Facets for Art Gallery Problems

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Abstract We discuss polyhedral methods for computing optimal solutions for large instances of the Art Gallery Problem (AGP). We extend our previous work [7], which uses a primal-dual linear programming approach to solve the fractional AGP to optimality, using cutting planes that eliminate fractional solutions.

We identify two classes of facets of the associated polytopes, based on Edge Cover (EC) and Set Cover (SC) inequalities. Solving the separation problem for these facets is NP-complete, but exploiting the underlying geometric structure of the AGP, we show that large subclasses of fractional SC solutions cannot occur for the AGP. This allows us to separate the relevant subset of facets in polynomial time. Finally, we characterize all facets for finite AGP relaxations with coefficients in $\{0, 1, 2\}$.

1 Introduction

The Art Gallery Problem (AGP) asks for the minimum number of points that can guard a given polygonal region P with n vertices. Chvátal [3] (and Fisk [6]) showed that $\lfloor \frac{n}{3} \rfloor$ guards are sometimes necessary and always sufficient for a simple polygon P. See O'Rourke [9] for a classical survey.

Algorithmically, the AGP is closely related to the Set Cover (SC) problem; it is NP-hard, even for simple polygons [8]. However, there are two differences to a discrete SC problem. On the one hand, geometric variants of problems may be easier to solve or approximate than their discrete, graph-theoretic counterparts; on the other hand, the AGP is far from being discrete: both the set that is to be covered (all points in P) and the covering family (all visibility polygons within P) usually are uncountably infinite.

Amit, Mitchell and Packer [1] have considered purely combinatorial primal and dual heuristics for general AGP instances. Only very recently have researchers begun to combine methods from integer linear programming with non-discrete geometry in order to obtain optimal solutions. As we showed in [7], it is possible to combine an iterative primal-dual relaxation approach with structures from computational geometry in order to solve AGP instances with unrestricted guard positions. Couto et al. [5] used a similar approach for the AGP with vertex guards. Closely related to this paper, Balas and Ng [2] describe all facets with coefficients in $\{0, 1, 2\}$ of the discrete SC polytope.

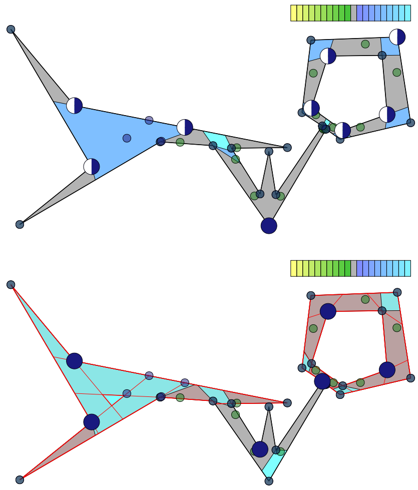

Figure 1: (Top) An optimal fractional solution of an AGP instance. The half-filled circles indicate $\frac{1}{2}$-guards. (Bottom) Cutting planes from Sections 3 and 4 force 2 guards in the left and 5 in the right part of the polygon respectively, thus yielding an integer optimum.

Formal Description We consider a polygonal region P, possibly with holes, with n vertices. For a point $p \in P$, we denote by $\mathcal{V}(p)$ the *visibility polygon* of p in P. P is *star-shaped* if $P = \mathcal{V}(p)$ for some $p \in P$. The set of all such points is the *kernel* of P. For a set $S \subseteq P$, $\mathcal{V}(S) := \cup_{p \in S} \mathcal{V}(p)$. A set $C \subseteq P$ is a *guard cover*, if $\mathcal{V}(C) = P$. The AGP asks for a guard cover of minimum cardinality.

Our Results In this paper, we extend and deepen our previous work on iterative primal-dual relaxations, by proving a number of polyhedral properties of the resulting AGP polytopes.

- We show how to employ cutting planes for an iterative primal-dual framework for solving the AGP. This is interesting in itself, as it provides an approach to tackling optimization problems with infinitely many constraints and variables. The particular challenge is to identify constraints that

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remain valid for *any* choice of infinitely many possible primal and dual variables, as we are solving an iteratively refined sequence of LPs.

- Based on a geometric study of the involved SC constraints, we characterize all facets of involved AGP polytopes that have coefficients in $\{0, 1, 2\}$. We also provide an additional class based on Edge Cover (EC) constraints.
- One class of discussed facets originates from the SC polytope. In that setting, the separation problem is NP-complete. We exploit the geometry to prove that the majority of these facets cannot cut off fractional solutions in an AGP setting (under reasonable assumptions), allowing us to avoid the NP-complete separation problem.
- We demonstrate the practical usefulness of our results with experiments.

2 Mathematical Programming Formulation and LP-Based Solution Procedure

Let $G \subseteq P$ be a set of possible guard locations, and $W \subseteq P$ a set of *witnesses*, i. e., points to be guarded. We assume $W \subseteq \mathcal{V}(G)$. In previous work [7], we have presented an LP-based procedure for the original AGP. It can be formulated as an integer linear program AGP(G, W):

$$\min \sum_{g \in G} x_g \tag{1}$$

$$\text{s.t.} \sum_{g \in G \cap \mathcal{V}(w)} x_g \geq 1 \quad \forall w \in W \tag{2}$$

$$x_g \in \{0, 1\} \qquad \forall g \in G \tag{3}$$

The original problem, AGP(P, P), has uncountably many variables and constraints and thus cannot be solved directly, especially, because a finite witness set generally cannot ensure coverage of P [4]. So we consider finite $G, W \subset P$. For dual separation and to generate lower bounds, we require the LP relaxation AGR(G, W) obtained by relaxing the integrality constraint (3):

$$0 \leq x_g \leq 1 \quad \forall g \in G\ . \tag{4}$$

We have shown that AGR(P, P) can be solved optimally for many problem instances by using finite G and W and solving the primal and dual separation problems, see [7]:

1. Given a solution $(x_g)_{g \in P}$, decide if it is feasible for AGR(G, P), i. e., completely covers the polygon, or prove infeasibility by presenting an insufficiently covered point w. In the latter case a new witness w is added to W, and the LP is re-solved. Otherwise, $(x_g)_{g \in P}$ is optimal for AGR(G, P), and its objective value is an upper bound for AGR(P, P).
2. Given a solution $(y_w)_{w \in P}$ for the dual LP of AGR(G, W), decide whether it is feasible for the dual of AGR(P, W), or prove infeasibility by presenting a violated dual constraint. This coincides with presenting an additional guard point g that will improve the solution. If such a g does not exist, $(y_w)_{w \in P}$ is an optimal dual solution for AGR(P, W) and the objective value is a lower bound for AGR(P, P).

If the upper and the lower bound meet, we have an optimal solution of the fractional AGP, AGR(P, P).

Both separation problems can be solved efficiently using the overlay of the visibility polygons of all points $g \in G$ with $x_g > 0$ (for the primal case) and all $w \in W$ with $y_w > 0$ (for the dual case), which decomposes P into a planar arrangement of bounded complexity.

Our approach may produce fractional solutions as in Fig. 1. In this paper, we use cutting planes to eliminate such fractional solutions. The cuts must remain feasible in all iterations of our algorithm, so feasibility for AGP(G, W) is insufficient; we require them not to cut off integer solutions of AGP(G', P) for any $G' \supset G$.

3 Set Cover Facets

We transfer known facets [2] of the SC polytope to the AGP polytope, and show that the underlying geometry greatly reduces their impact on the involved AGP polytopes.

3.1 A Family of Facets

Consider a finite non-empty subset $\emptyset \subset S \subseteq W$ of witness positions. Every feasible cover of P is a cover of S. Analogous to what Balas and Ng [2] did for the SC polytope, we partition $P = J_0 \dot\cup J_1 \dot\cup J_2$, as follows, see Fig. 2: $J_2 := \{g \in P \mid S \subseteq \mathcal{V}(g)\}$, the set of positions that cover all of S; $J_0 := \{g \in P \mid \mathcal{V}(g) \cap S = \emptyset\}$, the set of positions that see none of S; $J_1 := P \setminus (J_2 \cup J_0)$ the set of positions that cover a non-trivial subset of S. Thus, it takes one guard in J_2, or at least two guards in J_1 to cover S. This can be captured in the following inequality:

$$\sum_{g \in J_2 \cap G} 2x_g + \sum_{g \in J_1 \cap G} x_g \geq 2\ . \tag{5}$$

Sufficient coverage of S is necessary for sufficient coverage of P, so (5) is valid for any feasible solution of AGP(G, P). However, covering S may require more than two guards in J_1, so (5) does not always provide a supporting hyperplane of conv(AGP(G, W)). For $|S| \leq 2$, the inequality never cuts off any point of AGR(G, W); hence, we only consider the case $|S| \geq 3$.

In order to show when Inequality (5) defines a facet of conv(AGP(G, W)), we apply a result of [2] to the

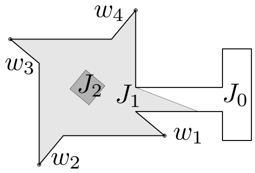

Figure 2: Witness selection $S = \{w_1, w_2, w_3, w_4\}$ and resulting partition $P = J_0 \dot\cup J_1 \dot\cup J_2$.

AGP setting. All proofs in this paper are omitted due to space limitations.

Lemma 1 *Let P be a polygon and $G, W \subset P$ finite sets of guard and witness positions. Then* $\mathrm{conv}(\mathrm{AGP}(G, W))$ *is full-dimensional, if and only if* $|\mathcal{V}(w) \cap G| \geq 2 \ \forall w \in W$.

We require some terminology adapted from [2]. Two guards $g_1, g_2 \in J_1$ form a *2-cover* of S, if $S \subseteq \mathcal{V}(\{g_1, g_2\})$. The *2-cover graph* is the graph with nodes in $J_1 \cap G$ and an edge between g_1 and g_2 iff g_1, g_2 are a 2-cover. Finally, we define $T(g) := \{w \in \mathcal{V}(g) \cap W \mid \mathcal{V}(w) \cap G \cap (J_0 \setminus \{g\}) = \emptyset\}$.

Theorem 2 *Given a polygon P and finite $G, W \subset P$, let* $\mathrm{conv}(\mathrm{AGP}(G, W))$ *be full-dimensional, and let S be maximal, i.e., there is no $w \in W \setminus S$ with $\mathcal{V}(w) \subseteq \mathcal{V}(S)$. Then Inequality* (5) *is facet-defining for* $\mathrm{conv}(\mathrm{AGP}(G, W))$*, if and only if:*

1. *Every 2-cover graph component has an odd cycle.*
2. *$\forall g \in J_0 \cap G$ with $T(g) \neq \emptyset$ there exists either*
   (a) *some $g' \in J_2 \cap G$ such that $T(g) \subseteq \mathcal{V}(g')$, or*
   (b) *$g', g'' \in J_1 \cap G$ with $T(g) \cup S \subseteq \mathcal{V}(g') \cup \mathcal{V}(g'')$.*

3.2 Geometric Properties.

For any size $|S|$, there are SC instances where the general, abstract variant of Inequality (5) actually cuts off fractional solutions [2]. In this section, we show that for the AGP, only a very reduced number of these actually occur.

Lemma 3 *Let P be a polygon and $G, W \subset P$ be finite. Assume $\emptyset \subset S \subseteq W$ is minimal for G, i.e., there is no proper subset $R \subsetneq S$ inducing the same Inequality* (5) *as S. Then the LP coefficient matrix of* $\mathrm{AGP}(G, S)$ *contains a permutation of the full circulant of order $k = |S|$, which is defined as*

$$C_k^{k-1} := \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \in \{0,1\}^{k \times k} . \quad (6)$$

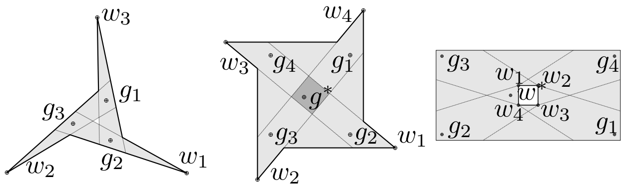

Figure 3: P_3^2 (left) and two attempts for P_4^3 (middle and right). In the left case, Ineq. (5) enforces using two guards instead of three $\frac{1}{2}$-guards. The attempts for P_4^3 are star-shaped (middle) or invalid, as $x_{g_1} = \ldots = x_{g_4} = \frac{1}{3}$ is infeasible (right, at w^*).

This motivates a formal definition of polygons that correspond to C_k^{k-1}.

Definition 1 (Full Circulant Polygon) *A polygon $P = P_k^{k-1}$ along with $G = \{g_1, \ldots, g_k\} \subset P$ and $W = \{w_1, \ldots, w_k\} \subset P$ for $k \geq 3$ is called* Full Circulant Polygon, *if*

$$\mathcal{V}(g_i) \cap W = W \setminus \{w_i\} \quad \forall 1 \leq i \leq k, \text{ and} \quad (7)$$
$$|\mathcal{V}(w) \cap G| \geq k - 1 \quad \forall w \in P . \quad (8)$$

Note that P_k^{k-1} is defined such that C_k^{k-1} completely describes the visibility relations between G and W. This implies that the optimal solution of $\mathrm{AGR}(G, W)$ is $\frac{1}{k-1}\mathbb{1}$ (i.e., assigns a value of $\frac{1}{k-1}$ to each $g \in G$), with total cost $\frac{k}{k-1}$. It is feasible for $\mathrm{AGR}(G, P_k^{k-1})$ by Property (8).

Figure 3 captures construction attempts for models of C_k^{k-1}. P_3^2 exists, but for $k \geq 4$, the polygons are either star-shaped or not full circulant. If they are star-shaped, the optimal solution is to place one guard within the kernel. If they are not full circulant polygons, the optimal solution of $\mathrm{AGR}(G, W)$ is infeasible for $\mathrm{AGR}(G, P_k^{k-1})$. In this case, the current fractional solution is intermittent, i.e., it will be cut off by the algorithm by introducing new witnesses. Both cases eliminate the need for a cutting plane. In the following we argue that P_k^{k-1} is indeed star-shaped for $k \geq 4$, allowing us to avoid the NP-complete separation problem of finding all permutations of all full circulants in the matrix of $\mathrm{AGP}(G, W)$ by reducing our search to $k = 3$.

The first step is Lemma 4, which restricts the possible structure of P_k^{k-1}. It provides the basis for our main theorem.

Lemma 4 *For $k \geq 4$, every full circulant polygon is simple, i.e., it has no holes. This is not true for $k < 4$.*

Theorem 5 *For $k \geq 4$, every full circulant polygon is star-shaped.*

By Theorem 5, no cuts of type (5) are necessary to cut off fractional solutions for a full circulant polygon P_k^{k-1} with $k \geq 4$. It is still possible to embed

P_k^{k-1} in larger polygons, where these cuts play a role. However, our experiments, see Section 5, suggest that these cases rarely occur in practice.

3.3 All Facets with Coefficients in $\{0, 1, 2\}$

Balas and Ng [2] identified all SC facets with coefficients in $\{0, 1, 2\}$. The only nontrivial facet class corresponds to Ineq. (5). As for finite $G, W \subset P$, $\mathrm{AGP}(G, W)$ is an SC instance, we have identified all nontrivial AGP facets with coefficients in $\{0, 1, 2\}$.

4 Edge Cover Facets

Solving $\mathrm{AGR}(G, W)$ for finite $G, W \subset P$ in which no guard sees more than two witnesses is equivalent to solving a fractional edge cover instance on the following graph: The nodes correspond to the witnesses, each guard is represented by an edge or a loop. This is the case in the 5-gonal corridor in the right part of Fig. 1. As outlined in [7], the inequality

$$\sum_{g \in \mathcal{V}(W) \cap G} x_g \geq \left\lceil \frac{k}{2} \right\rceil \tag{9}$$

can cut off such fractional solutions, provided $|W|$ is odd. It is a valid constraint if no guard exists that sees more than two witnesses in W. Inequality (9) is facet-defining for $\mathrm{conv}(\mathrm{AGP}(G, W))$ under some conditions that we leave out due to space restrictions.

5 Computational Experience

A variety of experiments on benchmark polygons show the usefulness of our cutting planes. We employed the same four classes of test polygons as in [7] with approximately 60, 200, 500 and 1000 vertices.

The experiments keep track of the gap between the smallest (integer) upper bound and largest (fractional) lower bound. They were conducted on 3.0 GHz Intel dual core PCs with 2 GB of memory. Our algorithms used CGAL 4.0 and CPLEX 12.1.

Due to space restrictions, we only present the relative gap over time for so-called *von Koch* polygons with 1000 vertices. Fig. 4 shows the distribution of relative gaps over time for different combinations of cutting planes in the third quartile.

Clearly, using no separation yields the largest gaps. The EC cuts are successful in reducing the gap and the SC cuts for $k = 3$ close the gaps faster than all other cut separators. The SC cuts for $k \in \{3, 4\}$ are weaker than those limited to $k = 3$, supporting the practical relevance of our arguments from Section 3: The separation for $k = 4$ takes time, but has no benefits for the gap. Combining SC cuts for $k = 3$ and EC cuts is slightly slower than SC cuts for $k = 3$ only due to an overlap of the facet classes.

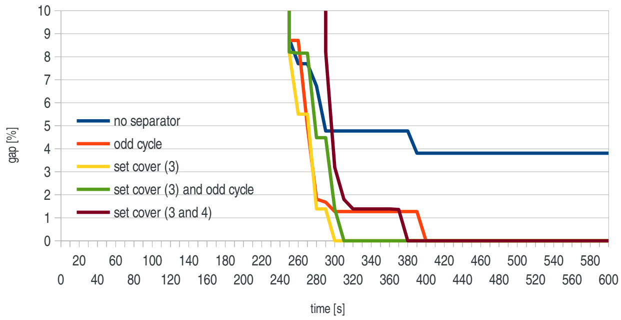

Figure 4: Separator performance.

6 Conclusion

In this paper, we have shown how we can exploit both geometric properties and polyhedral methods of mathematical programming to solve a classical and natural, but highly challenging problem from computational geometry. We have shown that an NP-complete separation problem for the SC case can mostly be avoided in the AGP scenario by considering its underlying geometric structure.

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