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Reallocation Problems in Scheduling

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Abstract In traditional on-line problems, such as scheduling, requests arrive over time, demanding available resources. As each request arrives, some resources may have to be irrevocably committed to servicing that request. In many situations, however, it may be possible or even necessary to *reallocate* previously allocated resources in order to satisfy a new request. This reallocation has a cost. This paper shows how to service the requests while minimizing the reallocation cost. We focus on the classic

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problem of scheduling jobs on a multiprocessor system. Each unit-size job has a time window in which it can be executed. Jobs are dynamically added and removed from the system. We provide an algorithm that maintains a valid schedule, as long as a schedule with sufficient slack exists. The algorithm reschedules only a total number of $O(\min\{\log^* n, \log^* \Delta\})$ jobs for each job that is inserted or deleted from the system, where n is the number of active jobs and Δ is the size of the largest window.

Keywords Scheduling · Online problems · Reallocation

1 Introduction

Imagine you are running a doctor's office. Every day, patients call and try to schedule an appointment, specifying a time period in which they are free. You respond by agreeing to a specific appointment time. Sometimes, however, there is no available slot during the period of time specified by the patient. What should you do? You might simply turn the patient away. Or, you can reschedule some of your existing patients, making room in the schedule.[1] Unfortunately, patients do not like being rescheduled. How do you minimize the number of patients whose appointments are rescheduled?

While scheduling a doctor's office may (or may not) seem a somewhat contrived motivating example, this situation arises with frequency in real-world applications. Almost any scenario that involves creating a schedule also requires the flexibility to change that schedule later, and those changes often have real costs (measured in equipment, computation, or tempers). For example, in the computational world, scheduling jobs on multiprocess machines and scheduling computation on the cloud lead to rescheduling. In the physical world, these problems arise with depressing regularity in scheduling airports and train stations. Real schedules are always changing.

In a tightly packed schedule, it can be difficult to perform this rescheduling efficiently. Each task you reschedule risks triggering a cascade of other reschedulings, leading to high costs (and unhappy patients or passengers). It is easy to construct an example where each job added or removed changes $\Omega(n)$ other jobs, even with constant-sized tasks. In this paper, we show that if there is slack in the schedule, then these rescheduling cascades can be collapsed down to $O(\log^* n)$ for unit-size jobs.

1.1 Reallocation Problems

In this paper, we formalize a class of problems that ask the question: how should we modify resource allocationa to respond to a changing situation. We call such problems ***reallocation problems***. A reallocation problem is online in the sense that requests arrive and the system responds. Unlike in the standard online setting where resources are irrevocably assigned, in a reallocation problem, allocations may change. These reallocations, however, have a cost.

[1] Before you get too skeptical about the motivation, this is exactly what M. F-C's ophthalmologist does.

Reallocation lies somewhere between traditional notions of offline and online resource allocation. If the reallocation cost is 0, then there is no penalty for producing an optimal allocation after each request. In this case, a reallocation problem can be viewed as a sequence of offline problems. If the cost of reallocation is ∞, then no finite-cost reallocation is possible and the result is a traditional online problem. When there is a bounded but non-zero cost for reallocation, then there is a trade-off between the quality of an allocation and the cost of reallocation.

1.2 Our Problem

We focus on the reallocation version of a classical multiprocessor scheduling problem [23] (described more fully in Sect. 2). We are given a set of unit-length jobs to process on m machines. Each job has an arrival time and a deadline. The job must be assigned to a machine and processed at some point within the specified time window. Jobs are added and removed from the schedule dynamically. The goal is to maintain a feasible schedule despite job insertions and deletions.

In order to process a request, it may be necessary to reschedule some previously scheduled jobs. There are two ways in which a job may be rescheduled: it may be ***reallocated*** to another time on the same machine, or it may be ***migrated*** to a different machine. The ***migration cost*** is the total number of jobs that are moved to different machines when new jobs are added or removed. The ***reallocation cost*** is the total number of jobs that are rescheduled, regardless of whether they are migrated or retained on the same machine. Our goal is to minimize both the migration cost and the total reallocation cost. We bound these costs separately, since we expect that a reallocation might be more expensive if it also entails a migration. (See [5,7] for other work that considers migrations separately from other scheduling considerations, such as preëmptions.)

We call an algorithm that processes such a sequence of scheduling requests a ***reallocating scheduler***. We show in Sect. 6 that a reallocating scheduler must allow for some job migrations and that there is no efficient reallocating scheduler without some form of resource augmentation; here we consider speed augmentation [26,33]. We say that an instance is ***γ-underallocated*** if it is feasible even when all jobs sizes (processing times) are multiplied by γ. In other words, the offline scheduler is γ times slower than the online scheduler.

1.3 Results

This paper gives an efficient m-machine reallocating scheduler for unit-sized jobs with arrival times and deadlines. Informally, the paper shows that as long as there is sufficient slack (independent of m) in the requested schedule, then every request is fulfilled, the reallocation cost is small, and at most one job migrates across machines on each request. Specifically, this paper establishes the following theorem:

Theorem 1 *There exists a constant γ as well as a reallocating scheduler for unit-length jobs such that for any m-machine γ-underallocated sequence of scheduling*

requests, we achieve the following performance. Let n_i denote the number of jobs in the schedule and Δ_i the largest window size when the ith reallocation takes place. Then the ith reallocation:

has reallocation cost $O(\min\{\log^* n_i, \log^* \Delta_i\})$ *and migration cost at most* 1.

We prove Theorem 1 in stages. In Sects. 3 and 4, we assume that job windows are all nicely "aligned," by which we mean that all job windows are either disjoint, or else one is completely contained in the other. In Sect. 3, we show that the multi-machine aligned case can be reduced to the single-machine aligned case, sacrificing a constant-factor in the underallocation. In Sect. 4, we establish Theorem 1, assuming the windows are aligned and that $m = 1$. Finally, in Sect. 5, we remove the alignment assumption from Sects. 3 and 4, again sacrificing a constant-factor in the underallocation.

The crux of our new approach to scheduling appears in Sect. 4. This section gives a simple scheduling policy that is robust to changes in the scheduling instances. By contrast, most classical scheduling algorithms are brittle, where small changes to a scheduling instance can lead to a cascade of job reallocations even when the system is highly underallocated. This brittleness is certainly inherent to earliest-deadline-first (EDF) and least-laxity-first (LLF) scheduling policies, the classical greedy algorithms for scheduling with arrival times and deadlines. In fact, we originally expected that any greedy approach would necessarily be fragile. We show that this is not the case.

Our new scheduler is based upon a simple greedy policy ("reservation-based pecking-order scheduling"). Instead of explicitly engineering redundancy, the resiliency of our scheduler derives from a basic combinatorial property of the underlying "reservation" system.

1.4 Related Work

We now review related work. Reallocation-style problems have been studied explicitly in scheduling, combinatorial optimization, and memory and storage allocation. There are many other domains in which it is possible to identify reallocation problems, even though these problems have not been studied explicitly in the context of reallocation.

Scheduling and planning Many reallocation-related questions have been explored in the scheduling community. One such category of problems is known as "robust scheduling" (or "robust planning"). In this setting, the goal is to design schedules that can tolerate some level of uncertainty. See [29,32] for surveys and [15,16,25,30] for applications to train and airline scheduling. The assumption in these papers is that the scheduling instance is approximately static, but there are small changes to the job requirements, or else some error or uncertainty. The schedule should remain near optimal, despite the error or changes [38]. By contrast, we focus on an arbitrary, worst-case, sequence of requests that may lead to significant changes in the overall allocation of resources.

Many other papers in the literature work within the similar setting of job scheduling with reallocations, but with different goals, restrictions, or scheduling problems in mind. Unal et al. [39] study a problem wherein an initial feasible schedule consisting of jobs with deadlines must be augmented to include a set of newly added

jobs, minimizing some objective function on only the new jobs without violating any deadline constraints on the initial schedule. As in the present paper they observe that slackness in the original schedule facilitates a more robust schedule, but outside of the hard constraints they do not count the reallocation cost. Hall and Potts [21] allow a sequence of updates and aim to restrict the change in the schedule, but they evaluate the quality of their algorithm incrementally, rather than with respect to a full sequence of updates or an offline objective.

More closely related to our setting, Westbrook [41] considers the total cost of migrating jobs across machines in an online load-balancing problem while also keeping the maximum machine load competitive with the current offline optimal, which is a different scheduling problem in a similar framework. Unlike in the present paper, Westbrook considers only migration costs and does not include the reallocation cost of reordering jobs on machines. Sanders et al. [35] consider a similar load-balancing problem with migration costs but no other reallocation costs; their goal is to study the tradeoff between migration costs and the instantaneous competitive ratio. See Verschae [40] for more work on job migrations, as well as more details on robust optimization.

Combinatorial optimization Resource reallocation has also been studied in other combinatorial optimization problems. For example, Jansen et al. [24] look at robust algorithms for online bin packing that minimize migration costs.

Some combinatorial reallocation problems are known as "reoptimization problems." Consider a hard instance for which you have found a solution. If the problem instance changes, how hard is it to find a new optimal solution? That is, given an optimal solution for an input, the goal is to compute a near-optimal solution to a closely related input [1–4,14]. These papers typically focus on the computational complexity of incremental optimization. By contrast, we focus here on the cost of changing the schedule (i.e., the cost incurred by the changes themselves, rather than the cost for computing the new schedule).

Shachnai et al. [36] introduce a model that is closely related to ours. They considered computationally intractable problems that admit approximation algorithms. When the problem instance changes, they would like to change the solution as little as possible in order to reestablish a desired approximation ratio. One difference between their model and ours is that we measure the ratio of reallocation cost to allocation cost, whereas there is no notion of initial cost for them. Rather, they measure the ratio of the transition cost to the optimal possible transition cost that will result in a good solution.

Davis et al. [18] propose a resource reallocation problem where the allocator must assign resources with respect to a user-determined set of constraints. The constraints may change, but the allocator is only informed when the solution becomes infeasible. The goals is to minimize communication between the allocator and the users.

Memory allocation Memory allocation [28,31,34,45] is a classic example of resource allocation within operating systems; programs request and free blocks of memory, and there is some mechanism by which the O/S satisfies these requests. If the allocated blocks can be moved (e.g., to a new location in physical memory), then the problem of memory allocation is a reallocation problem.

The focus is typically on the trade-off between time and space: how much time is spent reallocating memory (and how much performance is sacrificed) versus how

much space is wasted due to poor allocations. This trade-off is explored in papers by Ting [37], Błażewicz et al. [13], Bendersky and Petrank [12], and Cohen and Petrank [17]. Bender et al. [9] studied the problem of storage reallocation, i.e., the problem of dynamic memory reallocation where the blocks are stored on external storage. In this context, the cost of reallocation is much harder to determine, given the many parameters associated with a disk access. They develop *cost oblivious* algorithms that ensure near optimal reallocation cost for a large family of reallocation cost functions.

Reallocation in other domains Many existing algorithms, when looked at in the right way, can be viewed as reallocation problems, e.g., reconfiguring FPGAs [19], maintaining a sparse array [11,22,42–44], or maintaining an on-line topological ordering (e.g., [10,20,27]).

2 Reallocation Model

Formally, an *on-line execution* consists of a sequence of scheduling requests of the following form: ⟨INSERTJOB, *name*, *arrival*, *deadline*⟩ and ⟨DELETEJOB, *name*⟩. A job j has integral arrival time a_j and deadline $d_j > a_j$, meaning that it must be scheduled in a timeslot no earlier than time a_j and no later than time d_j. We call the time interval $[a_j, d_j]$ the job's ***window*** W. We call $d_j - a_j$, denoted by $|W|$, the ***window W's span***. We use ***job j's span*** as a shorthand for its window's span. Each job takes exactly one unit of time to execute.

At each step, we say that the ***active jobs*** are those that have already been inserted, but have not yet been deleted. Before each scheduling request, the scheduler must output a feasible schedule for all the active jobs. A feasible schedule is one in which each job is properly scheduled on a particular machine for a time in the job's available window, and no two jobs on the same machine are scheduled for the same time. Notice that we are not concerned with actually *running* the schedule; rather, we construct a sequence of schedules subject to an on-line sequence of requests.

We define the ***migration cost*** of a request r_i to be the number of jobs whose machine changes when r_i is processed. We define the ***reallocation cost*** of a request r_i to be the number of jobs that must be rescheduled when r_i is processed.

When the scheduling instances do not have enough "slack" it may become impossible to achieve low reallocation costs. In fact, if there are n jobs currently scheduled, a new request may have reallocation cost $\Theta(n)$. Even worse, it may be that most reallocations require most jobs to be moved, as is shown in Lemma 8: for large-enough s, there exist length-s request sequences, in which $\Theta(s^2)$ reallocations are necessary. Moreover, for large-enough s, there exist length-s request sequences in which $\Theta(s)$ machine migrations are necessary (see Lemma 7).

2.1 Underallocated Schedules and Our Result

To cope with Lemmas 7 and 8, we consider schedules that contain sufficient slack, i.e., that are not fully subscribed. We say that a set of jobs is ***m-machine γ-underallocated***, for $\gamma \geq 1$, if there is a feasible schedule for those jobs on m machines even when the job length (processing time) is multiplied by γ. This is equivalent to giving the offline

scheduler a processing speed that is γ times slower than the online scheduler. When m is implied by context, we simply say ***γ-underallocated***.

Overloading terminology, we say that a sequence of scheduling requests is ***γ-underallocated*** if after each request the set of active jobs is γ-underallocated.

2.2 Aligned-Windows Assumption

The assumption of aligned windows is used in Sects. 3 and 4, but it is dropped in Sect. 5 to prove the full theorem. We say that a window W is ***aligned*** if for any nonnegative integer i, (i) W has span 2^i, and (ii) W has a starting time that is a multiple of 2^i. If a job's window is aligned, we say that the job is ***aligned***. We say that a set of windows (or jobs) are ***recursively aligned*** if every window (or job) is aligned.

Notice that recursive alignment implies that two jobs' windows are either equal, disjoint, or one is contained in the other (i.e., the windows are laminar). Dealing with recursively aligned windows is convenient in part due to the following observation.

Lemma 1 *If a recursively aligned set of jobs is m-machine γ-underallocated, then for any aligned window W there are at most $m|W|/\gamma$ jobs with span at most $|W|$ whose windows overlap W.*

Proof The window W comprises $|W|$ timeslots on each of m machines, for a total of $m|W|$ timeslots. By definition, a γ-underallocated instance is feasible even if the jobs' processing times are increased to γ. Thus, there can be at most $m|W|/\gamma$ jobs restricted to window W. Since the set of jobs is recursively aligned, if a job has window W' that overlaps W and $|W'| \leq |W|$, then W' is fully contained by W. There can be at most $m|W|/\gamma$ such jobs. □

3 Reallocating Aligned Jobs on Multiple Machines

This section reduces the multiple-machine scheduling problem to a single-machine scheduling problem, assuming recursive alignment. The reduction uses at most one migration per request.

The algorithm is as follows. For every window W, record the number n_W of jobs having window W. (This count need only be recorded for windows that actually exist in the current instance, so there can be at most n relevant windows for n jobs.) Number the processors from $0, \ldots, m-1$.

Our algorithm maintains the following invariant:

Invariant 1 *Every machine has between $\lfloor n_W/m \rfloor$ and $\lceil n_W/m \rceil$ jobs with window W, with the extra jobs being assigned to the smallest-numbered machines.*

For each window W, we maintain this invariant by assigning jobs to processors round-robin. If there are n_W jobs with window W, a new job with window W is assigned to machine $(n_W + 1) \mod m$. When a job with window W is deleted from some machine m_i, then a job is removed from machine $(n_W \mod m)$ and migrated

to machine m_i. All job reallocations and migrations are performed via delegation to the single-machine scheduler on the specified machine(s).

The following lemma says that by assigning jobs to processors according to Invariant 1, the instances assigned to each machine are feasible.

Lemma 2 *Consider any m-machine 5γ-underallocated, recursively aligned set of jobs J, where γ is an integer. Consider a subset of jobs J' such that if J contains n_W jobs of window W, then J' contains at most $\lceil n_W/m \rceil$ jobs of window W. Then J' is 1-machine γ-underallocated.*

Proof The proof proceeds in two stages. In the first stage, we show that because the m machines are 5γ-underallocated, an aligned window W on any given machine contains at most $3|W|/(5\gamma)$ jobs. In the second stage, we show that if a machine has at most $3|W|/(5\gamma)$ jobs in any aligned window W, then that machine is γ-underallocated. If γ happens to be a power of 2, then that machine is, in fact $5\gamma/3$-underallocated, which is slightly better. (This second stage proves the converse of Lemma 1, but with the loss of a factor of 5/3.)

We now proceed with stage one, which means showing that despite the ceilings in Invariant 1, an aligned window W on any given machine contains at most $3|W|/(5\gamma)$ jobs. Since J is 5γ-underallocated and recursively aligned, by Lemma 1, for any aligned window W, there are at most $m|W|/(5\gamma)$ jobs that have windows nested inside W (i.e., overlapping W and with span at most $|W|$).

By the definition of underallocation, no jobs can have windows smaller than 5γ. Thus, there are fewer than $2|W|/(5\gamma)$ windows with jobs nested inside W. (The factor of two comes from a geometric series: at most one window of size $|W|$, two of size $|W|/2$, four of size $|W|/4$, and so forth, up to $|W|/5\gamma$ of size 6γ.)

The ceilings in Invariant 1 add at most one job for each of these windows. So the total number of jobs in J' with windows nested inside W is at most $|W|/(5\gamma) + 2|W|/(5\gamma) = 3|W|/(5\gamma)$, as promised.

We now continue to stage two, which means describing the previous bound in terms of γ-underallocation. We impose the restriction that all jobs run only at multiples of γ. This means that, if γ is a power of 2, then we are already done, and the instance is $5\gamma/3$-underallocated.

If γ is not a power of 2, however, then the job windows may not line up with multiples of γ. Specifically, a job window might be placed one slot after a multiple of γ and end just before another, so that the $\gamma - 1$ slots in the beginning and of the window cannot hold jobs. Thus, a window of length $|W|\gamma$ may only have $|W| - 2$ slots that are multiples of γ, and it may need to house $3|W|/5$ jobs. Since $|W| \geq 5\gamma$ and by a simple induction on window size, there are always enough slots for the jobs. □

4 Reallocating Aligned Jobs on One Machine

This section describes a single machine, reallocating scheduler for unit-sized jobs.

4.1 Naïve Pecking-Order Scheduling is Logarithmic

We first give the naïve solution, which requires a logarithmic number of reallocations per job insert/delete.

This solution uses what we call ***pecking-order scheduling***, which means that a job k schedules itself without regard for jobs with longer span and with complete deference to jobs with shorter span. A job k with window W may get displaced by a job j with a shorter window (nested inside W), and k may subsequently displace a job ℓ with longer window.[2]

Lemma 3 *Let n denote the maximum number of jobs in any schedule and let Δ denote the longest window span. There exists a greedy reallocating scheduler such that for every feasible sequence of recursively aligned scheduling requests, the reallocation cost of each insert/delete is $O(\min\{\log n, \log \Delta\})$.*

Proof To insert a job j with span 2^i, find any empty slot in j's window, and place j there. Otherwise, select any job k currently scheduled in j's window that has span $\geq 2^{i+1}$. If no such k exists, the instance is not feasible (as every job currently scheduled in j's window *must* be scheduled in j's window). If such a k exists, replace k with j and recursively insert k. This strategy causes cascading reallocations through increasing window spans, reallocating at most one job with each span. Since there are at most $\log \Delta$ distinct window spans in the aligned case, and moreover all jobs can fit within a window of span n, the number of cascading reallocations is $O(\min\{\log n, \log \Delta\})$. □

4.2 Pecking-Order Reallocation via Reservations Costs $O(\min\{\log^* n, \log^* \Delta\})$

We now give a more efficient reallocating scheduler, which matches Theorem 1 when the scheduling requests are recursively aligned. The algorithm is summarized for job insertions in Fig. 1.

The intuition behind reservation scheduling manifests itself in the process of securing a reservation at a popular restaurant. If higher-priority diners already have reservations, then our reservation is waitlisted. Even if our reservation is "confirmed," a celebrity (or the President, for DC residents) may drop in at the last moment and steal our slot. If the restaurant is empty, or full of low-priority people like graduate students, then our reservation is fulfilled. The trick to booking a reservation at a competitive restaurant is to make several reservations in parallel. If multiple restaurants grant the reservation, we can select one to eat at. If a late arrival steals our slot, no problem, we have another reservation waiting.

Back to our scheduling problem, by spreading out reservations carefully, jobs will only interfere if they have drastically different spans. Our algorithm handles jobs with "long" windows and "short" windows separately, and only a "short" job can displace a "long" job. The scheduler itself is recursive, so "very short" jobs can displace "short" jobs which can displace "long" jobs, but the number of levels of recursion here will be $\log^* \Delta$, as opposed to $\log \Delta$ in the naïve solution.

[2] At first glance, Lemma 3 seems to contradict the underallocation requirement given in Lemma 8. That lower bound, however, applies to the general case, whereas this lemma applies to the aligned case.

```
• Initially, each level-ℓ ≥ 1 window W has one reservation in each
    level-ℓ interval contained in W.
• Initially, each interval I has allowance(I) = I.
• To insert a new level-ℓ ≥ 1 job j with window W:
    1. Identify the two underloaded intervals I_1 and I_2 according to Invariant 2
    2. Call RESERVE(I_1, W) and RESERVE(I_2, W)
    3. Call PLACE(j)

RESERVE(I, W)                          // make a reservation in I for level-ℓ window W
1  if there is a slot s ∈ allowance(I) that has not been assigned
2     then fulfill the reservation, assigning slot s to window W and return
3  let W' be the longest window with a fulfilled reservation in I,
   and let s be one of its slot
4  if |W'| ≤ |W|
5     then waitlist the reservation for W
6     else waitlist the reservation for W' and take slot s from W'
7          if there is a level-ℓ job j' in slot s
8             then MOVE(j')
9          fulfill the reservation, assigning slot s to W
           // Note that though the reservation is fulfilled, the slot
           // may be occupied by a higher-level job

MOVE(j')                               // level-ℓ job j' lost the reservation to a slot it occupies
10 let W' be the window of j', and let s be the slot it occupies
11 let s' be a fulfilled slot, assigned to W', not containing any level-ℓ job
   // exists by Lemma 4
12 for all ancestor intervals I' containing W'
13       do swap s and s' with regards to reservations and allowances for I'
            // both slots are inside I'
            if a higher-level job h occupies s'
               then schedule h in s instead of s'
14          schedule j' in slot s'

PLACE(j)                               // let W be j's window and let ℓ be j's level
15 let s be a fulfilled slot, assigned to W, not containing any level-ℓ job
   // exists by Lemma 4
16 schedule j in s, potentially displacing a higher-level job h
17 remove s from the allowance of all higher-level intervals
18 for each ancestor interval whose allowance decreases
   // s is only in allowances up to h's level
19       do adjust the reservations to reflect a smaller allowance,
            possibly waitlisting one reservation
20          if a newly waitlisted reservation is for a slot containing a job j'
21             then MOVE(j')
22 if there is a displaced job h
23    then PLACE(h)
```

Fig. 1 Pecking-order scheduling with reservations, for levels $\ell \geq 1$

There are two components to the scheduler. The first component uses reservations to guarantee that jobs cannot displace (many) other jobs having "similar" span, so the reallocation cost, if all jobs have similar spans, is $O(1)$. These (over-)reservations, however, consume timeslots and amplify the underallocation requirements. Applying the scheduler recursively at this point is trivial to achieve a good reallocation cost, but the required underallocation would become nonconstant. The second component of

the scheduler is to combine levels of granularity so that their effects on underallocation do not compound. This more efficient combining is achieved by augmenting the reservations with an allowance system, which allows long jobs to entirely ignore the reservations made by short jobs—only their schedule matters.

The remainder of the section is organized as follows. We first discuss an interval decomposition to separate jobs into different "levels" according to their spans. Then we present the scheduler with regards to a single job level. Finally we discuss how to incorporate multiple levels simultaneously.

Interval Decomposition

Our scheduler operates nearly independently at multiple levels of granularity. More precisely, view these levels from bottom up by defining the thresholds $L_1, L_2, \ldots$ by

$$L_{\ell+1} = \begin{cases} 2^5 & \text{if } \ell = 0 \\ 2^{L_\ell/4} & \text{if } \ell > 0 \end{cases}.$$

The base case is chosen to be $L_1 = 32$ because any smaller power of 2 would yield $L_1 \geq L_2 \geq \cdots$, whereas we want increasing thresholds. It is not hard to see that L_ℓ is always a power of 2, growing as a tower function of $\sqrt[4]{2}$. It is often convenient to use the equivalent relationship $L_\ell = 4 \lg(L_{\ell+1})$—each threshold is roughly the lg of the next.

For $\ell \geq 1$, we define a recursive ***level- ℓ scheduler*** that handles jobs and windows W with span $L_\ell < |W| \leq L_{\ell+1}$. The level-0 scheduler handles the base case of windows with span at most 32, using the naïve scheduler.

We call a job (or window) a ***level- ℓ job (window)*** if its span falls in the range handled by the level-ℓ scheduler. The following observation is useful in our analysis.

Observation 2 *For $\ell \geq 1$, there are fewer than $L_\ell/4$ distinct spans that constitute level-ℓ windows.*

This observation follows from the fact that window spans are powers of 2, so there are fewer than $\lg L_{\ell+1} = L_\ell/4$ spans possible.

For $\ell \geq 1$, we partition level-ℓ windows into nonoverlapping, aligned subwindows called ***level-ℓ intervals***, each consisting of $L_\ell = 4 \lg L_{\ell+1}$ timeslots. The reallocation scheduler operates recursively within each interval to handle lower-level jobs. Because this is pecking-order scheduling, the recursive scheduler makes decisions without paying attention to the location of the higher-level jobs, guaranteeing only that each lower-level job is assigned a unique slot within its appropriate window. In doing so, it may displace a long job and invoke the higher-level scheduler.

Schedule Level-ℓ Jobs via Reservations

Consider a level-$\ell \geq 1$ window W with span $2^k L_\ell$, for some integer $k \geq 1$ (i.e., W contains 2^k level-ℓ intervals). Let x denote the number of jobs having exactly window W.

The window W maintains a set of ***reservations*** for these x jobs, where each reservation is a *request for a slot in a given level-ℓ interval*. A reservation made by W can be ***fulfilled***; this means that one slot from the requested interval is ***assigned to W***, and the only level-ℓ jobs that may ***occupy*** that slot are any of the x jobs with window exactly W. Alternatively, a reservation can be ***waitlisted***; this means that all the slots in the requested interval are already assigned to smaller windows than W. Which reservations are fulfilled and which are waitlisted may change over time as jobs get allocated and removed.

We now explain how these reservations are made. Initially, a level-ℓ window W makes one reservation for each enclosed level-ℓ interval. It makes two additional reservations for each job having window W. These reservations are spread out round-robin among the intervals within W (and independently of any jobs with any different windows). We maintain the following invariant:

Invariant 2 *For $\ell \geq 1$, if there are x jobs having level-ℓ window W with $|W| = 2^k L_\ell$, then W has exactly $2x + 2^k$ reservations in level-ℓ intervals.*

- *These reservations are assigned in round-robin order to the intervals in W.*
- *Each of the enclosed intervals contains either $\lfloor 2x/2^k \rfloor + 1$ or $\lfloor 2x/2^k \rfloor + 2$ of W's reservations, where the leftmost intervals have the most reservations and the rightmost intervals have the least reservations.*

To maintain Invariant 2, when a new job with window W is allocated, W makes two new reservations, and these are sent to the leftmost intervals that have the least number ($\lfloor 2x/2^k \rfloor + 1$) of W's reservations. When a job having window W is deleted, W removes one reservation each from the two rightmost intervals that have the most reservations.

We now describe the reservation process from the perspective of the interval, which handles reservation requests from the $< L_\ell/4$ level-ℓ windows that contain the interval (see Observation 2). The interval decides whether to fulfill or waitlist a reservation, prioritizing reservations made by shorter windows. Each interval I has an ***allowance*** $allowance(I)$, specifying which slots it may use to fulfill reservations. In the absence of lower-level jobs, the $|allowance(I)| = L_\ell$, since the interval has span L_ℓ. (When lower-level jobs are introduced, however, the allowance decreases—the allowance contains all those slots that are not *occupied* by lower-level jobs.) Thus, the interval sorts the window reservations with respect to span from shortest to longest, and fulfills the $|allowance(I)| \leq L_\ell$ reservations that originate from the shortest windows. A fulfilled reservation is assigned to a specific slot in the interval, while a waitlisted reservation has no slot. The interval maintains a list of these waitlisted reservations.

The set of fulfilled reservations changes dynamically as insertions/deletions occur. When a new reservation is made by window W, a longer window W' may lose a reserved slot as one of its fulfilled reservations is moved to the waitlist; if there is a job (of the same level) in that slot, it must be moved. When a job with window W is deleted, W has two fewer reservations, and so may lose two fulfilled slots. If there is a job in either of these slots, then that job must be moved. (In this case, a longer window W' may gain a fulfilled slot, but this does not require any job movement.)

The following invariant is needed to establish the algorithm's correctness.

Invariant 3 *When a job having window W is newly allocated, W makes two new reservations. Then the job is assigned to any empty slot for which W has a fulfilled reservation. There will always be at least one such slot (proved by Lemma 4).*

Interestingly, as a consequence of pecking-order scheduling combined with round-robin reservations:

Observation 3 *Which reservations in which intervals are fulfilled and which are wait-listed is history independent, meaning that which reservations are made is a function of only the current active jobs, not the preceding sequence of insertions/deletions. The actual placement of the jobs, however, is not history independent.*

Unlike for levels $\ell \geq 1$, a level-0 window is not decomposed into intervals, and it does not employ the reservation process. Instead level-0 jobs can be scheduled using the naïve pecking-order scheduler.

Scheduling Across All Levels

Consider inserting a level-ℓ job j. Suppose j's window is contained in a higher-level interval I'. We schedule j at its own level according to the pecking-order scheduler, without regard to higher-level schedulers.

Let us first consider the more complicated case that $\ell \geq 1$. Recall that the first step of the insertion process is to place two new reservations in level-ℓ intervals. Whenever the reservations cause another level-ℓ job j' to move from slot s to slot s', the allowance of all higher-level intervals must be updated to reflect the change in slot usage. However, since both $s \in I'$ and $s' \in I'$, and j' vacates the original slot s, there is no net change to $\left|\mathit{allowance}(I')\right|$. It is thus sufficient to swap s and s' for all higher-level intervals I', which may result in a total of one higher-level job movement.

After updating the reservations, the new job j is placed in one of its assigned slots s. This slot may either be empty, or it may contain a higher-level job h—the scheduler chooses s without regard to these possibilities. In either case, the slot s will be used by j, so it must be removed from $\mathit{allowance}(I')$ for any ancester interval I'—meaning the higher-level scheduler cannot use this slot. If the slot s was empty, then the job j is assigned to that slot and the insertion terminates. If the slot s was previously occupied by a higher-level job h, then h is displaced and a new slot must be found. Unlike in the case of reservations, $\left|\mathit{allowance}(I')\right|$ decreases here and we do not immediately have a candidate slot into which to place h. Instead, we reinsert h recursively using the scheduler at its (higher) level. This displacement and reinsertion may cascade to higher levels.

In the case that $\ell = 0$, the reservation step is skipped. Instead, the job is placed directly into an empty slot if possible, or a slot occupied by a job with a strictly longer window. This process may cause up to $\lg 32 = O(1)$ level-0 jobs to move before the last job is finally assigned a slot that is not occupied by level-0 jobs. Since the number of level-0 jobs in the enclosing level-1 interval I' has increased by 1, the net effect is the same as above—$\left|\mathit{allowance}(I')\right|$ decreases by 1. If the level-0 scheduler terminates by displacing a higher-level job h, then h must be reinserted recursively as in Fig. 2.

```
• To insert a new level-0 job j with window W:
      Call BASEPLACE(j)

BASEPLACE(j)
 1  let W be j's level-0 window.
 2  if all slots in W are occupied by level-0 jobs
 3     then choose a slot s occupied by level-0 job j' with strictly longer window
 4          schedule j in s, displacing j'
 5          BASEPLACE(j')
 6          return
    // otherwise, there is a slot available in W
 7  choose any slot s in W that is either empty or contains a higher-level job
 8  schedule j in s, potentially displacing a higher-level job h
 9  remove s from the allowance of all higher-level intervals
10  for each ancestor interval whose allowance decreases
    // s is only in allowances up to h's level
11       do adjust the reservations to reflect a smaller allowance,
             possibly waitlisting one reservation
12          if a newly waitlisted reservation is for a slot containing a job j'
13            then MOVE(j')
14  if there is a displaced job h
15     then PLACE(h)
```

Fig. 2 The pecking-order scheduler for level-0 jobs

Observe that the higher-level scheduler is unaware of the reservation system employed by the lower-level scheduler. It only knows which slots are in its allowance. These slots are exactly those that are not *occupied* by short-window jobs. The interval does not observe the reservations occurring within nested intervals—only actual job placement matters. When a lower-level job is deleted, the allowance of the containing interval increases to include the slot that is no longer occupied.

4.3 Reservation Analysis

We now use the following lemma to establish Invariant 3, which claims that there are always enough fulfilled reservations. Since the reservations fulfilled by each interval are history independent (see Observation 3), this proof applies at all points during the execution of the algorithm.

Lemma 4 *Suppose that a sequence of recursively aligned scheduling requests 8-underallocated. If there are x jobs each having the same window W, then W has at least $x + 1$ fulfilled reservations.*

Proof Let $|W| = 2^k L_\ell$ for level-ℓ window W. Let y be the number of level-ℓ jobs with windows nested inside W. Each of those windows makes 2 reservations for each job, plus an extra reservation to each of the 2^k intervals. So the total number of reservations in W is at most $2(x+y)+2^k \lg W$. In addition, let z be the number of lower-level jobs nested inside W. Since we are 8-underallocated, we have $2(x + y) + z \leq 2(x + y + z) \leq |W|/4$ by Lemma 1. By Observation 2, we have $\lg |W| \leq L_\ell/4$, and

hence $2^k \lg W \leq (2^k L_\ell)/4 = |W|/4$. Summing these up, we have that at most $|W|/2$ slots consumed by lower-level jobs and these reservations.

In order for a particular interval to waitlist even one of W's reservation requests, it would need to have strictly more than L_ℓ of these reservations or lower-level jobs assigned to it. But there are only $|W|/2$ slots consumed in total, so strictly less than $1/2$ the intervals can waitlist even one of W's reservations. Since window W reserves at least $\lfloor 2x/2^k \rfloor + 1$ slots in every one of the 2^k intervals by Invariant 2, it must therefore be granted strictly more than $(\lfloor 2x/2^k \rfloor + 1)(1/2)(2^k) \geq x$ fulfilled reservations. □

Since each window W containing x jobs has at least $x + 1$ fulfilled reservations at intervals within W, there is always an appropriate slot to schedule a new belonging to this window. This ensures that each operation leads to only $O(1)$ reallocations at each level.

4.4 Trimming Windows to n_i and Deamortization

As described thus far, the reallocation cost of our scheduler depends only on the span of windows, and we would obtain a reallocation cost for the ith request of $O(\log^* \Delta_i)$, where Δ_i is the longest window span of all active jobs when the ith request is made. This span Δ_i, however, could be significantly larger than n_i, the number of active jobs when the request is made. To obtain a better performance bound when $n_i \ll \Delta_i$, we augment the scheduler to trim windows according to an estimate on the number of currently active jobs, constructively making the trimmed span obey $\Delta_i = O(n_i)$.

Note that the naïve pecking-order scheduler obtains a bound with respect to n_i (see Lemma 3), without trimming windows, by preferentially choosing empty slots over occupied slots. Due to the complexity of the reservation system and waitlisting, however, it is not obvious that such a simple analysis applies here.

To trim window spans to $O(n_i)$, the number of active jobs after the ith request, we maintain a value n^* such that $n^*/4 \leq n_i \leq n^*$. For every job that has a window larger than $2\gamma n^*$, we trim its window—reducing it arbitrarily to span $2\gamma n^*$. The adjusted instance remains γ-underallocated, since there are at most n^* other jobs scheduled in the trimmed window with span $2\gamma n^*$.

We update n^* using standard techniques from amortized algorithms. In particular, whenever the number of active jobs exceeds n^*, double n^*; whenever the number of active jobs drops below $n^*/4$, halve n^*. To achieve good reallocation cost, it is sufficient to rebuild the schedule from scratch every time the value of n^* changes. During such a global rebuild, jobs need not be re-inserted one by one. Rather, first calculate the new schedule (e.g., by simulating a one-by-one reinsertion), then move every job once. Since this rebuilding involves moving $n_i \leq n^*$ jobs, and it occurs only after $\Theta(n^*)$ insertions or deletions, rebuilding incurs an amortized reallocation cost of $O(1)$.

This amortized solution can be deamortized, resulting in good worst-case reallocation costs. The main idea is to rebuild the schedule gradually, performing a little of the update every time a new reallocation request is serviced. This approach is reminiscent of how one deamortizes the rebuilding of a hash table that is too full or too empty. Whenever n^* doubles or halves, the existing schedule becomes ***old***, and an initially

empty ***new*** schedule is created. All subsequent insertion requests are made directly into the new schedule trimming windows according to the new value of n^*, and subsequent deletion requests are made into whichever version contains the deleted job. Active jobs are gradually moved from the old schedule to the new schedule—with two such movements occurring on each subsequent request—by deleting any job from the old schedule and re-inserting it into the new schedule. Each new request thus translates into $O(1)$ requests—the new request plus two additional insertions and deletions for the movements. By the time n^* changes again, the old schedule is empty.

The remaining question is how the old and new schedule can coexist within the same timeline. Our solution is to interleave both schedules, using all even timeslots for one version and all odd timeslots for the other. As long as the instance of jobs is 2γ-underallocated, then the interleaved schedule is effectively γ-underallocated.

4.5 Wrapping Up

We conclude with the following lemma, which puts together the various results in this section. One factor of 2 in the underallocation comes from the aforementioned deamortization; if an amortized bound is acceptable, then being 8-underallocated is sufficient.

Lemma 5 *For any* 1*-machine* 16*-underallocated sequence of recursively aligned scheduling requests, we achieve the following performance. Let* n_i *denote the number of jobs in the schedule and* Δ_i *the largest window span when the* i*th reallocation takes place. Then the* i*th reallocation has cost* $O(\min\{\log^* n_i, \log^* \Delta_i\})$.

Proof We consider the performance of the pecking-order scheduler with reservations, where we maintain an estimate n^* via deamortized shrinking and doubling and trim all windows to $2\gamma n^*$, for $\gamma = 16$.

Lemma 4 shows that there is always a slot available to put a job (Invariant 3), and hence we observe that there are at most $O(1)$ reallocations at each level of the scheduler. Specifically, on insertion, the two reservations may result in two calls to MOVE for jobs at the same level as the one being inserted. Each MOVE results in one reallocation of the job being moved, plus at most one reallocation at a higher level. Then the call to PLACE may cascade across all levels, but it in aggregate it only includes one MOVE per level, each causing at most two reallocations. The total number of reallocations for a single job insertion/deletion is thus $O(1)$ per level for a total of $O(\log^* \Delta_i)$. The deamortization process multiplies this cost by a constant.

Since the estimate n^* is always bounded by $n^*/4 \le n_i \le n^*$, and all windows are trimmed to $2\gamma n^*$, we have the maximum trimmed span of $2\gamma(4n_i) = O(n_i)$. It follows that the number of levels is $O(\log^* n_i)$, which completes the proof. □

5 Reallocating Unaligned Jobs on Multiple Machine

In this section, we generalize to jobs that are not aligned, removing the alignment assumptions that we made in Sects. 3 and 4. We show that if S is a γ-underallocated

sequence of scheduling requests, then we can safely *trim* each of the windows associated with each of the jobs, creating an aligned instance. Since the initial sequence of scheduling requests is underallocated, the resulting aligned sequence is also underallocated, losing only a constant factor.

We first define some terminology. If W is an arbitrary window, we say that ALIGNED(W) is a largest aligned window that is contained in W. (If there is more than one largest window, choose arbitrarily.) Notice that $|\text{ALIGNED}(W)| \geq |W|/4$. If J is a set of jobs, then ALIGNED(J) is the set of jobs in which the window W associated with each job is replaced with ALIGNED(W).

Lemma 6 *Consider any m-machine 4γ-underallocated set of jobs J, where γ is an integer. Then* ALIGNED(J) *is m-machine γ-underallocated.*

Proof Assume for the sake of contradiction that ALIGNED(J) is not γ-underallocated. This implies that there must exist a window W that has $> m|W|/\gamma$ jobs with trimmed windows contained in W (as otherwise we could schedule the size γ jobs via a simple inductive argument). Let $J' \subseteq J$ be the jobs whose trimmed windows are contained in W.

Since J is 4γ-underallocated, we now examine an (unaligned) scheduling of the jobs in J' that satisfies the 4γ-underallocation requirement. We observe that all the jobs in J' are scheduled in a region of size at most $4|W|$.

However, since the schedule is 4γ-underallocated, there can be at most $4m|W|/(4\gamma)$ jobs in this region of size $4|W|$. That is $|J'| \leq m|W|/\gamma$, which is a contradiction. □

From this we can conclude with the proof of Theorem 1:

Proof of Theorem 1. Jobs are scheduled as follows: first, a new job has its window aligned; second, it is delegated to a machine in round-robin fashion; finally, it is scheduled via single-machine pecking-order scheduling with reservations. When a job is deleted, it is removed by the appropriate single-machine scheduler, and then there is at most one migration to maintain the balance of jobs across machines. This is the only time that jobs migrate.

Let $\gamma = 320$. Then Lemma 6 shows that the set of aligned jobs is m-machine 80-underallocated, and Lemma 2 shows that the aligned jobs assigned to each machine are 1-machine 16-underallocated. Finally, Lemma 5 shows that each single-machine scheduler operation has cost $O(\min\{\log^* n_i, \log^* \Delta_i\})$—and each job addition or deletion invokes $O(1)$ single-machine scheduler operations. □

6 What Happens Without Underallocation?

This section explains what happens without underallocation and why migrations are necessary at all.

If migration cost is to be bounded only by reallocation cost and since jobs have unit size, it is trivial to transform a parallel instance to a single-machine instance by making a single machine go m times faster. Since migration cost across machines could be more expensive than rescheduling a single machine, we are interested in providing a tighter bound on the migration cost. The question then is: *are migrations necessary?*

The following lemma shows that they are. In fact, the per-request migration cost must be $\Omega(1)$ in the worst-case for any deterministic algorithm.

Lemma 7 *There exists a sufficiently large sequence of s job insertions/deletions on $m > 1$ machines, such that any deterministic scheduling algorithm has a total migration cost of $\Omega(s)$.*

Proof Without loss of generality, assume $6m$ divides s. Divide the s requests into $s/(6m)$ consecutive subsequences of $6m$ requests each. Each subsequence is as follows:

1. Insert $2m$ span-2 jobs with window $[0, 2]$.
2. Delete the m jobs scheduled on the first $m/2$ machines.
3. Insert m span-1 jobs with windows $[0, 1]$.
4. Delete all $2m$ remaining jobs.

After step 1, the only feasible schedule is to put two jobs on each machine. After step 2, half the machines have two jobs, and the other half of the machines have no jobs. The only feasible schedule after step 3 is to have on each machine a span-1 job starting at time 0, and a span-2 job starting at time 1. This means that half of the span-2 jobs must migrate across machines, causing $m/2$ migrations. There are thus $m/2$ migrations for every $6m$ requests, or a total of $s/12$ migrations. □

It is also easy to see that for some sequences of scheduling requests, if they are not underallocated, it is impossible to achieve low reallocation costs, even if there exists a feasible schedule.

Lemma 8 *There exists a sequence of s job inserts/deletions, such that any scheduling algorithm has a rescheduling cost of $\Omega(s^2)$.*

Proof Consider for example $\eta = s/2$ jobs numbered $0, 1, \ldots, \eta - 1$, where job j has window $[j, j + 2]$. With the insertion of one additional job having window $[0, 1]$, forcing the job to be scheduled at time 0, all η other jobs are forced to schedule during their later slot. If that job is deleted and another unit-span job with window $[\eta, \eta + 1]$ is inserted, then all jobs are forced to schedule during their earlier slot. By toggling between these two options, all jobs are forced to move, resulting in cost $\Omega(\eta)$ to handle each request. Repeating η times gives a total cost of $\Omega(\eta^2) = \Omega(s^2)$. □

7 Conclusions and Open Questions

The results in this paper suggest several followup questions. First, is it possible to generalize this paper's reallocation scheduler for the case where jobs are not unit-sized? Observe that we are limited by the computational difficulty of scheduling with arrival times and deadlines when jobs are not unit size; see [6] for recent results with resource augmentation. We are also limited by the following observation:

Observation 4 *Suppose there exist jobs of size 1 and jobs of size k, for any $k > 1$. For any reallocation scheduler, there is a sequence of $\Theta(n)$ scheduling requests that has aggregate reallocation cost $\Omega(kn)$, for $k \leq n$, even if the requests are γ-underallocated for any constant γ.*

Proof Consider a schedule of length $m = 2\gamma k$. Assume there are k unit-sized jobs that are each scheduled with a window beginning at 0 and ending at m. In addition, consider a single large job p that has size k and a window of span exactly k.

Initially, all k unit-size jobs are scheduled and they remain in the system throughout. The large job p is initially scheduled at time slot 0. It is then deleted from time slot 0 and re-inserted at time slot k, and then again at time slot $2k, 3k, \ldots, m-k$. The same sequence of 2γ insertions and deletions is then repeated n times.

During a single sequence of 2γ insertions and deletions, each of the k unit-sized jobs has to be rescheduled at least once, resulting in $\Omega(kn)$ reallocation cost. □

Does there exist a reallocation scheduler that handles jobs whose sizes are integers less than or equal to k and matching the bounds in Observation 4? There could be applications where jobs are not unit size, but where k is relatively small.

What happens if other types of reallocations are allowed, such as if new machines can be added or dropped from the schedule, or if machine speeds can change?

In this paper, γ is impracticably large, and the paper does not attempt to optimize this constant, preferring clarity of exposition. How much can this constant be improved? Is there a reallocation scheduler where $\gamma = 1 + \varepsilon$?

Finally, what other scheduling and optimization problems lend themselves to study in the context of reallocation?

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