Matching regions in the plane using non-crossing segments

Greg Aloupis[1], Esther M. Arkin[2], David Bremner[3], Erik Demaine[4], Martin Demaine[4], Sándor P. Fekete[5], Bahram Kouhestani[6], and Joseph S. B. Mitchell[2]

[1]Tufts University
[2]Stony Brook University
[3]University of New Brunswick
[4]MIT
[5]TU Braunschweig
[6]Queens University

Abstract

Given a set $S = \{R_1, R_2, \ldots, R_{2n}\}$ of $2n$ disjoint regions in the plane, we examine the problem of computing a *non-crossing perfect region-matching*: determine if there exists a perfect matching on S that is realized by a set of non-crossing line segments, with the segments disjoint from the regions. We study the complexity of this problem, showing that, in general, it is NP-hard. We also show that a perfect matching always exists and can be computed in polynomial time if the regions are unit disks (or nearly equal-sized disks or squares). We consider also the bipartite version of the problem in which there are n red regions and n blue regions; in this case, even for unit disk (or unit square) regions, the problem is NP-hard.

1 Introduction

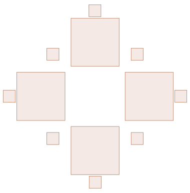
We consider a natural geometric matching problem on planar regions. Given a set $S = \{R_1, R_2, \ldots, R_{2n}\}$ of $2n$ disjoint regions in the plane, we examine the problem of computing a *non-crossing perfect region-matching*: determine if there exists a set of n pairs of regions, $\{(P_1, Q_1), (P_2, Q_2), \ldots, (P_n, Q_n)\}$, of S ($P_i, Q_i \in S$) such that there exist non-crossing (disjoint interiors) line segments, $p_i q_i$, that realize the matching, with $p_i \in P_i$, $q_i \in Q_i$, and $p_i q_i$ disjoint from the interiors of the regions of S (i.e., the regions of S are *obstacles* through which the edges of the matching are not allowed to pass).

Related Work. Aloupis et al. [4] considered the problem of realizing a *given* matching (i.e., with a pairing of the regions specified) by a set of non-crossing line segments connecting each pair of regions in the matching. They particularly studied the problem in which one region of each given pair is a single point, while the other region is either a discrete point set or a line segment, possibly in a special configuration. The bottleneck version of the problem has been studied by Abu-Affas et al. [3].

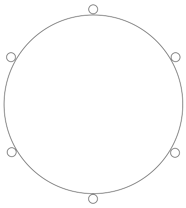
Addressing a problem posed by Ferran Hurtado at a workshop in Barcelona, Ábrego et al. [1, 2] studied a variant of geometric matching, called *$\mathcal{C}$-matching*, in which a set S of points are to be matched using regions of a specified type (e.g., squares) from a set $\mathcal{C}$; the regions serve as the "edges" of the $\mathcal{C}$-matching. In a perfect $\mathcal{C}$-matching, each point of S lies in exactly one of the $|S|/2$ regions of the $\mathcal{C}$-matching, and each such region contains exactly two points of S. If the regions are pairwise-disjoint, the matching is said to be *strong*.

Preliminaries. We are given a set $S = \{R_1, \ldots, R_{2n}\}$ of $2n$ disjoint (open) regions; most of our attention is focused on the case of regions that are circular disks or axis-aligned squares. The complement, $\Re^2 \setminus (\cup_i R_i)$ of the set of regions is a (closed) connected set, which we call the *free space*. Consider the *region visibility graph*, G, whose nodes are the regions S and whose edges E correspond to pairs of regions, R_i and R_j, that are weakly visible, meaning that there exist points $p_i \in \partial R_i$ and $p_j \in \partial R_j$ such that the line segment $p_i p_j$ lies fully within the free space. A set of line segments is a *non-crossing matching* for S if the segments all lie within the free space, are pairwise non-crossing (no point lies in the relative interior of two distinct segments), and there is a matching in the graph G for which the segments are a geometric realization. Note that even if the graph G on S has a perfect matching, it may not be possible to realize it with non-crossing line segments; see Figure 1(a), where the only realization by straight segments of a perfect matching on the set of small and large squares results in two edges crossing in the middle of the figure. In Figure 1(b), we show a simple case of circular regions (small and large) for which the graph G is a star (so only one pair of regions can be matched).

Summary of Results. We prove that determining if a non-crossing perfect matching exists is NP-complete for disjoint regions that are squares or disks, not all the same size. In contrast, we prove that if the regions are (disjoint) unit disks/squares, a non-crossing perfect matching always exists and can be computed efficiently. If the re-

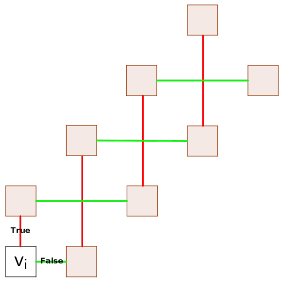


(a) While the graph G has a perfect matching, it has no realization with non-crossing straight segments.

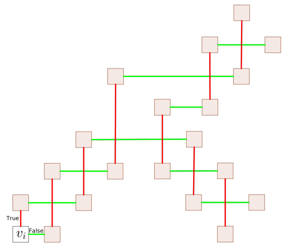


(b) The graph G is a star; at most two regions can be matched.

Figure 1: Examples.



(a) Variable gadget.



(b) A splitting gadget.

Figure 2: Variable gadgets used in the proof of Theorem 1.

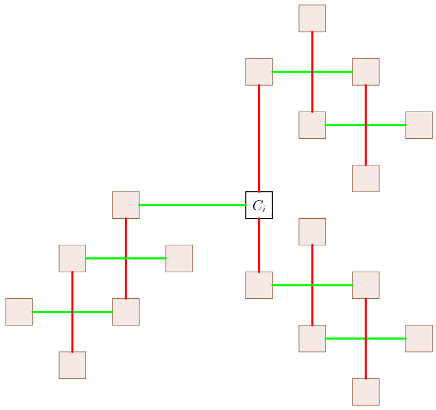
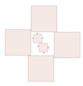
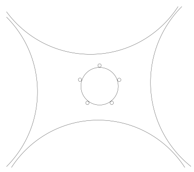


Figure 3: Clause gadget: Three variables arrive at the clause.



(a) Blocker using squares.



(b) Blocker using circular disks.

Figure 4: Blocker gadgets.

gions are disks/squares with a bounded ratio of largest to smallest, then one can always find a non-crossing matching that matches a constant fraction of the input regions. We also consider the bipartite case, in which the input is n "red" regions and n "blue" regions; we prove that it is NP-complete to decide if there is a non-crossing perfect bipartite matching for unit disks/squares.

2 Hardness

Theorem 1 *Given a set S of $2n$ disjoint circular disks or axis-aligned squares in the plane, deciding if there exists a non-crossing perfect matching on S is NP-complete.*

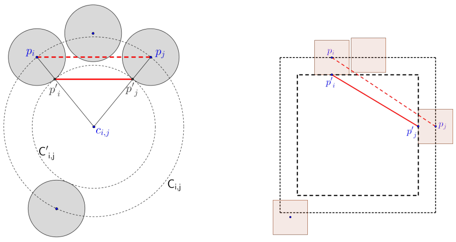
Proof. [Sketch] Our proof is from Planar-Exactly-1-in-3-SAT. We focus on the case of axis-aligned squares. In Figures 2,3,4 we show some of the gadgets; not shown are the (polynomially) numerous "blockers", which fill the space around the squares and red/green edges shown, making it so that the only edges possible to consider for the matching are (essentially) those red/green edges shown, as there are no other combinatorially distinct free-space connecting segments between pairs of squares. Gadgets for blockers are shown in Figure 4 (both for squares and circular disks). A blocker is designed in such a way that the only way to pair up the objects in the blocker is to make internal connections that leave the outer bounding objects unavailable for matching.

In the variable gadget of Figure 2(a), using the red (vertical) edges corresponds to setting the variable to True, while using the green (horizontal) edges corresponds to setting the variable to False. Once we commit to the type of edge (red or green) matching to the square labelled v_i, we are commited to this choice along the "variable chain". Figure 2(b) shows a splitting gadget that allows the signal from a variable to be split, so that it can propagate to multiple different clauses. Figure 3 shows three variable chains connecting to a clause gadget (a single square). Note that the clause square may have to be larger than shown, in order to accommodate configurations in which, say, the three edges incident to the clause are all red (i.e., all three variables appear as positive literals). We claim that there is a non-crossing perfect matching if and only if it is possible to satisfy all clauses using exactly one true literal per clause. Further, we claim that the entire construction uses a polynomial number of squares. □

3 Matching Unit Disks and Squares

Theorem 2 *Given a set $S = \{R_1, \ldots, R_{2n}\}$ of $2n$ disjoint unit-radius disks or axis-aligned unit squares in the plane, there is always a non-crossing perfect matching on S, and it can be computed in polynomial time.*

Proof. For disks R_i, we construct the Euclidean Delaunay triangulation of the disk centers, p_i, in time $O(n \log n)$. We know, from Dillencourt [5], that there is a perfect matching in the Delaunay triangulation. We match pairs



(a) Circular regions. (b) Square regions.

Figure 5: Proof of Theorem 2.

of disks according to this matching. We then realize the connections as follows (see Figure 5(a)): for a Delaunay edge (p_i, p_j), with corresponding witness circle $C_{i,j}$ (which passes through p_i and p_j and has no other center points interior to it), centered at $c_{i,j}$, we connect point $p'_i \in \partial R_i$ to the point $p'_j \in \partial R_j$, where p'_i (resp., p'_j) is the "shifted" point on the segment $p_i c_{i,j}$ at distance 1 from p_i (resp., on the segment $p_j c_{i,j}$ at distance 1 from p_j). Then, the circle $C'_{i,j}$ centered at $c_{i,j}$ of radius 1 less than the radius of $C_{i,j}$ has an interior disjoint from all other unit disks R_k of S (since $C_{i,j}$ is empty of unit disk centers, and the radius of $C'_{i,j}$ is 1 less than that of $C_{i,j}$). Thus, the segment $p'_i p'_j$, which lies within $C'_{i,j}$, does not intersect any other unit disk. Further, the segments $p'_i p'_j$ obtained from Delaunay edges in this way are pairwise non-crossing, since each such segment has a corresponding witness circle $C'_{i,j}$, whose interior contains no other shifted points p'_k. Thus, we have obtained a non-crossing perfect matching on the set S of unit disks.

For squares, we construct the L_∞ Delaunay triangulation of the centers of the regions S. We know that the Delaunay triangulation has a perfect matching (in fact, it also has a Hamiltonian path; one simple proof is given in [2]). We match pairs of squares according to such a matching/path. We then realize the connections as shown in Figure 5(b). □

Theorem 3 *Computing a non-crossing perfect matching on $2n$ unit disks or axis-aligned unit squares has an $\Omega(n \log n)$ lower bound.*

Proof. Our reduction is from sorting. Given n distinct integers $\{x_1, x_2, \ldots, x_n\}$ that are to be sorted, we create an instance of region matching on a set of $2n - 2$ disjoint small squares, each of side length $1/4$, centered on the points $x_{i_{min}}$, $x_{i_{max}}$, and $x_i \pm 1/4$, for $i \neq i_{min}, i_{max}$, along the x-axis. Here, $x_{i_{min}} = \min_i x_i = x_{\pi_1}$ and $x_{i_{max}} = \max_i x_i = x_{\pi_n}$ are the smallest and largest of the input integers, whose sorted sequence (unknown to us) is given by the permutation π: $(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$. (The values $x_{i_{min}}$ and $x_{i_{max}}$ are easily computed in time $O(n)$.) For this set of disjoint squares, the only non-crossing perfect matching is that which joins the square centered at $x_{i_{min}}$ with the square centered at $x_{\pi_2} - 1/4$, the square centered at $x_{\pi_2} + 1/4$ with the square centered at $x_{\pi_3} - 1/4$, etc. Thus, the result of the matching (which square is matched to which square) determines the sorted order of the input x_i. □

Note that if the radii of the regions (disks) can be arbitrary, then it may not be possible to match more than a single pair of regions; see Figure 1(b). It is interesting to consider for what ratio of large to small radius can we say that a perfect matching always exists. We know that a perfect matching always exists if the ratio is one (i.e., all disks are of the same radius). Let r_{max} (resp., r_{min}) denote the radius of the largest (resp., smallest) disk/square. Let $\rho = r_{max}/r_{min}$ be the ratio of the size of the largest to smallest object. We denote by ρ^*_D the critical ratio for disks, and by ρ^*_S the critical ratio for squares, so a non-crossing matching exists for any ratio $\rho \geq \rho^*_D$ or $\rho \geq \rho^*_S$, respectively, but not for $\rho < \rho^*_D$, $\rho < \rho^*_S$. It follows from Theorem 2 that ρ^*_D and ρ^*_S exist, because there is a non-crossing matching whenever $\rho = 1$. The examples in Figure 6 show that $\rho^*_D \geq 1/3$ for disjoint disks, and $\rho^*_S \geq \phi$ for disjoint squares, where $\phi = 0.618\ldots$ is the Golden Ratio.

We now consider the question of achieving a matching (with non-crossing edges) of at least a certain fraction of the input regions.

Theorem 4 *If $r_{max}/r_{min} \leq C$, then there always exists a non-crossing matching of $\Omega(n/C)$ pairs.*

Proof. [Sketch] We shrink disks to r_{min} and do a non-crossing matching on the equal-radius disks. Then, we argue that no original (larger) disk can block more than $O(C)$ of the matched edges. □

4 Bipartite Matchings

Theorem 5 *Given a set S of $2n$ disjoint axis-aligned unit squares in the plane, n of them "red" and n of them "blue", deciding if there exists a non-crossing perfect bipartite matching between red and blue squares is NP-complete.*

Proof. [Sketch] We reduce from planar Constraint Graph Satisfaction [6]: given a planar graph with edge weights

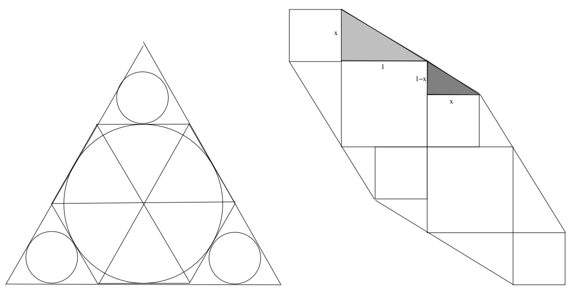
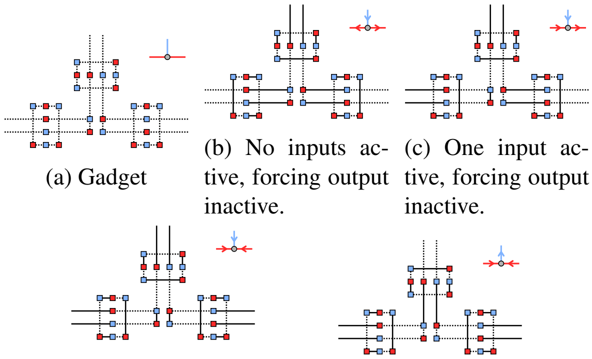


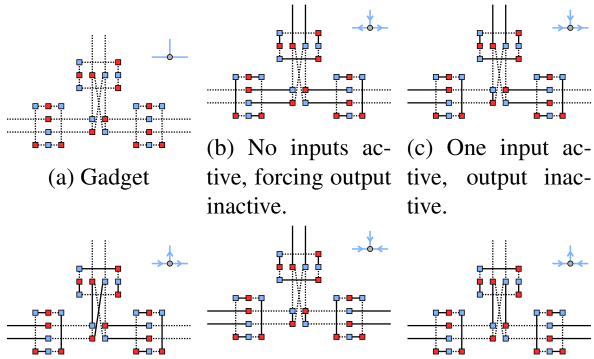
Figure 6: Lower bounds for the critical ratio: Examples for which no non-crossing perfect matchings exist.

of 1 (denoted red) or 2 (denoted blue), where each vertex is red-red-blue (called AND) or blue-blue-blue (called OR), decide whether there is an orientation such that every vertex has a total incoming weight of at least 2. Given such a graph, we embed it orthogonally in a grid, and replace each AND vertex, each OR vertex, and each turn with Figures 7a, 8a, and 9a, respectively. Dashed lines denote candidate matching connections; all other connections are suitably blocked by a bipartite unit-square blocker (details appear in the full paper). Figures 7, 8, and 9 show all valid solutions of these gadgets. Each 6-cycle forces the contained two points to either both match into this gadget (representing an outgoing edge in the orientation) or into the adjacent gadget (representing an incoming edge in the orientation). The central points force the appropriate behavior by blocking certain connections. □



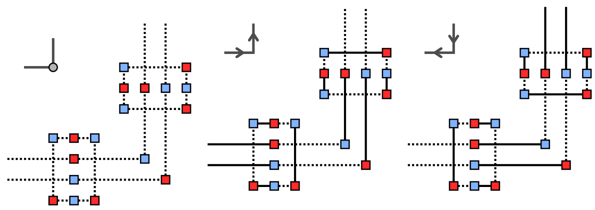
(a) Gadget (b) No inputs active, forcing output inactive. (c) One input active, forcing output inactive. (d) Two inputs active, output inactive. (e) Two inputs active, output active.

Figure 7: Constraint-graph AND vertex represented as bipartite region matching. Here we view the red edges as inputs and the blue edge as an output; we call an input *active* if it is incoming, and call an output *active* if it is outgoing.



(a) Gadget (b) No inputs active, forcing output inactive. (c) One input active, output inactive. (d) One input active, output active. (e) Two inputs active, output inactive. (f) Two inputs active, output active.

Figure 8: Constraint-graph OR vertex represented as bipartite region matching. Here we view the horizontal blue edges as inputs and the vertical blue edge as an output; we call an input *active* if it is incoming, and call an output *active* if it is outgoing.



(a) Gadget (b) Directed from left to top. (c) Directed from top to left.

Figure 9: Constraint-graph 90° turn represented as bipartite region matching.

References

[1] B. M. Ábrego, E. Arkin, S. Fernández-Merchant, F. Hurtado, M. Kano, J. S. Mitchell, and J. Urrutia. Matching points with geometric objects: Combinatorial results. In *Proc. 8th Jap. Conf. Discrete Comput. Geometry, JCDCG04, Springer-Verlag*, 2005.

[2] B. M. Ábrego, E. M. Arkin, S. Fernández-Merchant, F. Hurtado, M. Kano, J. S. Mitchell, and J. Urrutia. Matching points with squares. *Discrete & Computational Geometry*, 41(1):77–95, 2009.

[3] A. K. Abu-Affash, P. Carmi, M. J. Katz, and Y. Trabelsi. Bottleneck non-crossing matching in the plane. *Comput. Geom.*, 47(3):447–457, 2014.

[4] G. Aloupis, J. Cardinal, S. Collette, E. D. Demaine, M. L. Demaine, M. Dulieu, R. F. Monroy, V. Hart, F. Hurtado, S. Langerman, M. Saumell, C. Seara, and P. Taslakian. Non-crossing matchings of points with geometric objects. *Comput. Geom.*, 46(1):78–92, 2013.

[5] M. B. Dillencourt. A non-Hamiltonian, nondegenerate Delaunay triangulation. *Inf. Process. Lett.*, 25(3):149–151, 1987.

[6] R. A. Hearn and E. D. Demaine. *Games, Puzzles, and Computation*. A K Peters, July 2009.