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The complexity of economic equilibria for house allocation markets

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Abstract

We prove NP-completeness of deciding the existence of an economic equilibrium in so-called house allocation markets. House allocation markets are markets with indivisible goods in which every agent holds exactly one copy of some good.

Keywords: Computational complexity; Economic equilibrium; Mathematical economics

1. Introduction

1.1. Markets with divisible goods

A celebrated result by Arrow and Debreu [1] establishes the existence of price equilibria for markets with *divisible* goods: A market consists of n agents $A_1, \ldots, A_n$ and of m continuously divisible goods $G_1, \ldots, G_m$. In the initial allocation, agent A_i is endowed with $e_{i,j} \geqslant 0$ units of good G_j (where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$); the vector $e_i = (e_{i,1}, \ldots, e_{i,m})$ is called the *endowment vector* for agent A_i. Moreover, every agent A_i has a concave utility function $u_i : \mathbb{R}^m \to \mathbb{R}$; the value $u_i(x)$ measures the degree of desirability of the bundle x for agent A_i. The Arrow–Debreu theorem [1] states that for every market of the described form, there exists a price vector $p = (p_1, \ldots, p_m)$ and allocation vectors $x_i = (x_{i,1}, \ldots, x_{i,m})$ for all agents A_i $(i = 1, \ldots, n)$ with the following properties.

(i) All prices p_j are positive real numbers. All allocations $x_{i,j}$ are non-negative real numbers.
(ii) For every agent A_i, the optimization problem "maximize the utility function $u_i(x)$ subject to the constraint $p \cdot x \leqslant p \cdot e_i$" is solved by the vector $x = x_i$.
(iii) The equation $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ holds.

Let us briefly discuss these conditions: Condition (i) is a technical condition on prices and allocations. Next, suppose that every agent A_i sells all his initial goods under the price vector p; this provides him with $p \cdot e_i$

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units of money. Condition (ii) states that the allocation vector x_i is the optimal reinvestment of this money $p \cdot e_i$ according to the utility function u_i of agent A_i. Condition (iii) states that when all agents sell and reinvest, then the market will clear exactly. As an immediate consequence of conditions (ii) and (iii), every agent must reinvest all his money:

(iv) For every agent A_i, the equation $p \cdot x_i = p \cdot e_i$ holds.

Such a price mechanism is said to constitute an *economic equilibrium* for the market. Arrow and Debreu [1] prove the *existence* of such an equilibrium via Kakutani's fixpoint theorem. It is an outstanding open problem, whether such an equilibrium can be *computed* in polynomial time.

1.2. Markets with indivisible goods

In a market with *indivisible* goods, the goods are discrete and only available in integer amounts. Consequently, all the endowment vectors e_i and all the allocation vectors x_i in the above formulation now become integer vectors. It is well known in Mathematical Economics that such markets with indivisible goods do not necessarily possess an economic equilibrium:

Example 1. Consider a market with $n = 3$ agents A_1, A_2, A_3 and $m = 2$ indivisible goods G_1 and G_2. In the initial allocation, agents A_1 and A_2 each hold one copy of good G_1, whereas agent A_3 holds one copy of good G_2. Agents A_1 and A_2 only want G_2 (for them good G_1 has utility zero), and agent A_3 only desires good G_1 (for him good G_2 has utility zero).

Consider a price mechanism that assigns prices p_1 and p_2 to goods G_1 and G_2, respectively. If $p_1 \geqslant p_2$, then agents A_1 and A_2 both could afford a copy of good G_2, whereas there is only one such copy available. If $p_1 < p_2$, then agents A_1 and A_2 cannot afford a copy of G_2 and must stay with their initial endowment G_1. However, agent A_3 could also afford a copy of G_1, and there would be no demand for G_2. In either case the market will not clear, and so this market does not possess an economic equilibrium.

In their seminal paper [2] Deng, Papadimitriou and Safra have proved that it is NP-complete to decide whether a given market with indivisible goods has an economic equilibrium, even if the utility functions of all agents are linear functions. The argument in [2] constructs markets in which every agent may own an arbitrarily high number of objects.

1.3. House allocation markets

In a house allocation market, every agent starts and ends by holding a single indivisible object (a house, for instance). House allocation markets were first introduced and investigated in 1974 by Shapley and Scarf [7], and in 1977 by Roth and Postlewaite [6]. It may be argued that house allocation markets form the most simple and the most primitive family of non-trivial markets. There is a huge number of papers in Mathematical Economics that study house allocation markets; see for instance Takamiya [8] and Ehlers, Klaus and Pápai [3] for some recent contributions.

In house allocation markets, the possible utility functions are particularly easy to describe: Since every agent always owns exactly *one* object, his utility function only needs to specify his personal linear ranking of all objects. Note that this ranking may have tied objects; it is a complete, reflexive, transitive (but not necessarily anti-symmetric!) relation.

Example 2. Shapley and Scarf [7, Section 6] describe "*David Gale's top trading cycle algorithm*" for house allocation markets in which every good occurs exactly once:

If there are n agents that hold n pairwise distinct houses, then let every agent point to the agent who owns his first-ranked house; ties are broken arbitrarily. The underlying directed graph has n vertices (= agents) and n arcs (= pointing hands), and hence it contains a directed cycle C. Assign the same price of n to all the houses on this cycle C, and assign to every agent on this cycle C his favorite house. Then remove these agents together with their alloted houses, and repeat the whole process for the remaining $n' < n$ agents and n' houses. Clearly, this procedure yields an economic equilibrium for this type of market.

The market in Example 1 is a house allocation market with duplicate houses that does *not* possess an economic equilibrium.

1.4. Contribution of this paper

We show that it is NP-complete to decide whether a given house allocation market (with duplicate houses) has an economic equilibrium. Our hardness result and Gale's polynomial time algorithm (see Example 2) together draw a sharp separation line between markets for which this decision problem is hard and markets for which this problem is easy. Our result also answers an open question posed by Papadimitriou [5].

2. The result

Our NP-completeness argument is done via a reduction from the following variant of the NP-complete HITTING SET problem; see Garey and Johnson [4].

Problem: HITTING SET
Instance: A set Q; a set T of three-element subsets over Q.
Question: Does there exist a subset $S \subseteq Q$ that has exactly one element in common with every triple $t \in T$?

From a given instance of HITTING SET, we construct a house allocation market with $n = 4|T|$ agents and $m = 3|Q| + |T|$ goods. For every element $q \in Q$, there are three corresponding goods $G^+(q)$, $G^-(q)$, and $G^0(q)$. For every triple $t \in T$, there is one corresponding good $G^*(t)$. There are four types of agents.

(A*) For every triple $t = (q_x, q_y, q_z) \in T$, there is a corresponding triple-agent $A^*(t)$. $A^*(t)$ holds one copy of good $G^*(t)$. He values the three goods $G^+(q_x)$, $G^+(q_y)$, $G^+(q_z)$ equally, and he has no use for any other good.

(A+) For every occurrence of an element $q \in Q$ in some triple $t \in T$, there is a corresponding agent $A^+(q,t)$. Agent $A^+(q,t)$ holds one copy of good $G^+(q)$. Agent $A^+(q,t)$ desires good $G^0(q)$ the most, desires good $G^*(t)$ a little bit less, and has no use for any other good.

(A−) For every occurrence of an element $q \in Q$ in some triple $t \in T$, there is a corresponding agent $A^-(q,t)$. Agent $A^-(q,t)$ holds one copy of good $G^-(q)$. Agent $A^-(q,t)$ desires good $G^0(q)$ the most, desires good $G^-(q)$ a little bit less, and has no use for any other good.

(A0) For every occurrence of an element $q \in Q$ in some triple $t \in T$, there is a corresponding agent $A^0(q,t)$. Agent $A^0(q,t)$ holds one copy of good $G^0(q)$. Agent $A^0(q,t)$ values the two goods $G^+(q)$ and $G^-(q)$ equally, and he has no use for any other good.

Lemma 3. *If the constructed house allocation market has an economic equilibrium, then the instance of* HITTING SET *has answer YES.*

Proof. Consider some economic equilibrium for the constructed house allocation market. Note first that by condition (iv) (as stated in the Introduction), in this equilibrium every agent must sell his item for some price and buy an item at the same price.

For every element $q \in Q$, we denote the prices of the three corresponding goods $G^+(q)$, $G^-(q)$, $G^0(q)$ by $p^+(q)$, $p^-(q)$, $p^0(q)$, respectively. For every $t \in T$, we denote the price of $G^*(t)$ by $p^*(t)$. Now consider some fixed element q that occurs in exactly d of the triples in T. Note that for each of the goods $G^+(q)$, $G^-(q)$, $G^0(q)$, there are exactly d copies available, that initially are held by corresponding agents of types (A+), (A−), and (A0). We distinguish three cases on the prices $p^-(q)$ and $p^0(q)$.

- In the first case, we assume that $p^0(q) < p^-(q)$ holds. Then the d agents of type (A−) sell their copies of $G^-(q)$ and buy copies of $G^0(q)$ instead. Since this leaves them with $p^-(q) - p^0(q) > 0$ units of money, we have a contradiction to condition (iv) from the Introduction.
- In the second case, we assume that $p^0(q) = p^-(q)$ holds. Then all the d corresponding agents of type (A−) sell their copies of $G^-(q)$, get a copy of $G^0(q)$ instead, and end up satisfied and with an empty account. Moreover, there are d corresponding agents of type (A+) who own $G^+(q)$, and who desire to get a copy of $G^0(q)$. Since all these copies have already been assigned to agents of type (A−), the equilibrium prices $p^0(q)$ must be too expensive for agents of type (A+). Summarizing, this yields

$$p^0(q) = p^-(q) > p^+(q). \tag{1}$$

Now the following is easy to see: The d copies of good $G^-(q)$ must end up with the d corresponding agents of type (A^0), since they are the only remaining agents who want $G^-(q)$. The d corresponding agents of type (A^+) must receive copies of the d goods $G^*(t)$ that correspond to triples t that contain the element q; no other good is acceptable for them. The corresponding d agents $A^*(t)$ of type (A^*) must receive the d copies of good $G^+(q)$; they are the only remaining agents who want this good.

To summarize, the corresponding d agents of type (A^*) swap their goods with the d agents of type (A^+), and the d agents of type (A^-) swap their goods with the agents of type (A^0).

- In the third case, we assume that $p^0(q) > p^-(q)$ holds. Then the agents of type (A^-) start with good $G^-(q)$, and cannot afford goods $G^0(q)$. Therefore, they first sell their copy of $G^-(q)$, and then buy it back. For the agents of type (A^0), the only remaining acceptable goods are $G^+(q)$; they sell their copies of $G^0(q)$ and buy $G^+(q)$ instead. Summarizing, this yields

$$p^0(q) = p^+(q) > p^-(q). \tag{2}$$

Then the d copies of good $G^0(q)$ must end up with the d corresponding agents of type (A^+), since they are the only remaining agents who want this good.

To summarize, in this case the corresponding agents of type (A^-) keep their goods, whereas the d agents of type (A^+) swap their goods with the agents of type (A^0). The corresponding d agents $A^*(t)$ of type (A^*) that correspond to triples t containing q must get some other good.

Finally, let us define a set $S \subseteq Q$ that contains an element $q \in Q$ if and only if the prices $p^+(q)$, $p^-(q)$, $p^0(q)$ satisfy the inequalities in (1).

Consider an arbitrary $t \in T$, say $t = (q_x, q_y, q_z)$. In the economic equilibrium, agent $A^*(t)$ must end up with one of the three goods $G^+(q_x)$, $G^+(q_y)$, $G^+(q_z)$, since only these goods are acceptable to him. Let us say, $A^*(t)$ ends up with $G^+(q_x)$. Then the discussion in the second and third case above yields $q_x \in S$, $q_y \notin S$, and $q_z \notin S$. Therefore, $|t \cap S| = 1$ holds, and the set S is a hitting set for T. □

Lemma 4. *If the instance of* HITTING SET *has answer YES, then the constructed house allocation market has a economic equilibrium.*

Proof. Consider a hitting set $S \subset Q$ for T. Define the following prices:

- All goods $G^*(t)$ with $t \in T$ have price 1.
- All goods $G^0(q)$ with $q \in Q$ have price 2.
- If $q \in S$, then $G^-(q)$ has price 2 and $G^+(q)$ has price 1. If $q \notin S$, then $G^-(q)$ has price 1 and $G^+(q)$ has price 2.

Moreover, define the following allocation:

- If $q \in S$, then $A^0(q)$ receives $G^-(q)$, and $A^-(q)$ receives $G^0(q)$. Every agent $A^+(q)$ receives a good $G^*(t)$ (where $q \in t$), and the corresponding triple agent $A^*(t)$ receives $G^+(q)$.
- If $q \notin S$, then $A^-(q)$ receives $G^-(q)$. Moreover, every agent $A^0(q)$ receives $G^+(q)$, and every agent $A^+(q)$ receives a good $G^0(t)$.

Since S is a hitting set for T, every agent $A^*(t)$ receives exactly one good. It is straightforward to verify that the described prices and allocations satisfy conditions (i)–(iii), and hence constitute an economic equilibrium. □

Lemmas 3 and 4 together yield the following theorem.

Theorem 5. *It is* NP-*complete to decide whether a house allocation market possesses an economic equilibrium.*

References

[1] K.J. Arrow, G. Debreu, Existence of an equilibrium for a competitive economy, Econometrica 22 (1954) 265–290.

[2] X. Deng, Ch. Papadimitriou, S. Safra, On the complexity of equilibria, in: Proc. 34th ACM Symp. on Theory of Computation (STOC'2002), 2002, pp. 67–71.

[3] L. Ehlers, B. Klaus, Sz. Pápai, Strategy-proofness and population-monotonicity for house allocation problems, J. Math. Econ. 38 (2002) 329–339.

[4] M.R. Garey, D.S. Johnson, Computers and Intractability, W.H. Freeman, New York, 1979.

[5] Ch. Papadimitriou, Games, computation, and economic equilibria, Plenary lecture presented at the 10th Annual European Symposium on Algorithms (ESA'2002), Rome, Italy, September 2002.
[6] A.E. Roth, A. Postlewaite, Weak versus strong domination in a market with indivisible goods, J. Math. Econ. 4 (1977) 131–137.
[7] L. Shapley, H. Scarf, On cores and indivisibility, J. Math. Econ. 1 (1974) 23–37.
[8] K. Takamiya, Coalition strategy-proofness and monotonicity in Shapley–Scarf housing markets, Math. Social Sci. 41 (2001) 201–213.