On the Reflexivity of Point Sets

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Abstract. We introduce a new measure for planar point sets S. Intuitively, it describes the combinatorial distance from a convex set: The *reflexivity* $\rho(S)$ of S is given by the smallest number of reflex vertices in a simple polygonalization of S. We prove various combinatorial bounds and provide efficient algorithms to compute reflexivity, both exactly (in special cases) and approximately (in general). Our study naturally takes us into the examination of some closely related quantities, such as the *convex cover* number $\kappa_1(S)$ of a planar point set, which is the smallest number of convex chains that cover S, and the *convex partition* number $\kappa_2(S)$, which is given by the smallest number of disjoint convex chains that cover S. We prove that it is NP-complete to determine the convex cover or the convex partition number, and we give logarithmic-approximation algorithms for determining each.

1 Introduction

In this paper, we study a fundamental combinatorial property of a discrete set, S, of points in the plane: What is the minimum number, $\rho(S)$, of *reflex vertices* among all of the *simple polygonalizations* of S? A *polygonalization* of S is a closed tour on S whose straight-line embedding in the plane defines a connected cycle without crossings, i.e., a simple polygon. A vertex of a simple polygon is *reflex* if it has interior angle greater than π. We refer to $\rho(S)$ as the *reflexivity* of S.

In general, there are many different polygonalizations of a point set, S. There is always at least one: simply connect the points in angular order about the center of mass. A set S has precisely one polygonalization, if and only if it is in convex position, but usually there is a great number of them. Studying the set of polygonalizations (e.g., counting them, enumerating them, or generating a random element) is a challenging active area of investigation in computational geometry.

The reflexivity $\rho(S)$ quantifies, in a combinatorial sense, the degree to which the set of points S is in convex position. See Figure 1 for an example.

F. Dehne, J.-R. Sack, and R. Tamassia (Eds.): WADS 2001, LNCS 2125, pp. 192–204, 2001.
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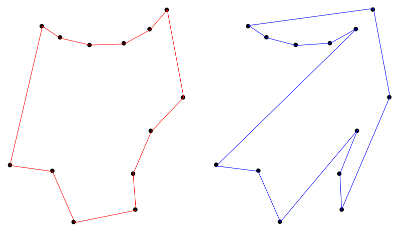

Fig. 1. Two polygonalizations of a point set, one (left) using 7 reflex vertices and one (right) using only 3 reflex vertices.

We conduct a formal study of reflexivity, both in terms of its combinatorial properties and in terms of an algorithmic analysis of the complexity of computing it, exactly or approximately. Some of our attention is focussed on the closely related *convex cover number* of S, which gives the minimum number of convex chains (subsets of S in convex position) that are required to cover all points of S. For this question, we distinguish two cases: The *convex cover number*, $\kappa_1(S)$, is the smallest number of convex chains to cover S; the *convex partition number*, $\kappa_2(S)$, is the smallest number of convex chains with pairwise-disjoint convex hulls to cover S. (Note that nested chains are feasible for a convex cover, but not for a convex partition.)

Motivation. In addition to the fundamental nature of the questions and problems we address, we are also motivated to study reflexivity for several other reasons:

(1) An application motivating our original investigation is that of meshes of low stabbing number and their use in performing ray shooting efficiently. If a point set S has low reflexivity or convex partition number, then it has a triangulation of low stabbing number, which is much lower than the general $O(\sqrt{n})$ upper bound guaranteed to exist ([1,12]). (e.g., a constant reflexivity implies a logarithmic stabbing number triangulation)

(2) Classifying point sets by their reflexivity may give us some structure for dealing with the famously difficult question of counting and exploring the set of all polygonalizations of S. See [11] for some references to this problem.

(3) There are several applications in computational geometry in which the number of reflex vertices of a polygon can play an important role in the complexity of algorithms. If one or more polygons are *given* to us, there are many problems for which more efficient algorithms can be written with complexity in terms of "r" (the number of reflex vertices), instead of "n" (the total number of vertices), taking advantage of the possibility that we may have $r \ll n$ for some practical instances. (See, e.g., [13,16].) The number of reflex vertices also plays an important role in convex decomposition problems for polygons (see, e.g., [17]).

(4) Reflexivity is intimately related to the issue of convex cover numbers, which has roots in the classical work of Erdős and Szekeres [8,9], and has been studied more recently by Urabe [22,23].

(5) Our problems are related to some problems in curve (surface) reconstruction, where the goal is to obtain a "good" polygonalization of a set of sample points. (See [3,5,6].)

Related Work. The study of convex chains in finite planar point sets is the topic of classical papers by Erdős and Szekeres [8,9], who showed that any point set of size n has a convex subset of size $t = \Omega(\log n)$. This is closely related to the convex cover number κ_1, since it implies an asymptotically tight bound on $\kappa_1(n)$, the worst-case value for sets of size n. There are still a number of open problems related to the exact relationship between t and n; see, for example, [21] for recent developments. Other issues have been considered, such as the existence and computation (see [7]) of large "empty" convex subsets (i.e., with no points of S interior to their hull); this is related to the convex partition number, $\kappa_2(S)$. It was shown by Horton [15] that there are sets with no empty convex chain larger than 6, so this implies that $\kappa_2(n) \geq n/6$. Tighter worst-case bounds were given by Urabe [22,23].

Another possibility is to consider a simple polygon having a given set of vertices, that is "as convex as possible". This has been studied in the context of TSP tours of a point set S, where convexity of S provides a trivial optimal tour. Convexity of a tour can be characterized by two conditions. If we drop the global condition (i.e., no crossing edges), but keep the local condition (i.e., no reflex vertices), we get "pseudoconvex" tours. In [10] it was shown that any set with $|S| \geq 5$ has such a pseudoconvex tour. It is natural to require the global condition of simplicity instead, and minimize the number of local violations – i.e., the number of reflex vertices. As in the paper [10], this draws a close connection to angles in a tour, a problem that has also been studied by Aggarwal et al. [2]. We will see in Section 4 that there are further connections.

The number of polygonalizations on n points is in general exponential in n. To give tight bounds on the maximum value attainable for a given n has also been object of intensive research ([11]). The minimum number of reflex vertices among all the polygonalizations of a point set S is the *reflexivity* of S, a concept we introduce in this work.

Main Results. The main results of this work include:

- Tight bounds on the worst-case reflexivity in a number of cases, including the general case and the case of onion depth 2.
- Upper and lower bounds on reflexivity, convex cover number, convex partition number, and their relative behavior. We obtain exact worst-case values for small cardinalities.
- Proofs of NP-completeness for computing convex cover and convex partition numbers.
- Algorithmic results yielding $O(\log n)$ approximations for convex cover number, convex partitioning number, and (Steiner) reflexivity. We also give efficient exact algorithms for cases of low reflexivity.

Throughout this extended abstract we omit many proofs and details, due to space limitations. We refer the reader to the full paper, available on the internet.

2 Preliminaries

Throughout this paper, S will be a set of n points in the plane $\Re^2$. Let P be any polygonalization of S. We say that P is *simple* if edges may only share common endpoints, and each endpoint is incident to exactly two edges. Let $\mathcal{P}$ be the set of all polygonalizations of S. Note that $\mathcal{P}$ is not empty, since any point set S having $n \geq 3$ points has at least one polygonalization (e.g., the star-shaped polygonalization obtained by sorting points of S angularly about a point interior to the convex hull of S).

A simple polygon P is a closed Jordan curve, subdividing the plane into an unbounded and a bounded component. We say that the bounded component is the *interior* of P. A *reflex vertex* of a simple polygon is a common endpoint of two edges, such that the interior angle between these edges is larger than π. We say that an angle is *convex* if it is not reflex. We define $r(P)$ to be the number of reflex vertices in P, and $c(P)$ to be the number of non-reflex, i.e., convex vertices in P. We define the *reflexivity* of a planar point set S to be $\rho(S) = \min_{P \in \mathcal{P}} r(P)$. Similarly, the *convexivity* of a planar point set S is defined to be $\chi(S) = \max_{P \in \mathcal{P}} c(P)$. Note that $\chi(S) = n - \rho(S)$.

The *convex hull* $CH(S)$ of a set S is the smallest convex set that contains all elements of S; the convex hull elements of S are the members of S that lie on the boundary of the convex hull. The *layers* of a point set S are given by repeatedly removing all convex hull elements, and considering the convex hull of the remaining set. We say that S has k *layers* or *onion depth* k if this process terminates after precisely k layers. A set S forms a *convex chain* (or is in *convex position*) if it has only one layer. A *Steiner point* is a point not in the set S that may be added to S in order to improve some structure of S. We define the *Steiner reflexivity* $\rho'(S)$ to be the minimum number of reflex vertices of any simple polygon with vertex set $V \supset S$. Similarly, we can define the *Steiner convexivity* $\chi'(S)$. (Furthermore, Steiner points can be required to lie within the convex hull, or be arbitrary. In this abstract, we do not elaborate on this difference.)

We use the notation $\max_{S:|S|=n} \rho(S) = \rho(n)$ and $\chi(n) = \min_{S:|S|=n} \chi(S)$ for the worst-case values for point sets of size n. For a given finite set S, let $\mathcal{C}_1$ be the family of all sets of convex chains, such that each element of S is part of at least one chain. We say that a set of chains $C \in \mathcal{C}_1$ is called a *convex cover* of S. Similarly, $\mathcal{C}_2 \subset \mathcal{C}_1$ is the family of all convex covers of S for which the convex hulls of any two chains are mutually disjoint. Then we define the *convex cover number* $\kappa_1(S) = \min_{C \in \mathcal{C}_1} |C|$ as the smallest size of a convex cover of S, and the *convex partition number* $\kappa_2(S) = \min_{C \in \mathcal{C}_2} |C|$. Again, we denote by $\kappa_1(n)$ and $\kappa_2(n)$ the worst-case values for sets of size n.

Finally, we state a basic property of polygonalizations of point sets. The proof is straightforward.

Lemma 1. *In any polygonalization of S, the points on the convex hull of S are always convex vertices, and they occur in the polygonalization in the same order in which they occur along the convex hull.*

3 Combinatorial Bounds

In this section we establish several combinatorial results on reflexivity and convex cover numbers.

One of our main combinatorial results establishes an upper bound on the reflexivity of S that is tight in terms of the number n_I of points *interior* to the convex hull of S. Given that the points of S that are vertices of the convex hull are required to be convex vertices in any (non-Steiner) polygonalization of S, the bound in terms of n_I seems to be quite natural.

Theorem 1. *Let S be a set of n points in the plane, n_I of which are interior to the convex hull $CH(S)$. Then, $\rho(S) \leq \lceil n_I/2 \rceil$.*

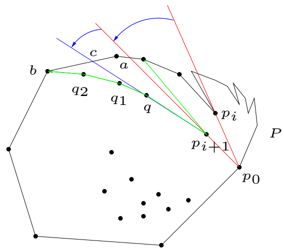

Fig. 2. Computing a polygonalization with at most $\lceil n_I/2 \rceil$ reflex vertices.

Proof. We describe a polygonalization in which at most half of the interior points are reflex. We begin with the polygonalization of the convex hull vertices that is given by the convex polygon bounding the hull. We then iteratively incorporate other (interior) points of S into the polygonalization. Fix a point p_0 that lies on the convex hull of S. At a generic step of the algorithm, the following invariants hold: (1) our polygonalization consists of a simple polygon, P, whose vertices form a subset of S; and (2) all points $S' \subset S$ of S that are not vertices of P lie interior to P; in fact, the points S' all lie within the subpolygon, Q, to the left of the diagonal p_0p_i, where p_i is a vertex of P such that the subchain of ∂P from p_i to p_0 (counter-clockwise) together with the diagonal p_0p_i forms a convex polygon (Q). Define p_{i+1} to be the first point of S' that is encountered when sweeping the ray p_0p_i counter-clockwise about its endpoint p_0. Then, we sweep the subray with endpoint p_{i+1} further counter-clockwise, about p_{i+1}, until we encounter another point, q, of S'. (If $|S'| = 1$, we can readily incorporate p_{i+1} into the polygonalization, increasing the number of reflex vertices by one.) Now, the ray $p_{i+1}q$ intersects the boundary of P at some point $c \in ab$ on the boundary of Q.

We now modify P to include interior points p_{i+1} and q (and possibly others as well) by replacing the edge ab with the chain $(a, p_{i+1}, q, q_1, \ldots, q_k, b)$, where the points q_i are interior points that occur along the chain we obtain by "pulling taut" the chain (q, c, b) (i.e., by continuing, in the "gift wrapping" fashion, to rotate rays counter-clockwise about each interior point q_i that is hit until we encounter b). In this way we incorporate at least two new interior points (of S') into the polygonalization P, while creating only one new reflex vertex (at p_{i+1}). It is easy to check that the invariants (1) and (2) hold after this step. □

In fact, the upper bound of Theorem 1, $\rho(S) \leq \lceil n_I/2 \rceil$, is *tight*, as we now argue based on the special configuration of points, $S = S_0(n)$, in Figure 3. The set $S_0(n)$ is defined for any integer $n \geq 6$, as follows: $\lceil n/2 \rceil$ points are placed in convex position (e.g., forming a regular $\lceil n/2 \rceil$-gon), forming the convex hull $CH(S)$, and the remaining $n_I = \lfloor n/2 \rfloor$ interior points are also placed in convex position, each one placed "just inside" $CH(S)$, near the midpoint of an edge of $CH(S)$. The resulting configuration $S_0(n)$ has two layers in its convex hull. Lemma 2, below, shows that $\rho(S_0(n)) \geq \lceil n_I/2 \rceil \geq \lfloor n/4 \rfloor$.

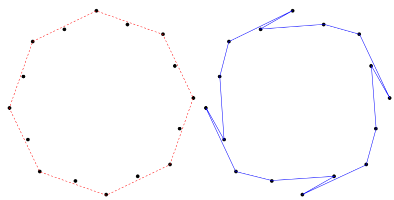

Fig. 3. Left: The configuration of points, $S_0(n)$, which has reflexivity $\rho(S_0(n)) \geq \lceil n_I/2 \rceil$. Right: A polygonalization having $\lceil n_I/2 \rceil$ reflex vertices.

Lemma 2. *For any $n \geq 6$, $\rho(S_0(n)) \geq \lceil n_I/2 \rceil \geq \lfloor n/4 \rfloor$.*

Proof. Denote by x_i the points on the convex hull, $i = 1, ..., \lceil n/2 \rceil$, and v_i the points "just inside" the convex hull, $i = 1, ..., \lfloor n/2 \rfloor$, where v_i is along the convex hull edge (x_i, x_{i+1}).

From Lemma 1 we know that points x_i are connected in their order around the convex hull, and are all convex vertices in any polygonalization. Consider an arbitrary pocket of a polygonalization, having lid (x_j, x_{j+1}) and let m_j denote the number of interior points that go to this pocket (if the convex hull edge (x_j, x_{j+1}) belongs to the polygonalization, with a slight abuse of notation we can consider it as a pocket with $m_j = 0$). Observe that $m_j > 0$ implies that v_j belongs to this particular pocket. If $m_j = 1$ the pocket contains a single interior point, namely v_j, and then v_j is a reflex vertex in this polygonalization. To complete our proof we will show that if this pocket contains more interior points, among them only v_j will be a convex point.

We use the following simple fact: Given a set of points, all but one of which is in convex position, all polygonalizations of this set have a unique reflex vertex, namely the point not on the convex hull.

The pocket with lid (x_j, x_{j+1}) includes points x_j, x_{j+1}; if v_j is not the only interior point included in this pocket, then this pocket together with the lid is a polygon as in the simple fact. Therefore the polygon which is the pocket has only one reflex vertex, v_j, and when considered "inside-out" as a pocket of the original polygon, only v_j among the interior points is a convex vertex.

Therefore the number of reflex vertices in a pocket is in any case at least $\lceil m_j/2 \rceil$, and we have

$$\begin{aligned}\rho(S_0(n)) &\geq \sum \lceil m_j/2 \rceil \\ &\geq \left\lceil \sum (m_j/2) \right\rceil = \lceil n_I/2 \rceil \geq \lfloor n/4 \rfloor .\end{aligned}$$

□

Remark. Since $n_I \leq n$, the corollary below is immediate from the theorem. The gap in the bounds for $\rho(n)$, between $\lfloor n/4 \rfloor$ and $\lceil n/2 \rceil$, remains an intriguing open problem. While our combinatorial bounds are tight in terms of n_I (the number of points of S whose convexivity/reflexivity is not forced by the convex hull of S), they are not yet tight in terms of n.

Corollary 1. $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$.

Steiner Points. If we allow Steiner points in the polygonalizations of S, the reflexivity of S may decrease; in fact, we have examples of point sets with reflexivity $\rho(S) = r$, where the introduction is reduced by a factor of 2 from the no-Steiner case: $\rho'(S) = r/2$. At this point, it is unclear whether this estimate characterizes a worst case. We believe it does:

Conjecture 1. For a set S of points in the plane, we have $\rho'(S) \geq \rho(S)/2$.

In terms of the cardinality n of S, we obtain the following combinatorial bounds; a proof can be found in the full version of the paper:

Theorem 2. *For a set S of n points in the plane, we have $\rho'(S) \leq \lceil n/3 \rceil$.*

By a careful analysis of our example in Figure 3, we can show the following:

Theorem 3. *If one only allows Steiner points that are interior to $CH(S)$, then any Steiner polygonalization of $S_0(n)$ has at least $\lceil n/4 \rceil$ reflex vertices.*

Two-Layer Point Sets. Let S be a point set that has onion depth 2. It is clear from our repeated use of the example in Figure 3 that this is a natural case that is a likely candidate for worst-case behavior. With a very careful analysis of this case, we are able to obtain tight bounds on the worst-case reflexivity in terms of n:

Theorem 4. *Let S be a set of n points having two layers. Then $\rho(S) \leq \lceil n/4 \rceil$, and this bound is tight.*

A proof can be found in the full version of our paper.

Furthermore, we can prove the following lower bound for polygonalizations with a very special structure that may be useful in an inductive proof in more general cases:

Lemma 3. *For a point set with onion depth* 2 *a polygonalization with at most* $\lceil \frac{n}{3} \rceil$ *reflex vertices exists such that none of the edges exist in the interior of the inner layer of the onion.*

Convex Cover Numbers. As we noted in the introduction, it was shown by Erdős and Szekeres [8,9] that any set of n points in the plane has a convex chain of size $O(\log n)$. Moreover, they have shown that there are sets of size $2^t + 1$ without a convex chain of $t + 3$ points. This implies the following:

Theorem 5. $\kappa_1(n) = \Theta(n/\log n)$.

Proof. For point sets with $\kappa_1(n) = \Omega(n/\log n)$, consider the sets constructed by Erdős and Szekeres. These have a largest chain of size $O(\log n)$, and the lower bound follows.

To see that there always is a cover with $O(n/\log n)$ chains, consider a greedy cover in the following way. Let $S_0 = S$, and for each S_i, remove a largest chain, yielding S_{i+1}. By the result of Erdős and Szekeres, each removed chain has size $\Omega(\log |S_i|)$; the lower bound for the size of S_i remains constant until $\lceil \log |S_i| \rceil$ decreases, i.e., until we have removed at least half of the points. Furthermore, any largest convex chain in S_i has at least 3 points, so the iteration must terminate after removing a series of chains of size at least 3. This yields a total number of at most $O(\sum_{i=3}^{\log n} \lceil \frac{n}{2^{\log n + 1 - i} i} \rceil) = O(\sum_{i=3}^{\log n} \lceil \frac{2^{i-1}}{i} \rceil)$ chains. A straightforward induction over q shows that $\sum_{i=3}^{q} \frac{2^{i-1}}{i} \leq 2\frac{2^q}{q}$, so the claim follows. □

Even better bounds are known for the disjoint cover number:

Theorem 6 (Urabe [22]).
$\lceil (n-1)/4 \rceil \leq \kappa_2(n) \leq \lceil 2n/7 \rceil$.

The lower bound can be seen directly from our Figure 3; the upper bound is the result of a detailed construction in [22].

Now we discuss the relationship between the different measures for a set S.

The ratio $\kappa_1(S)$ and $\kappa_2(S)$ for a set S may be as big as $O(n)$: Our example in Figure 3 has $\kappa_1(S) = 2$, but $\kappa_2(S) \geq n/4$. However, there is a very tight lower bound of $\kappa_2(S)$ in terms of $\rho(S)$:

Theorem 7. *For a planar set S, we have the estimates* $\kappa_1(S) \leq \kappa_2(S) \leq \rho(S) + 1$*, and these upper bounds are best possible.*

Proof. The upper bound for $\kappa_1(S)$ by $\kappa_2(S)$ is trivial. The upper bound for $\kappa_2(S)$ by $\rho(S)$ can be found in [4]. □

One can construct examples with $2\kappa_2(S) = \rho(S)$, which is the worst example we know of, see the full paper. On the other hand, it is a surprisingly difficult open problem to prove that there is *some* bounded ratio between $\kappa_2(S)$ and $\rho(S)$:

Conjecture 2. For a set S of points in the plane, we have $\rho(S) = O(\kappa_2(S))$.

However, it is not hard to see that the estimate $\rho'(S) = O(\kappa_2(S))$ holds (see the proof of Corollary 2), so a proof of Conjecture 2 would follow from the validity of Conjecture 1.

Small Point Sets. It is natural to consider the exact values of $\rho(n)$, $\kappa_1(n)$, and $\kappa_2(n)$ for small values of n. Table 1 below shows some of these values, which we obtained through (sometimes tedious) case analysis. Oswin Aichholzer has recently applied his software that enumerates point sets of size n of all distinct order types to verify our results; in addition, he has obtained the result that $\rho(10) = 3$. (Values of $n \geq 11$ seem to be intractable for enumeration.)

n	$\rho(n)$	$\kappa_1(n)$	$\kappa_2(n)$
≤ 3	0	1	1
4	1	2	2
5	1	2	2
6	2	2	2
7	2	2	2
8	2	2	3
9	3	3	3

Table 1. Worst-case values of ρ, κ_1, κ_2 for small values of n.

4 Complexity

Theorem 8. *It is NP-complete to decide whether for a planar point set S the convex partition number $\kappa_2(S)$ is below some threshold k.*

Proof. We give a reduction of PLANAR 3 SAT, which was shown to be NP-complete by Lichtenstein (see [18]). See Figure 4 for the overall proof idea, and the full paper for proof details. □

Theorem 9. *It is NP-complete to decide whether for a planar point set S the convex cover number $\kappa_1(S)$ is below some threshold k.*

Proof. Our proof uses a reduction of the problem 1-in-3 SAT. (It is inspired by the hardness proof for the ANGULAR METRIC TSP given in [2].) See Figure 5 for the proof idea, with bold lines corresponding to appropriate dense point sets. Proof details can be found in the full paper. □

So far, the complexity status of determining the reflexivity of a point set remains open. However, the close relationship between convex cover number and reflexivity leads us to believe the following:

Conjecture 3. It is NP-complete to determine the reflexivity $\rho(S)$ of a point set.

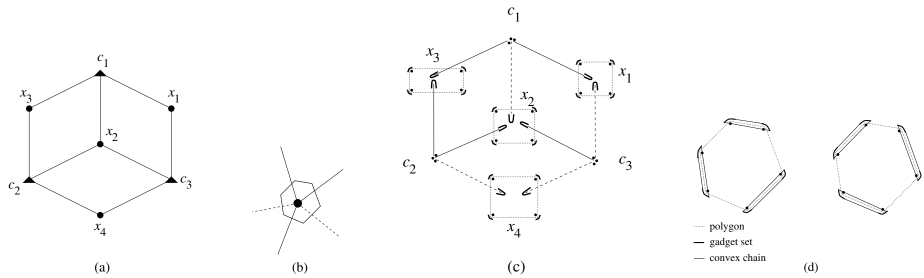

Fig. 4. (a) A straight-line embedding of the occurrence graph for the 3 SAT instance $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3 \vee \overline{x_4}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_4})$; (b) a polygon for a variable vertex; (c) a point set S_I representing the PLANAR 3 SAT instance I; (d) joining point sets along the odd or even polygon edges.

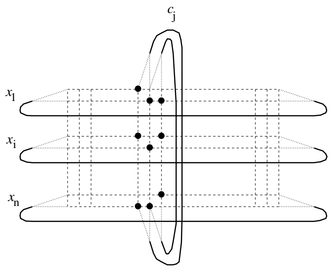

Fig. 5. A point set S_I for a 1-in-3 SAT instance I. Pivot points are shown for the clause $(x_1 \vee \overline{x_i} \vee x_n)$.

5 Algorithms

In this section, we provide a number of algorithmic results. Since some of the methods are rather technical, we can only give proof sketches in this extended abstract.

Theorem 10. *Given a set S of n points in the plane, in $O(n \log n)$ time one can compute a polygonalization of S having at least $\chi(S)/2$ convex vertices, where $\chi(S)$ is the convexivity of S.*

Proof. (sketch) The algorithm of Theorem 1 is constructive, producing a polygonalization of S having at most $n_I/2 \leq n/2$ reflex vertices, and thus at least $n/2$ convex vertices (thereby giving a 2-approximation for convexivity). In order to obtain the stated time bound, we must implement the algorithm efficiently. We utilize a dynamic convex hull data structure ([14]), to be able to obtain the points q, $q_1, \ldots, q_k$, efficiently (in amortized $O(\log n)$ time per point). □

Theorem 11. *Given a set S of n points in the plane, the convex cover number, $\kappa_1(S)$, can be computed approximately, within a factor of $O(\log n)$, in polynomial time.*

Proof. (sketch) We use a greedy set cover heuristic. At each stage, we need to compute a largest convex subset among the remaining (uncovered) points of S. This can be done in polynomial time using the dynamic programming methods of [20]. □

Theorem 12. *Given a set S of n points in the plane, the convex partition number, $\kappa_2(S)$, can be computed approximately, within a factor of $O(\log n)$, in polynomial time.*

Proof. (sketch) Let $C^* = \{P_1, \ldots, P_{k^*}\}$ denote an optimal solution, consisting of $k^* = \kappa_2(S)$ disjoint convex polygons whose vertices are the set S. Following the method of [19], we partition each of these polygons into $O(\log n)$ vertical trapezoids whose x-projection is a "canonical interval" (one of the $O(n)$ such intervals determined by the segment tree on S).

For the algorithm, we use dynamic programming to compute a minimum-cardinality partition of S into a disjoint set, C', of (empty) convex subsets whose x-projections are canonical intervals. Since the optimal solution, C^*, can be converted into at most $k^* \cdot O(\log n)$ such convex sets, we know we have obtained an $O(\log n)$-approximate solution to the disjoint convex partition problem. □

Corollary 2. *Given a set S of n points in the plane, its Steiner reflexivity, $\rho'(S)$, can be computed approximately, within a factor of $O(\log n)$, in polynomial time.*

A proof can be found in the full paper.

Special Cases. For small values of r, we have devised particularly efficient algorithms that check if $\rho(S) \leq r$ and, if so, produce a witness polygonalization having at most r vertices. Of course, the case $r = 0$ is trivial, since that is equivalent to testing if S lies in convex position (which is readily done in $O(n \log n)$ time, which is worst-case optimal). It is not particularly surprising that for any fixed r one can obtain an $n^{O(r)}$ algorithm, e.g., by enumerating over all r-element subsets that correspond to reflex vertices, along with all possible neighboring segments incident on these vertices, etc. The factor in front of r in the exponent, however, is not so trivial to reduce. In particular, the straightforward method applied to the case $r = 1$ gives $O(n^5)$ time. With a more careful analysis of the cases $r = 1, 2$, we obtain:

Theorem 13. *Given a set S of n points in the plane, in $O(n \log n)$ time one can determine if $\rho(S) = 1$, and, if so, produce a witness polygonalization. Furthermore, $\Omega(n \log n)$ is a lower bound.*

For $r = 2$, a careful analysis of how two pockets can interact also yields a very efficient algorithm; in the full paper we prove:

Theorem 14. *Given a set S of n points in the plane, in $O(n^2)$ time one can determine if $\rho(S) = 2$, and, if so, produce a witness polygonalization.*

Acknowledgments. We thank Adrian Dumitrescu for valuable input on this work. The collaboration between UPC and SUNY Stony Brook was made possible by a grant from the Joint Commission USA-Spain for Scientific and Technological Cooperation Project 98191. E. Arkin acknowledges support from the NSF (CCR-9732221) and HRL Laboratories. S. Fekete acknowledges travel support from the Hermann-Minkowski-Minerva Center for Geometry at Tel Aviv University. F. Hurtado, M. Noy, and V. Sacristán acknowledge support from CUR Gen. Cat. 1999SGR00356, and Proyecto DGES-MEC PB98-0933. J. Mitchell acknowledges support from HRL Laboratories, NSF (CCR-9732221), NASA (NAG2-1325), Northrop-Grumman, Sandia, Seagull Technology, and Sun Microsystems.

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