Online Exploration and Triangulation in Orthogonal Polygonal Regions

Sándor P. Fekete, Sophia Rex, and Christiane Schmidt

Department of Computer Science, TU Braunschweig,
D–38116 Braunschweig, Germany
{s.fekete,c.schmidt}@tu-bs.de, mail.s.rex@gmail.com

Abstract. We consider the problem of exploring and triangulating a region with a swarm of robots with limited vision and communication range. For an unknown polygonal region P, the Online Minimum Relay Triangulation Problem (OMRTP) asks for an exploration strategy that maintains a triangulation with limited edge length and achieves a minimum number of robots (relays), such that the triangulation covers P; for a given number n of robots, the Online Maximum Area Triangulation Problem (OMATP) asks for maximizing the triangulated area. Both problems have been studied before, with a competitive factor of 3 for the OMRTP in general polygons, and an unbounded competitive factor for the OMATP; the latter holds for polygons with very narrow corridors.

In this paper, we study the OMRTP for polygons without such bottlenecks: polyominoes, i.e., orthogonal polygons with integer edge lengths. Based on optimal solutions for small squares, we establish a competitive factor of $\frac{17\sqrt{3}}{16+\sqrt{3}} \approx 1.661$ for polyominoes with and $\frac{19\sqrt{3}}{20+\sqrt{3}} \approx 1.514$ for polyominoes without holes. We also give a lower bound of $\frac{38}{37} \approx 1.027$ for any deterministic strategy for the OMRTP in polyominoes. For the OMATP, we establish a competitive factor of $\frac{2}{3\sqrt{3}} \approx 0.3849$, and argue that this is asymptotically optimal.

1 Introduction

Consider a swarm of robots that has to explore a region P. Each robot has limited capabilities: both vision and communication are restricted in range. Incrementally, the swarm has to build a rigid, stable formation that covers all of P. This gives rise to the Minimum Relay Triangulation Problem (MRTP): find a triangulation T with limited edge length, such that P is fully covered by T and the number of relays is minimized. In the online version of this problem (OMRTP), the polygon is unknown in advance. Closely related is the Maximum Area Triangulation Problem (MATP), and its online version OMATP, in which the number n of available robots is fixed, and the enclosed area needs to be maximized. Note that considering the problem as a domain decomposition we give solutions for mesh generation with triangle elements with a bounded edge length, such that the number of Steiner points is minimized.

S.K. Ghosh and T. Tokuyama (Eds.): WALCOM 2013, LNCS 7748, pp. 29–40, 2013.

Both of these problems have been considered before for the case of arbitrary polygonal regions that need to be explored. As Fekete et al. [6] showed, there is a 3-competitive strategy for the OMRTP, while the OMATP does not allow any constant competitive factor. Limiting factors for both of these results are sharp turns along the boundary, as well as tight bottlenecks; however, when exploring buildings, we are typically faced with orthogonal walls, as well as corridors of reasonable dimensions that are multiples of some underlying size. This makes it natural to consider orthogonal polygons with integer dimensions, i.e., *polyominoes*. In this paper, we provide a number of refined results for this class of environments. Our results are as follows.

– We give a strategy with a competitive factor of $\frac{17\sqrt{3}}{16+\sqrt{3}} \approx 1.661$ for polyominoes with holes.
– We give a strategy with a competitive factor of $\frac{7}{8}\sqrt{3} \approx 1.516$ and sketch one of $\frac{19\sqrt{3}}{20+\sqrt{3}} \approx 1.514$ for polyominoes without holes.
– We establish a lower bound of $\frac{38}{37} \approx 1.027$ for any deterministic algorithm for the OMRTP in polyominoes.
– We show that the OMATP in polyominoes does allow a constant competitive ratio of $\frac{2}{3\sqrt{3}} \approx 0.3849$.
– We argue that the value of $\frac{2}{3\sqrt{3}} \approx 0.3849$ is asymptotically optimal.

Related Work. There exists a broad spectrum of work on **triangulations**, both in theory and in practical applications. Here we just mention the work by Bern and Eppstein [1], who investigated triangulations with certain characteristics, e.g., Steiner points, bounds on angles, and even minimizing the sum of edge lengths. For **online robot exploration**, Hoffman et al. [11] presented a 26.5-competitive algorithm for exploring a simple polygon with a single robot with continuous vision. For discrete vision, the problem was studied by Fekete and Schmidt [9]. Another variant with discrete vision and limited visibily range was investigated by Fekete, Mitchell and Schmidt [8]. A related problem on polyominoes was considered by Icking et al. [13].

Strongly related to exploration problems is the task of **deploying a robot swarm into an unknown environment**. Hsiang et al. [12] studied the problem for polyominoes, but they placed one robot per unit square instead of triangulating the area. Practical aspects of deploying strategies were investigated by McLurkin and Smith [15]. Brunner et al. [3] examined the minimum set of abilities a robot needs to perform a certain task, see also Suri et al. [17] for exploration and triangulation algorithms with such robots.

Offline **relay placement** has also been studied, especially in the context of network properties. Bredin et al. [2] considered deploying a minimal number of sensors with limited communication range in an outdoor area such that they form a network with k-connectivity. Differences to our scenario include the absence of defined boundary and holes as well as the requirement that sensors need to stay connected during deployment. Moreover, their aim is to guarantee the connectivity and not to form a triangulation, so some faces of the final network graph may not be triangles. They also considered adding sensors to an existing network with lower connectivity in order to achieve k-connectivity. Kashyap et al. [14] studied the problem for $k = 2$ in higher dimensions. For $k = 1$ and two dimensions, a similar problem was studied by Degener et al. [4] with the added difficulty of finding shortest paths from the entry point to the final locations of the robots carrying the sensors. Moreover, the robots only perceive their current environment and have to make decisions based on this local information without knowing the global situation. Another variant studied by Efrat et al. [5] is to equip sensors with short-range communication devices and then place a minimum number of relays with long-range communication devices in the sensor network to establish connectivity.

The MRTP and OMRTP discussed in this paper were first considered by Fekete et al. [7] and a competitive ratio of 6 on polygons was achieved for the OMRTP. The results were refined by Schmidt [16] and Fekete et al. [6]; they include an NP-hardness proof and a PTAS for the MRTP and the MATP, as well as a lower bound of 1.2 and a 3-competitive algorithm for the OMRTP on polygons. Moreover, they showed that no competitive online algorithm for the online MATP can exist.

Finally, Friedman [10] has applied results similar to our optimality results for unit squares and two-by-two squares in his proofs of lower bounds on *packing* unit squares into squares. However, their optimality was neither mentioned nor proved. Note, however, that these (offline) packing problems are notoriously difficult, and there are still major gaps in what is known.

2 Preliminaries

A *triangulation* T is a set of relays and edges that subdivide a given polygon P into triangles. All edges must lie in P, i.e., they must not cross the boundary. All edges and relays of the triangulation must belong to triangles. The most common form of triangulation places relays of T on all vertices of P only. A *unit triangulation* is a triangulation in which all edges have at most length one. In order to achieve a unit triangulation, it is usually necessary to place relays in the interior and on boundary edges of P in addition to the vertices placed on P's vertices. These extra relays are called *Steiner points*.

For the *Minimum Relay Triangulation Problem* (MRTP), we are given a polygon P with vertex set V and a point $z \in P$. We want to compute a set, R (with $z \in R$ and $V \subset R$), of relays within P such that there exists a (Steiner) triangulation of P whose vertex set is exactly the set R and whose edges have length at most 1, i.e., a unit triangulation, that covers P. For the *Maximum Area Triangulation Problem* (MATP), we are given a polygon P, a point $z \in P$, and a number n of available relays. We want to compute a set, R (with $z \in R$), of n relays within P such that there exists a (Steiner) unit triangulation of P whose vertex set is exactly the set R, such that the total area of all triangles is maximized.

For the online version of the MRTP, the polygon P is unknown in advance. We want to compute a set R as for the MRTP. Here, the relays move into the polygon, starting from z. A relay extending the yet established subset $R' \subset R$ must stay within a distance of 1 to at least one relay $r \in R'$. Once it fixed its position it will not move again. The OMATP is defined analogously. For this paper P is an unknown polyomino, so we consider the online versions of MRTP and MATP. A strategy needs to place relays one by one, such that at any time there are at most two edges and two relays that are not part of a triangle.

We now state several useful lemmas that are related to triangulations.

Lemma 1. *The number of relays n in any triangulation of a polygon P with h holes is equal to $\frac{1}{2}(t+b)+1-h$, where t is the number of triangles and b the number of relays on the boundary. (The boundary of the holes is part of the polygon's boundary.)*

Proof. For any triangulation of P, let t denote the number of triangles, e the number of edges and n the number of relays, of which b are on the boundary of the polygon P. Counting the triangle-edge incidences we find that every triangle has three edges and every edge belongs to two triangles if it is inside the polygon P, or one triangle if it is on the boundary of P. The number of boundary edges is equal to b, the number of boundary relays. This yields $3t = 2e - b$. Not counting the one face outside the Polygon, Euler's formula gives $n + t + h = e + 1$ or $e = n + t - 1$. Substituting this into the previous equation yields $3t = 2n + 2t - 2 + 2h - b$, which can now be solved for $n = \frac{1}{2}(t+b)+1-h$. □

In this work, we consider unit triangulations. We therefore state a simple lower bound for the number of relays in such a triangulation.

Lemma 2. *Let A be the area, h the number of holes and B be the length of the boundary of the polygon P. Then the minimum number of relays in a unit triangulation of P is $n_{OPT} \geq \frac{1}{2}\left(\frac{4A}{\sqrt{3}}+B\right)+1-h$.*

Proof. We use the formula $n = \frac{1}{2}(t+b)+1-h$ from Lemma 1. $b \geq B$ and $t \geq \frac{4A}{\sqrt{3}}$ must hold for any triangulation, because the maximum side length in each triangle of the triangulation is 1, so the maximum area of a triangle is $\frac{\sqrt{3}}{4}$, and we have to cover the whole area and boundary of the Polygon P. We get $n_{OPT} \geq \frac{1}{2}\left(\frac{4A}{\sqrt{3}}+B\right)+1-h$ for the optimal solution. □

3 Solutions for Squares

We briefly sketch some results for the case in which P is a square of limited size. These results serve as stepping stones for the following online strategies.

Lemma 3. *The optimal solution of the MRTP is five for a unit square, twelve for a 2×2-square, and 21 for a 3×3-square.*

In addition, Fig. 1, left, shows the best solution we could find for a 4×4-square. It places the minimum number of 16 relays on the boundary. The relays of the second layer are placed greedily towards the middle. The four center relays are rotationally symmetric. Note that our solution for a 4×4-square can be enhanced into four optimal 2×2-solutions, or 16 optimal 1×1-solutions, by adding relays on the unit grid positions inside the square. This construction, which is shown in Fig. 1, plays a key role in our online algorithm.

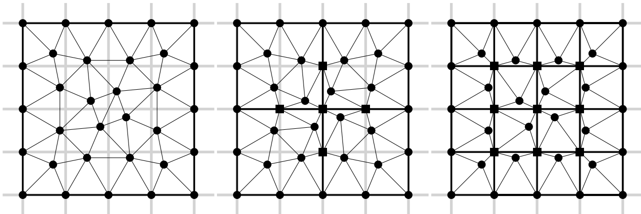

Fig. 1. Decomposition of our best solution for side length four into four optimal solutions of side length two and 16 optimal solutions for unit squares

4 Minimum Relay Triangulation in Polyominoes

4.1 A Strategy Using Optimal 1×1-Squares

Our first strategy is to apply the optimal solution for a unit square for each grid square of the polyomino, as shown in Fig. 2. Algorithm 4.1 picks a square, places the center relay, connects it to all existing grid point relays, places and connects the remaining grid point relays and moves on to the next square.

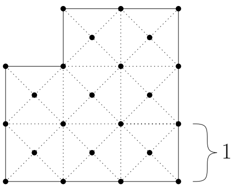

Fig. 2. Example of a triangulation as a result of Algorithm 4.1

Lemma 4. *Algorithm 4.1 achieves a competitive ratio of $\sqrt{3}$ for the OMRTP on any polyomino P.*

Algorithm 4.1. Deploying relays with the optimal unit square strategy

Input : Starting point on integer coordinates on the boundary of a yet unknown polyomino P
Output: Triangulation of P with at most OPT$\cdot\sqrt{3}$ relays
while *P is not completely triangulated* **do**
  pick an empty grid square bordering the triangulated area;
  place one relay in its center;
  connect it to all existing relays on grid vertices in range;
  while *the grid square is not completely triangulated* **do**
    place one relay on an empty vertex of the grid square;
    connect it with the existing relays in range;
  end
end

Proof. We apply Lemma 1. Our algorithm uses triangles with an area of $\frac{1}{4}$, thus $t = 4A$, and places all boundary relays with the maximum distance of one to their neighbors, thus $b = B$. Overall the algorithm places $n_{ALG} = \frac{1}{2}(t + b) + 1 - h = \frac{1}{2}(4A + B) + 1 - h$ relays. According to Lemma 2, the optimal solution needs at least $n_{OPT} \geq \frac{1}{2}\left(\frac{4A}{\sqrt{3}} + B\right) + 1 - h$ relays.

The smallest hole in the polyomino P must itself be a polyomino and thus have a boundary length of at least 4. Therefore, $h < \frac{1}{4}B$ and, in particular, $B + 2 - 2h \geq 0$. Thus, the competitive ratio is

$$\frac{n_{ALG}}{n_{OPT}} \leq \frac{\frac{1}{2}(4A + B) + 1 - h}{\frac{1}{2}\left(\frac{4A}{\sqrt{3}} + B\right) + 1 - h} = \frac{4A + B + 2 - 2h}{\frac{4A}{\sqrt{3}} + B + 2 - 2h} \leq \sqrt{3}. \quad \square$$

4.2 A Strategy Using Optimal 2 × 2-Squares

Now we refine the previous strategy. The basic idea is to introduce an imaginary grid of width two and place the optimal solutions for squares of side length two according to this new grid, see Fig. 3, left. If such a 2×2 grid square turns out to be intersected by the boundary (because one of its unit squares is missing), an additional relay is placed in its middle and the strategy locally reverts to the 1×1-strategy. See Fig. 3 for an illustration; a detailed description is given by Algorithm 4.2.

We exploit the following properties of polyominoes.

Lemma 5. *In a polyomino without holes, $r = c - 4$, where r is the number of reflex vertices and c is the number of convex vertices.*

Lemma 6. *In a polyomino with $A > 1$ and $h = 0$, Algorithm 4.2 places at most $u \leq B - 4$ unit squares.*

Lemma 7. *In a polyomino with $A > 1$ and $h \geq 0$, Algorithm 4.2 places at most $u \leq B + 4h - 4$ unit squares.*

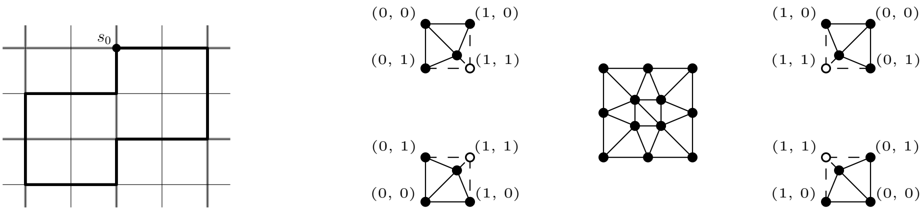

Fig. 3. A polyomino with its natural unit grid and the gray grid of side length two determined by the unit grid and the starting point. In this case, only one optimal 2×2-square will be placed, even though one could place two.

Algorithm 4.2. Deploying relays with the 2×2-square strategy

Input : Starting point on integer coordinates on the boundary of a yet unknown polyomino P
Output: Triangulation of polyomino P with at most OPT·1.661 relays
while *P is not completely triangulated* **do**
  pick a grid square adjacent to the triangulated area;
  place one relay inside, according to the rules in Fig. 3;
  if *there is boundary at the square's (1,1) vertex* **then**
    connect the relay to all grid relays in range;
    while *the grid square is not completely triangulated* **do**
      place one relay on the vertex of the grid square closest to the existing triangulation;
      connect it with the existing relays in range;
    end
  else
    connect the relay to all relays in range, without crossing existing connections;
    while *(0,0) or (0,1) or (1,0) does not have a relay* **do**
      place one relay on the vertex of the grid square closest to the existing triangulation except (1,1);
      connect it with the existing relays in range;
    end
  end
end

From these lemmas we gain $u \leq B$ and a new bound on the competitive ratio.

Theorem 8. *Algorithm 4.2 achieves a competitive ratio of* $\frac{7}{8}\sqrt{3} \approx 1.516$ *for the OMRTP in polyominoes with* $h = 0$, $A > 1$.

Proof. Lemma 1 yields $n_{ALG} = \frac{1}{2}(\frac{1}{2}u + \frac{7}{2}A + B) + 1$. In order to compare n_{ALG} with the lower bound for the optimal solution n_{OPT}, we have to eliminate u: $u \leq A$, because we cannot have more unit squares than area. Lemma 6 yields $u \leq B$. We proceed by another case-by-case analysis.

Case $\mathbf{A} \leq \mathbf{B}$: In this case, we use $u \leq A$ and obtain

$$n_{ALG} = \frac{1}{2}(\frac{1}{2}u + \frac{7}{2}A + B) + 1 \leq \frac{1}{2}(4A + B) + 1 \Rightarrow \frac{n_{ALG}}{n_{OPT}} \leq \frac{\frac{1}{2}(4A+B)+1}{\frac{1}{2}\left(\frac{4A}{\sqrt{3}}+B\right)+1}.$$

We observe that the ratio increases if the area is large in comparison to the boundary length. Because we have $A \leq B$, we can obtain

$$\frac{n_{ALG}}{n_{OPT}} \leq \frac{\frac{1}{2}(4A+B)+1}{\frac{1}{2}\left(\frac{4A}{\sqrt{3}}+B\right)+1} \leq \frac{\frac{1}{2}(5B)+1}{\frac{1}{2}\left(\frac{4B}{\sqrt{3}}+B\right)+1} \leq \frac{5}{\frac{4}{\sqrt{3}}+1} < \frac{7}{8}\sqrt{3} \approx 1.516$$

Case $\mathbf{A} \geq \mathbf{B}$: In this case, we use $u \leq B$ and obtain

$$n_{ALG} = \frac{1}{2}(\frac{1}{2}u + \frac{7}{2}A + B) + 1 \leq \frac{1}{2}(\frac{7}{2}A + \frac{3}{2}B) + 1.$$

Thus,

$$\frac{n_{ALG}}{n_{OPT}} \leq \frac{\frac{7}{4}A + \frac{3}{4}B}{\frac{2}{\sqrt{3}}A + \frac{1}{2}B} \leq \frac{\frac{7}{4}A + \frac{7}{8}B}{\frac{2}{\sqrt{3}}A + \frac{1}{2}B} \leq \frac{\frac{7}{4}A + \frac{7}{8}B}{\frac{2}{\sqrt{3}}A + \frac{1}{\sqrt{3}}B} = \frac{7}{8}\sqrt{3} \approx 1.516.$$

A similar analysis provides the following result for polyominoes with holes. A detailed proof is omitted due to space constraints.

Theorem 9. *Algorithm 4.2 achieves a competitive ratio of* $\frac{17\sqrt{3}}{16+\sqrt{3}} \approx 1.661$ *for the OMRTP in polyominoes with holes.*

4.3 A Strategy Using Good Solutions for 4 × 4-Squares

We now proceed to give a slight improvement for the case of polyominoes without holes. As described in Section 3 and illustrated by Fig. 1, our best solution for the 4×4-square can be divided into smaller optimal solutions by placing relays on the grid points inside the 4×4-square. see Fig. 1. As in the 2×2-approach, our 4×4-solution can be adjusted when boundary pixels are encountered before a 4×4-solution is encountered. An example is shown in Fig. 4.

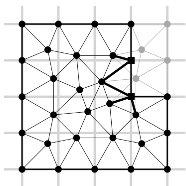

Fig. 4. A 4×4-square is cut by a polyomino's boundary. Adjustments are only necessary on the new boundary: adjusted edges are shown in bold, while the edges and relays of the original 4×4-solution are indicated in gray.

Our algorithm places the 4×4-solution on a grid of width four, starting at the entry point s_0, and adds boundary relays if necessary. That is, only if we cannot place the complete 4×4-square, as the boundary interesects this module, we switch to 2×2-squares and 1×1-squares. The layout of the 4×4-solution allows to do so with the limit on the vision range. The algorithm triangulates as much of each 4×4-grid square as possible before moving on to the next 4×4-square. Inside the 4×4-square, the inner unit squares are treated only after all boundary unit squares around them have been processed. This specified order of triangulation ensures that at most two relays are not part of the triangulation. Correctness is provided by Lemma 10.

Lemma 10. *At all times during the execution of the algorithm using* 4×4-*squares, at most two relays and edges are not part of the triangulation.*

For the competitive ratio estimate of this algorithm, we assume that a 4×4-square intersected by boundary is divided into four 2×2-squares and each 2×2-square intersected by boundary is divided into four unit squares, as in Section 4.2. The result is that if we place any additional relay at all, we will automatically assume that the additional relays at $(1,2), (2,1), (2,2), (2,3)$, and $(3,2)$ have also been placed. Having assumed these extra relays, we can reuse some of the analysis techniques from Section 4.2.

For the case of polyominoes without holes, we can improve upon the result of Section 4.2. Proof details are omitted due to space constraints.

Theorem 11. *The algorithm using* 4×4-*squares achieves a competitive ratio of* $\frac{19\sqrt{3}}{20+\sqrt{3}} \approx 1.514$ *for the OMRTP in polyominoes with* $h = 0$ *and* $A > 1$.

For polyominoes with holes, we achieve the same result as in Section 4.2.

4.4 Lower Bound

Theorem 12. *For deterministic online algorithms, a lower bound on the competitive ratio for the OMRTP on polyominoes is* $\frac{38}{37} \approx 1.027$.

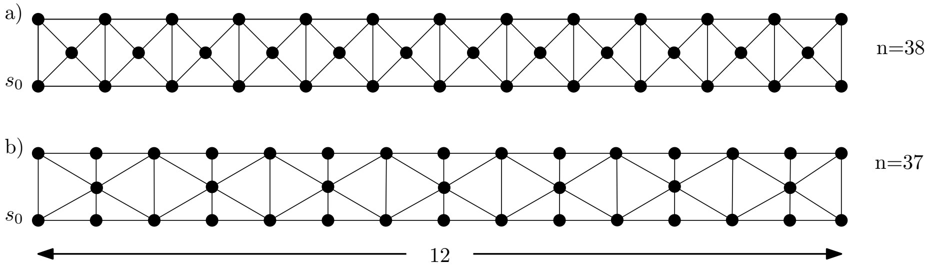

Fig. 5. The adversary's relay placement for a strip of length 12

Proof. We consider a unit strip with starting point at the lower left corner; details are due to space constraints. As shown in Fig. 5, it is possible to save a relay compared to a unit-square solution for a strip of length 12; however, if an algorithm tries this approach, an adversary can choose a shorter strip, for which the unit-square solution is still better.

5 Online Maximum Area Triangulation in Polyominoes

In the Online Maximum Area Triangulation Problem (OMATP) we are given a fixed number of relays, n, and our goal is to triangulate as much area as possible. As mentioned in the introduction, no competitive algorithm can exist for arbitrary polygons. However, for polyominoes we can achieve a factor of $\frac{2}{3\sqrt{3}}$. We use Algorithm 4.1, because it guarantees triangles of size $\frac{1}{4}$.

Theorem 13. *With $n \geq 5$ relays, Algorithm 4.1 achieves a competitive factor of $\frac{2}{3\sqrt{3}}$, which is asymptotically best possible for any deterministic online algorithm.*

Proof. We assess the number of triangles using Lemma 1: $n = \frac{1}{2}(t+b)+1-h \Rightarrow t = 2n-2+2h-b$. Note that the boundary edges and holes are those of the triangulation and not of the polyomino. With $n \geq 4$, the optimal triangulation must have at least four outside boundary edges, because placing the fourth relay inside an already formed triangle cannot be optimal. Moreover, each hole must have more than two boundary edges, so $b \geq 2h+4$ and $2h+4-b \leq 0$ so $t = 2n-2+2h-b \leq 2n-6 = 2(n-3)$. Since the maximal size of each triangle is $\frac{\sqrt{3}}{4}$ we obtain $A_{OPT} = \frac{\sqrt{3}}{2}(n-3)$ for the area covered by an optimal strategy.

For $n \geq 5$, Algorithm 4.1 fills at least one unit square, plus one unit square for at least every three additional relays, plus at least $\frac{1}{4}$ for each relay in the last, not completely triangulated unit square. Therefore $A_{ALG} \geq 1+1\cdot\lfloor\frac{n-5}{3}\rfloor + \frac{1}{4}\left(\frac{n-5}{3} - \lfloor\frac{n-5}{3}\rfloor\right) \geq \frac{n-4}{3}+\frac{1}{2} \geq \frac{n-3}{3}$ (the penultimate inequality is derived by a case distinction for different remainders by the division by 3 and the resulting relay placements).

For the lower bound, see Figure 6. □

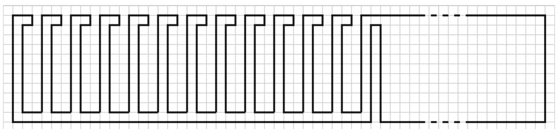

Fig. 6. An online algorithm will fail to find the large polygonal region, where the offline optimum can places a large number of unit triangles. Thus, an online algorithm uses triangles of area $\frac{1}{4}$, while the optimum uses only an asymptotically small fraction of them, with the bulk being unit triangles.

6 Conclusion

We have given a number of competitive strategies for the OMRTP in orthogonal integral polygons. Our refined algorithms rely on optimal solutions for subsquares of limited size. A possible improvement could arise from a refined analysis of the algorithm using 4×4-square, where we do overestimate the number of placed relays. A further interesting extension could be an optimality proof for the 4×4-square; studying triangulations of general $k \times k$-squares is interesting in itself, but can be expected to serious difficulties even for moderate values of k, as it is similar in nature to the notoriously difficult problem of packing and covering with unit disks. For the OMRTP, it may be possible to raise the general lower bound; for the offline problem, the complexity is open, but we believe it to be NP-hard. Other possible extensions ask for a biconnected network when deploying the relays for the OMRTP in addition to the property that every relay and edge is part of a triangle.

There are also some open problems for the case of general polygons, where it may not only be possible to improve on the competitive factor of 3 for the OMRTP, but also achieve finite bounds for the OMATP, assuming bounded feature size.

References

1. Bern, M., Eppstein, D.: Mesh Generation and Optimal Triangulation. Computing in Euclidean Geometry 1, 23–90 (1992)
2. Bredin, J., Demaine, E., Hajiaghayi, M., Rus, D.: Deploying Sensor Networks with Guaranteed Fault Tolerance. IEEE/ACM Transactions on Networking (TON) 18(1), 216–228 (2010)
3. Brunner, J., Mihalák, M., Suri, S., Vicari, E., Widmayer, P.: Simple Robots in Polygonal Environments: A Hierarchy. In: Fekete, S.P. (ed.) ALGOSENSORS 2008. LNCS, vol. 5389, pp. 111–124. Springer, Heidelberg (2008)
4. Degener, B., Fekete, S., Kempkes, B., Meyer auf der Heide, F.: A Survey on Relay Placement with Runtime and Approximation Guarantees. Computer Science Review 5(1), 57–68 (2011)
5. Efrat, A., Fekete, S.P., Gaddehosur, P.R., Mitchell, J.S.B., Polishchuk, V., Suomela, J.: Improved Approximation Algorithms for Relay Placement. In: Halperin, D., Mehlhorn, K. (eds.) ESA 2008. LNCS, vol. 5193, pp. 356–367. Springer, Heidelberg (2008)
6. Fekete, S.P., Kamphans, T., Kröller, A., Mitchell, J.S.B., Schmidt, C.: Exploring and Triangulating a Region by a Swarm of Robots. In: Goldberg, L.A., Jansen, K., Ravi, R., Rolim, J.D.P. (eds.) APPROX/RANDOM 2011. LNCS, vol. 6845, pp. 206–217. Springer, Heidelberg (2011)
7. Fekete, S.P., Kamphans, T., Kröller, A., Schmidt, C.: Robot Swarms for Exploration and Triangulation of Unknown Environments. In: Proceedings of the 25th European Workshop on Computational Geometry, pp. 153–156 (2010)
8. Fekete, S.P., Mitchell, J., Schmidt, C.: Minimum Covering with Travel Cost. Journal of Combinatorial Optimization, 393–402 (2010)
9. Fekete, S.P., Schmidt, C.: Polygon Exploration with Time-Discrete Vision. Computational Geometry 43(2), 148–168 (2010)

10. Friedman, E.: Packing Unit Squares in Squares: A Survey and New Results. The Electronic Journal of Combinatorics (2009)
11. Hoffmann, F., Icking, C., Klein, R., Kriegel, K.: The Polygon Exploration Problem. SIAM Journal on Computing 31(2), 577–600 (2002)
12. Hsiang, T., Arkin, E., Bender, M., Fekete, S., Mitchell, J.: Algorithms for Rapidly Dispersing Robot Swarms in Unknown Environments. In: Algorithmic Foundations of Robotics V, pp. 77–94 (2004)
13. Icking, C., Kamphans, T., Klein, R., Langetepe, E.: Exploring Simple Grid Polygons. In: Wang, L. (ed.) COCOON 2005. LNCS, vol. 3595, pp. 524–533. Springer, Heidelberg (2005)
14. Kashyap, A., Khuller, S., Shayman, M.: Relay Placement for Fault Tolerance in Wireless Networks in Higher Dimensions. Comp. Geom. 44(4), 206–215 (2011)
15. McLurkin, J., Smith, J.: Distributed Algorithms for Dispersion in Indoor Environments using a Swarm of Autonomous Mobile Robots. In: Distributed Autonomous Robotic Systems 6, pp. 399–408 (2007)
16. Schmidt, C.: Algorithms for Mobile Agents with Limited Capabilities. Ph.d. thesis, Braunschweig Institute of Technology (2011)
17. Suri, S., Vicari, E., Widmayer, P.: Simple Robots with Minimal Sensing: From Local Visibility to Global Geometry. The International Journal of Robotics Research 27(9), 1055–1067 (2008)