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What is the optimal shape of a city?

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Abstract

If one defines the distance between two points as the Manhattan distance (the sum of the horizontal distance along streets and the vertical distance along avenues) then one can define a city as being optimal if the average distance between pairs of points is a minimum. In this paper a nonlinear differential equation for the boundary curve of such a city is determined. The problem solved here is the continuous version of an optimization problem on how to design efficient allocation algorithms for massively parallel supercomputers. In the language of continuum mechanics, the shape of the optimal city is that taken by a blob of incompressible fluid composed of molecules whose pairwise interactions are described by an attractive potential proportional to the Manhattan distance between the particles.

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1. Introduction

An outstanding and practical problem in the field of computer science concerns the allocation of processors to parallel programs in a supercomputer grid consisting of a large number of processors [1–7]. This is a nontrivial problem because ordinarily some processors are already allocated to other running programs. Thus, it is usually necessary to allocate from an irregular set of free processors. The sum of the pairwise distances between the processors allocated to a task strongly correlates with the running time required to complete the task [3, 5]. Therefore, a good strategy for processor allocation minimizes this quantity. An algorithm for approximately minimizing the pairwise distances has already been implemented on Cplant, a class of Sandia supercomputers [7]. The most closely related work approximately minimizing pairwise distances for general metric spaces and pairwise Manhattan distances may be found in [8] and [9], respectivly. We emphasize that the pairwise distance between processors is the number of vertical communication hops plus the number of horizontal communication hops. Such a distance is commonly called the *Manhattan distance*.

In this paper we solve the continuum version of this allocation problem. We treat the grid of processors as having a continuous two-dimensional planar distribution. For simplicity we call such a distribution a *city*. Our objective is to find the shape of a city that minimizes the average pairwise Manhattan distance between all points in that city. The techniques that we use to solve this problem are those that are commonly used in the solution of continuum mechanics and classical and quantum field theory problems, namely variational methods [10].

The problem being solved here has a simple physical interpretation in continuum mechanics in terms of self-gravitating fluids. Imagine a two-dimensional fluid composed of particles whose pairwise interactions are described by an attractive potential that is proportional to the Manhattan distance between fluid particles. If this fluid is incompressible, then a given blob of this fluid will pull itself into a configuration of the same area, whose shape minimizes the total potential energy. This lowest energy shape is the optimal city shape.

In section 2 of this paper we formulate this optimization problem in mathematical terms. Then, in section 3 we consider a one-parameter family of cities and find the shape of the optimal city in this limited family. From this calculation we can see that the optimal city is not circular. Next, in section 4 we carry out a full variational calculation and determine an interesting nonlinear differential equation satisfied by the boundary of the optimal city. The optimal city is nearly circular in shape but is not a disc.

Finally, in section 5 we offer some concluding observations and suggest some further avenues for investigation. Our principal conclusions are that so long as an array of computer processors is clustered in a tight shape having no holes or only small holes, the array will be extremely close to optimal.

2. Formulation of the problem

Let $w(x)$ be the upper boundary of the region occupied by the city. Without loss of generality we may assume that $w(x)$ is positive. Because the Manhattan distance is north–south symmetric we may assume that the lower bound of the region is $-w(x)$. Also, because the Manhattan distance is east–west symmetric we may assume that $w(x) = w(-x)$. The boundary of the city must be continuous, so we assume that $w(x)$ crosses the x-axis at the point $x = a$: $w(a) = w(-a) = 0$.

Furthermore, because the north–south and east–west directions are of equal weight we expect the shape of the city to be symmetric about the 45° lines $y = x$ and $y = -x$. This means that we may decompose the function $w(x)$ into two functions, $w(x) = g(x)$ below the line $y = x$ and $w(x) = h(x)$ above the line $y = x$:

$$w(x) = \begin{cases} h(x) & (0 \leqslant x \leqslant b) \\ g(x) & (b \leqslant x \leqslant a) \end{cases} \tag{1}$$

where b marks the point on the x-axis where the boundary curve $w(x)$ crosses the line $y = x$. The symmetry about the line $y = x$ implies that $h(x)$ is the *functional* inverse of $g(x) : h(x) = g^{-1}(x)$.

Finally, we define the length scale of this problem by choosing, without loss of generality, to work in units such that $b = 1$. We summarize the properties of the functions $w(x)$, $h(x)$ and $g(x)$ as follows:

$$w(0) = h(0) = a$$
$$w(1) = h(1) = g(1) = 1$$

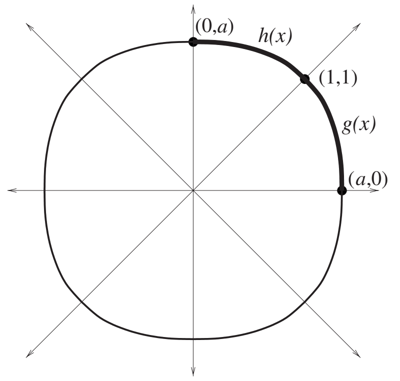

Figure 1. Definition of the notation used in this paper. The curve that outlines the city is symmetric with respect to reflections about the x axis $y = 0$, the y axis $x = 0$ and the 45° lines $y = \pm x$. In the upper-half plane this outline curve is called $w(x)$. Note that $w(x)$ crosses the y and x axes at the points $(0, a)$ and $(\pm a, 0)$. The slope of $w(x)$ is 0 at $x = 0$ and infinite at $x = a$. Also, $w(x)$ crosses the line $y = x$ at the point $(1, 1)$, and at this point the slope is -1. The portion of the function $w(x)$ in the range $0 \leqslant x \leqslant 1$ is called $h(x)$ and the portion of $w(x)$ in the range $1 \leqslant x \leqslant a$ is called $g(x)$. The functions $h(x)$ and $g(x)$ are inverses of one another because $w(x)$ is symmetric about the line $y = x$.

$$h(x) = g^{-1}(x) \qquad (0 \leqslant x \leqslant 1)$$
$$w(x) \geqslant 0 \qquad (-a \leqslant x \leqslant a). \tag{2}$$

The functions $g(x)$ and $h(x)$ are illustrated in figure 1.

Let the functional $A[w]$ represent the area of the city whose boundary curve is shown in figure 1. Note that $A[w]$ is four times the area in the positive quadrant:

$$A[w] = 4 \int_{x=0}^{a} \mathrm{d}x\, w(x). \tag{3}$$

Let the functional $M[w]$ represent the integrated sum of the distances between all pairs of points in the city. Then,

$$M[w] = 2 \int_{x=-a}^{a} \mathrm{d}x \int_{y=-w(x)}^{w(x)} \mathrm{d}y \int_{u=-a}^{a} \mathrm{d}u \int_{v=-w(u)}^{w(u)} \mathrm{d}v |x - u|. \tag{4}$$

The factor of two in this equation arises because we have summed only over east–west distances. North–south distances make an equal contribution.

Our objective is to describe the shape of a region of given area for which the average pairwise distance between points in the region is minimized. That is, our objective is to find the function $w(x)$ that minimizes the functional $M[w]$ subject to the constraint that the area $A[w]$ is held fixed. Note that $M[w]$ has units of $[\text{length}]^5$ and that $A[w]$ has units of $[\text{length}]^2$. Thus, a *dimensionless* measure of the average pairwise distance is given by the ratio $D[w]$:[5]

$$D[w] = \frac{M[w]}{(A[w])^{5/2}}. \tag{5}$$

To prepare for the calculations to be done in the next two sections we simplify the functional $M[w]$. First, we perform the y and v integrals in (4), and then we decompose

[5] Note that the optimization scheme in this paper minimizes the value of $D[w]$ for a city of any given fixed area. Because $D[w]$ is dimensionless, the size of area A is irrelevant; the shape of the city that minimizes $D[w]$ simply scales with A. Thus, the shape of the city that minimizes $D[w]$ also minimizes the average pairwise Manhattan distance for any area A.

the integration region to eliminate the absolute value signs. Finally, we split apart the terms, change variable names, and merge integrands so that all limits are the same. Our final result is

$$M[w] = 64 \int_{x=0}^{a} dx\, x w(x) \int_{u=0}^{x} du\, w(u). \tag{6}$$

3. Special cases and optimization of a one-parameter family of boundary curves

Let us begin by illustrating the calculation of the dimensionless functional $D[w]$ in (5) in terms of $A[w]$ in (3) and $M[w]$ in (6) for some elementary shapes.

3.1. Square

Consider a square city defined by the function $w(x) = 1 (-1 \leqslant x \leqslant 1)$. For this geometry $a = 1, h(x) = 1$ and $g(x)$ is a vertical line connecting the points (1, 1) to (1, 0). The area of this square is $A = 4$. We calculate M in (6) as follows:

$$M = 64 \int_{x=0}^{1} dx\, x w(x) \int_{u=0}^{x} du\, w(u) = 64 \int_{x=0}^{1} dx\, x \int_{u=0}^{x} du = \frac{64}{3}. \tag{7}$$

Thus,

$$D = \frac{M}{A^{5/2}} = \frac{2}{3} \approx 0.666\,667. \tag{8}$$

3.2. Diamond

Next, consider a diamond-shaped city for which $w(x) = 2 - |x| (-2 \leqslant x \leqslant 2)$. For this case $a = 2$, the area of the city is $A = 8$, and

$$M = 64 \int_{x=0}^{2} dx\, x w(x) \int_{u=0}^{x} du\, w(u) = 64 \int_{x=0}^{2} dx\, x(2-x) \int_{u=0}^{x} du\, (2-u) = \frac{1792}{15}. \tag{9}$$

Thus, for a diamond-shaped city

$$D = \frac{M}{A^{5/2}} = \frac{7}{15}\sqrt{2} \approx 0.659\,966. \tag{10}$$

3.3. Disc

Now let us consider a circular city, whose boundary is given by $w(x) = \sqrt{2 - x^2}$ with $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$. Here, $a = \sqrt{2}$, the area of the city is $A = 2\pi$, and

$$M = 64 \int_{x=0}^{\sqrt{2}} dx\, x w(x) \int_{u=0}^{x} du\, w(u) = 64 \int_{x=0}^{\sqrt{2}} dx\, x(2-x^2) \int_{u=0}^{x} du\, (2-u^2) = \frac{2048}{45}\sqrt{2}. \tag{11}$$

Thus, for a circular city

$$D = \frac{M}{A^{5/2}} = \frac{512}{45\pi^{5/2}} \approx 0.650\,403. \tag{12}$$

3.4. The family of curves $w(x) = (2 - |x|^d)^{1/d}$

Evidently, a disc is a better shape for a city than a square or a diamond. Apparently, the optimal city shape is rounded even though the Manhattan distance is composed of vertical and horizontal measurements. On the basis of these calculations one might wonder if a disc is indeed the optimal shape for a city. In fact, a disc is not optimal, as we now show. Consider a one-parameter family of boundary functions of the form $w(x) = (2 - |x|^d)^{1/d}(-2^{1/d} \leqslant x \leqslant 2^{1/d})$. For this family of functions we have $a = 2^{1/d}$. Note that this family includes a square ($d = \infty$), a diamond ($d = 1$) and a disc ($d = 2$) as special cases.

It is noteworthy that the function $D(d)$ is quite insensitive to the value of d: the values of D for three such different city shapes, a square, a diamond and a disc, differ by only 3%. We now show that a disc is not the optimal shape for a city by studying how the value of D depends on d. The area of the city, as a function of d, is given by

$$A(d) = 4\int_{x=0}^{a} \mathrm{d}x\, (a^d - |x|^d)^{1/d} = \frac{2a^2}{d}\frac{\Gamma^2(1/d)}{\Gamma(2/d)} \tag{13}$$

where the gamma function is defined by $\Gamma(\alpha) \equiv \int_{x=0}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x} x^{\alpha-1}$ and we have used the standard integration formula [11]

$$\int_{x=0}^{1} \mathrm{d}x\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{14}$$

The expression for $M(d)$ cannot be evaluated in closed form and may be left as a double integral:

$$M(d) = 64a^5 \int_{x=0}^{1} \mathrm{d}x\, x(1-x^d)^{1/d} \int_{u=0}^{x} \mathrm{d}u\, (1-u^d)^{1/d}. \tag{15}$$

For numerical purposes we convert this expression for $M(d)$ to an infinite sum. We do so by expanding the expression $(1-u^d)^{1/d}$ into a binomial series and using (14) to perform the double integration:

$$M(d) = 16\frac{\Gamma(1+1/d)}{d\Gamma(-1/d)} \sum_{n=0}^{\infty} \frac{\Gamma(n-1/d)\Gamma(n+3/d)}{n!(1+nd)\Gamma(n+1+4/d)}. \tag{16}$$

This series converges like $n^{-3-2/d}$ for large n and is therefore useful for obtaining numerical results.

We have evaluated $D(d) = M(d)A^{-5/2}(d)$ numerically for many values of d in the range $0 \leqslant d < \infty$. (See table 1 and figure 2.) The function $D(d)$ is infinite at $d = 0$, falls to a minimum just below $d = 2$ and then rises and levels off at $d = \infty$.

We expect the shape of an optimal city to be convex. The function $w(x) = (a^d - x^d)^{1/d}$ is convex when $d \geqslant 1$ and concave when $d < 1$. Thus, we expect the optimal value of d for this class of functions to be in the range $d > 1$. Indeed, by interpolating the values in table 1 we find that $D(d)$ attains its minimum of 0.650 247 634 near $d = 1.815\,46$.

Nevertheless, it is interesting to study the function $D(d)$ for values of d less than 1 to see what happens when the city is not convex. For example, when $d = \frac{2}{3}$, the boundary curve $w(x)$ becomes an *astroid*[6]. It is remarkable that even for this concave city shape the value of $D(2/3) \approx 0.686\,397$ does not differ much from the value of $D(2) \approx 0.650\,403$ for a circular city.

[6] An astroid is the path of the centre of mass of a ladder as it slides down a wall with one end of the ladder in contact with the wall and the other in contact with the floor.

Table 1. Numerical values for $D(d) = M(d)A^{-5/2}(d)$ for the one-parameter family of boundary curves $w(x) = (a^d - x^d)^{1/d}$ with $a^d = 2$. These curves are concave for $d < 1$ and convex for $d \geqslant 1$. For this family of boundary curves the minimum value of $D(d)$ is 0.650 247 634, and this value is attained when $d \approx 1.815\,46$. It is remarkable that this one-parameter family of curves gives an extremely close approximation that differs by only 2.6×10^{-4}% from the true minimum of the functional $D[w]$. The true minimum, as obtained in section 4, is 0.650 245 952 951.

Boundary shape	d	$D(d)$
Concave boundaries		
Plus sign	0	∞
	1/64	16 134.274
	1/32	82.153 831
	1/16	6.122 774
	1/8	1.719 920
	1/4	0.940 692
	1/2	0.723 183
Astroid	2/3	0.686 397
Convex boundaries		
Diamond	1	0.659 966
	1.81	0.650 247 797
	1.812	0.650 247 700
	1.814	0.650 247 646
Optimal	1.815 46	0.650 247 634
	1.816	0.650 247 636
	1.818	0.650 247 670
	1.82	0.650 247 746
Disc	2	0.650 403
Square	∞	0.666 667

The growth of the function $D(d)$ for small d can be obtained by using Laplace's method to evaluate the double integral for $M(d)$ in (15) in the limit of small d. The asymptotic behaviour of $M(d)$ is given by

$$M(d) \sim \frac{64\pi}{d\sqrt{3}} 27^{-1/d} \qquad (d \to 0+).$$

Using the Stirling formula to approximate the gamma functions in $A(d)$ in (13) we obtain

$$D(d) \sim \frac{2d^{1/4}}{\pi^{1/4}\sqrt{3}} \left(\frac{32}{27}\right)^{1/d} \qquad (d \to 0+). \tag{17}$$

This asymptotic approximation is extremely accurate. At $d = \frac{1}{32}$ the numerical value of $D(d)$ is 82.153 831, while this asymptotic formula predicts the value 83.763 and at $d = \frac{1}{64}$ the numerical value of $D(d)$ is 16 134.274 096, while this asymptotic formula predicts the value 16 179. The relative error in the latter prediction is about 0.3%.

4. Variational determination of the boundary curve

We now use variational methods to determine a differential equation satisfied by the boundary curve. We begin by simplifying the functionals $A[w]$ and $M[w]$.

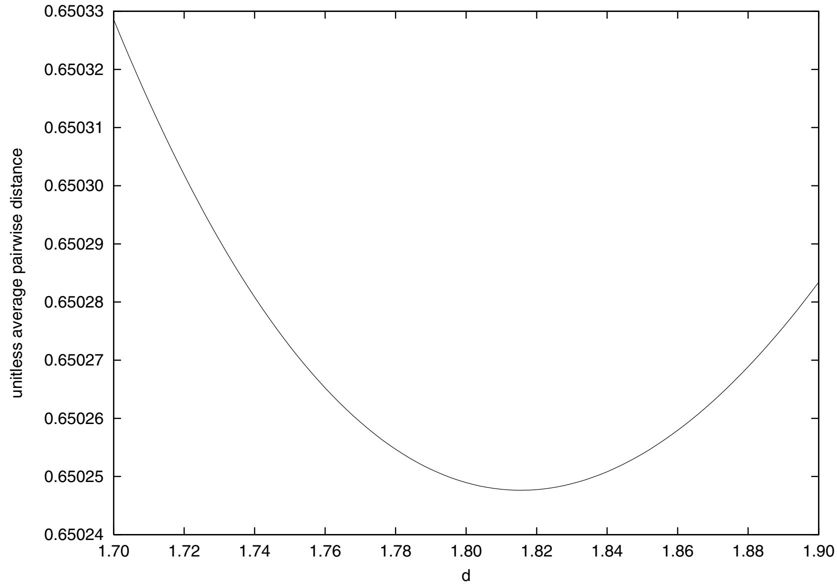

Figure 2. Plot of $D(d) = M(d)A^{-5/2}(d)$ for the class of boundary curves $w(x) = (a^d - x^d)^{1/d}$ with $a^d = 2$ in the range $1.7 \leqslant d \leqslant 1.9$. The optimal city, that is, the city for which $D(d)$ is minimized, is nearly circular, but is definitely not a disc $(d = 2)$. The minimum value of $D(d)$ for this one-parameter family of boundary curves is obtained when $d \approx 1.815\,46$.

4.1. Exploiting the symmetry across $y = x$

To prepare for our variational calculation we decompose $w(x)$ into its separate components $g(x)$ and $h(x)$. Then, using symmetry arguments, we eliminate all dependence on $g(x)$. Our optimization procedure will then determine the value of a. To begin we note that the area $A[w]$ in (3) is eight times the area in the positive octant above the line $y = x$:

$$A[h] = 8\int_{x=0}^{1} \mathrm{d}x[h(x) - x] = 8\int_{x=0}^{1} \mathrm{d}x\, h(x) - 4. \tag{18}$$

Next we turn to the quantity $M[w]$ in (6) and decompose $w(x)$ into $h(x)$ and $g(x)$:

$$M[g,h] = 64\int_{x=0}^{1} \mathrm{d}x\, xh(x)\int_{u=0}^{x} \mathrm{d}u\, h(u) + 64\int_{x=1}^{a} \mathrm{d}x\, xg(x)\left[\int_{u=0}^{1} \mathrm{d}u\, h(u) + \int_{u=1}^{x} \mathrm{d}u\, g(u)\right]. \tag{19}$$

Note that from (18) we can evaluate one of the integrals in terms of $A[h]$:

$$\int_{u=0}^{1} \mathrm{d}u\, h(u) = \frac{A}{8} + \frac{1}{2}. \tag{20}$$

Thus, $M[g,h]$ is

$$M[g,h] = 64\int_{x=0}^{1} \mathrm{d}x\, xh(x)\int_{u=0}^{x} \mathrm{d}u\, h(u) + (8A[h] + 32)\int_{x=1}^{a} \mathrm{d}x\, xg(x)$$
$$+ 64\int_{x=1}^{a} \mathrm{d}x\, xg(x)\int_{u=1}^{x} \mathrm{d}u\, g(u). \tag{21}$$

To replace $g(x)$ by $h(x)$, we exploit the inverse relationship between the two functions: $h(x) = g^{-1}(x)$. Thus, if $g(x) = \gamma$ and $x : 1 \to a$, we have $\gamma : 1 \to 0$. Hence, we can write

$$x = g^{-1}(\gamma) = h(\gamma) \tag{22}$$

and

$$dx = h'(\gamma)\,d\gamma = \frac{1}{g'(\gamma)}\,d\gamma. \tag{23}$$

Similarly, we let $g(u) = \alpha$, for $\alpha : 1 \to 0$ (and $u : 1 \to a$), implying that $u = g^{-1}(\alpha) = h(\alpha)$. Therefore, $du = h'(\alpha)\,d\alpha = \frac{1}{g'(\alpha)}\,d\alpha$. We then change the integration variables to obtain

$$M[h] = 64\int_{x=0}^{1} dx\, xh(x)\int_{u=0}^{x} du\, h(u) - (4A[h]+16)\int_{x=0}^{1} dx\, x[h^2(x)]' \\ +32\int_{x=0}^{1} dx\, x[h^2(x)]'\int_{u=x}^{1} du\, uh'(u). \tag{24}$$

4.2. Preparing M[h] for functional differentiation

It will be necessary to calculate the functional derivative of $M[h]$. To simplify this procedure we use integration by parts to remove all instances of the derivative function $h'(x)$. Integrating by parts four times and simplifying gives the following expression:

$$M[h] = 64\int_{x=0}^{1} dx\, xh(x)\int_{u=0}^{x} du\, h(u) - 4A[h] - \frac{16}{3} + (4A[h]-16)\int_{x=0}^{1} dx\, h^2(x) \\ +\frac{32}{3}\int_{x=0}^{1} dx\, xh^3(x) + 32\int_{x=0}^{1} dx\, h^2(x)\int_{u=x}^{1} du\, h(u). \tag{25}$$

4.3. Performing the functional differentiation

We must minimize $\frac{M[h]}{(A[h])^{5/2}}$, subject to the constraint that $h(1) = 1$. We impose this constraint by introducing a Lagrange multiplier λ:

$$D[h] = \frac{M[h]}{(A[h])^{5/2}} + [h(1)-1]\lambda.$$

Note that if we differentiate with respect to λ, we recover the original constraint,

$$\frac{\partial D[h]}{\partial\lambda} = 0 \quad\Rightarrow\quad h(1) = 1.$$

We will also need the functional derivative of $A[h]$:

$$\frac{\delta A[h]}{\delta h(z)} = \frac{\delta}{\delta h(z)}\left[8\int_{x=0}^{1} dx\,[h(x)-x]\right] = 8. \tag{26}$$

4.3.1. Calculating $\delta M[h]/\delta h(z)$. We now calculate the functional derivative of $M[h]$ in (25). To prepare for functional differentiation we introduce the Heaviside step function to remove all variables from the limits of integration. The Heaviside step function $H(x)$ is defined as follows:

$$H(x) \equiv \begin{cases} 1 & (x \geqslant 0) \\ 0 & (x < 0). \end{cases} \tag{27}$$

We perform the following differentiation:

$$\frac{\delta M[h]}{\delta h(z)} = \frac{\delta}{\delta h(z)}\Bigg[64\int_{x=0}^{1} \mathrm{d}x\, xh(x)\int_{u=0}^{1}\mathrm{d}u\, h(u)H(x-u) + (4A[h]-16)\int_{x=0}^{1}\mathrm{d}x\, h^2(x)$$
$$-4A[h] - \frac{16}{3} + \frac{32}{3}\int_{x=0}^{1}\mathrm{d}x\, xh^3(x) + 32\int_{x=0}^{1}\mathrm{d}x\, h^2(x)\int_{u=0}^{1}\mathrm{d}u\, h(u)H(u-x)\Bigg]$$
$$= 64z\int_{u=0}^{z}\mathrm{d}u\, h(u) + 64\int_{x=0}^{1}\mathrm{d}x\, xh(x)\int_{u=0}^{1}\mathrm{d}u\, \delta(z-u)H(x-u) - 32$$
$$+32\int_{x=0}^{1}\mathrm{d}x\, h^2(x) + (8A[h]-32)h(z) + 32zh^2(z)$$
$$+64h(z)\int_{u=0}^{1}\mathrm{d}u\, h(u)H(u-z) + 32\int_{x=0}^{1}\mathrm{d}x\, h^2(x)H(z-x). \tag{28}$$

Finally, we remove the Heaviside function and restore the limits on the integrals:

$$\frac{\delta M[h]}{\delta h(z)} = 64z\int_{u=0}^{z}\mathrm{d}u\, h(u) + 64\int_{x=z}^{1}\mathrm{d}x\, xh(x) - 32 + 32\int_{x=0}^{1}\mathrm{d}x\, h^2(x)$$
$$+ (8A[h]-32)h(z) + 32zh^2(z) + 64h(z)\int_{u=z}^{1}\mathrm{d}u\, h(u) + 32\int_{x=0}^{z}\mathrm{d}x\, h^2(x). \tag{29}$$

4.3.2. Calculating $\delta D[h]/\delta h(z)$. Now that we have obtained the functional derivative of $M[h]$ in (29), we can calculate the functional derivative of $D[h]$:

$$\frac{\delta D[h]}{\delta h(z)} = \frac{1}{(A[h])^{5/2}}\frac{\delta M[h]}{\delta h(z)} - \frac{5}{2(A[h])^{7/2}}8M[h] + \lambda\delta(z-1). \tag{30}$$

Since the only term containing a delta function is the last term, when we set $\frac{\delta D[h]}{\delta h(z)} = 0$, we learn that $\lambda = 0$ and that $h(1) = 1$.

Substituting the expressions for $\frac{\delta M[h]}{\delta h(z)}$ in (29) and $M[h]$ in (25) into this equation, we obtain

$$0 = 8A[h]z\int_{u=0}^{z}\mathrm{d}u\, h(u) + 8A[h]\int_{x=z}^{1}\mathrm{d}x\, xh(x) + 6A[h] - 6A[h]\int_{x=0}^{1}\mathrm{d}x\, h^2(x)$$
$$+ A[h](A[h]-4)h(z) + 4A[h]zh^2(z) + 8A[h]h(z)\int_{u=z}^{1}\mathrm{d}u\, h(u)$$
$$+4A[h]\int_{x=0}^{z}\mathrm{d}x\, h^2(x) - 160\int_{x=0}^{1}\mathrm{d}x\, xh(x)\int_{u=0}^{x}\mathrm{d}u\, h(u) + \frac{40}{3}$$
$$+40\int_{x=0}^{1}\mathrm{d}x\, h^2(x) - \frac{80}{3}\int_{x=0}^{1}\mathrm{d}x\, xh^3(x) - 80\int_{x=0}^{1}\mathrm{d}x\, h^2(x)\int_{u=x}^{1}\mathrm{d}u\, h(u). \tag{31}$$

4.4. Expressing the boundary as a differential equation

The integro-differential equation (31) expresses the shape of the boundary of the optimal city. Our objective now is to convert this equation to an ordinary differential equation. We begin by differentiating (31) with respect to z

$$0 = 8zh(z)h'(z) + (A[h]-4)h'(z) + 8\int_{u=0}^{z}\mathrm{d}u\, h(u) + 8h'(z)\int_{u=z}^{1}\mathrm{d}u\, h(u). \tag{32}$$

Next, we introduce the function $f(x)$, whose derivative is $h(x)$:

$$f(x) \equiv \int_{u=0}^{x} \mathrm{d}u\, h(u) \tag{33}$$

so that $f'(x) = h(x)$ and

$$f(0) = 0. \tag{34}$$

Note that from (2) we have

$$f'(1) = h(1) = 1. \tag{35}$$

Also, (33) and (18) imply that

$$A[f] = 8f(1) - 4. \tag{36}$$

Finally, substituting the function $f(z)$ into (32) we obtain the following differential equation:

$$0 = zf'(z)f''(z) + [2f(1) - 1]f''(z) + f(z) - f''(z)f(z). \tag{37}$$

Note that if we evaluate this differential equation at $z = 1$ and $z = 0$, then we obtain the two conditions

$$f''(1) = h'(1) = -1 \tag{38}$$

and

$$f''(0) = h'(0) = 0. \tag{39}$$

The first of these conditions shows that the slope of the boundary curve at the point (1, 1), where $h(x)$ joins onto $g(x)$ in the positive quadrant, is -1. This result implies that the boundary curve has a continuous derivative at this point and thus there is no cusp there. Similarly, the second of these conditions shows that the boundary curve at the point $(0, a)$ is level. Thus, again there is no cusp at this point. In short, the boundary of the optimal city has no dimples.

4.5. Substituting $f(z)$ into $M[h]$

We now show that the functional $M[h]$ simplifies dramatically when $M[h]$ is evaluated at the optimal boundary curve. We substitute $f(x)$ into (25). Then we use the differential equation (37) to simplify the result:

$$M[f] = 64\int_{x=0}^{1} \mathrm{d}x\, xf'(x)\int_{u=0}^{x} \mathrm{d}u\, f'(u) - 4A[f] - \frac{16}{3} + (4A[f] - 16)\int_{x=0}^{1} \mathrm{d}x\, f'^2(x)$$
$$+ \frac{32}{3}\int_{x=0}^{1} \mathrm{d}x\, xf'^3(x) + 32\int_{x=0}^{1} \mathrm{d}x\, f'^2(x)\int_{u=x}^{1} \mathrm{d}u\, f'(u). \tag{40}$$

Performing the indicated integrations and doing repeated integration by parts, we simplify this equation to

$$M[f] = 64\int_{x=0}^{1} \mathrm{d}x\, xf'(x)f(x) - 4A[f] - 32f(1) + 16 + \{4A[f]$$
$$+ 32f(1) - 16\}\int_{x=0}^{1} \mathrm{d}x\, f'^2(x) - 64\int_{x=0}^{1} \mathrm{d}x\, x^2 f'^2(x)f''(x)$$
$$+ 64\int_{x=0}^{1} \mathrm{d}x\, xf(x)f'(x)f''(x). \tag{41}$$

Next, in the last two integrals we use the differential equation (37) to replace the expression $f''(x)f(x) - xf'(x)f''(x)$, which is cubic in f, by the quadratic expression

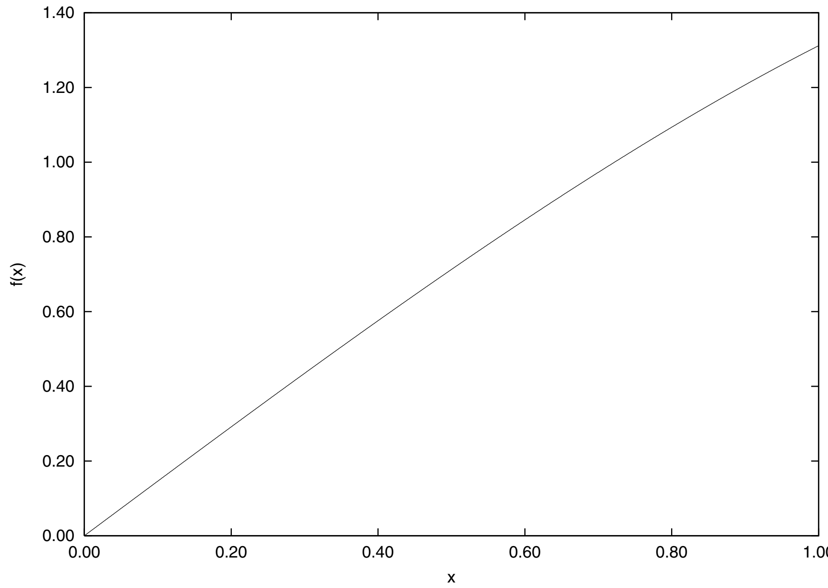

Figure 3. Plot of $f(x)$ for $0 \leqslant x \leqslant 1$. Note that $f(0) = 0$ and that $f'(1) = 1$. By repeated iteration (shooting) we find that $f(1) = 1.311\,794\,482\,332$. The derivative of the plotted function $f(x)$ gives the boundary of the optimal city and for this city the value of D is $0.650\,245\,952\,951$.

$[2f(1) - 1]f''(x) + f(x)$. The resulting formula for $M[f]$ simplifies, after further integration by parts, to

$$M[f] = -64 \int_{x=0}^{1} \mathrm{d}x\ f^2(x) + 64 f^2(1) \tag{42}$$

where we have used the formula $A[f] = -4 + 8f(1)$ in (36).

4.6. Numerical results

We have now completed the analytical work and reduced the problem of finding the boundary of the optimal city to a numerical computation. The numerical procedure is a follows: first, we guess a value, say c, for $f(1)$. Also, we know from (35) that $f'(1) = 1$. From these two initial conditions we solve the differential equation (37) numerically to obtain the value of $f(0)$. But, from (34) we know that $f(0) = 0$. By repeated shooting, we determine the value of c that gives the result $f(0) = 0$. When we achieve this condition, we then learn that the value of $a = f'(0) = h(0) = 1.463\,110\,117\,728$ (see (2)). The shape of the boundary $h(x)$ is given by the derivative of $f(x) : h(x) = f'(x)$. Once we have found $f(x)$, we then compute the value of $D[f]$ from

$$D[f] = \frac{M[f]}{(A[f])^{5/2}} = \frac{64f^2(1) - 64\int_{x=0}^{1} \mathrm{d}x\ f^2(x)}{[8f(1) - 4]^{5/2}}. \tag{43}$$

Our numerical analysis reveals that the minimum value of $D[f]$ is $0.650\,245\,952\,951$, which is about a 2.6 parts per million improvement in the optimal value of D obtained by using the one-parameter family of functions $w(x) = (2 - |x|^d)^{1/d}$. The optimal function $f(x)$ is displayed in figure 3, and the optimal boundary curve $w(x)$ is shown in figure 4.

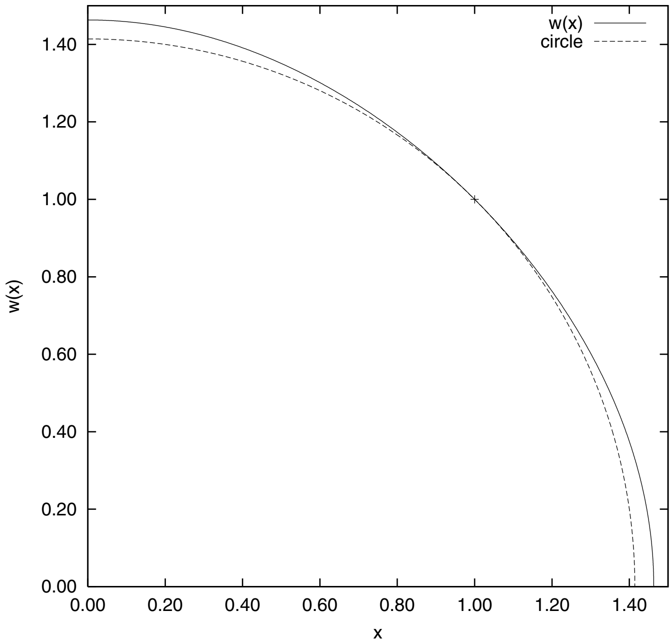

Figure 4. Plot of the function $w(x)$ for $x \geqslant 0$. The function $w(x)$ is composed of $h(x)$ for $0 \leqslant x \leqslant 1$ and $g(x) = h^{-1}(x)$ for $x \geqslant 1$. Note that $h(0) = f'(0) = a = 1.463\,110\,117\,728$. The number a is the value of x at which $g(x)$ crosses the x axis. The function h continues onto the function g at the point $(1, 1)$, which is marked by a + sign. For this optimal boundary curve the (minimum) numerical value of the functional D is $0.650\,245\,952\,951$. For purposes of comparison, the dotted line is a quarter circle, illustrating that the optimal city is close to circular.

5. Discussion and conclusions

The principal conclusion that can be drawn from the calculations in this paper is that if a city is tightly clustered and convex, then it is extremely close to optimal; that is, its value of $D[w]$, where w is the boundary curve of the city, is close (just a few per cent off) to the optimal value $0.650\,245\,952\,951$. Thus, an allocation of computer processors from an array of processors, so long as they are tightly bunched and convex, will be close to optimal as measured by the average pairwise Manhattan distance metric.

We already know from table 1 that if a city departs markedly from convexity (say, it is shaped like an astroid) then it will be far from optimal. It is also interesting that if the city has holes then the corresponding value of the functional D can also increase significantly. To illustrate we consider the case of an annular city, whose outside diameter is 1 and whose inside diameter is r. Note that the hole of radius r in the centre of the city may be one of two possible types, a park or a lake. If the hole is a park, then the notion of the Manhattan distance between two points in the city does not change (one can walk through the park). If the hole is a lake, one must walk around it, and the definition of the distance between two points in the city changes accordingly.

Let us consider the simpler case of a park. The area of this city is $A(r) = \pi(1 - r^2)$. Also, we calculate that the pairwise Manhattan distance is

$$M(r) = \tfrac{512}{45}(1 + r^5) - \tfrac{8}{3}\pi(2r^2 + r^4)\,{}_2F_1\left(-\tfrac{1}{2}, \tfrac{1}{2}; 3; r^2\right) + \tfrac{2}{9}\pi(r^4 - r^6)\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{3}{2}; 4; r^2\right)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function [11]. The value of $M(r)$ for this city reduces to that in table 1 for a disc when $r = 0$ and it vanishes when $r = 1$ (the annulus becomes a circle). For this city, the value of $D(r)$ in (5), which measures the optimality of the city shape, grows rapidly with r : $D(0) \approx 0.650, D(0.3) \approx 0.713$, $D(0.6) \approx 0.947$, $D(0.9) \approx 1.998$, $D(0.99) \approx 6.531$ and $\lim_{r\to 1} D(r) = \infty$.

There are many interesting possible continuations of this study. For example, one can consider cities whose dimension is higher than two. For such cities, the boundary is a surface rather than a curve, and thus the boundary is described by a partial differential equation rather than an ordinary differential equation. Hence, the calculations are much more elaborate and have not been considered here.

Another possible generalization of this work is to consider metrics other than the Manhattan distance (p-norms where $p \neq 1$). (Of course, the Euclidean metric, for which $p = 2$ is trivial, and in this case the optimal city is a disc.) Additionally, one could minimize the average of the *squares* of the pairwise distances for the city. Squares of distances are commonly considered by computer scientists and statisticians in clustering applications.

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References

[1] Lo V, Windisch K, Liu W and Nitzberg B 1997 *IEEE Trans. Parallel Distrib. Comput.* **8** 712–26
[2] Mache J, Lo V and Windisch K 1997 *Proc. 10th Int. Conf. on Parallel and Distributed Computing Systems* pp 120–4
[3] Mache J and Lo V 1997 *Proc. 3rd Joint Conference on Information Sciences, Sessions on Parallel and Distributed Processing* vol 3 pp 223–6
[4] Chang C and Mohapatra P 1998 *J Parallel Distrib. Comput.* **52** 40–68
[5] Leung V J, Arkin E, Bender M A, Bunde D P, Johnston J, Lal A, Mitchell J, Phillips C A and Seiden S 2002 *Proc. 4th IEEE Int. Conf. on Cluster Computing* pp 296–304
[6] Subramani V, Kettimuthu R, Srinivasan S, Johnson J and Sadayappan P 2002 *Proc. 4th IEEE Int. Conf. on Cluster Computing*
[7] Bender M A, Bunde D P, Demaine E D, Fekete S P, Leung V J, Meijer H and Phillips C A Communication-aware processor allocation for supercomputers unpublished
[8] Krumke S O, Marathe M V, Noltemeier H, Radhakrishnan V, Ravi S S and Rosenkrantz D J 1997 *Theor. Comp. Sci.* **181** 379–404
[9] Fekete S and Meijer H *Algorithmica* to appear
[10] See, for example, Marsden J E and Ratiu T S 1999 *Introduction to Mechanics and Symmetry* (New York: Springer)
Mercier A 1963 *Variational Principles of Physics* (New York: Dover)
[11] See Gradsteyn I S and Ryzhik I M 1965 *Table of Integrals, Series, and Products* (New York: Academic)
Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)