Extending Partial Suborders

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Abstract

We consider the problem of finding a transitive orientation T of a comparability graph $G = (V, E)$, such that a given partial order P is extended. Existing algorithms for this problem require the full knowledge of E, so they are of limited use in the context of a branch-and-bound algorithm, where only parts of E may be known at any stage. We present a new approach to the problem by describing a pair of necessary and sufficient conditions for the existence of an orientation T, based on two simple forbidden subconfigurations. This allows it to solve higher-dimensional packing and scheduling problems of interesting size to optimality. We have implemented this approach and the computational results are convincing.

1 Introduction

Higher-dimensional packing and scheduling problems occur in many practical contexts; see [1,9] for overviews. When dealing with precedence constraints, we encounter the following natural order-theoretic problem, called *transitive ordering with precedence constraints* (TOP):

Consider a partial order $P = (V, \prec)$ and a comparability graph $G = (V, E)$, such that all relations in P are represented by edges in G. Is there a transitive orientation $D = (V, A)$ of G, such that P is contained in D?

Korte and Möhring [7] have given a linear-time algorithm for deciding TOP. However, their approach can only be used when the full set of edges in G is known. When running a branch-and-bound algorithm for solving a packing or scheduling problem, these edges of G are only known partially, but they may already prohibit the existence of a feasible solution for a given partial order P. This makes it desirable to come up with structural characterizations that

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are already useful when only parts of G are known.

In this paper, we give a pair of necessary and sufficient conditions for the existence of a solution for the problem TOP on graphs G in terms of forbidden substructures. Using the concept of packing classes developed by Fekete and Schepers [3], this characterization can be used quite effectively in the context of a branch-and-bound framework, because it can recognize infeasible subtrees at "high" branches of the search tree, where only some of the edges of G are known. The usefulness of these concepts and results has been validated by implementation and computational experiments.

2 Packing Problems with Precedence Constraints

When considering a feasible packing of d-dimensional boxes into a container, we can describe it by a set of d interval graphs, which arise by projecting the packing onto the d coordinate axes. Fekete and Schepers [3] gave a set of easy combinatorial necessary and sufficient conditions for when a set of graphs corresponds to a feasible packing. They also showed that this leads to useful results in practice. See Figure 1 for the underlying idea.

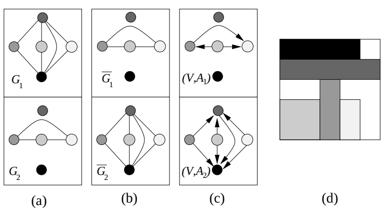

Fig. 1. *(a) A pair of interval graphs, characterizing overlap in the coordinate projections. (b) The complements are the corresponding comparability graphs. (c) A transitive orientation, describing the relative positioning in the two directions. (d) A feasible packing corresponding to the orientation.*

In a situation where we need to satisfy a partial order $P = (V, A_P)$ of precedence constraints in some dimension t, it follows that each arc $a = (u, w) \in A_P$ in this partial order forces the corresponding undirected edge $e = \{u, w\}$ to be excluded from E_t. As the example in Figure 2 shows, this causes an additional problem: Even if we know that the graph $\overline{G_t}$ has a transitive orientation, and all arcs $a = (u, w)$ of the precedence order (V, A_P) are contained in $\overline{E_t}$ as $e = \{u, w\}$, it is not clear that there is a transitive orientation that contains all arcs of A_P.

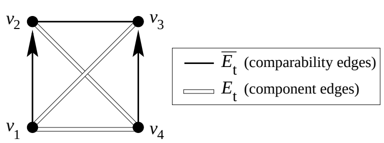

Fig. 2. *A comparability graph $\overline{\boldsymbol{G_t}} = (\boldsymbol{V}, \overline{\boldsymbol{E_t}})$ with a partial order $\boldsymbol{P}$ contained in $\overline{\boldsymbol{E_t}}$, such that there is no transitive orientation of $\overline{\boldsymbol{G_t}}$ that extends $\boldsymbol{P}$.*

In the following, we discuss how to deal with this problem by means of forbidden subconfigurations.

2.1 Implied Orientations

There are three basic states for any edge while performing a tree search: (1) edges that have been fixed to be in E_t, i.e., *component edges*; (2) edges that have been fixed to be in $\overline{E_t}$, i.e., *comparability edges*; (3) *unassigned edges*. In order to deal with precedence constraints, we also consider orientations of the comparability edges. Therefore, we can have three different possible states for each comparability edge: (2a) one possible orientation; (2b) the opposite possible orientation; (2c) no assigned orientation.

A stepping stone for this approach arises from considering the following two configurations – see Figure 3:

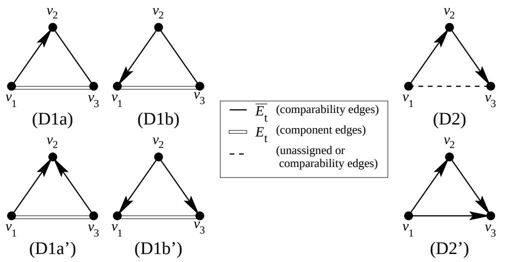

Fig. 3. *Implications for edges and their orientations: Above are path implications (D1, left) and transitivity implications (D2, right); below the forced orientations of edges.*

The first configuration consists of two comparability edges $\{v_1, v_2\}, \{v_2, v_3\} \in \overline{E_t}$, such that the third edge $\{v_1, v_3\}$ has been fixed to be an edge from the component graph E_t. Now any orientation of one of the comparability edges forces the orientation of the other comparability edge, as shown in the left part of the figure. Since this configuration corresponds to an induced path in $\overline{G_t}$, we call this arrangement a *path implication*. Any set of edges connected by a series of path implication forms an *implicaton class*.
The second configuration consists of two directed comparability edges (v_1, v_2), (v_2, v_3). In this case we know that the edge $\{v_1, v_3\}$ must also be a comparability edge, with an orientation of (v_1, v_3). Since this configuration arises directly from transitivity in $\overline{G_t}$, we call this arrangement a *transitivity implication*. We call a violation of a path implication a *path conflict*. and a violation of a transitivity implication a *transitivity conflict*. Summarizing, we have the following necessary conditions for the existence of a transitive orientation that extends a given partial order P:

D1: Any path implication can be carried out without a conflict.
D2: Any transitivity implication can be carried out without a conflict.

These necessary conditions are also sufficient:

Theorem 1 (Fekete, Köhler, Teich) *Let $P = (V, A_P)$ be a partial order with arc set A_P that is contained in the edge set E of a given comparability graph $G = (V, E)$. A_P can be extended to a transitive orientation of G, iff all arising path implications and transitivity can be carried out without creating a path conflict or a transitivity conflict.*

A full proof and further mathematical details are described in our report [2]. The interested reader may take note that we are extending previous work by Gallai [4], who extensively studied implication classes of comparability graphs. See Kelly [6], Möhring [8] for informative surveys on this topic.
The main idea of the proof is the following. Consider a partial orientation of the edges of G consisting of all arcs of P together with the arcs implied by path and transitivity implications. By definition this is a complete orientation of some of the implication classes of G, and by D1 and D2 this partial orien-

tation of G is conflict-free.
Gallai [4] has defined a decomposition scheme, called modular decomposition, which is closely related to the concept of implication classes. Using this decomposition one can prove that the partial orientation described above can be extended to a transitive orientation of G if and only if for each so-called decomposition graph this partial orientation can be extended to a transitive orientation. The latter can be deduced from D1 and D2.

2.2 Application: Solving Problems with Precedence Constraints

We start by fixing for all arcs $(u,v) \in A$ the edge $\{u,v\}$ as an edge in the comparability graph $\overline{G_t}$, and we also fix its orientation to be (u,v). In addition to the tests for enforcing the conditions for unoriented packing classes described in [3], we employ the implications suggested by conditions D1 and D2. For this purpose, we check directed edges in $\overline{G_t}$ for being part of a triangle that gives rise to either implication. Any newly oriented edge in $\overline{G_t}$ gets added to a queue of unprocessed edges. Like for packing classes, we can again get cascades of fixed edge orientations. If we get an orientation conflict or a cycle conflict, we can abandon the search on this tree node. The correctness of the overall algorithm follows from Theorem 1; in particular, the theorem guarantees that we can carry out implications in an arbitrary order.
Results and further details are reported in our paper [1].

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