Universal Guards: Guarding All Polygonalizations of a Point Set in the Plane

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1 Introduction

The Art Gallery Problem (AGP) seeks to find the fewest guards to see all of a given domain; in its classic combinatorial variant (posed by Victor Klee), it asks for the number of guards that always suffice and are sometimes necessary to guard any simple n-gon: the answer is the well known $\lfloor n/3 \rfloor$ [1, 3].

While Klee's question was posed about guarding an n-vertex *simple polygon*, a related question about *point sets* was posed at the 2014 Goodman-Pollack Fest (NYU, November 2014): Given a set S of n points in the plane, how many guards always suffice to guard any simple polygon with vertex set S? A set of guards that guard every polygonalization of S is said to be a set of *universal guards* for the point set. The question is how many universal guards are always sufficient, and sometimes necessary, for any set of n points? We give the first set of results on universal guarding. We focus here on the case in which guards must be placed at a subset (the *guarded points*) of the input set S and thus will be vertex guards for any polygonalization of S.

Due to space limitations, we outline here two selected cases of results: The UGPI (universal guard problem using interior guards), in which guards are placed only at points of S that are not on the convex hull of S, and the UGPG (universal guard problem on grids), in which the input set S is a regular grid of points. We then mention results for the general UGP (guards placed at any points of S) and cases in which S has a bounded number of convex layers. For the UGPI and UGP, it turns out that a fraction smaller than 1 is not possible: essentially all of the points of S require guards for universal coverage of all polygonalizations of S. For the UGPG (on grids) and for cases with bounded convex layers, fractions less than 1 are possible, as we show. Details and further results appear in the paper [2].

Preliminaries. We say that three points $a, b, c \in S$ form a *spike* if there exists a subset $S' \subseteq S$ with $a, b, c \in S'$ and a simple polygonal chain, π, having vertex set S' such that not all of $\triangle abc$ is seen by the points $S \backslash \{a, b, c\}$ when treating π as a set of opaque edges. Refer to Figure 1. A point set S is said to be in a *safe configuration with respect to spikes* if no 3 points of S form a spike.

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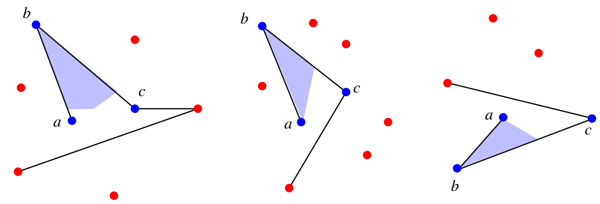

Figure 1: Examples of spikes on a, b, c; guards at red points fail to see all of $\triangle abc$.

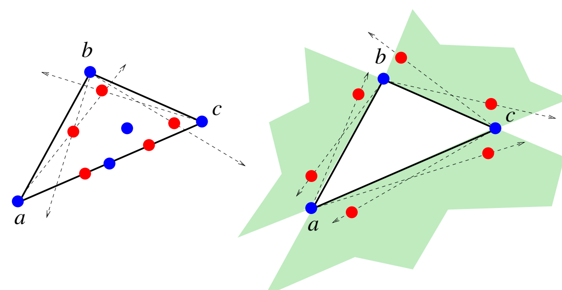

Figure 2: Safe conditions: Rules 1 and 2.

For three unguarded points $a, b, c \in S$ we say that they *satisfy the safe condition* if they satisfy either one of the following rules (refer to Figure 2):

Rule 1: There are points inside (or on the boundary of) $\triangle abc$, and within $\triangle abc$ a ray with apex in $\{a, b, c\}$ rotated inwards, starting from each incident edge to the apex, hits a guarded interior point before hitting an unguarded point.

Rule 2: There is no point of S inside (or on the boundary of) $\triangle abc$, and, further, a ray with apex in $\{a, b, c\}$ rotated outwards, starting from each incident edge to the apex hits a guarded point that is within the corresponding "wedge" (shown in green in the figure), before hitting an unguarded point.

A key fact (proof omitted here) is the equivalence:

Lemma 1 *A point set S with guards at $G \subseteq S$ is in safe configuration with respect to spikes if and only if any three unguarded points of it satisfy the safe conditions.*

2 The UGPI: Using Interior Guards

In the UGPI we allow guards to be placed only at points of S that are interior to the convex hull, $CH(S)$. Note that placing guards at *all* interior points is sufficient to guard any polygonalization of S, since the $CH(S)$ vertices are convex vertices in any polygonalization of S, and a simple fact is that the reflex vertices of any simple polygon see all of the polygon. Our main result in this section is a proof that it is sometimes necessary to place guards at all interior points, in order to have a universal guard set:

Theorem 2 *There exist configurations of n points S, for arbitrarily large n, for which $CH(S)$ is a triangle, and the only universal guard set using only interior guards is the set of* all *$n-3$ interior points.*

Proof sketch: We utilize Lemma 1 and construct a careful configuration of points whose general structure is shown in Figure 3: The points $a, b, c \in S$ are the vertices of $CH(S)$. Six additional points (in red) are placed just inside each edge of $\triangle abc$, so that each is first hit by rays rotating inwards from the edges of $\triangle abc$. Then, very carefully located points are placed (in a sequence of "rounds") along each of three line segments (thick green in the figure), in such a way that all of these interior points must be guarded in order to avoid a spike (created by the unguarded point, together with two vertices of $\triangle abc$). (Each of the potential spikes is such that, in this configuration S, we can argue that there exists a polygonalization of S that includes the spike.)

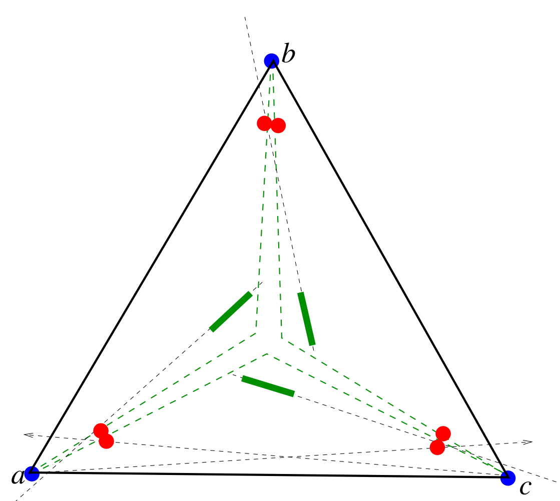

Figure 3: The overall configuration for the proof of Theorem 2.

3 The UGPG: Guarding Full Grids

Theorem 3 *For $n = ab$ points S on a regular $a \times b$ grid, $\lfloor \frac{n}{2} \rfloor$ guards are sufficient to guard all polygonalizations on n points. Further, $\lfloor \frac{a}{2} \rfloor \cdot b$ guards are necessary to guard every polygonalization on $a \times b$ grid points ($a \leq b$) with size at least 4×5.*

The proof of sufficiency is based on either of two patterns of guard selection: (1) place guards at the odd posititions on odd-numbered rows and at even positions on even-numbered rows of the grid (i.e., place guards in the grid according to white squares on a checkboard); or (2) place guards at all positions on the even-numbered rows. We argue that with either placement strategy, any triangle with vertices at grid points, and no other grid points on the boundary or interior of the triangle, must have at leat one of its vertices guarded. This implies that the $\lfloor \frac{n}{2} \rfloor$ guards see every point in any polygonalization P of S, since any such P can be triangulated, and every triangle in any triangulation has at least one guard at a vertex. The proof of necessity is based on analyzing possible spikes in the grid, using the fact that in a solution an unguarded interior grid point cannot have both of its horizontal and vertical neighbors unguarded at the same time.

4 The General UGP, Bounded Layers, k-UGP

In [2] we prove a bound for the general UGP:

Theorem 4 *For any $m = 2^h \geq 8$ such that $h \in \mathbb{N}$, there is a point set P with $|P| = n = m^2 + 2 * m - 21$ that requires at least $(1 - \frac{5}{\sqrt{n-1}})n$ universal guards.*

The proof of this theorem is based on having the points evenly distributed on multiple convex layers in such a way that on each layer at most 4 points can be unguarded. We also consider sets S on m layers:

Theorem 5 *$(1 - \frac{1}{16n^{\frac{2m-1}{2m}}})n$ are always sufficient to guard all polygonalizations for n points that lie on m convex layers.*

In [2] we also give results on the k-*universal* guarding problem, in which the guards must perform visibility coverage for a set of k different polygonalizations of the input points (instead of *all* polygonalizations).
The complexity of deciding if a given set S has a universal guard set of size at most m is open; it is also open to obtain approximation algorithms for universal guarding.

References

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