Matching as the Intersection of Matroids

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Abstract

This paper deals with the problem of representing the matching independence system in a graph as the intersection of finitely many matroids. After characterizing the graphs for which the matching independence system is the intersection of two matroids, we study the function $\mu(n)$, which is the minimum number of matroids that need to be intersected in order to obtain the set of matchings on a graph with n vertices. We describe an integer programming formulation for deciding whether $\mu(n) \leq m$. Using combinatorial arguments, we prove that $\mu(n) \in \Omega(\log\log n)$. On the other hand, we establish that $\mu(n) \in O(\log n/\log\log n)$. Finally, we prove that $\mu(n) = 4$ for $n = 5, \ldots, 12$, and $\mu(n) = 5$ for $n = 13, \ldots, 20$.

1 Introduction

Many combinatorial optimization problems can be viewed as optimization problems over independence systems. Some of them are polynomially solvable, e.g., spanning trees in graphs, the branching problem in digraphs, or the matching problem. Others are known to be NP-complete, like the traveling salesman problem (TSP) or the stable set problem (cf. [1,7,8]). Among the problems with a polynomial-time algorithm, the matching problem is generally recognized as one of the "hardest", and the famous blossom algorithm by Edmonds [2] is one of the highlights of combinatorial optimization.

Another seminal result on the optimization in independence systems was also given by Edmonds [3], who proved that the optimization problem over the intersection of two matroids is solvable in polynomial time. Algorithms for this problem were given by Edmonds [4], Frank [6], and Lawler [9,10]. Unfortunately, this cannot be generalized to the case of three or more matroids: As the NP-complete TSP can be written as an optimization problem over the intersection of three matroids, it is highly unlikely that a polynomial-time algorithm exists.

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It is natural to suspect that the minimum number of matroids that need to be intersected for defining a given independence system is an indicator for the complexity of the related optimization problem. This motivates the study conducted in this paper. As it turns out, describing the matchings of a graph as the intersection of matroids is an interesting and challenging combinatorial problem in its own right.

2 Preliminaries

Let S be a finite set and $\mathcal{I}$ a family of subsets of S. $\mathcal{I}$ is an *independence system* on S if $\varnothing \in \mathcal{I}$ and if $J' \subseteq J$ and $J \in \mathcal{I}$ then $J' \in \mathcal{I}$. The subsets of S belonging to $\mathcal{I}$ are called *independent*, otherwise *dependent*. The minimal dependent subsets of S are the *circuits* of $\mathcal{I}$. The *circuit system* $\mathcal{C}$ of $\mathcal{I}$ is the set of circuits of $\mathcal{I}$ and $\mathcal{I} = \{J \subseteq S : C \not\subseteq J \text{ for all } C \in \mathcal{C}\}$. A maximal independent subset of a set $A \subseteq S$ is a *basis* of A. An independence system $\mathcal{I}$ on S is a matroid if for every subset $A \subseteq S$ all its bases have the same cardinality. For further background, see Oxley [11] and Welsh [12].
Any independence system is the intersection of finitely many matroids: There can only be a finite number of circuits C of $\mathcal{I}$. Each one of them can be used to define a matroid M_C, and $\mathcal{I} = \bigcap\{M_C : C \in \mathcal{C}\}$. This, however, may not be the most economical way to describe $\mathcal{I}$, since we may be able to cover several circuits by the same matroid. In the following, we write $\mu(\mathcal{I})$ for the minimum number of matroids necessary for this task.

3 Matching as the Intersection of Matroids

Now consider a finite graph $G = (V, E)$. The set $\mathcal{M}(G)$ of matchings in G forms an independence system on E. In this context, $\mu(n)$ is used for the maximum $\mu(\mathcal{M})$ on graphs with $|V| \leq n$. Obviously, the circuits of $\mathcal{M}(G)$ are the sets that consist of two intersecting edges. (Throughout the rest of this paper, the term *circuit* refers to such a pair of edges.) We call a circuit an *i-circuit* if its edges intersect in vertex i. We denote the circuit $\{ij, ik\}$ with the two edges ij and ik by i^{jk}.
The following easy lemma implies that $\mu(n) = \mu(K_n)$.

Lemma 1 *Let $G' = (V', E')$ be a subgraph of $G = (V, E)$. Then $\mu(\mathcal{M}(G')) \leq \mu(\mathcal{M}(G))$.*

3.1 IP-Formulation

In the full paper [5] we give an integer programming formulation for deciding whether $\mu(\mathcal{M}(G)) \leq m$ for any graph G and any integer m. The facets in this formulation are based on four observations:

(1) Any circuit has to be contained in at least one of the matroids.
(2) For no induced *path* of length three can both circuits be contained in the same matroid.

(3) For no *claw*, i.e., three edges sharing one vertex, can exactly two of the three resulting circuits be contained in the same matroid.
(4) For no three edges forming a *triangle* can exactly two of the three resulting circuits be contained in the same matroid.

3.2 When is Matching the Intersection of Two Matroids?

A proof of the following theorem can be found in the full paper [5]:

Theorem 2 *The set of matchings $\mathcal{M}$ of a graph $G = (V, E)$ is the intersection of two matroids if and only if G contains no odd cycle of cardinality ≥ 5 and each triangle of G has at most one vertex with degree > 2.*

3.3 General Upper Bounds

Using a recursive construction, we can show that m matroids suffice to generate the matchings of any graph with $\Omega(\sqrt[3]{m!}) = 2^{\Omega(m \log m)}$ vertices. (Space does not permit to describe full technical details, so the interested reader is referred to the full paper [5].) This implies the following general upper bound:

Theorem 3 $\mu(n) \in O(\log n / \log \log n)$.

3.4 General Lower Bounds

Theorem 4 $\mu(n) \in \Omega(\log \log n)$.

Proof sketch: First we consider the situation in the absence of "isolated triangles", where all three circuits of a triangle are covered by the same matroid. For a vertex i, consider all (directed) edges ij. If an i-circuit with edge ij is covered by matroid M_ℓ, we say that ij has color ℓ. For m different matroids, there are only 2^m different colorings of an edge, which we call *color classes*. Moreover, there are at most 2^{2^m} different combinations of color classes for the edges ij from a vertex i, which we call the *color type* of vertex i. Any i-circuit formed by a pair of edges ij, ik with $j \neq k$ must be covered by some matroid, so no two color classes in any valid color type can be disjoint. In the absence of isolated triangles, this implies that a feasible covering of the circuits of K_n by m matroids cannot have two vertices of the same color type, since their connecting edge would violate condition (ii) above. This implies $n < 2^{2^m}$, or $m \in \Omega(\log \log n)$.
If there are any isolated triangles in a circuit cover with m matroids, we can construct a triangle-free circuit cover with $3m$ matroids: For each isolated triangle in matroid ℓ, cover one circuit by matroid ℓ, the other two by the additional matroids ℓ' and ℓ''. By the previous argument, we conclude $n < 2^{2^{3m}}$, and again $m \in \Omega(\log \log n)$ holds. □

3.5 Tight Bounds for $\mu(n)$

Lawler mentioned in [10] that the nonbipartite matching problem can be formulated as an intersection problem involving two partition matroids, but with additional constraints in the form of symmetry conditions. However, we can give an elementary proof for the following:

Theorem 5 $\mu(n) = 4$ *for* $n = 5, \ldots, 12$.

Using refined versions of the techniques for the lower bound described in Section 3.4 and the construction for the proof of Theorem 3, we can show the following:

Theorem 6 $\mu(n) = 5$ *for* $n = 13, \ldots, 20$.

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