Conflict-Free Coloring of Intersection Graphs

Sándor P. Fekete* Phillip Keldenich*

Abstract

A *conflict-free k-coloring* of a graph $G = (V, E)$ assigns one of k different colors to some of the vertices such that, for every vertex v, there is a color that is assigned to exactly one vertex among v and v's neighbors. Such colorings have applications in wireless networking, robotics, and geometry, and are well studied in graph theory. Here we study the conflict-free coloring of geometric intersection graphs. We demonstrate that geometric objects without fatness properties and size restrictions have intersection graphs with unbounded conflict-free chromatic number. For unit-disk intersection graphs, we prove that it is NP-complete to decide the existence of a conflict-free coloring with one color; we also show that six colors always suffice, using an algorithm that colors unit disk graphs of restricted height with two colors. We conjecture that four colors are sufficient, which we prove for unit squares instead of unit disks.

1 Introduction

Coloring the vertices of a graph is one of the fundamental problems in graph theory, both scientifically and historically. The notion of proper graph coloring can be generalized to hypergraphs in several ways. One natural generalization is *conflict-free coloring*, which asks to color the vertices of a hypergraph such that every hyperedge has at least one uniquely colored vertex. This has applications in wireless communication, where "colors" correspond to different frequencies. The notion can be transported back to simple graphs by considering hypergraphs induced by the neighborhoods of vertices.

In current work with Abel et al. [2], we prove a conflict-free variant of Hadwiger's conjecture, which implies planar graphs have conflict-free chromatic number at most 3; see that paper for a more detailed overview of related work. In the geometric context, motivated by frequency assignment problems, the study of conflict-free coloring of hypergraphs was initiated by Even et al. [5] and Smorodinsky [11]. For disk intersection hypergraphs, Even et al. [5] prove that $\mathcal{O}(\log n)$ colors suffice. For disk intersection hypergraphs with degree at most k, Alon and Smorodinsky [3] show that $\mathcal{O}(\log^3 k)$ colors are sufficient. If every edge of a disk intersection hypergraph must have k distinct unique colors, Horev et al. [8] prove that $\mathcal{O}(k \log n)$ suffice. Moreover, for unit disks, Lev-Tov and Peleg [9] present an $\mathcal{O}(1)$-approximation algorithm for the conflict-free chromatic number. Abam et al. [1] consider the problem of making a conflict-free coloring robust against removal of a certain number of vertices, and prove worst-case bounds for the number of colors required.

Conflict-free coloring also arises in the context of the conflict-free variant of the chromatic art gallery problem, where a simple polygon P has to be guarded by colored guards such that each point in P sees at least one uniquely colored guard. Regarding complexity, Fekete et al. [6] prove that computing the chromatic number is NP-hard in this context. On the positive side, Hoffman et al. [7] give tight bounds for the conflict-free chromatic art gallery problem under rectangular visibility in orthogonal polygons: $\Theta(\log \log n)$ colors are sometimes necessary and always sufficient. For the more common straight-line visibility, Bärtschi et al. [4] prove that $\mathcal{O}(\log n)$ colors always suffice.

2 Preliminaries

In the following, $G = (V, E)$ will denote a graph on $n := |V|$ vertices. For a vertex v, $N(v)$ denotes its *open neighborhood* and $N[v] = N(v) \cup \{v\}$ denotes its *closed neighborhood*. A *conflict-free k-coloring* of a graph $G = (V, E)$ is a coloring $\chi_C : V' \to \{1, \dots, k\}$ of a subset $V' \subseteq V$ of the vertices of G, such that each vertex v has at least one *conflict-free neighbor* $u \in N[v]$, i.e., a neighbor u whose color $\chi_C(u)$ occurs only once in $N[v]$. The *conflict-free chromatic number* $\chi_C(G)$ is the minimum number of colors required for a conflict-free coloring of G.

A graph G is called *disk graph* iff G is the intersection graph of disks in the plane. A disk graph G is a *unit disk graph* iff G is the intersection graph of disks with fixed radius $r = 1$ in the plane. A graph G is a *unit square graph* iff G is the intersection graph of axis-aligned squares with side length 2 in the plane. A unit disk (square) graph is of *height h* iff G can be modeled by the intersection of unit disks (squares) with center points in $(-\infty, \infty) \times [0, h]$. In the following, when dealing with intersection graphs, we assume that we are given a geometric model. In the case of unit disk and unit square graphs, we identify the vertices of the graph with the center points of the corresponding geometric objects in this model.

*Department of Computer Science, TU Braunschweig, Germany, {s.fekete,p.keldenich}@tu-bs.de

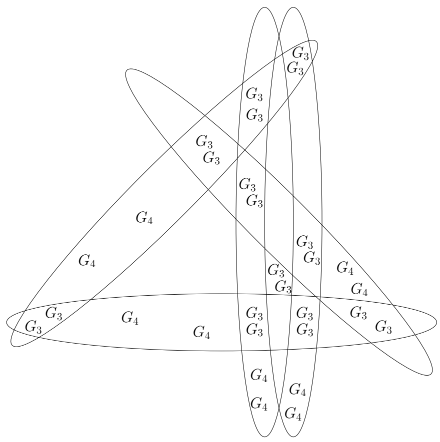

Figure 1: The graph G_5, shown as an intersection graph of ellipses, requires 5 colors.

3 General Objects

For general objects like freely scalable ellipses or rectangles, it is possible to model a complete graph K_n of arbitrary size n such that the following conditions hold: (1) For every object v, there is some non-empty area of v not intersecting any other objects. (2) For every pair of objects v, w, there is a non-empty area common to these objects disjoint from all other objects.

In this case, the conflict-free chromatic number is unbounded, because we can inductively build a family G_n of intersection graphs with $\chi_C(G_n) = n$ as follows. Starting with $G_1 = (\{v\}, \emptyset)$ and $G_2 = C_4$, we construct G_n by starting with a K_n modeled according to conditions (1) and (2). For every object v, we place two scaled-down non-intersecting copies of G_{n-1} into an area covered only by v. For every pair of objects v, w, we place two scaled-down non-intersecting copies of G_{n-2} into an area covered only by v and w. The resulting graph requires n colors, as every vertex of the underlying K_n has to receive a unique color. Figure 1 depicts the construction of G_5 for ellipses.

4 Unit-Disk Graphs

4.1 Complexity: One Color

While it is trivial to decide whether a graph has a regular chromatic number of 1 and straightforward to check a chromatic number of 2, it is already NP-complete to decide whether a conflict-free coloring with a single color exists, even for unit-disk intersection graphs with maximum degree 3. This is a refinement of Theorem 4.1 in Abel et al. [2], which shows the same results for general planar graphs.

Theorem 1 *It is NP-complete to decide whether a unit-disk intersection graph $G = (V, E)$ has a conflict-free coloring with one color.*

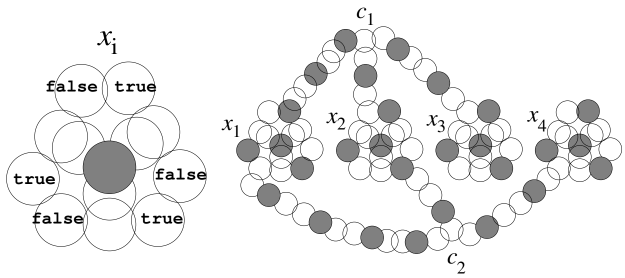

Figure 2: (Left) A variable gadget; note that the central disk must be part of any solution, leaving only the choices labeled `true` and `false` for the other disks. (Right) The overall construction for the instance $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_4})$.

Proof. We sketch a reduction from Planar 1-in-3-SAT, see Mulzer and Rote [10]. For a given instance I, we build a unit disk intersection graph G_I, in which variables x_i are represented by the gadget shown in Figure 2 (Left), consisting of an exterior cycle of $3n_i$ vertices, for some number $n_i \in \mathbb{N}$, and an auxiliary internal tree. A clause c_j is represented by a single unit disk; we connect it to each of the three involved variable gadget with $3n_\ell$ unit disks, as shown in Figure 2 (Right).

Now a satisfying truth assignment for I induces a conflict-free coloring of G_I with a single color in a straightforward manner. Conversely, in a conflict-free coloring of G with one color, the set $S \subseteq V$ of colored disks is both an independent and a dominating set in G, so any two disks in S must have distance at least 3. This implies that for each exterior cycle in a variable, every third vertex must belong to S, inducing a truth assignment. Similarly, along each connecting path, every third disk must belong to S. As it turns out, no clause disk can be picked, implying that precisely one of its neighbors must be in S; this requires a solution for I. Full details are omitted for lack of space. ☐

4.2 A Worst-Case Upper Bound: Six Colors

On the positive side, we show that the conflict-free chromatic number of unit disk graphs is bounded by 6. We do not believe this result to be tight. In particular, we conjecture that the number is bounded by 4; in fact, we do not even know an example where two colors are insufficient. One of the major obstacles towards obtaining tighter bounds is the fact that a simple graph-theoretic characterization of unit disk graphs is not available, as recognizing unit disk graphs is complete for the existential theory of the reals. This makes it hard to find unit disk graphs with high conflict-free chromatic number, especially considering the size such a graph would require: The smallest graph with conflict-free chromatic number 3 we know has 30 vertices, and by enumerating all graphs on 12 vertices one can show that at least 13 vertices are necessary, even without the restriction to unit disk graphs.

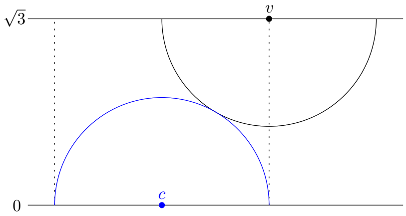

Figure 3: Every colored point c induces a vertical strip of width 2 (dashed lines); all points v within this strip are adjacent to c.

One approach to conflict-free coloring of unit disk graphs is by subdividing the plane into strips, coloring each strip independently. We conjecture the following.

Conjecture 2 *Unit disk graphs of height* 2 *are conflict-free* 2*-colorable.*

If this conjecture holds, every unit disk graph is conflict-free 4-colorable. In this case, one can subdivide the plane into strips of height 2, and then color the subgraphs in all even strips using colors $\{1, 2\}$ and the subgraphs in odd strips using colors $\{3, 4\}$. Instead of Conjecture 2, we prove the following weaker result.

Theorem 3 *Unit disk graphs G of height $\sqrt{3}$ are conflict-free* 2*-colorable.*

Proof. Given a realization of G consisting of unit disks with center points with y-coordinate in $[0, \sqrt{3}]$, we compute a conflict-free 2-coloring of G using the following greedy approach. We iterate through the disk centers in lexicographical order, choosing a set C of points to be colored. At every iteration, let c be the current and n be the next point. Let C be the set of selected colored points and let $S = N[C]$ be the points that already have a colored neighbor. We select c to be colored iff coloring n instead of c would leave a previous point uncovered, i.e., iff there is a point $c' \notin S$, $c' \leq c$ adjacent to c but not to n. Thus, starting from the leftmost point, we always color the rightmost point that does not leave any previous points without a colored neighbor. We alternatingly assign colors 1 and 2 to the selected points.

In this procedure, any point v is assigned a colored neighbor $w \in N[v]$. This leaves the following three cases. (1) a colored point v is adjacent to another point w of the same color, (2) an uncolored point is adjacent to two or more points of one color and none of the other color, (3) an uncolored point is adjacent to two or more points of both colors.

To this end, we use the following. Each colored point c induces a closed vertical *strip* of width 2 centered around c. As shown in Figure 3, every point v in this strip is adjacent to c. Thus, the horizontal distance between two colored points must be greater than 1. For case (1), assume there was a point v of color 1 adjacent to a point $w > v$ of color 1. This cannot occur, because between v and w, there must be a point x of color 2; therefore, the horizontal distance between v and w must be greater than 2, a contradiction.

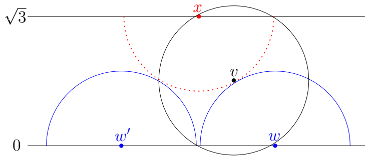

Figure 4: The configuration in case (2); there must be a point x of color 2 adjacent to v.

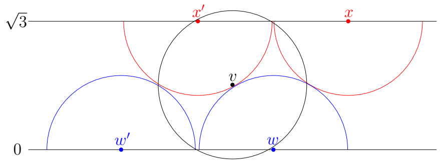

Figure 5: The configuration in case (3); the algorithm would have chosen v or a larger point instead of x'.

Regarding case (2), assume there was an uncolored point v adjacent to two points $w' < v < w$ of color 1. Between points w' and w, there must be a point x of color 2, and v must not be adjacent to x. There are two possible orderings: $w' < v < x < w$ and $w' < x < v < w$. W.l.o.g., let $v < x$; the other case is symmetric. In this situation, the x-coordinates of the points have to satisfy $x(v) < x(x) - 1$, $x(x) < x(w) - 1$, and thus $x(v) < x(w) - 2$ in contradiction to the assumption that v and w are adjacent.

Regarding case (3), assume there was an uncolored point v adjacent to two points $w' < v < w$ of color 1 and two points $x' < v < x$ of color 2. W.l.o.g., assume $w' < x' < v < w < x$ as depicted in Figure 5; the case $x' < w'$ is symmetric. Because w' and v are adjacent, the vertical strip induced by v intersects the strip induced by w'. Thus, there cannot be a point y with $w' < y < v$ not adjacent to w' or v. This is a contradiction to the choice of x': The algorithm would have chosen v or a larger point instead of x'. ☐

The next Corollary 4 follows by subdividing the plane into strips of height $\sqrt{3}$; Moreover, applying the proof of Theorem 3 to unit square graphs of height 2 instead of $\sqrt{3}$ yields Corollary 5.

Corollary 4 *Unit disk graphs are conflict-free* 6*-colorable.*

Corollary 5 *Unit square graphs of height* 2 *are conflict-free* 2*-colorable. Unit square graphs are conflict-free* 4*-colorable.*

Unfortunately, the proof of Theorem 3 does not appear to have a straightforward generalization to strips of larger height. Further reducing the height to find strips that are colorable with one color is also impossible, because unit interval graphs, which correspond to

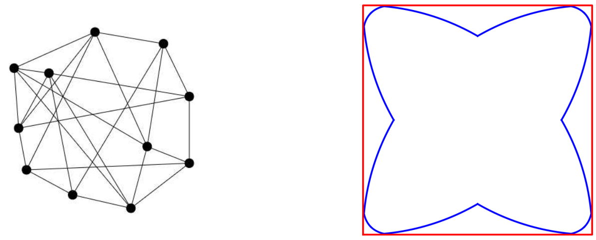

Figure 6: Left: A vertex-minimal graph satisfying (1) and (2). Right: In any unit disk graph G embeddable in a 2×2-square with $\gamma(G) = 3$, no points lie in the depicted area.

unit disk graphs with all centers lying on a line, already may require two colors in a conflict-free coloring; the Bull Graph is such an example. In this case, the bound of 2 is tight: By Theorem 3, unit interval graphs are conflict-free 2-colorable. By adapting the algorithm used in the proof to always choose the interval extending as far as possible to the right without leaving a previous interval uncovered, this can be extended to interval graphs with non-unit intervals.

4.3 Unit-Disk Graphs of Bounded Area

Proving Conjecture 2 is non-trivial, even when all center points lie in a 2×2-square. In this setting, a circle packing argument can be used to establish the sufficiency of three colors. If a unit disk graph with conflict-free chromatic number 3 can be embedded into a 2×2-square, the following are necessary. (1) Every minimum dominating set D has size 3, and every pair of dominating vertices must have a common neighbor not shared with the third dominating vertex. Thus, every minimum dominating set lies on a 6-cycle without chords connecting a vertex with the opposite vertex. (2) G has diameter 2; otherwise, one could assign the same color to two vertices at distance 3.

Using the domination number, one can further restrict the position of the points in the 2×2-square: There is an area in the center of the square, depicted in Figure 6, that cannot contain the center of any disk because this would yield a dominating set of size 2.

The smallest graph satisfying constraints (1) and (2) has 11 vertices and is depicted in Figure 6. It is not a unit disk graph and it is still conflict-free 2-colorable, but every coloring requires at least four colored vertices, proving that coloring a minimum dominating set can be insufficient. This implies that a simple algorithm like the one used in the proof of Theorem 3 will most likely be insufficient for strips of greater height. We are not aware of any unit disk graph satisfying these constraints.

5 Conclusion

There are various directions for future work. In addition to closing the worst-case gap for unit disks (and proving Conjecture 2), it is interesting to study the conflict-free chromatic number of non-unit disk graphs. Other questions include a tight bound for unit square graphs, square intersection graphs of general squares, and a necessary criterion for a family of geometric objects to have intersection graphs with unbounded conflict-free chromatic number.

References

[1] M. A. Abam, M. de Berg, and S.-H. Poon. Fault-tolerant conflict-free colorings. In *Proc. CCCG'08*, pages 13–16, 2008.

[2] Z. Abel, V. Alvarez, E. D. Demaine, S. P. Fekete, A. Gour, A. Hesterberg, P. Keldenich, and C. Scheffer. Three colors suffice: Conflict-free coloring of planar graphs. In *Proc. SODA17*, 2017. To appear.

[3] N. Alon and S. Smorodinsky. Conflict-free colorings of shallow discs. In *Proc. SoCG06*, pages 41–43, 2006.

[4] A. Bärtschi, S. K. Ghosh, M. Mihalák, T. Tschager, and P. Widmayer. Improved bounds for the conflict-free chromatic art gallery problem. In *Proc. SoCG14*, page 144, 2014.

[5] G. Even, Z. Lotker, D. Ron, and S. Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM J. Comp.*, 33(1):94–136, 2003.

[6] S. P. Fekete, S. Friedrichs, M. Hemmer, J. B. M. Mitchell, and C. Schmidt. On the chromatic art gallery problem. In *Proc. CCCG14*, pages 1–6, paper 11, 2014.

[7] F. Hoffmann, K. Kriegel, S. Suri, K. Verbeek, and M. Willert. Tight bounds for conflict-free chromatic guarding of orthogonal art galleries. In *Proc. SoCG15*, pages 421–435, 2015.

[8] E. Horev, R. Krakovski, and S. Smorodinsky. Conflict-free coloring made stronger. In *Proc. SWAT10*, volume 6139, pages 105–117, 2010.

[9] N. Lev-Tov and D. Peleg. Conflict-free coloring of unit disks. *Discrete Applied Mathematics*, 157(7):1521–1532, 2009.

[10] W. Mulzer and G. Rote. Minimum-weight triangulation is NP-hard. *J. ACM*, 55(2):11, 2008.

[11] S. Smorodinsky. *Combinatorial Problems in Computational Geometry.* PhD thesis, School of Computer Science, Tel-Aviv University, 2003.