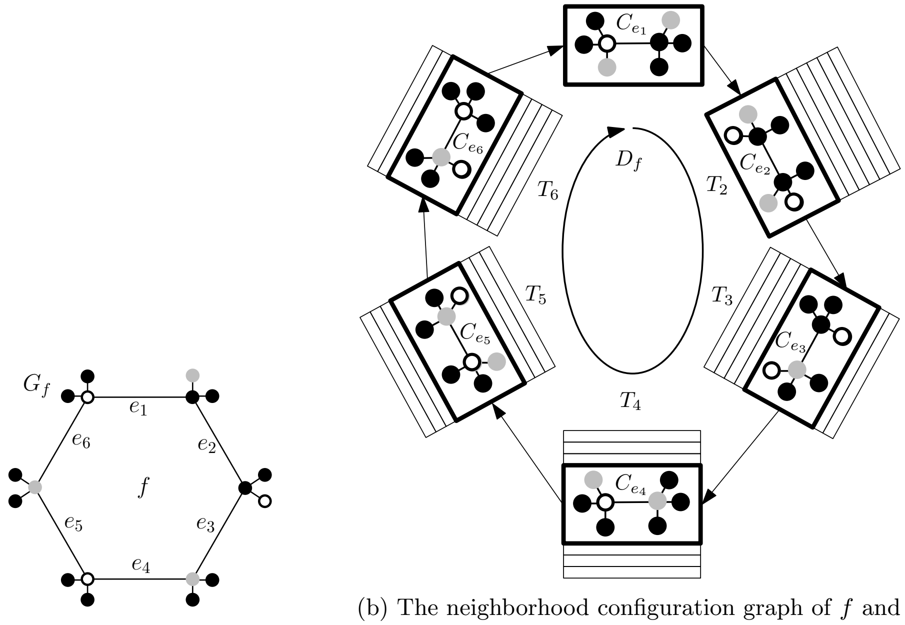
Our algorithm works by dynamic programming on an arborescence $T(G)$ derived from a decomposition of G along vertex separators and edge separators into components called *atoms*.

Definition 5.7. *A* vertex separator *of G is an articulation point of G, i.e., a vertex whose removal disconnects G. An* edge separator *of G is an edge uv of G such that removing u and v disconnects G. An* atom *of G is either an edge atom (formed by an edge) or a face atom (induced chord-free cycle of G).*

Observe that, because G is outerplanar, any connected induced subgraph of G with at least two vertices either is an atom or contains a separator. The vertex set $V(T(G))$ of the arborescence $T(G)$ consists of atoms of G and is defined by induction on the induced subgraphs G' of G as follows. If an induced subgraph G' of G is an atom, $V(T(G')) = \{G'\}$. If G' is no atom, let $s = \{v\}$ be a vertex separator of G' if one exists; otherwise, let $s = \{u, v\}$ be an edge separator of G'. Let $V_1', \ldots, V_\ell'$ be the vertex sets of the connected components of $G' - s$, and let $G_1', \ldots, G_\ell'$ be the subgraphs induced by $V_1' \cup s, \ldots, V_\ell' \cup s$. Then $V(T(G')) = \bigcup_{1 \le i \le \ell} V(T(G_i'))$ is the set of all atoms obtained by further subdividing G'.

There is an arc between two vertices of $T(G)$ if the two atoms share a separator. To avoid cycles in $T(G)$, if more than two atoms share a vertex separator, instead of introducing an arc between every pair of them, we pick an arbitrary atom among them and connect it to all other atoms sharing the vertex separator. Because G is outerplanar, this yields a tree of atoms of G; we turn this tree into the arborescence $T(G)$ by picking an arbitrary root vertex and orienting all edges away from this root. Each vertex a of $T(G)$ except for the root has a unique incoming arc corresponding to a unique separator, called the *incoming separator* of the atom a. For the root atom r, we pick an arbitrary vertex of r as incoming separator; in this way, each atom a has exactly one incoming separator s_a. See Figure 8 for an example of the construction. For an atom $a \in T(G)$, we denote by $T(G, a)$ the subtree of $T(G)$ rooted at a. Moreover, let $S(G, a)$ be the subgraph of G induced by all vertices occurring in any atom in $T(G, a)$.

5.1.2. Description of the algorithm. For each vertex and each edge, our algorithm keeps a list of *feasible* neighborhood configurations. At any point in the algorithm, we know that any neighborhood configuration not on this list cannot be extended to a conflict-free k-coloring of G. Whenever we remove a neighborhood configuration from the list of feasible neighborhood configurations for a vertex, we also remove all corresponding neighborhood configurations from its incident edges. Similarly, when we remove the last neighborhood configuration of an edge that contains a certain vertex neighborhood configuration, we also remove that neighborhood configuration from the list of feasible neighborhood configurations of the vertex. In this way, deleting a feasible neighborhood configuration may cause a cascade of further deletions; however, a careful implementation of our algorithm can handle these deletions in $\mathcal{O}(1)$ time per deleted neighborhood configuration. Because each neighborhood configuration is deleted at most once, this does not affect our asymptotic running time.



(a) A neighborhood configuration of a face f of G.

(b) The neighborhood configuration graph of f and the cycle in the neighborhood configuration graph corresponding to the neighborhood configuration of Figure 12a.

FIG. 12. *A face of G and the corresponding neighborhood configuration graph.*

We initialize the lists of feasible neighborhood configurations by computing, for each vertex and each edge, the list of all possible neighborhood configurations according to Definitions 5.3 and 5.4. We proceed by refining, for each atom $a \in V(T(G))$, the list of feasible neighborhood configurations of the incoming separator s_a. This process starts in the leaves of $T(G)$ and works its way up towards the root, terminating once the root has been processed. Processing an atom $a \in V(T(G))$ means removing all neighborhood configurations $\mathcal{C}_{s_a}$ of its incoming separator that cannot be extended to a conflict-free k-coloring of $S(G,a)$. Note that in this conflict-free k-coloring, the vertices of s_a need not have conflict-free neighbors in $S(G,a)$ if $\mathcal{C}_{s_a}$ is such that all their conflict-free neighbors are outside of $S(G,a)$. Moreover, for each atom a and each feasible neighborhood configuration $\mathcal{C}_{s_a}$, the algorithm computes and stores the minimum number of colored vertices required for a conflict-free k-coloring of $S(G,a)$ extending $\mathcal{C}_{s_a}$.

If the list of feasible neighborhood configurations of any vertex or edge becomes empty at any point, the algorithm aborts and reports that the graph is not conflict-free k-colorable. Otherwise, after processing the root r, the algorithm checks all feasible neighborhood configurations of s_r to find a neighborhood configuration for which the number of colored vertices is minimal. Starting with this neighborhood configuration, the algorithm backtracks and reconstructs a conflict-free k-coloring of G with a minimal number of colored vertices.

It remains to describe how to process an atom of $T(G)$. In case of a face atom f, the incoming separator can be either a vertex separator or an edge separator. We assume that it is an edge separator $e_1 = uv$; vertex separators can be handled analogously. The face f may contain vertices and edges that are not part of any separator. For those vertices and edges, we have already computed the set of feasible neighborhood configurations in the first step of the algorithm. All other vertices and edges except for the incoming separator correspond to children of f in $T(G)$; therefore, we have already computed the set of feasible neighborhood configurations for each of them.

For each neighborhood configuration $\mathcal{C}_{e_1}$ still in the list of feasible neighborhood configurations of e_1, we build the directed *neighborhood configuration graph* $G^* = (V^*, E^*)$ as depicted in Figure 12b.

The vertex set of G^* consists of the neighborhood configuration $\mathcal{C}_{e_1}$ and each feasible neighborhood configuration $\mathcal{C}_{e_i}$ of each edge $e_i \neq e_1$ of f. There is an edge between two neighborhood configurations $\mathcal{C}_{uv}, \mathcal{C}_{vw}$ iff they are compatible. We choose a direction of the edges around the face and direct all edges in G^* accordingly; see Figure 12b. Each simple directed cycle D_f in G^* must contain $\mathcal{C}_{e_1}$ and thus corresponds to a selection of one neighborhood configuration for each edge of f; these neighborhood configurations are pairwise compatible. Therefore, there is a conflict-free k-coloring of $S(G, f)$ extending $\mathcal{C}_{e_1}$ iff there is a simple directed cycle in G^*.

Moreover, we add weights to the edges of G^* such that the weight of a simple directed cycle corresponds to the minimum number of colored vertices in such a coloring. In order to compute the weights, for vertices and edges of f that are separators corresponding to children of f in $T(G)$, we make use of the minimum number of colored vertices in their corresponding subtrees that we computed earlier.

We can find a minimum-weight cycle in G^* or decide there is no such cycle in time $\mathcal{O}(|V^*| + |E^*|)$, using an algorithm similar to Dijkstra's shortest path algorithm. We can do this in linear time because we can expand the vertices in fixed order, expanding all vertices corresponding to an edge of f before moving on to all vertices of the next edge around the face. If our algorithm finds a minimum-weight cycle, we store its weight as the minimum number of vertices colored in any conflict-free k-coloring of $S(G, f)$ extending $\mathcal{C}_{e_1}$. Otherwise, $\mathcal{C}_{e_1}$ is removed from the list of feasible neighborhood configurations of e_1. Repeating this procedure for each feasible neighborhood configuration of e_1 concludes the processing of a face atom f.

In the following, we describe how to handle an edge atom $e \in V(T(G))$. In this case, the incoming separator s_e is a vertex separator v. For each neighborhood configuration in the list of feasible neighborhood configurations of v, there is at least one neighborhood configuration in the list of feasible neighborhood configurations of e; otherwise, we would have already deleted the neighborhood configuration. To compute the minimum number of colored vertices in $S(G, e)$ for some neighborhood configuration $\mathcal{C}_v$, we check for each neighborhood configuration of e containing $\mathcal{C}_v$, the minimum number of colored vertices, taking into account the color of u and the minimum number of vertices computed for the children of e in $T(G)$. Repeating this for each feasible neighborhood configuration of v concludes the processing of an edge atom e.

5.1.3. Correctness of the algorithm.

Next, we argue that our algorithm is correct, i.e., it finds a conflict-free k-coloring with minimum number of colored vertices iff one exists. In this section, we call a neighborhood configuration *valid* if it can be extended to a conflict-free k-coloring of G.

There are only two reasons for deleting a neighborhood configuration $\mathcal{C}$ from a list of feasible neighborhood configurations. In the first case, the deletion of $\mathcal{C}$ is a consequence of a deletion of another neighborhood configuration $\mathcal{C}'$. In this case, $\mathcal{C}$ is deleted because deleting $\mathcal{C}'$ has led to an incident vertex or edge without a feasible neighborhood configuration compatible to $\mathcal{C}$. This can never cause a valid neighborhood configuration to be deleted unless we deleted a valid neighborhood configuration $\mathcal{C}'$ first.

In the second case, the deleted neighborhood configuration $\mathcal{C}$ belongs to an incoming separator s_f of a face atom f for which the algorithm finds that there is no conflict-free k-coloring of $S(G, a)$ extending it. By induction on $T(G)$, we assume that when we start processing f, no valid neighborhood configurations have been deleted from the list of feasible neighborhood configurations for any vertex or edge of f. Assume there was a valid neighborhood configuration $\mathcal{C}$ of s_f deleted by our algorithm. Because $\mathcal{C}$ is valid, there is a conflict-free k-coloring of $S(G, f)$ extending $\mathcal{C}$. This yields a set of compatible neighborhood configurations for the edges of f and thus a cycle in the corresponding neighborhood configuration graph G^*. This is a contradiction, because the algorithm only deletes $\mathcal{C}$ if there is no such cycle. We conclude that no valid neighborhood configuration is ever deleted from the list of feasible neighborhood configurations of any vertex or edge. Therefore, the algorithm will always find the graph to be conflict-free k-colorable if it is. In a similar manner, we can argue that the number of colored vertices used by the coloring produced by our algorithm is minimal.

In the remainder of the section, we prove that our algorithm never produces an invalid conflict-free k-coloring of G. Again, the proof is by induction on $T(G)$. We discuss an inductive step for the case that the current atom is a face f with an incoming edge separator e_1; the induction base and the remaining cases are analogous. We assume by induction that for each neighborhood configuration $\mathcal{C}_{s_a}$ of the incoming separator s_a of each child a of f, there is a conflict-free k-coloring of $S(G, a)$ extending $\mathcal{C}_{s_a}$. Let $\mathcal{C}$ be a neighborhood configuration of e_1 that remains feasible after the processing of f. This is because there is a cycle in the corresponding neighborhood configuration graph G^*. This cycle corresponds to a set of pairwise compatible edge neighborhood configurations. We can construct a conflict-free k-coloring of $S(G, f)$ by combining the colorings induced by these neighborhood configurations and the corresponding colorings of the graphs $S(G, a)$ for children a of f. At the root r of $T(G)$, this yields a conflict-free k-coloring of G, because all neighbors of the incoming separator of r are part of $S(G, r) = G$. Therefore, our algorithm never produces an invalid conflict-free coloring.

5.1.4. Runtime of the algorithm. Finally, we need to analyze the running time of our dynamic programming approach. We begin by observing that $T(G)$ has $\mathcal{O}(n)$ atoms. Moreover, we observe that the number of vertex neighborhood configurations $\mathcal{C}_v = [\chi(v), S_v, \rho_v]$ of a vertex v is in $\mathcal{O}(n^k)$, as there are at most $\binom{|N[v]|}{k} \cdot k!$ possibilities for S_v and ρ_v. Therefore, the number of edge neighborhood configurations $\mathcal{C}_e = [\mathcal{C}_u, \mathcal{C}_v]$ of an edge $e \in E$ is in $\mathcal{O}(n^{2k})$.

Let $f = e_1 e_2 \ldots e_m$ be a face atom; the running time for processing face atoms dominates the running time for all other computation steps of the algorithm. For f, we build the neighborhood configuration graph G^* that has $\mathcal{O}(n^{2k+1})$ vertices, because f has at most n edges, each with $\mathcal{O}(n^{2k})$ neighborhood configurations. The number of edges between the neighborhood configurations of two incident edges $e_i = uv, e_{i+1} = vw$ along f is at most $\mathcal{O}(n^{3k})$ because there are only $\mathcal{O}(n^k)$ neighborhood configurations for each of the vertices u, v, and w. Therefore, the number of edges in G^* is $\mathcal{O}(n^{3k+1})$. This leads to a running time of $\mathcal{O}(n^{5k+2})$, because we run a graph scan on G^* for each of the $\mathcal{O}(n^{2k})$ neighborhood configurations of the incoming separator and each of the $\mathcal{O}(n)$ face atoms.

Streamlining this approach leads to a runtime of $\mathcal{O}(n^{4k+1})$. In particular, we modify our subroutine processing a face atom f that has an incoming edge separator $e = uv$ as follows. For each neighborhood configuration $\mathcal{C}_v$ of v we extend the neighborhood configuration graph G^* of f by considering all feasible neighborhood configurations $\mathcal{C}_{e_1}$ of e_1 such that $\mathcal{C}^v_{e_1} = \mathcal{C}_v$ holds and compute minimum-weight cycles in G^*. For each neighborhood configuration $\mathcal{C}_{e_1}$ of e_1 that is reached during an application of the shortest path algorithm, we obtain the minimum number of vertices colored in any conflict-free coloring of $S(G, f)$ extending $\mathcal{C}_{e_1}$. As the number of all edge and vertex neighborhood configurations of G is $\mathcal{O}(n^{3k+1})$, we obtain an overall runtime of $\mathcal{O}(n^{4k+1})$.

This concludes the proof of Theorem 5.2.

5.2. Approximability for three or more colors.

In section 4.2 we stated that every planar graph is conflict-free 3-colorable. In this section we deal with conflict-free 3-colorings of planar graphs that, additionally, minimize the number of colored vertices.

THEOREM 5.8. *Let $k \geq 3$, and let G be a planar graph. The following hold:*

(1) *Unless P = NP, there is no polynomial-time approximation algorithm providing a constant-factor approximation of $\gamma^3_{CF}(G)$ for planar graphs.* 3-CONFLICT-FREE DOMINATING SET *is NP-complete for planar graphs.*

(2) *For $k \geq 4$, k-*CONFLICT-FREE DOMINATING SET *is NP-complete. Also, $\gamma^k_{CF}(G) = \gamma(G)$, and the problem is FPT with parameter $\gamma^k_{CF}(G)$. Furthermore, there is a PTAS for $\gamma^k_{CF}(G)$.*

(3) *If G is outerplanar, then $\gamma^k_{CF}(G) = \gamma(G)$, and there is a linear-time algorithm to compute $\gamma^k_{CF}(G)$.*

The proof of Theorem 5.8 is based on the following polynomial-time algorithm, which transforms a dominating set D of a planar graph G into a conflict-free k-coloring of G, coloring only the vertices of D: Let D be a dominating set of a planar graph G. Every vertex $v \in V(G) \setminus D$ is adjacent to at least one vertex in D. Pick any such vertex $u \in D$ and contract the edge $uv \in E(G)$ towards u. Repeat this until only the vertices from D remain. Because G is planar, the graph $G' = (D, E')$ obtained in this way is planar, as G' is a minor of G. By the 4-coloring theorem, we can compute a proper 4-coloring of G'.

LEMMA 5.9. *The 4-coloring generated by this procedure induces a conflict-free 4-coloring of G.*

Proof. Every vertex $u \in D$ is a conflict-free neighbor to itself as its color does not appear in $N_G(u)$. Let $v \in V(G) \setminus D$ be some uncolored vertex, and let $u \in D$ be the vertex that v was contracted towards by the algorithm. In G', this contraction made u adjacent to all other vertices in $N_G(v) \cap D$, which guarantees that the color of u is unique in $N_G(v) \cap D$. As $V(G) \setminus D$ remains uncolored, the color of u is thus unique in $N_G[v]$. □

Proof of Theorem 5.8. Proposition (1) follows from Theorem 3.7 of section 3.3: The reduction used there preserves planarity and proper planar 3-coloring is NP-

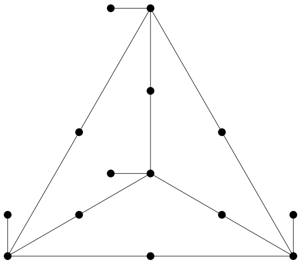


FIG. 13. *The graph G' resulting from applying the reduction to K_4. This bipartite planar graph has $\chi_O(G') = 4$.*

complete. For (2), $\gamma_{CF}^k(G) = \gamma(G)$ implies NP-hardness in planar graphs because planar minimum dominating set is NP-hard. Moreover, the coloring algorithm allows us to apply any approximation scheme for planar dominating set to conflict-free k-coloring. We obtain a PTAS for the conflict-free domination number by applying our coloring algorithm to the dominating set produced by the PTAS of Baker and Hill [9]. Additionally, Alber, Fellows, and Niedermeier [4] proved that planar dominating set is FPT with parameter $\gamma(G)$, implying that computing the planar conflict-free domination number for $k \geq 4$ is FPT with parameter $\gamma_{CF}^k(G)$. For (3), the class of outerplanar graphs is properly 3-colorable in linear time and closed under taking minors. Kikuno, Yoshida, and Kakuda [23] present a linear time algorithm for finding a minimum dominating set in a series-parallel graph, which includes outerplanar graphs. The result follows by combining this linear time algorithm with the coloring algorithm mentioned above, but using just three colors instead of four. □

6. Open neighborhoods: Planar conflict-free coloring. In this section we discuss the problem of conflict-free coloring with open neighborhoods. Recall that an open-neighborhood conflict-free coloring is a coloring of some vertices of a graph G such that every vertex has a conflict-free neighbor in its open neighborhood $N(v)$. In some settings, this problem is a natural alternative to the closed-neighborhood variant; for instance, when guiding a robot from one location to another, a uniquely colored beacon at the robot's current position may be insufficient.

Note that isolated vertices are problematic for this variant of conflict-free coloring; therefore, in the following, we assume that G does not contain isolated vertices. Moreover, we observe the following.

Observation 6.1. Let G be a graph, $v, w \in V(G)$, and $\deg(v) = 1$, $\deg(w) = 2$. Then, for any number k of colors, in any conflict-free k-coloring, the unique neighbor of v must be colored. Moreover, the two neighbors of w cannot have the same color.

This leads to a straightforward reduction from proper coloring to conflict-free coloring. Given a graph G, adding an otherwise isolated neighbor to each original vertex and placing a vertex with degree 2 on every original edge yields a graph G' with $\chi_O(G') = \chi_P(G)$. See Figure 13 for an example of this reduction. The resulting graph G' is bipartite. Furthermore, the reduction preserves planarity, implying that bipartite planar graphs may require at least four colors in a conflict-free coloring. Moreover, even though this reduction does not necessarily preserve outerplanarity, applying it to a K_3 yields an outerplanar graph that requires at least three colors. For *bipartite* planar and outerplanar graphs, these bounds are tight.

Corollary 6.2. *It is NP-complete to decide whether a bipartite planar graph G is open-neighborhood conflict-free* 3*-colorable.*

Theorem 6.3. *Every bipartite planar graph is open-neighborhood conflict-free* 4*-colorable. For bipartite outerplanar graphs, three colors are sufficient.*

Proof. Let $G = (V_1 \cup V_2, E)$ be a bipartite planar graph with partitions V_1 and V_2; the proof proceeds analogously for outerplanar graphs. We construct two minors G_1 and G_2 of G, to each of which we apply the planar four-color theorem. We build G_1 by merging all vertices $v \in V_2$ into an arbitrarily chosen neighbor $v_1(v) \in V_1$. Because G is bipartite and does not contain isolated vertices, it is possible to continue this process until no vertices from V_2 remain. G_2 is constructed analogously, merging all vertices $v \in V_1$ into an arbitrarily chosen neighbor $v_2(v) \in V_2$. Each of the two resulting graphs G_i contains exactly the vertices from V_i. Moreover, as a minor of G, G_i is planar and therefore has a proper coloring with four colors. We assign the colors from this coloring to the vertices in V_i.

It remains to show that this induces an open-neighborhood conflict-free coloring of G. Let v be a vertex of G. W.l.o.g., assume $v \in V_1$. During the construction of G_2, v was merged into its neighbor $v_2(v) \in V_2$. Therefore, in G_2, $v_2(v)$ is adjacent to all other neighbors of v in G. Because all neighbors of v are in V_2, this implies that the color of $v_2(v)$ is unique in $N_G(v)$, and $v_2(v)$ is a conflict-free neighbor of v. □

On the other hand, for nonbipartite planar graphs, we can show the following upper bound on the number of colors.

Theorem 6.4. *Every planar graph has an open-neighborhood conflict-free coloring using at most eight colors.*

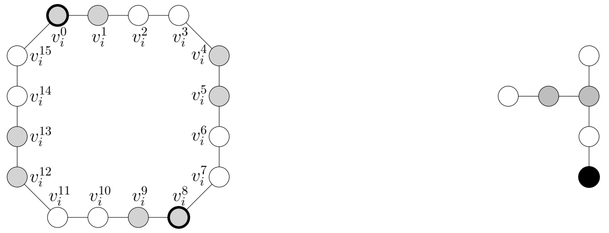
Proof. Let $G = (V, E)$ be a planar graph. Analogously to the proof of Theorem 6.3, we proceed by producing two minors G_1 and G_2 of G, to each of which we apply the planar four-color theorem. However, without the assumption of bipartiteness, we cannot use the same set of four colors for G_1 and G_2, leading to a conflict-free coloring with eight colors.

We start by constructing an independent dominating set V_1 of G. Let $V_2 := V \setminus V_1$. We construct the minor G_i of G by contracting each vertex $v \in V_{3-i}$ into an arbitrarily chosen neighbor $v_i(v) \in V_i$. Then we apply the planar four-color theorem to G_1 and G_2 with colors $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$. To build a conflict-free coloring of G, we assign to each $v \in V_i$ its color in the proper coloring of G_i. This results in a conflict-free coloring because $v_{3-i}(v)$ is a conflict-free neighbor of v. □

Similar to the situation for closed neighborhoods, open-neighborhood conflict-free coloring is hard even for $k = 1$ and $k = 2$. For closed neighborhoods, a conflict-free 1-coloring corresponds to a dominating set consisting of vertices at pairwise distance at least 3. For open neighborhoods, a conflict-free 1-coloring corresponds to a matching whose vertices form a dominating set and are at pairwise distance at least 3 (except for those adjacent in the matching).

Theorem 6.5. *It is NP-complete to decide whether a bipartite planar graph G is open-neighborhood conflict-free* 1*-colorable.*

Proof. We prove hardness using a reduction from Positive Planar 1-in-3-SAT. In a manner similar to the proof of Theorem 4.1, from a positive planar 3-CNF formula ϕ with clauses $C = \{c_1, \ldots, c_l\}$ and variables $X = \{x_1, \ldots, x_n\}$ and its plane formula graph $G(\phi)$, we construct in polynomial time a bipartite planar graph $G_1'(\phi)$ such that ϕ is 1-in-3-satisfiable iff $\chi_O(G_1'(\phi)) = 1$. The graph $G_1'(\phi)$ has one *variable cycle*



(a) A *variable cycle*, with a conflict-free 1-coloring that corresponds to setting the variable to `true`. All conflict-free 1-colorings of a variable cycle result from this coloring by shifting the groups of colored vertices around the cycle. The vertices v_i^0 and v_i^8 that may be connected to the clause gadgets are drawn with a bold outline.

(b) A clause gadget. The gray vertices must be colored in any conflict-free 1-coloring. The white vertices cannot be colored. The black vertex cannot be colored, but does not have a conflict-free neighbor within the gadget. It is connected to the variables occurring in the clause, thus enforcing that exactly one of them is set to true.

FIG. 14. *Variable and clause gadgets for the reduction.*

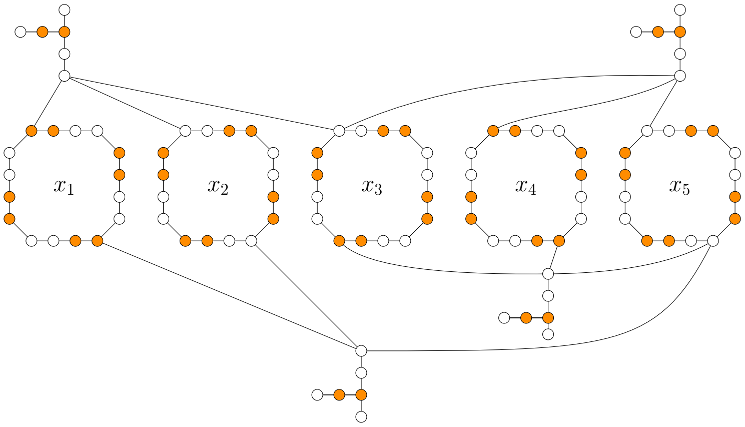


FIG. 15. *The graph $G'_1(\phi)$ resulting from applying the reduction to $\{\{x_1, x_2, x_3\}, \{x_1, x_2, x_5\}, \{x_2, x_4, x_5\}, \{x_3, x_4, x_5\}\}$, and an open-neighborhood conflict-free 1-coloring corresponding to setting x_1 and x_4 to* `true`.

$v_i^0 \cdots v_i^{15}$ of length 16 for each variable x_i. There are exactly four ways to color a variable cycle; see Figure 14. Two of these color v_i^0 and v_i^8; using one of these colorings for the variable cycle of x_i corresponds to setting x_i to `true`. Leaving v_i^0 and v_i^8 uncolored corresponds to setting x_i to `false`. For each clause c_j, $G'_1(\phi)$ contains a copy of the clause gadget depicted in Figure 14. We can compute an embedding of the formula graph $G(\phi)$ in which the variable vertices are placed on a horizontal line. The clause vertices are embedded above and below this horizontal line. If a clause c_j is embedded below the variables, we connect its black vertex to vertex v_i^8 of all variables occurring in c_j; otherwise, we use v_i^0. An example of this construction is depicted in Figure 15.

If ϕ is 1-in-3-satisfiable, coloring the variable cycles according to a satisfying assignment and the clause gadgets according to Figure 14 yields a coloring of $G_1'(\phi)$ in which the black vertex of each clause is adjacent to exactly one colored neighbor. This coloring is an open-neighborhood conflict-free 1-coloring of ϕ. On the other hand, let $G_1'(\phi)$ have an open-neighborhood conflict-free 1-coloring χ. In each clause gadget, χ colors exactly the two gray vertices from Figure 14. Therefore, the black vertex of each clause has to be adjacent to exactly one colored variable vertex. Setting the variables corresponding to variable cycles with colored vertices v_i^0 and v_i^8 to `true` thus yields a 1-in-3-satisfying assignment for ϕ. □

The same holds for $k = 2$ colors, but the restriction to bipartite planar graphs requires a slightly more sophisticated argument.

THEOREM 6.6. *It is NP-complete to decide whether a bipartite planar graph G is open-neighborhood conflict-free* 2*-colorable.*

Proof. Again we prove hardness using a reduction from POSITIVE PLANAR 1-IN-3-SAT. From a positive planar 3-CNF formula ϕ with clauses $C = \{c_1, \ldots, c_l\}$ and variables $X = \{x_1, \ldots, x_n\}$ and its plane formula graph $G(\phi)$, we construct in polynomial time a bipartite planar graph $G_2'(\phi)$ such that ϕ is 1-in-3-satisfiable iff $\chi_O(G_2'(\phi)) \leq 2$. The graph $G_2'(\phi)$ has a *variable path* $v_i^1 v_i^2 v_i^3$ of length 3 for each variable x_i. For each clause c_j, there is a clause gadget as depicted in Figure 16; this gadget contains a distinguished *clause vertex*. The gadget prevents the clause vertex from being colored and cannot be used to provide a conflict-free neighbor to the clause vertex. We connect vertex v_i^1 to the clause vertex of c_j with an edge iff x_i occurs in c_j; the other vertices of clause gadgets and variable gadgets are not connected to any vertex outside their respective gadget. Therefore, variable vertex v_i^1 can provide a conflict-free neighbor to the clause vertex of c_j iff x_i occurs in c_j.

We still have to enforce that the color of the conflict-free neighbor of the clause vertex is the same for all clauses. To this end, we connect the clause vertices using the *equality gadget* depicted in Figure 17. This gadget ensures that the conflict-free neighbors of the two clause vertices connected by it have the same color in any conflict-free 2-coloring. We cannot add this gadget between all pairs of clause vertices because this would destroy planarity. Instead, we compute a spanning tree T on the clause vertices that could be added to $G_2'(\phi)$, preserving planarity. Then, for each edge $c_a c_b$ of T, we add a copy of the equality gadget to $G_2'(\phi)$, using it to connect the clause vertices c_a and c_b. Because adding the edges of T preserves planarity, the graph resulting from adding the gadgets is planar as well. Moreover, because the equality gadget works transitively and T is connected, the conflict-free neighbors of all clause vertices must receive the same color in any conflict-free 2-coloring.

It remains to prove that such a T always exists. For this purpose, consider the plane formula graph $G(\phi)$, including the backbone of the formula. Because only one vertex of each variable or clause gadget is connected to vertices outside the gadget, these gadgets do not influence the planarity of $G_2'(\phi)$. Therefore, if adding T preserves the planarity of $G(\phi)$, it also preserves the planarity of $G_2'(\phi)$. As the root of T, we choose an arbitrary clause vertex r on the boundary of the unbounded face of $G(\phi)$. We add an edge from r to all other clause vertices on the boundary of the unbounded face to T. Now we consider the connected component R of r in T. Either $R = V(T)$, in which case we are done, or there must be a vertex $v \in R$ that lies on a face whose boundary contains a vertex $w \notin R$. For each such vertex v, we add an edge to all such vertices $w \notin R$. We iterate this procedure until we are done.

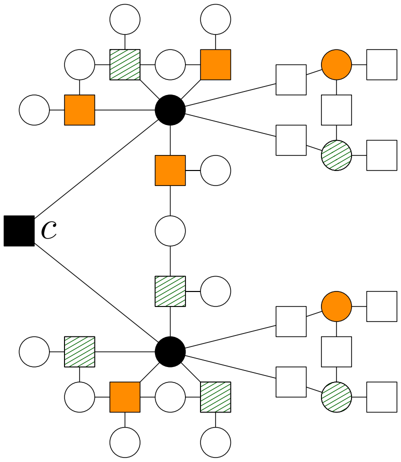


FIG. 16. *The bipartite clause gadget with clause vertex c; the components of the bipartition are indicated using squares and circles. Black vertices cannot receive a color. Vertices drawn with shaded or solid color must be colored. Except for automorphisms and swapping colors, shaded vertices have to receive color* 1 *and solidly colored vertices have to receive color* 2*. White vertices may be colored or may remain uncolored; it is straightforward to extend the depicted coloring to a conflict-free* 2*-coloring of the gadget (except for c) by coloring the white vertices of degree* 1*. By construction, one of c's neighbors has three neighbors of color* 1 *and a conflict-free neighbor of color* 2 *(and vice versa for c's other neighbor). In total, the gadget guarantees that c remains uncolored and cannot have a colored neighbor within the gadget.*

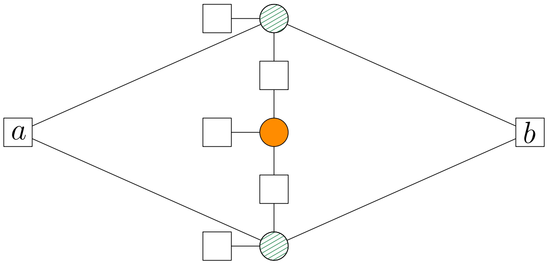


FIG. 17. *The* equality gadget *that can be used to connect two terminal vertices (marked a and b) in the same partition of a bipartite graph. It adds two occurrences of the same color to the neighborhoods of a and b, thereby forcing the conflict-free neighbor of a and b to have the same color.*

Let ϕ be 1-in-3-satisfiable, and let Γ be the set of true variables in a 1-in-3-satisfying assignment of ϕ. We construct a conflict-free 2-coloring of $G_2'(\phi)$ by assigning color 1 to v_i^1 and v_i^2 for all $x_i \in \Gamma$ and to v_i^3 and v_i^2 for $x_i \notin \Gamma$. The vertices in equality gadgets that are adjacent to clause vertices receive color 2. All other vertices in the gadgets are colored as sketched in Figures 16 and 17. All clause vertices are adjacent to exactly one variable vertex carrying color 1 and thus have a conflict-free neighbor. Therefore, the coloring constructed in this way is a valid conflict-free 2-coloring.

Now assume that $G_2'(\phi)$ has a conflict-free 2-coloring χ. By the argument above, the conflict-free neighbor of each clause vertex is a variable vertex v_i^1. Moreover, all clause vertices have a conflict-free neighbor of the same color—w.l.o.g., color 1. Therefore, each clause vertex is adjacent to exactly one variable vertex with color 1, and the set of variables x_i where $\chi(v_i^1) = 1$ induces a satisfying assignment of ϕ. □

7. Conclusion.

A spectrum of open questions remains. Many of them are related to general graphs, in particular to our sufficient condition for general graphs. For every $k \geq 2$, G_{k+1} provides an example showing that excluding K_{k+2} as a minor is not sufficient to guarantee k-colorability. However, for $k \geq 2$ we have no example where excluding K_{k+3}^{-3} as a minor does not suffice.

With respect to open-neighborhood conflict-free coloring, several open questions remain. Are four colors always sufficient for general planar graphs? Are three colors always sufficient for outerplanar graphs?

Another variant of our problem arises from requiring that *all* vertices must be colored. It is clear that one extra color suffices for this purpose; however, it is not always clear that this is also necessary, in particular, for planar graphs. Adapting our argument to this situation does not seem straightforward, especially because there are outerplanar graphs requiring three colors in this setting.

In addition, there is a large set of questions related to geometric versions of the problem. What is the worst-case number of colors for straight-line visibility graphs within simple polygons? It is conceivable that $\Theta(\log \log n)$ is the right answer, just like for rectangular visibility, but this is still an open problem, just like complexity and approximation. Other questions arise from considering geometric intersection graphs, such as unit-disk intersection graphs, for which necessary and sufficient conditions, just like upper and lower bounds, would be quite interesting.

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