On the Traveling Salesman Problem in Solid Grid Graphs

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Abstract

We consider the Traveling Salesman Problem in solid grid graphs, whose complexity is one of the long-standing problems in *The Open Problems Project*. We disprove a conjecture that a component-minimal 2-factor, i.e., a minimum cardinality set of disjoint cycles that cover all vertices, would yield an optimal tour. Instead, we show that it is sufficient to find a longest cycle to obtain optimal tours—at least in solid grid graphs that contain a 2-factor.

1 Introduction

The Traveling Salesman Problem (TSP) is one of the classic problems of optimization. Easy to describe but provably hard, it shows up in a wide range of application fields, e.g., planning roadtrips, guiding industrial machines, organizing data or mowing a lawn. Discretizations to grid points are common, so many scenarios deal with *grid graphs*, in which vertices are points in the orthogonal integer grid, and edges connect grid points at unit distance.

We consider the TSP in *solid grid graphs* $G = (V, E)$ that do not have any *holes*, i.e., graphs for which the set $\mathbb{N}^2 \setminus V$ with unit-length connections is connected. In 1997, Umans and Lenhart showed that the Hamiltonian Cycle Problem (HCP) is polynomially solvable in these graphs; the more general problem of finding a shortest tour (for which distances between non-adjacent vertices are induced by shortest-path distances in the graph, corresponding to Manhattan distances) has defied all attempts at resolving its complexity. As Problem #54 in *The Open Problems Project*[1] (TOPP), this belongs to a prominent list of long-standing open problems. The complexity of the related Longest Cycle Problem (LCP) in these graphs is also an open question.

Related Work. Itai et al. [5] showed that the HCP is NP-complete for general grid graphs. Umans and Lenhart [8] proved that the HCP is decidable in polynomial time for solid grid graphs. Their algorithm is based on merging components of an initial 2-factor, which is done by flipping the edge parities of a specific class of alternating cycles between different components. Kunas [6] showed that even if there is no Hamiltonian cycle, the edge-flipping algorithm terminates with a 2-factor consisting of a minimum number of components. She also conjectured that these component-minimal 2-factors can be used to solve the TSP in solid grid graphs, which would be sufficient to resolve Problem #54 of TOPP. Arkin, Fekete and Mitchell [2] showed that grid graphs with n vertices and without local cut vertices allow a tour of length at most $6n/5$; the results by Arora [3] and Mitchell [7] imply the existence of polynomial-time approximation schemes. For other classes of grid graphs, e.g., triangular and hexagonal grid graphs, Arkin et al. [1] proved that the HCP is NP-complete in the general case, but decidable in polynomial time for solid triangular grids. The complexity for solid hexagonal grids is an open problem. Asgharian-Sardroud et al. [4] gave a simple 2/3-approximation algorithm for the LCP in solid grid graphs.

Our Contribution. We show that a component-minimal 2-factor is *not* necessarily a substructure of an optimal tour, disproving a conjecture that would have resolved the long-standing Problem #54 of TOPP. Instead, we show that it is sufficient to find a longest cycle to obtain optimal tours—at least in solid grid graphs that contain a 2-factor. We further give some geometric observations and bounds on the LCP, which lead to a number of interesting questions.

2 Solid Grid Graphs with 2-Factor

Any grid graph with a Hamiltonian cycle contains a 2-factor. As the first step in the edge-flipping algorithm of Umans and Lenhart, its existence is easily checked with matching techniques. Kunas [6] extends this by proving that for this class, the edge-flipping algorithm terminates with a component-minimal 2-factor.

2.1 Component-Minimal 2-Factors

The cells between two components of any 2-factor can be of the different types shown in Figure 1. A simple observation is that a type IV cell induces the occurrence of a type III cell, and that a type III cell can be used to decrease the number of components of a given 2-factor by flipping its edge parities. So it is easy to conclude that in any component-minimal 2-factor,

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[1] `http://cs.smith.edu/~orourke/TOPP/`

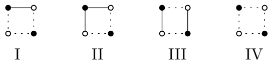

Figure 1: Different types of cells between any two adjacent components of a 2-factor, i.e., not all vertices of these cells belong to the same component. (Vertices are black and white according to their parity in the grid graph, while edges of a 2-factor are shown as solid edges.)

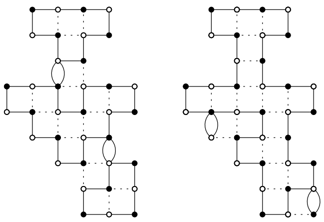

Figure 2: On the left we see a tour that has a component-minimal 2-factor as a substructure. This tour contains four vertices of degree four. On the right, we see that an optimal tour only contains two vertices of degree four.

only cells of type I and type II can occur between its components.

Furthermore, exactly two components are incident to each of these cells. We can construct the dual graph G_k of a component-minimal 2-factor F_k (with k components), where a vertex represents a component and an edge connects two vertices iff their respective components share at least one border cell. Due to the previous observations, it is clear that G_k is a tree.

2.2 First Approach for Using Edge Flips

Kunas suggests the following approach. Compute an initial 2-factor and decrease the number of components, using the edge-flipping algorithm of Umans and Lenhart, until it terminates with a component-minimal 2-factor F_k. Because G_k is a tree, we can connect the components of F_k by doubled edges in a tree-like manner to obtain a tour of length $|V| + 2\,(k-1)$.

Lemma 1 (see [6]) *Let F_k be a component-minimal 2-factor in any solid grid graph G. There exists a tour T in G of length at most $|V| + 2\,(k-1)$.*

This is an upper bound on the length of any tour. Kunas conjectured that it is also a lower bound: We insert additional edges if and only if we cannot merge another two components, so this approach produces—due to this specific substructure—a minimum number of vertices of degree four. It is easy to see that this yields an optimal solution for any solid grid graph with a 2-factor consisting of two cycles.

However, this is *not* true in general; even for graphs with a 2-factor with three components.

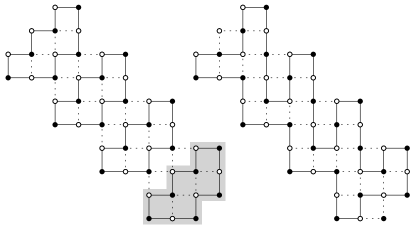

Figure 3: The graph can be extended by adding more W-shaped subgraphs. In the left case we need a doubled edge between any pair of adjacent components, whereas an optimal tour just has length $|V| + 2$.

Theorem 2 *A 2-factor with a minimum number of components is not necessarily a substructure of an optimal tour in solid grid graphs.*

Proof. See the graph in Figure 2. □

The gap between the length of a tour constructed by the previous approach to any optimal tour can get arbitrarily large. An example is given in Figure 3.

2.3 What Optimal Tours Look Like

As shown in Theorem 2, a 2-factor is not necessarily a substructure of optimal tours in solid grid graphs. In the following we work out some important properties of optimal tours. Putting them together establishes the following theorem.

Theorem 3 *Given a solid grid graph that contains a 2-factor. Then there is an optimal tour such that the longest cycle is a substructure of this tour.*

Lemma 4 *If an edge e is used more than twice in a tour T, then there is a shorter tour T', and e is used at most twice.*

The high-level idea is the following. Assume that such an edge e is used an even number of times. Then by removing e and all copies of e, the degree of both incident vertices remains even; we either obtain a tour or we just have to add e twice, which yields a shorter tour in both cases. A similar argument holds for any edge that is used an odd number of times. This directly implies that there is always a tour in which all vertices have degree at most eight.

Lemma 5 *In every solid grid graph that contains a 2-factor, an optimal tour cannot contain vertices of degree eight.*

Proof. We prove this by contradiction to the optimality of the tour. Due to space constraints we only show one case; the proof can be completed by similar arguments. As a consequence of Lemma 4, a vertex v of degree eight must be connected to each of its four neighbors by doubled edges. If v is no cut vertex,

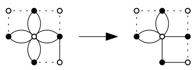

Figure 4: The modification from left to right yields a shorter tour, because the degree of the middle vertex decreases by two.

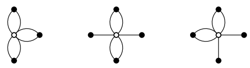

Figure 5: The different edge configurations of a degree-six vertex.

then at least three vertices of its 8-neighborhood must exist. Figure 4 shows one such possible edge configuration. One can see that the transformation from left to right yields a tour with shorter length. ☐

Lemma 6 *Every solid grid graph with a 2-factor has an optimal tour in which no vertex has degree six.*

Proof. This can be proven by similar arguments as in Lemma 5, checking the different configurations for a degree-six vertex (see Figure 5) in its 8-neighborhood. There are situations in which a degree-six vertex can occur in an optimal tour, but there are local modifications to this tour that move a doubled edge incident to this vertex, creating vertices of degree two and four. Details are omitted due to limited space. ☐

Figure 6: The two different types of a degree-four vertex. The dotted cycle in the right shows that the black vertex can have degree two or four. The left one is called a *cross vertex*, whereas the right one is an *isolated vertex*, iff the black vertex has degree two.

In order to prove Theorem 3, we have to show that we can modify the different types of degree-four vertices in an optimal tour (see Figure 6), such that we get a longest cycle. First consider the case in which two subtours are connected by a doubled edge. Without loss of generality, let the boundary between these subtours be neither a bridge nor a cut vertex. Then we can easily merge both subtours to get a single cycle that contains all but two vertices of the initial subtours. There is only one kind of unmergeable component, which is shown in Figure 7.

Lemma 7 *Let $G = (V, E)$ be solid grid graph without cut vertices and let F_k be a component-minimal 2-factor in G. Then there is a cycle of length at least $|V| - 2(k - 1 + \sigma)$, where σ denotes the number of unmergeable components.*

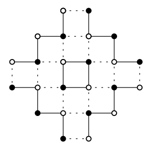

Figure 7: If all but the vertices of the nested C_4 are connected in a single cycle, then we cannot extend this cycle by some of these inner vertices.

In general, given an arbitrary 2-factor, this cycle is *not* a longest cycle (see Figure 3). However, if a doubled edge connects two cycles in an optimal tour, we can merge them into a single one—containing two isolated vertices. Now we only have to show that the *cross-shaped* vertex of degree four can be modified, such that we get a doubled edge between a single vertex and a bigger cycle, i.e., an isolated vertex.

Lemma 8 *Let G be solid grid graph without cut vertices that contains a 2-factor. In an optimal tour of G, a cross vertex is equivalent to an isolated vertex.*

Proof. See Figure 8 for the main idea; details are omitted due to limited space. ☐

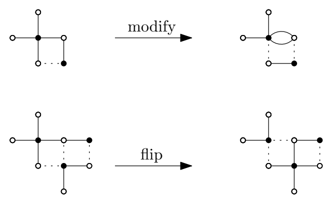

Figure 8: These are basically all possible configurations in the extended neighborhood of a cross vertex. Preserving the tour, we can either directly modify a cross vertex to an isolated one (Top), or we can *flip* it to obtain the upper configuration (Bottom).

2.4 Longest Cycles

Knowing that longest cycles yield optimal tours in solid grid graphs with a 2-factor, makes it interesting to exploit the structure of component-minimal 2-factors F_k. Recalling Figure 3 and Lemma 7, it is easy to argue that an upper bound for any cycle is $|V| - \lambda$, where λ denotes the number of leaves in G_k. It is plausible to conjecture that λ could also denote the number of odd vertices in G_k. Unfortunately, this is not true in general.

Instead, we analyze the color of the vertices that are left in the leaf-corresponding vertex sets to improve this bound. To this end, we count the leaves in G_k for which the corresponding component in F_k has black vertices adjacent to another component, and subtract one for each leaf in which these vertices are white. Let ζ denotes the absolute value of this sum, then $|V| - (\lambda + \zeta)$ is an upper bound for any cycle in G. The intuitive idea is that the number of vertices left in the leaf-corresponding sets does not suffice for connecting all vertices of the inner components; this is analogous to the Tutte-Berge formula from matching theory. Unfortunately, it is not easy to improve this bound because there are no specific structures that can be used to achieve this, i.e., some cutting sets of size two or the like (see Figure 9).

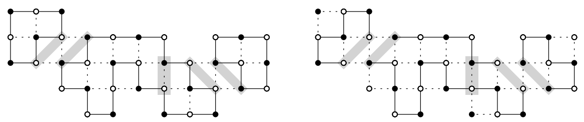

Figure 9: The upper bound of $|V| - (\lambda + \zeta)$ on any cycle results in $|V| - (2 + 0)$. Analyzing the shaded cutting sets yields an upper bound of $|V| - 4$ on any cycle in this graph.

3 General Solid Grid Graphs

A general solid grid graph can be decomposed into (A) vertex-clusters that contain a 2-factor, (B) clusters that are 2-connected, (C) 1-dimensional paths that connect different clusters or single vertices, and (D) clusters that contain a cut vertex (see Figure 10). It

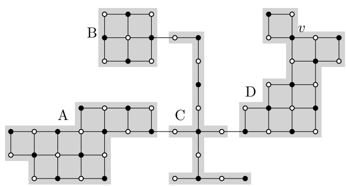

Figure 10: A general solid grid graph with the component classification given in Section 3.

is trivial to observe that all edges on a 1-dimensional path must be used twice in any optimal tour. It can also be shown that the Lemmas 5 and 6 can be adapted by similar arguments to the case that a vertex cluster is 2-connected but does not contain a 2-factor. Finally, it is easy to see that one can decompose a cluster at a cut vertex, solve the problem for each subcluster and merge them afterwards. This implies that many of the previous results can be reused.

Unfortunately, we were not able to extend the arguments of Lemma 7 and Lemma 8 for general 2-connected solid grid graphs. Without these, we were not able to show that a longest cycle is also part of an optimal tour in general solid grid graphs. Nevertheless, we are strongly convinced that this is the case, as stated in the following conjecture.

Conjecture 1 *In general solid grid graphs, a longest cycle is part of an optimal tour.*

4 Future Work

We have shown that a component-minimal 2-factor is not necessarily a substructure of an optimal tour in a solid grid graph with a 2-factor. We were able to prove that for these graphs, a longest cycle is part of an optimal tour. We also gave a number of geometric observations on longest cycles. Some of these results can be adapted to general solid grid graphs. We are convinced that a longest cycle is also part of an optimal tour in general solid grid graphs.

The computational complexity of both the TSP and the LCP in solid grid graphs is still open. It may be useful to know whether there always exists a longest cycle such that all vertices that are not part of this cycle lie on the boundary of the graph. If true, it may be possible to solve the question of how many vertices of the boundary must be deleted, such that the remaining graph is Hamiltonian. Another challenging question is whether there is always a longest cycle for which all unvisited vertices are adjacent to it.

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