Facets for Art Gallery Problems

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Abstract. We demonstrate how polyhedral methods of mathematical programming can be developed for and applied to computing optimal solutions for large instances of a classical geometric optimization problem with an uncountable number of constraints and variables.

The Art Gallery Problem (AGP) asks for placing a minimum number of stationary guards in a polygonal region P, such that all points in P are guarded. The AGP is NP-hard, even to approximate. Due to the infinite number of points to be guarded as well as possible guard positions, applying mathematical programming methods for computing provably optimal solutions is far from straightforward.

In this paper, we use an iterative primal-dual relaxation approach for solving AGP instances to optimality. At each stage, a pair of LP relaxations for a finite candidate subset of primal covering and dual packing constraints and variables is considered; these correspond to possible guard positions and points that are to be guarded.

Of particular interest are additional cutting planes for eliminating fractional solutions. We identify two classes of facets, based on Edge Cover and Set Cover (SC) inequalities. Solving the separation problem for the latter is NP-complete, but exploiting the underlying geometric structure of the AGP, we show that large subclasses of fractional SC solutions cannot occur for the AGP. This allows us to separate the relevant subset of facets in polynomial time.

Finally, we characterize all facets for finite AGP relaxations with coefficients in $\{0, 1, 2\}$. We demonstrate the practical usefulness of our approach with improved solution quality and speed for a wide array of large benchmark instances.

Keywords: Art Gallery Problem, geometric optimization, algorithm engineering, set cover polytope, solving NP-hard problem instances to optimality.

1 Introduction

The Art Gallery Problem (AGP) is one of the classical problems of geometric optimization: given a polygonal region P with n vertices, find as few stationary guards as possible, such that any point of the region is visible by one of the guards. As first proven by Chvátal [1] and then shown by Fisk [2]

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in a beautiful and concise proof (which is highlighted in the shortest chapter in "Proofs from THE BOOK" [3]), $\lfloor \frac{n}{3} \rfloor$ guards are sometimes necessary and always sufficient when P is a simple polygon. Worst-case bounds of this type are summarized under the name "Art-Gallery-type theorems", and used as a metaphor even for unrelated problems; see O'Rourke [4] for an early overview, and Urrutia [5] for a more recent survey.

Algorithmically, the AGP is closely related to the SET COVER (SC) problem; it is NP-hard, even for a simply connected polygonal region P [6]. There are, however, two differences to a discrete SC problem. On the one hand, it is well known that geometric variants of problems may be easier to solve or approximate than their discrete, graph-theoretic counterparts, so it is natural to explore ways to exploit the geometric nature of the AGP; on the other hand, the AGP is far from being discrete, as both the set to be covered (all points in P) as well as the covering family (all star-shaped subregions around some point of P) usually are uncountably infinite.

It is natural to consider more discrete versions of the AGP. Ghosh [7] showed that restricting possible guard positions to the n vertices, i.e., the AGP with vertex guards, allows an $O(\log n)$-approximation algorithm of complexity $O(n^5)$; conversely, Eidenbenz et al. [8] showed that for a region with holes, finding an optimal set of vertex guards is at least as hard as SC, so there is little hope of achieving a better approximation guarantee than $\Omega(\log n)$. While these results provide tight bounds in terms of approximation, they do by no means close the book on the arguably most important aspect of mathematical optimization: combining structural insights with powerful mathematical tools in order to achieve provably optimal solutions for instances of interesting size. Moreover, even a star-shaped polygon may require a large number of vertex guards, so general AGP instances may have significantly better solutions than the considerably simpler discretized version with vertex guards.

Computing optimal solutions for general AGP instances is not only relevant from a theoretical point of view, but has also gained in practical importance in the context of modeling, mapping and surveying complex environments, such as in the fields of architecture or robotics and even medicine, which are seeking to exploit the ever-improving capabilities of computer vision and laser scanning. Amit, Mitchell and Packer [9] have considered purely combinatorial primal and dual heuristics for general AGP instances. Only very recently have researchers begun to combine methods from integer linear programming with non-discrete geometry in order to obtain optimal solutions. As we showed in [10], it is possible to combine an iterative primal-dual relaxation approach with structures from computational geometry in order to solve AGP instances with unrestricted guard positions; this approach is based on considering a sequence of primal and dual subproblems, each with a finite number of primal variables (corresponding to guard positions) and a finite number of dual variables (corresponding to "witness" positions). Couto et al. [11,12,13] used a similar approach for the AGP with vertex guards. Due to space limitations, we omit a detailed discussion of the abundant work on the AGP. Highly relevant is the paper by Balas and Ng [14]

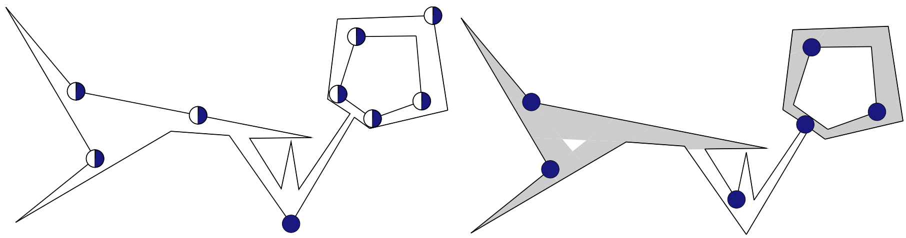


Fig. 1. An optimal fractional solution of value 5 without (left) and an optimal integer solution of value 6 with cutting planes (right). Circles show guards, fill-in indicates fractional amount. Cutting planes enforce at least two guards in the left and three in the right area, both marked in grey.

on the discrete SC polytope, which describes all its facets with coefficients in $\{0, 1, 2\}$.

Formal Description. We consider a polygonal region P with n vertices that may have holes, i.e., that does not have to be simply connected. For a point $p \in P$, we denote by $\mathcal{V}(p)$ the *visibility polygon* of p in P, i.e., the set of all $q \in P$, such that the straight-line connection $\overline{pq}$ lies completely in P. P is *star-shaped* if $P = \mathcal{V}(p)$ for some $p \in P$. The set of all such points is the *kernel* of P. For a set $S \subseteq P$, $\mathcal{V}(S) := \cup_{p \in S} \mathcal{V}(p)$. A set $C \subseteq P$ is a *guard cover*, if $\mathcal{V}(C) = P$. The AGP asks for a guard cover of minimum cardinality c; this is the same as covering P by a minimum number of star-shaped sub-regions of P. Note that Chvátal's Watchman Theorem [15] guarantees $c \leq \left\lfloor \frac{n}{3} \right\rfloor$.

Our Results. In this paper, we extend and deepen our recent work on iterative primal-dual relaxations, by proving a number of polyhedral properties of the resulting AGP polytopes. We provide the first study of this type, and give a full characterization of all facets with coefficients 0, 1, and 2. Remarkably, we are able to exploit geometry to prove that only a very restricted family of facets of the general SC polytope will typically have to be used as cutting planes for removing fractional variables. Instead, we are able to prove that many fractional solutions only occur in intermittent SC subproblems; thus, they simply vanish when new guards or witnesses are introduced. This saves us the trouble of solving an NP-complete separation problem. Computational results illustrate greatly reduced integrality gaps for a wide variety of benchmark instances, as well as reduced solution times. Details are as follows; due to space restrictions, proofs are omitted. Related SC results are described by Balas et al. [14].

– We show how to employ cutting planes for an iterative primal-dual framework for solving the AGP. This is interesting in itself, as it provides an approach to tackling optimization problems with infinitely many constraints and variables. The particular challenge is to identify constraints that remain valid for any choice of infinitely many possible primal and dual variables, as we are not solving one particular IP, but an iteratively refined sequence.

- Based on a geometric study of the involved SC constraints, we characterize all facets of involved AGP polytopes that have coefficients in $\{0, 1, 2\}$. In the SC setting, these facets are capable of cutting off fractional solutions, but the separation problem is NP-complete. We use geometry to prove that only some of these facets are able to cut off fractional solutions in an AGP setting under reasonable assumptions, allowing us to solve the separation problem in polynomial time.
- We provide a class of facets based on Edge Cover (EC) constraints.
- We demonstrate the practical usefulness of our results by showing greatly improved solution speed and quality for a wide array of large benchmarks.

2 Mathematical Programming Formulation and LP-Based Solution Procedure

Let P be a polygon and $G, W \subseteq P$ sets of points for possible guard locations and *witnesses*, i. e., points to be guarded, respectively. We assume $W \subseteq \mathcal{V}(G)$. The AGP can be formulated as an IP denoted by AGP(G, W):

$$\min \quad \sum_{g \in G} x_g \tag{1}$$

$$\text{s. t.} \quad \sum_{g \in G \cap \mathcal{V}(w)} x_g \geq 1 \quad \forall w \in W \tag{2}$$

$$x_g \in \{0, 1\} \qquad \forall g \in G \tag{3}$$

where Chvátal's Watchman Theorem [15] guarantees that only a finite number of variables are non-zero. The original AGP, AGP(P, P), has uncountably many variables and constraints, so it cannot be solved directly. Thus we consider finite $G, W \subset P$ and solve AGP(G, W). For dual separation and to generate lower bounds, we require the LP relaxation AGR(G, W) obtained by relaxing the integrality constraint (3):

$$0 \leq x_g \leq 1 \quad \forall g \in G \tag{4}$$

The relation between a solution of AGR(G, W) and AGR(P, P) is not obvious, see Figure 2. In [10], we show that AGR(P, P) can be solved optimally for many problem instances by using finite G and W. The procedure uses primal/dual separation (i. e., cutting planes and column generation) to connect AGR(G, W) to AGR(P, P). For some finite sets G and W, we solve AGR(G, W) using the simplex method. This produces an optimal primal solution x^* and dual solution y^* with objective value z^*. The primal is a minimum covering by guards, the dual a maximum packing of witnesses. We analyze x^* and y^* as follows:

1. If there exists a point $w \in P \setminus W$ with $x^*(G \cap \mathcal{V}(w)) < 1$, then w corresponds to an inequality of AGR(P, P) that is violated by x^*. The new witness w is added to W, and the LP is re-solved. If such a w cannot be found, then x^* is optimal for AGR(G, P), and z^* is an upper bound for AGR(P, P).

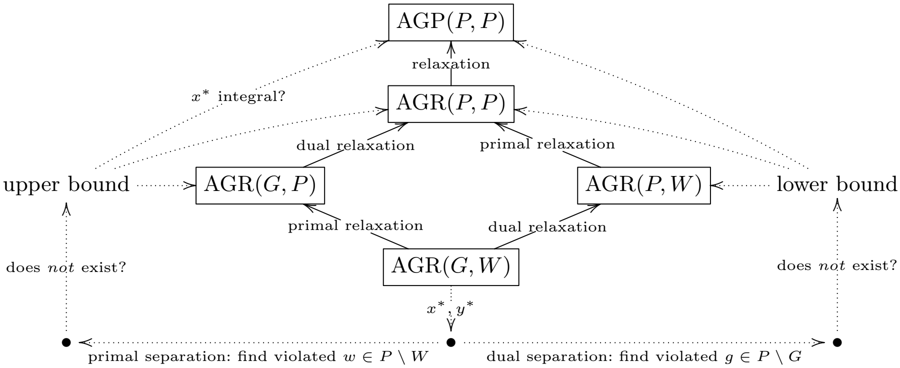


Fig. 2. The AGP and its relaxations for $G, W \subseteq P$. Dotted arrows represent which conclusions may be drawn from the primal and dual solutions x^* and y^*.

2. If there exists a point $g \in P \setminus G$ with $y^*(W \cap \mathcal{V}(g)) > 1$, then it corresponds to a violated dual inequality of AGR(P, P). We create the LP column for g and re-solve the LP. If such a g does not exist, y^* is an optimal dual solution for AGR(P, W) and z^* is a lower bound for AGR(P, P).

Both separation problems can be solved efficiently using the overlay of the visibility polygons of all points $g \in G$ with $x^*_g > 0$ (for the primal case) and all $w \in W$ with $y^*_w > 0$ (for the dual case), which decomposes P into a planar arrangement of bounded complexity.

Should the upper and the lower bound meet, we have an optimal solution of AGR(P, P) [10].

In this paper, we use cutting planes α that must remain feasible in all iterations of our algorithm, so feasibility for AGP(G, W) is insufficient; we require α not to cut off any $x \in \{0, 1\}^{G'}$ for an arbitrary $P \supseteq G' \supset G$, such that x is feasible for AGP(G', P). An LP with a set A of such additional constraints is denoted by AGR(G, W, A), its IP counterpart by AGP(G, W, A). Note that AGP(G, P) and AGP(G, P, A) are equivalent. By AGP(G, W), we sometimes denote the set of its feasible solutions rather than the IP itself, as in conv(AGP(G, W)).

3 Set Cover Facets

In this section, we discuss a family of SC facets, and show that the underlying geometry greatly reduces their impact on the involved AGP polytopes.

3.1 A Family of Facets

Let P be a polygon and $G, W \subset P$ finite sets of guard and witness positions. Consider a finite non-empty subset $\emptyset \subset S \subseteq W$ of witness positions; the overlay of visibility regions of S is called α_S. It contains the following partition $P =$

$J_0 \dot\cup J_1 \dot\cup J_2$, cf. Fig. 3; this is analogous to what Balas and Ng [14] did for the SC polytope.

1. $J_2 := \{g \in P \mid S \subseteq \mathcal{V}(g)\}$, the set of positions that cover all of S.
2. $J_0 := \{g \in P \mid \mathcal{V}(g) \cap S = \emptyset\}$, the set of positions that see none of S.
3. $J_1 := P \setminus (J_2 \cup J_0)$ the set of positions that cover a non-trivial subset of S.

Every feasible solution of the AGP has to cover S. Thus, it takes one guard in J_2, or at least two guards in J_1 to cover S. For any G, this induces the following constraint (5); for the sake of simplicity, we will also refer to this by α_S.

$$\sum_{g \in J_2 \cap G} 2x_g + \sum_{g \in J_1 \cap G} x_g \geq 2 \tag{5}$$

In the context of our iterative algorithm, α_S is represented by J_0, J_1 and J_2, independent of a specific set G; any guard $g \in J_i$ in current or future iterations simply gets the coefficient $\alpha_S(g) = i$.

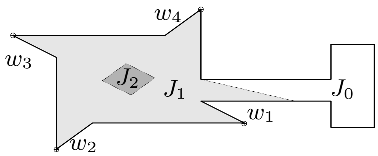


Fig. 3. Polygon and witness selection $S = \{w_1, w_2, w_3, w_4\}$. Guards located in J_2 can cover all of S, and those in J_1 some part of it, while those in J_0 cover none of S.

Sufficient coverage of S is necessary for sufficient coverage of P, so (5) is valid for any $x \in \{0,1\}^G$ that is feasible for AGP(G, P). However, covering S may require more than two guards in J_1, so (5) does not always provide a supporting hyperplane of conv(AGP(G, W)).

It is easy to see that for $|S| \leq 2$, (5) only yields constraints that are fulfilled by all feasible solutions of AGR(G, W). Thus, we consider $|S| \geq 3$.

In order to show when (5) defines a facet of conv(AGP(G, W)), we need to apply a result of [14] to the AGP setting.

Lemma 1. *Let P be a polygon and $G, W \subset P$ finite sets of guard and witness positions. Then* conv(AGP(G, W)) *is full-dimensional, if and only if*

$$\forall w \in W: \quad |\mathcal{V}(w) \cap G| \geq 2 \tag{6}$$

We require more terminology adapted from [14]. Two guards $g_1, g_2 \in J_1$ are a *2-cover* of α_S, if $S \subseteq \mathcal{V}(\{g_1, g_2\})$. The *2-cover graph* of G and α_S is the graph with nodes in $J_1 \cap G$ and an edge between g_1 and g_2 iff g_1, g_2 are a 2-cover of α_S. In addition, we have $T(g) = \{w \in \mathcal{V}(g) \cap W \mid \mathcal{V}(w) \cap G \cap (J_0 \setminus \{g\}) = \emptyset\}$.

Theorem 1. *Given a polygon P and finite $G, W \subset P$, let* conv(AGP(G, W)) *be full-dimensional and let α_S be as defined in* (5)*, such that S is maximal, i. e., there is no $w \in W \setminus S$ with $\mathcal{V}(w) \subseteq \mathcal{V}(S)$. Then the constraint induced by α_S defines a facet of* conv(AGP(G, W))*, if and only if:*

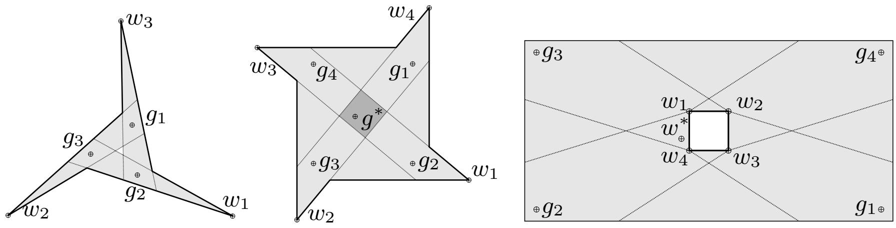


Fig. 4. P_3^2 (left) and two attempts for P_4^3 (middle and right). In the left case, Ineq. (5) enforces using two guards instead of three $\frac{1}{2}$-guards. The attempts for P_4^3 are star-shaped (middle) or invalid (right, at w^*, as $x_{g_1} = \cdots = x_{g_4} = \frac{1}{3}$).

1. *Every component of the 2-cover graph of α_S and G has an odd cycle.*
2. *For every $g \in J_0 \cap G$ such that $T(g) \neq \emptyset$ there exists either*
   (a) *some $g' \in J_2 \cap G$ such that $T(g) \subseteq \mathcal{V}(g')$;*
   (b) *some pair $g', g'' \in J_1 \cap G$ such that $T(g) \cup S \subseteq \mathcal{V}(g') \cup \mathcal{V}(g'')$.*

3.2 Geometric Properties of α_S

It is easy to construct SC instances for any choice of $|S| \geq 3$, such that the SC version of α_S cuts off a fractional solution, cf. [14]. In general, finding α_S is NP-complete. But in the following, we show that in an AGP setting, only α_S with $|S| = 3$ actually plays a role in cutting off fractional solutions under reasonable assumptions, allowing us to separate it in polynomial time.

Lemma 2. *Let P be a polygon, $G, W \subset P$ finite sets of guard and witness positions and $\emptyset \subset S \subseteq W$. If every guard in $J_1 \cap G$ belongs to some 2-cover of α_S and S is minimal for G, i. e., there is no proper subset $T \subset S$ such that α_T and α_S induce the same constraint for G, the matrix of* AGP(G, S) *contains a permutation of the full circulant of order $k = |S|$, which is*

$$C_k^{k-1} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \in \{0,1\}^{k \times k} \quad (7)$$

Lemma 2 holds, because the 2-cover property holds iff no guard's coefficient in α_S can be reduced without turning Inequality (5) invalid [14]. As S is minimal, removing w from S must increase coefficients, i. e., relocate a guard $g \in J_1 \cap G$ to J_2. So $\mathcal{V}(g) \cap S = S \setminus \{w\}$. Such a guard exists for every $w \in S$.

This motivates a formal definition of a polygon corresponding to C_k^{k-1}.

Definition 1 (Full Circulant Polygon). *A polygon P along with $G(P) = \{g_1, \ldots, g_k\} \subset P$ and $W(P) = \{w_1, \ldots, w_k\} \subset P$ for $3 \leq k \in \mathbb{N}$ is called* Full Circulant Polygon *or P_k^{k-1}, if*

$$\forall 1 \leq i \leq k: \quad \mathcal{V}(g_i) \cap W(P) = W(P) \setminus \{w_i\} \tag{8}$$

$$\forall w \in P: \quad |\mathcal{V}(w) \cap G(P)| \geq k-1 \tag{9}$$

We may refer to $G(P)$ and $W(P)$ by just G and W respectively.

Note that P_k^{k-1} is defined such that the full circulant C_k^{k-1} completely describes the visibility relations between G and W. This implies that the optimal solution of AGR(G, W) is $\frac{1}{k-1} \cdot \mathbb{1}$, with cost $\frac{k}{k-1}$. It is feasible for AGR(G, P_k^{k-1}) by Property (9), as any point $w \in P_k^{k-1}$ is covered by at least $(k-1) \cdot \frac{1}{k-1} = 1$.

Figure 4 captures construction attempts for models of C_k^{k-1}. P_3^2 exists, but for $k \geq 4$, the polygons are either star-shaped or not full circulant. If they are star-shaped, the optimal solution is to place one guard within the kernel. If they are not full circulant polygons, the optimal solution of AGR(G, W) is infeasible for AGR(G, P) and the current fractional solution is intermittent, i.e., cut off in the next iteration. Both cases eliminate the need for a cutting plane, and we may avoid the NP-complete separation problem by restricting separation to $k = 3$.

In the following we prove that P_k^{k-1} is star-shaped for $k \geq 4$. We start with Lemma 3, which shows that any pair of guards in G is sufficient to cover P_k^{k-1}.

Lemma 3. *Let P_k^{k-1} be a full circulant polygon. Then P_k^{k-1} is the union of the visibility polygons of any pair of guards in $G\left(P_k^{k-1}\right) = \{g_1, \ldots, g_k\}$:*

$$\forall 1 \leq i < j \leq k: \quad P_k^{k-1} = \mathcal{V}(g_i) \cup \mathcal{V}(g_j) \tag{10}$$

The next step is Lemma 4, which restricts the possible structure of P_k^{k-1}.

Lemma 4. *Let P_k^{k-1} be a full circulant polygon with $G\left(P_k^{k-1}\right) = \{g_1, \ldots, g_k\}$. Suppose $k \geq 4$. Then P_k^{k-1} has no holes.*

$k \geq 4$ is tight: a triangle with a concentric triangular hole is an example of P_3^2, with guards in the outside corners, and witnesses on the inside edges.

We require one final lemma before proceeding to the main Theorem 2.

Lemma 5. *Consider two disjoint non-empty convex polygons, described as the intersection of half-spaces: $P_1 = \bigcap_{i=1,\ldots,n} \mathcal{H}_i$ and $P_2 = \bigcap_{i=n+1,\ldots,n+m} \mathcal{H}_i$. Then some $\mathcal{H}_i$, $1 \leq i \leq n+m$ separates P_1 and P_2.*

Theorem 2. *A full circulant polygon P_k^{k-1} with $k \geq 4$ is star-shaped.*

Note that Theorem 2 does not rule out situations in which P_k^{k-1} is part of a larger polygon, as shown in Figure 5. This example has no integrality gap; placing at least five copies of P_4^3 around an appropriate central subpolygon with a hole can actually create one. However, such cases are much harder to come by, making these facets a lot less useful for cutting off fractional solutions; we demonstrate this in our experimental section.

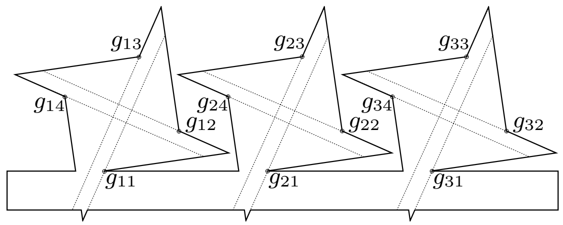


Fig. 5. Three instances of P_4^3 embedded into a larger polygon. Setting all guards to $\frac{1}{3}$ is feasible and optimal, even though no guard is placed in any of the P_4^3 kernels.

3.3 All Art Gallery Facets with Coefficients 0, 1, 2

Balas and Ng [14] identified all SC facets with coefficients in $\{0, 1, 2\}$; for finite $G, W \subset P$, AGP(G, W) is also an SC instance. Thus, all AGP facets with these coefficients must be among those facets. This includes three trivial facet classes, the constraints and the conditions under which they are facet-defining are easily translated into AGP terms; however, they are all satisfied by any feasible solution of AGR(G, W), so they do not play a role in cutting off fractional solutions. The only non-trivial AGP facet class with coefficients in $\{0, 1, 2\}$ is the one of type α_S, as discussed above.

4 Edge Cover Facets

Solving AGR(G, W) for finite $G, W \subset P$ such that no guard can see more than two witnesses is equivalent to solving fractional EC on the graph with nodes W, an edge between $v \neq w \in W$ for each $g \in G$ with $\mathcal{V}(g) \cap W = \{v, w\}$, and a loop for each $g \in G$ with $\mathcal{V}(g) \cap W = \{w\}$. The fractional EC polytope is known to be half-integral [16], which can be exploited to show that fractional solutions always form odd-length cycles of $\frac{1}{2}$-guards.

In the conclusions of [10], we proposed a class of valid inequalities motivated by this.

A fractional optimal solution has all guard values on the cycle at $\frac{1}{2}$. For an odd k,

$$\sum_{g \in \mathcal{V}(W)} x_g \geq \left\lceil \frac{k}{2} \right\rceil = \frac{k+1}{2} \tag{11}$$

separates these fractional solutions from feasible, integral solutions.

Obviously, for any choice of $G \subset P$, (11) does not cut off any feasible solution $x \in \{0, 1\}^G$ of AGP(G, P), as long as no point exists that sees more than two of these witnesses. So, analogously to the SC cuts, a cut can be kept in future iterations once it has been identified.

It is not hard to show that these are facet defining under relatively mild conditions.

Theorem 3. *Let P be a polygon with finite sets of guard and witness positions $G, W \subset P$, such that* $\mathrm{conv}(\mathrm{AGP}(G, W))$ *is full-dimensional. Let $\overline{W} = \{w_1, \ldots, w_k\} \subseteq W$ be an odd subset of $k \geq 3$ witnesses, such that*

1. *No guard sees more than two witnesses in $\overline{W}$:*
$$\forall g \in G: \quad \left|\mathcal{V}(g) \cap \overline{W}\right| \leq 2 \tag{12}$$
2. *If a guard sees two witnesses $w_i \neq w_j \in \overline{W}$, they are a* successive pair*, i. e., $i + 1 = j$ or $i = 1$ and $j = k$.*
3. *Each of the k successive pairs is seen by some $g \in G$.*
4. *No guard inside of $\mathcal{V}\left(\overline{W}\right)$ sees a witness outside of $\overline{W}$:*
$$\forall g \in G \cap \mathcal{V}\left(\overline{W}\right): \quad \mathcal{V}(g) \cap W \subseteq \overline{W} \tag{13}$$

Then the constraint
$$\sum_{g \in \mathcal{V}(\overline{W}) \cap G} x_g \geq \left\lceil \frac{|\overline{W}|}{2} \right\rceil \tag{14}$$
is a facet of $\mathrm{conv}(\mathrm{AGP}(G, W))$.

5 Computational Experience

A variety of experiments on benchmark polygons demonstrates the usefulness of our cutting planes. The test algorithm is a variation of the one introduced in [10]. In each test, the sets G and W are initialized with the vertices of P, while $A = \emptyset$. In the primal phase, we solve $\mathrm{AGP}(G, W, A)$. Should the solution be feasible for $\mathrm{AGP}(G, P, A)$, we have identified an upper bound. Otherwise, there are witnesses $W' \subseteq P \setminus W$ whose constraint is violated. In this case, the primal phase is continued and $\mathrm{AGP}(G, W \cup W', A)$ is solved. After an upper bound is found, the dual phase is entered. A lower bound is generated by iteratively solving the dual of $\mathrm{AGR}(G, W, A)$. If the solution is feasible for the dual of $\mathrm{AGR}(P, W, A)$ and the cut separators do not find a violated constraint either, we have a lower bound. Otherwise, guards with violated dual constraints are added to G, violated cut conditions are added to A, and the dual phase continues. This process is repeated until the upper and the lower bound meet, or a timeout occurs.

Just as in [10], we employed four different classes of benchmark polygons.

1. Random *von Koch* polygons are inspired by Koch curves, see Fig. 6, left.
2. Random floorplan-like *Orthogonal* polygons as in Fig. 6, second polygon.
3. Random *Spike* polygons (mostly with holes) as in Fig. 6, third polygon.
4. Random non-orthogonal *Simple* polygons as in Fig. 6, fourth polygon.

Each polygon class was evaluated for different sizes $n \in \{60, 200, 500, 1000\}$, where n is the approximate number of vertices in a polygon.

Different combinations of cut separators were also employed. The EC-related cuts from Section 4 are referred to as *EC cuts*, while the SC-related cuts of

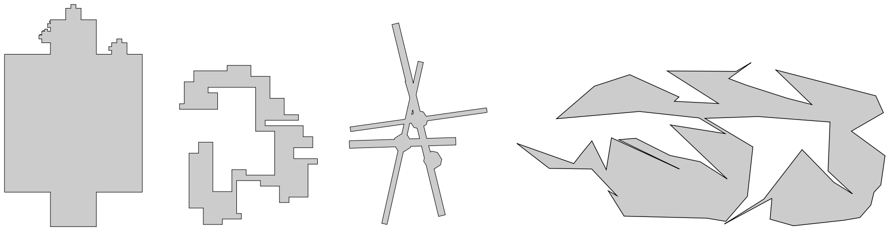


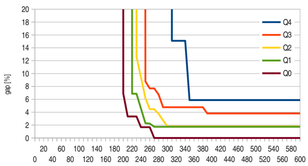
Fig. 6. Small *von Koch*, *Orthogonal*, *Spike* and *Simple* test polygons

Section 3 that rely on separating a maximum of $3 \leq k$ witnesses are denoted by *SCk* cuts. Note that for $k \leq m$, SCm cuts also include all SCk cuts.

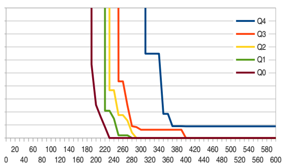
Whenever the above algorithm separates cuts, it applies all configured cut separators and we test the following combinations: no cut separation at all, SC3 cuts only, SC4 cuts only, EC cuts only, and SC3 and EC cuts at the same time.

In total, we have five combinations of separators, four classes of polygons and four polygon sizes; for each combination, we tested 10 different polygons. The experiments were run on 3.0 GHz Intel dual core PCs with 2 GB of memory, running 32 bit Debian 6.0.5 with Linux 2.6.32-686. Our algorithms were not parallelized, used version 4.0 of the "Computational Geometry Algorithms Library" (CGAL) and CPLEX 12.1. Each test run had a time limit of 600 s.

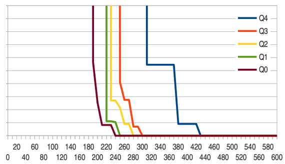
Below we present the relative gap over time for the five tested cut separator selections for the *von Koch*-type polygons with 1000 vertices. Fig. 7 shows the distribution of relative gaps over time for the different combination of cutting planes. Q_0, ..., Q_4 indicate the different quartiles; in particular, Q_0 is the best case, Q_2 the median, and Q_4 the worst case. The graphs for the other test polygon types have been omitted due to space restrictions. Their analysis allows the same interpretation as ours of Fig. 7.



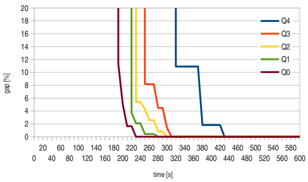
(a) No Cuts



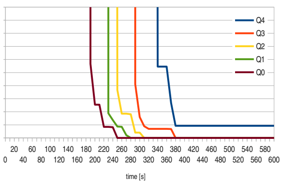
(b) EC



(c) SC3



(d) SC3 and EC



(e) SC4

Fig. 7. Relative gap over time in IP mode for 1000-vertex *von Koch*-type polygons

Fig. 7(a) shows the relative gap over time without cut separation. After about 400 s, gaps are fixed between 0% and 6%, the median gap being 2%. When applying the EC separator (Fig. 7(b)), 75% of the gaps drop to zero and the largest gap is 2%. Using the SC3 separator (Fig. 7(c)) yields an even better result in terms of both speed and relative gap. All gaps are closed, many of them earlier than with the EC separator. Combining both, see Fig. 7(d), yields a result comparable to using only SC3. Moving to the SC4 separator (Fig. 7(e)) yields a weaker performance: computation times go up, and not all gaps reach 0% within the allotted time, because separation takes longer without improving the gap. This illustrates the practical consequences of Theorem 2.

6 Conclusion

In this paper, we have shown how we can exploit both geometric properties and polyhedral methods of mathematical programming to solve a classical and natural, but highly challenging problem from computational geometry. This promises to pave the way for a range of practical AGP applications that have to deal with additional real-life aspects. We are optimistic that our basic approach can also be used for other geometric optimization problems related to packing and covering.

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