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RECTANGLE AND BOX VISIBILITY GRAPHS IN 3D

SÁNDOR P. FEKETE
Center for Parallel Computing, Universität zu Köln,
D–50923 Köln, Germany
E-mail: sandor@zpr.uni-koeln.de

and

HENK MEIJER
Department of Computer Science, Queen's University,
Kingston, Ont K7L 3N6, Canada
E-mail: henk@qucis.queensu.ca

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ABSTRACT

We discuss *rectangle* and *box visibility representations* of graphs in 3-dimensional space. In these representations, vertices are represented by axis-aligned disjoint rectangles or boxes. Two vertices are adjacent if and only if their corresponding boxes see each other along a small axis-parallel cylinder. We concentrate on lower and upper bounds for the size of the largest complete graph that can be represented. In particular, we examine these bounds under certain restrictions: What can be said if we may only use boxes of a limited number of shapes?

Some of the results presented are as follows:

- There is a representation of K_8 by unit boxes.
- There is no representation of K_{10} by unit boxes.
- There is a representation of K_{56}, using 6 different box shapes.
- There is no representation of K_{184} by general boxes.

A special case arises for *rectangle visibility graphs*, where no two boxes can see each other in the x- or y-directions, which means that the boxes have to see each other in z-parallel direction. This special case has been considered before; we give further results, dealing with the aspects arising from limits on the number of shapes.

Keywords: Graph representation, graph drawing, 3-dimensional geometry, visibility, orthogonal objects.

1. Introduction

A *visibility representation* of a graph G maps vertices of G to sets in Euclidean space. An edge (u, v) occurs in G if and only if the objects representing u and v

see each other according to some visibility rule. (In some investigations, the "if and only if" condition is relaxed to "only if".)

Application areas such as VLSI routing, circuit board layout, algorithm animation, and CASE tools have stimulated considerable research on visibility representations in $\mathbb{R}^2$. See for example, Refs. [8,14,18,19]. Recently, interest has developed in finding good 3-dimensional visualisations of graphs. See for example, Refs. [7,13,15].

Here we continue the study of a visibility representation, presented for example in Refs. [1,2,3,6,9,16], in which the objects representing vertices are 2-dimensional connected sets parallel to the x,y-plane. An edge (u,v) occurs in G if and only if the objects representing u and v see each other along a line of sight parallel to the z-axis. This line of sight must intersect the interiors of the objects; hence legitimate lines of sight (called thick lines) are extensible to tubes of small radius whose ends lie inside the objects. Furthermore, since G has an edge (u,v) if and only if u and v are mutually visible, the graph G is recoverable from the geometry of the representation. Throughout the paper, we use the term *ZPR* (for **Z**-**P**arallel visibility **R**epresentation) to refer to this specific model. It was shown in Ref. [3] that K_{20} has a ZPR by axis-parallel rectangles. Recently, Rote and Zelle[17] have found a ZPR of K_{22} by using simulated annealing techniques. In Ref. [9] it was shown that K_{56} does not have a ZPR by axis-parallel rectangles and that the largest complete graph with a ZPR by axis-parallel unit squares is K_7.

A different visibility representation is studied in Refs. [8,12]. Here vertices are represented by axis-parallel rectangles in 2-space. Two vertices are adjacent if and only if their respective rectangles can see each other parallel to the x- or y-axis.

This paper extends and generalizes these two representations by considering *box visibility representations (BR)*: vertices are represented by axis-parallel boxes in 3-space, such that edges correspond to axis-parallel thick lines of visibility between the respective boxes. It was shown in Ref. [5] that K_{42} has a BR. Box visibility graphs play a role in 3-dimensional packing algorithms, which are of importance for real-life logistics: when using certain local insertion heuristics for packing boxes into a container, the data structure for representing the packing is a box visibility graph. (See Refs. [10] and [20].)

We derive upper and lower bounds on the size of complete graphs with a BR by using results on graphs with a ZPR. We also study complete graphs with a ZPR or a BR when the number of different objects is limited – a question that is important when it comes to packing objects. Since there are different scenarios for orienting boxes in a packing, there are two ways in which two bodies can be considered equal:

(a) They are *isothetic*, i. e., they can be made identical by translations only.

(b) They are *congruent*, i. e., they can be made identical by translations and rotations.

We say that two isothetic objects have the same *size*, while two congruent objects have the same *shape*.

The rest of the paper is organized as follows: In Section 2, we give a description of lower and upper bounds for ZPR's with limited numbers of different shapes and sizes. In Section 3, we show some constructions that imply lower bounds for BR's.

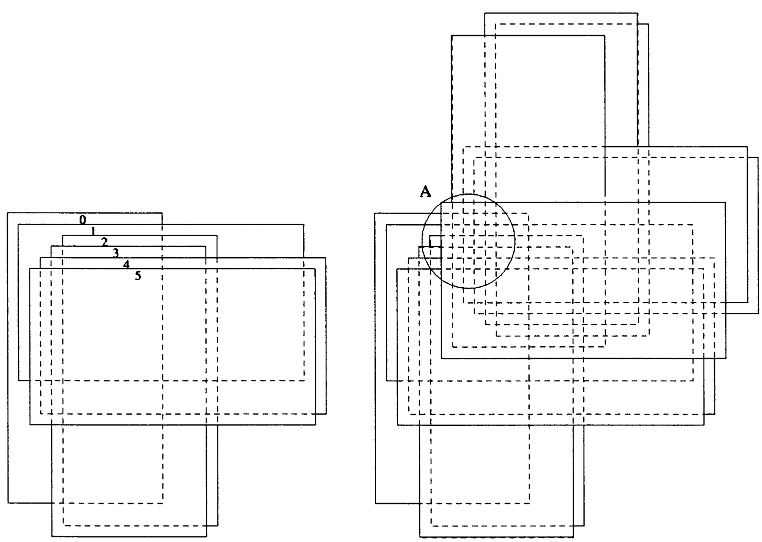
Section 4 deals with upper bounds for BR's using general boxes. Section 5 shows that there is no BR for K_{10} that uses only unit cubes.

2. ZPR's with Limited Numbers of Different Objects

Fekete, Houle, and Whitesides[9] show that the largest complete graph with a ZPR by unit squares has seven vertices. In the light of what we said at the end of the preceding section, the situation for unit squares corresponds to the case where we are only allowed to use rectangles of one size. On the other hand, the constructions in Ref. [4] (showing that there is a ZPR for K_{20}) and Ref. [17] (showing that there is a ZPR for K_{22}) do not consider the number of different shapes or sizes that are necessary. In the following, we examine these matters more closely.

2.1. Lower Bounds

Figure 1(a) shows a representation of a complete visibility graph using a single shape rectangle. The numbers written inside the rectangles are the z-coordinates of these rectangles. The top edges of the 6 rectangles are sorted, i. e., the top edge of rectangle i is higher than the top edge of rectangle $i+1$ for $0 \leq i < 5$. Therefore, we can represent a complete graph of size 12 by using a copy of the 6 rectangles and placing it on top of the first 6 as shown in Figure 1(b). Notice that if there is not enough space in the circle A in Figure 1(b) for the bottom 6 rectangles to see the top 6 rectangles, we can increase the length and width of all rectangles in Figure 1(a) by the same amount, while leaving the left-top corners in place.



(a) (b)

Fig. 1. ZPR of K_6 and K_{12} using 1 shape

A complete visibility graph of 10 rectangles with sorted top edges is shown in Figure 2. Two such sets of rectangles can therefore be used to create a ZPR of K_{20}.

The number of shapes used in this representation is 3:

- Rectangles 0, 4 and 9 are squares of equal size.
- Rectangle 3 is a square.
- Rectangles 1, 2, 5, 6, 7 and 8 are rectangles of the same shape.

We can remove rectangle 3 to create a K_{18} using 2 shapes.

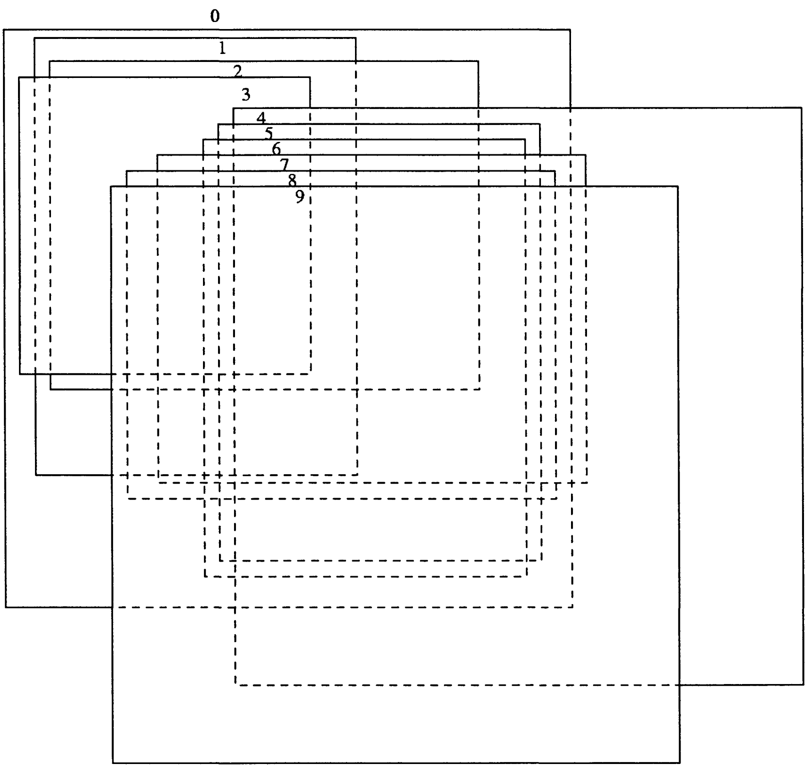


Fig. 2. ZPR of K_{10} using 3 shapes

Rote and Zelle[17] have shown that K_{22} has a ZPR. In Figure 3 it is shown how the ZPR of K_{22} can be contructed using rectangles of 6 different shapes:

- Rectangles 0 and 21 have size 55 by 55.
- Rectangles 1, 9, 16, 19 and 20 have size 21 by 30.
- Rectangles 2 and 4 have size 14 by 33.
- Rectangles 3, 7, 11 and 15 have size 7 by 35.
- Rectangles 8, 10, 13 and 14 have size 8 by 32.
- Rectangles 5, 6, 12, 17 and 18 have size 17 by 25.

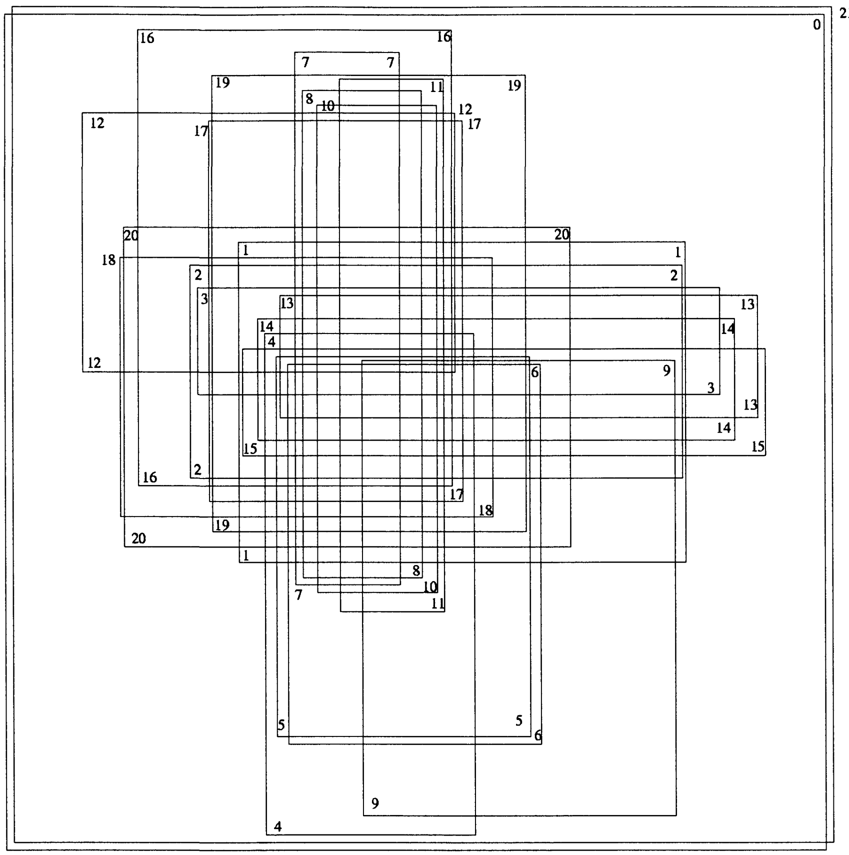


Fig. 3. ZPR of K_{22} using 6 shapes

The largest complete graph using only one size rectangle has size 7, as was shown in Ref. [9]. A representation of this graph is shown in Figure 4. If two sizes of rectangles are used, we can construct a complete graph of size 12, as shown in Figure 1(b). The ZPR in Figure 2 uses 4 sizes; since the set of these four rectangle sizes is closed under rotation by $\frac{\pi}{2}$, rectangles of 4 sizes are sufficient to construct K_{20}. Removal of rectangle 3 gives a K_{18} with 3 sizes. The construction in Figure 3 uses 11 sizes, since rectangles 0 and 21 are square.

2.2. Upper Bounds

It was shown by Fekete, Houle, and Whitesides in Ref. [9] that K_8 does not have a ZPR by unit squares. This implies a very simple upper bound for the maximal size of a K_n with a ZPR using k different sizes: Since any subset consisting of rectangles of the same size can be converted into a set of unit squares by scaling the coordinate axes appropriately, we have $n \leq 7k$, which implies for ℓ shapes that $n \leq 14\ell$. In the same paper[9] it was shown that $n \leq 55$ holds.

The following Table 1 summarizes the results for ZPR's; "min" indicates the best known lower bound, "max" the best known upper bound for the given number

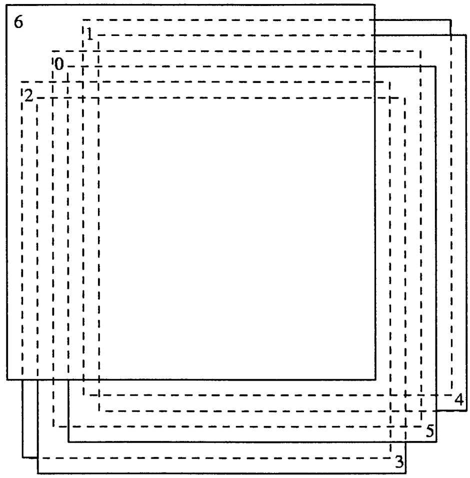


Fig. 4. ZPR of K_7 using one size

of shapes or sizes.

Table 1. Lower and upper bounds for ZPR's

shapes	min	max	sizes	min	max
1	12	14	1	7	7
2	18	28	2	12	14
3	20	42	3	18	21
4	20	55	4	20	28
5	20	55	5	20	35
6	22	55	6	20	42
..	..	..	7	20	49
			8	20	55
			9	20	55
			10	20	55
			11	22	55
			..	..	..

3. Lower Bounds on BR's

Figure 5 shows the projection of 8 unit cubes into the x,y-plane. The cubes have edges of length 1 and the number written inside each projected cube is the z-coordinate of the bottom face of the cube. It can easily be verified that the visibility graph of this arrangement of cubes is complete.

When we allow more shapes, we use some of the constructions of the previous section. For example, the ZPR of K_{22} is a box visibility representation if we transform the 22 rectangles into thin boxes. By placing two sets of these boxes side by side as illustrated in Figure 6, we create a complete box visibility graph of size 44.

Figure 7 shows the projection into the x,y-plane of the boxes of Figure 6. We added four boxes numbered 0, 1, 2 and 3 behind these forty-four boxes, as well

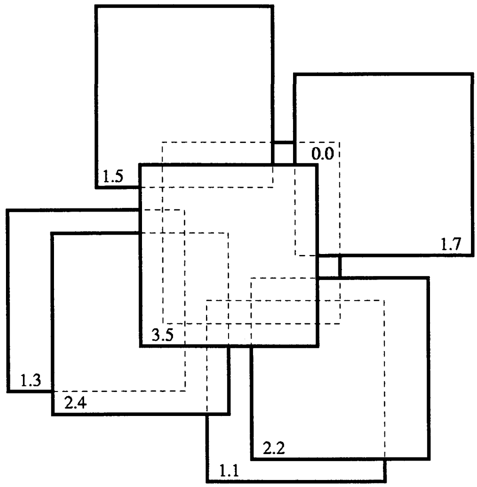


Fig. 5. BR of K_8 using unit cubes, i.e., one size

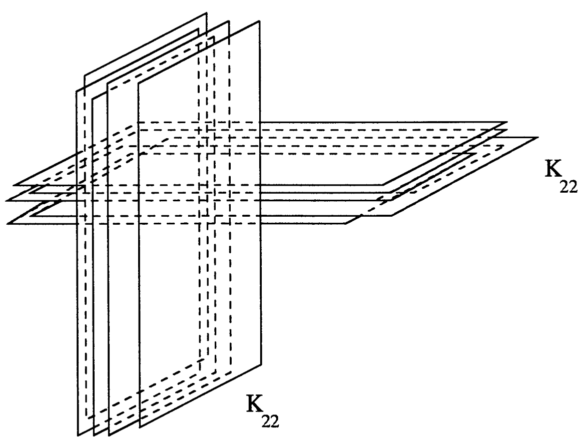


Fig. 6. BR of K_{44}

as three in front of it, numbered 48, 49 and 50. These 51 boxes form a complete visibility graph since

- Boxes 0-3 see boxes 4-25 in circle BM_2.
- Boxes 0-3 see boxes 26-47 in circles BM_1 and BM_3.
- Boxes 0-3 see boxes 48-50 in circle BF.
- Boxes 4-25 see boxes 26-47 in the centre of the drawing.
- Boxes 4-25 see boxes 48-50 in circles FM_1.
- Boxes 26-47 see boxes 48-50 in circle FM_2.

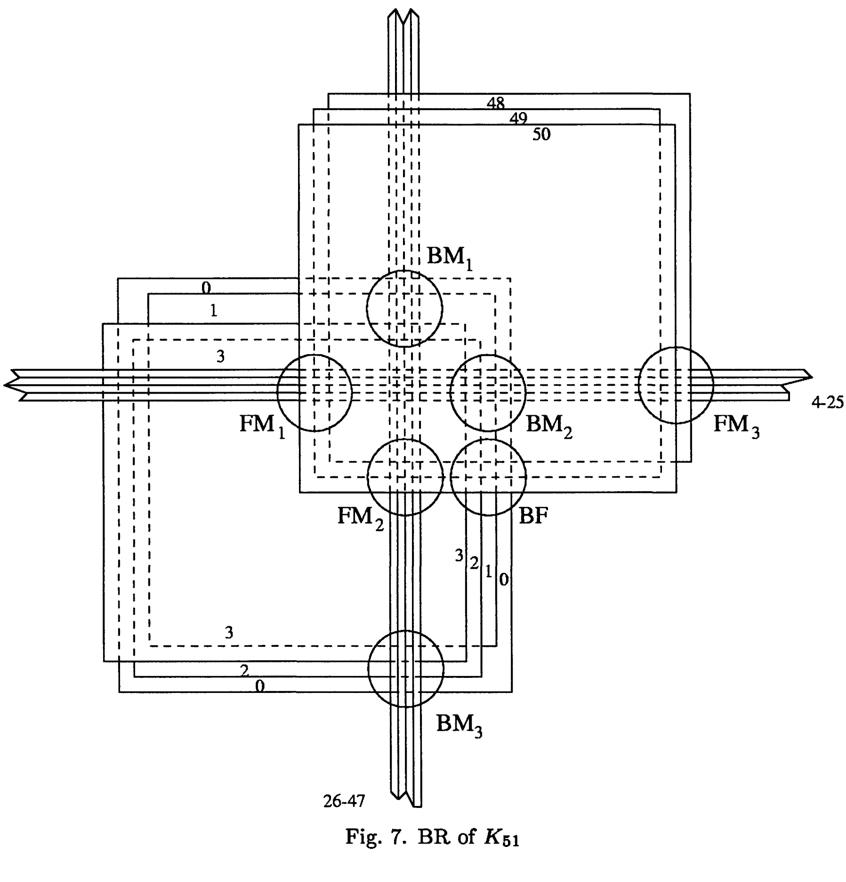


Fig. 7. BR of K_{51}

We use this type of arrangement to construct complete box visibility graphs. The construction uses four sets of boxes, which we call the back boxes (in Figure 7 numbered 0-3), lower middle (4-25) and upper middle (26-47) boxes and the front boxes (48-50) respectively. The lower and upper middle boxes that we use are based on ZPR constructions presented in the previous section. As can be seen from the drawing in Figure 7, the set of back boxes has to be such that all boxes are visible from the front along one side, so that the boxes in one of the middle sets and the boxes in the front set see the back boxes in circles BM_2 and BF respectively. We call this side of the back boxes the sorted side. The remaining middle boxes see the back boxes in circles BM_1 and BM_3, so in general the back boxes also have to be visible from the front along the two sides adjacent to the sorted side. Similarly, the front boxes have to have a sorted side, such that the adjacent sides also show all boxes.

Consider for example the boxes in Figure 8. Their projections into the x,y-plane are shown and in this projection the boxes have two adjacent sorted sides. So this construction can be used as a set of back boxes. The boxes in Figure 9 have a sorted side and the adjacent sides are such that all boxes are visible from the front along these two sides.

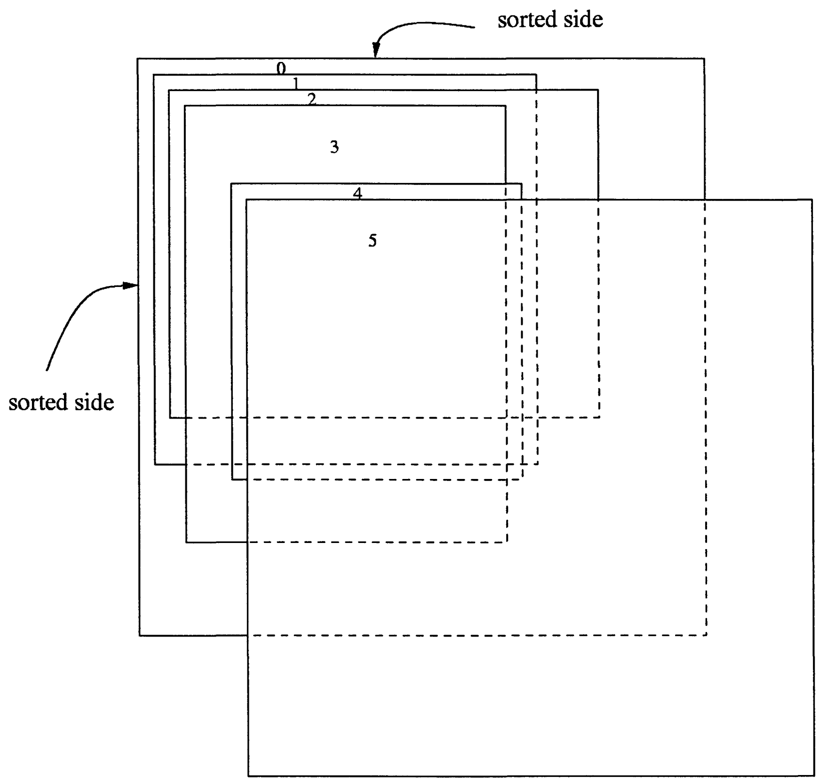


Fig. 8. BR of K_6 using 4 shapes for construction of BR of K_{56}

By combining the K_{20} of Figure 2 and the K_6 of Figure 8, we can construct a complete box visibility graph of size 52. Since the back and front boxes have two sorted sides, we do not need the visibilities in circles BM_3 and FM_3 in Figure 7. This enables us to use boxes in the front and the back with shapes already used in the middle boxes. The construction in Figure 8 is such that

- Squares 0, 4 and 9 in Figure 2 have the same shape as squares 0 and 5 in Figure 8.

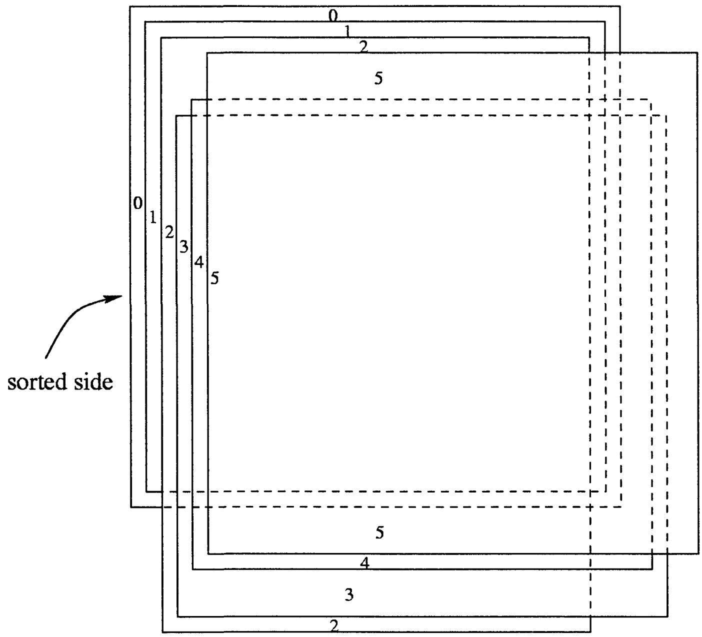


Fig. 9. BR of K_6 using 3 sizes for construction of BR of K_{56}

- Square 1 in Figure 8 is not used in Figure 2.
- Square 3 in Figure 2 has the same shape as square 4 in Figure 8.
- Rectangles 1, 2, 5, 6, 7 and 8 in Figure 2 have the same shape as rectangles 2 and 3 in Figure 8.

Removal of rectangle 1 from Figure 8 gives a K_{50} with 3 shapes. Subsequent removal of rectangle 3 from Figure 2 and rectangle 4 from Figure 8 gives a K_{44} with 2 shapes. The graph K_{56} can be represented by using the construction in Figure 3 and using its shapes to construct front and back boxes as shown in Figure 8. The complete graph K_{30} can be constructed with 1 shape by using the construction in Figure 1(b) together with 3 back and 3 front boxes as shown in Figure 10.

The boxes in Figure 9 have 3 different sizes. The projection into the x,y-plane is such that

- Rectangles 0, 3 and 5 are squares of equal size.
- Rectangles 1 and 4 are squares of equal size.
- Rectangle 2 is not a square.

If we combine the K_{20} of Figure 2 and the K_6 of Figure 9, we can create a complete box visibility graph of size 52 using 12 sizes, namely 3 sizes for the back boxes, one additional size for the front since box 2 has a rectangular projection, plus 4 sizes each for the lower and upper middle boxes. Removal of one or more sizes from the

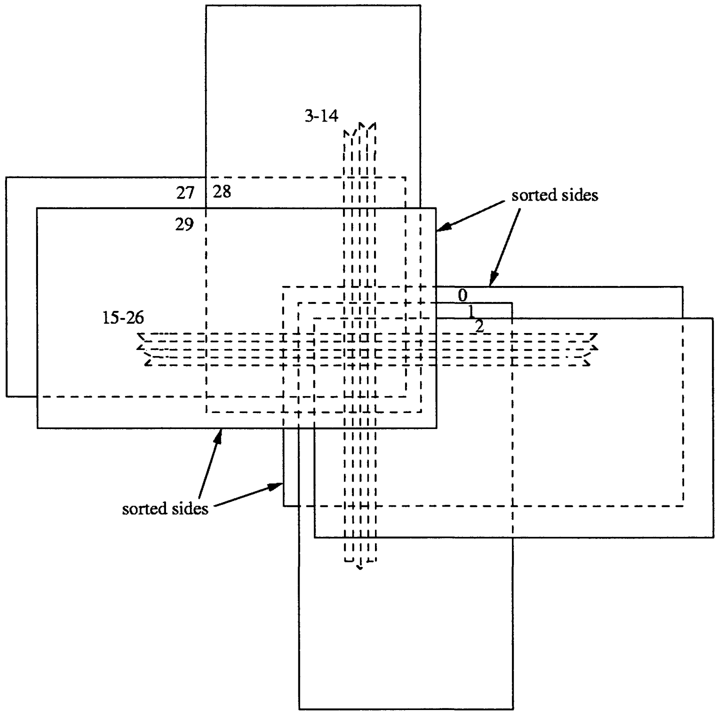


Fig. 10. BR of K_{30} using one shape

different sets give constructions of complete box visibility graphs with fewer sizes. For example, removal of box 2 from the back set shown in Figure 9 results in a K_5 with 2 sizes and therefore a K_{51} with 11 sizes. Removal of squares 1, 2 and 4 from the back set in Figure 9 gives a K_3 with 1 size.

In Table 2, the largest size complete box visibility graphs that we constructed using this approach are listed. For example, with 9 sizes, we can construct a K_{48} by using a K_5 for the back and front sets, using 2 sizes for the back and no additional sizes for the front, K_{20} for the lower middle set using 4 sizes (from Table 1) and a K_{18} for the upper middle set with 3 sizes.

4. Upper Bounds on BR's

In this section, we prove upper bounds for the size of complete graphs with box visibility representations. For this purpose, we show that certain subconfigurations cannot occur. Most of our attention is spent on considering forbidden configurations in the partial orders induced by arrangements of boxes forming a representation of a complete graph.

4.1. Preliminaries

We start with a number of definitions and easy observations.

Definition 1 *For a given axis-parallel box A, we denote by* $\min_x(A)$ *the smallest*

Table 2. Construction of BR's with limited numbers of shapes and sizes

shapes	min	sizes	n	back	lower	upper	front
1	30	1	8				
2	44	2	14	0, 0	1, 7	1, 7	0, 0
3	50	3	20	1, 3	1, 7	1, 7	0, 3
4	52	4	25	1, 3	2, 12	1, 7	0, 3
5	52	5	31	1, 3	3, 18	1, 7	0, 3
6	56	6	36	1, 3	3, 18	2, 12	0, 3
..	..	7	42	1, 3	3, 18	3, 18	0, 3
		8	46	2, 5	3, 18	3, 18	0, 5
		9	48	2, 5	4, 20	3, 18	0, 5
		10	50	2, 5	4, 20	4, 20	0, 5
		11	51	3, 6	4, 20	4, 20	0, 5
		12	52	3, 6	4, 20	4, 20	1, 6
		..	..	..	..	..	..
		17	52	3, 6	4, 20	4, 20	1, 6
		18	53	3, 6	11,22	4, 20	0, 5
		19	54	3, 6	11,22	4, 20	1, 6
		..	..	..	..	..	..
		24	54	3, 6	11,22	4 ,20	1, 6
		25	55	3, 6	11,22	11,22	0, 5
		26	56	3, 6	11,22	11,22	1, 6
		..	..	..	..	..	..

x-coordinate in the box, i. e., the coordinate of the face that is directed towards the negative x-direction. Similarly, we define $\max_x(A)$, $\min_y(A)$, $\max_y(A)$, $\min_z(A)$, *and* $\max_z(A)$. *We write $A \prec\succ B$ to indicate that two boxes A and B see each other. We say that two boxes are x-visible if they see each other in direction parallel to the x-axis and denote this by $A \prec_x\succ B$; if they see each other in the x- or y-direction, we write $A \prec_{xy}\succ B$. If* $\max_x(A) < \min_x(B)$ *and $A \prec\succ B$, we write $A \prec_x B$.*

In the following, we assume that any arrangement of boxes forms a visibility representation of a complete graph. This means that any two boxes must see each other along an axis-parallel line that stabs the interior of both boxes and the line segment connecting the boxes does not meet any points from another box. Without loss of generality we also assume that $\min_i(A) \neq \min_i(B)$ and $\max_i(A) \neq \max_i(B)$ for any two boxes A and B and for $i = x, y, z$. We note an easy observation:

Observation 2 *$\prec_x$, $\prec_y$, and $\prec_z$ induce partial orders on the BR of a complete graph.*

We use the terms *x-chain* or *x-antichain* to indicate a set of boxes that form a chain or antichain in the partial order $\prec_x$.

Lemma 3 *If two boxes A and B see each other, they are* either *x-visible* or *y-visible* or *z-visible, i. e., they cannot see each other in more than one direction.*

Proof. Suppose A and B are x-visible. This means that $\max_x(A) < \min_x(B)$ or $\max_x(B) < \min_x(A)$, so the projections of A and B into the $y = 0$ plane do not intersect, implying that A and B cannot be y-visible. By the same argument, they cannot be z-visible either. □

This means that any edge (a, b) of a complete graph K_n with a particular BR can be considered to have one of three "colors" (the boxes A and B representing a and b are either x-, y-, or z-visible) and a "direction" ($A \prec B$ or $B \prec A$ in the appropriate partial order.) In the following, we prove that

(a) particular ways of coloring and/or orienting edges cannot occur in representations of complete graphs

(b) excluding certain structures limits the size of complete graphs with a representation.

We make use of the following well-known theorem:

Proposition 4 (Dilworth's Theorem) *Let $(P, \prec)$ be a partial order with k being the maximum size of an antichain. Then $(P, \prec)$ can be partitioned into k chains.*

4.2. Forbidden Configurations

In the following, all figures show projections of boxes into the plane $z = 0$ with a horizontal x-axis and a vertical y-axis. Numbers written inside a projection refer to the smallest z-coordinate of the box.

Lemma 5 *If there are boxes C_0, C_1, D_0 and D_1 whose visibility graph is complete with $C_0 \prec_x C_1$, $D_0 \prec_x D_1$, $C_0 \prec_y D_0$ and $C_1 \prec_y\succ D_1$, then $C_1 \prec_y D_1$.*

Proof. If $D_1 \prec_y C_1$ then $\max_y(D_1) < \min_y(C_1) < \max_y(C_0) < \min_y(D_0)$, so $D_1 \prec_y D_0$. However, by assumption $D_0 \prec_x D_1$, so we have $C_1 \prec_y D_1$. □

Lemma 6 *If there are boxes C_0, C_1, D_0 and D_1 whose visibility graph is complete with $C_0 \prec_x C_1$, $D_0 \prec_x D_1$, $C_0 \prec_y D_0$ and $C_1 \prec_y D_1$, then ($C_1 \prec_y D_0$ and $C_0 \prec_x D_1$) or ($C_0 \prec_y D_1$ and $D_0 \prec_x C_1$).*

Proof. Suppose the boxes C_0, C_1, D_0 and D_1 are as described in the lemma. If $C_0 \prec_x\succ D_1$ or $C_0 \prec_z\succ D_1$ or $D_1 \prec_y C_0$ then $\max_y(C_1) < \min_y(D_1) < \max_y(C_0) < \min_y(D_0)$, so $C_1 \prec_y D_0$. In other words, either $C_1 \prec_y D_0$ or $C_0 \prec_y D_1$. If $C_1 \prec_y D_0$, then $\max_x(C_0) < \min_x(C_1) < \max_x(D_0) < \min_x(D_1)$, so $C_0 \prec_x D_1$. Similarly, if $C_0 \prec_y D_1$, then $D_0 \prec_x C_1$. □

Corollary 1 *If the four boxes as described in the lemma exist, then they form a z-antichain. The two possible configurations are shown in Figure 11.*

Lemma 7 *In a box visibility representation of a complete graph there cannot be boxes C_0, C_1, C_2, D_0, D_1 and D_2, such that $C_0 \prec_x C_1 \prec_x C_2$, $D_0 \prec_x D_1 \prec_x D_2$, $C_0 \prec_y\succ D_0$, $C_1 \prec_y\succ D_1$ and $C_2 \prec_y\succ D_2$.*

Proof. Suppose the six boxes as described in the lemma do exist. From Lemma 5 we derive that we may assume without loss of generality that $C_0 \prec_y D_0$, $C_1 \prec_y D_1$ and $C_2 \prec_y D_2$, as shown in Figure 12. We have $\max_x(C_0) < \min_x(C_1) < \max_x(D_1) < \min_x(D_2)$ so $C_0 \prec_x D_2$. Also $\max_x(D_0) < \min_x(D_1) < \max_x(C_1) < \min_x(C_2)$ so $D_0 \prec_x C_2$, which is impossible by Lemma 6. □

The following definition is needed for describing further configurations.

Definition 1 *Let C and D be two sets of boxes such that $|C| \geq |D|$ and $C \cap D = \emptyset$. We say that there is an x-matching (xy-matching) between C and D if for each element in D we can assign a distinct element of C visible in the x-direction (x- or y-direction).*

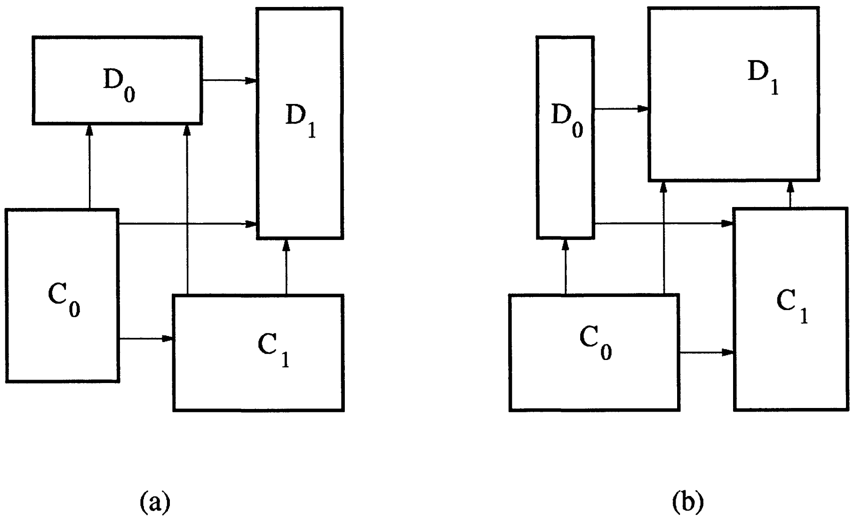


Fig. 11. A z-antichain consisting of two x-chains of length 2.

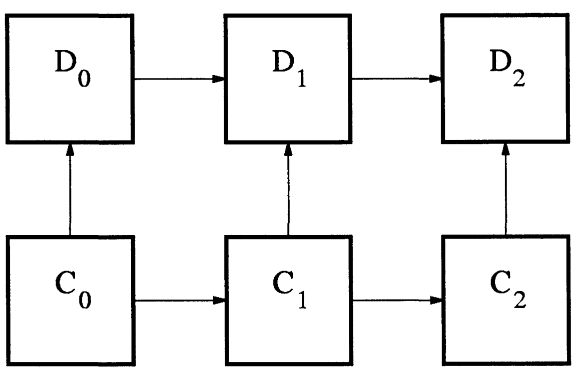


Fig. 12. A z-antichain consisting of two x-chains of length 3.

Lemma 8 *In a box visibility representation of a complete graph, let C be a maximum length x-chain. For any x-chain D with $|C| \geq |D|$ and $C \cap D = \emptyset$, there is a yz-matching between C and D.*

Proof. Let S be a subset of elements in D and let $N_{yz}(S)$ be the boxes in C that are visible in the y- or z-direction from an element in S. Suppose that there is a set S for which $|N_{yz}(S)| < |S|$. Then the set $S \cup C \setminus N_{yz}(S)$ contains only boxes that see each other in the x-direction and $|S \cup C \setminus N_{yz}(S)| > |C|$. So the boxes in this set form an x-chain which is larger than C, which contradicts the fact that C is a maximum length x-chain. Therefore $|N_{yz}(S)| \geq |S|$ for any S, which implies (see Ref. [11]) that there is a yz-matching between C and D. □

Corollary 2 *In a box visibility representation of a complete graph, let C be a maximum length x-chain. For any x-chain D such that $|C| \geq |D|$, $C \cap D = \emptyset$ and $C \cup D$ is a z-antichain, there is a y-matching between C and D.*

Lemma 9 *If there are two x-chains $C_0 \prec_x C_1 \prec_x \cdots \prec_x C_{k-1}$ and $D_0 \prec_x D_1 \prec_x \cdots \prec_x D_{k-1}$, such that there is a yz-matching between these chains, then this matching is unique and consists of edges (C_i, D_i) for $0 \leq i \leq k-1$.*

Proof. Suppose that the edge (C_i, D_j) with $i < j$ is in a yz-matching. Then $C_i \prec_{yz}\succ D_j$, so $\max_x(D_{j-1}) < \min_x(D_j) < \max_x(C_i) < \min_x(C_{i+1})$, in other words, for all D_a and C_b with $a < j$ and $i < b$ we have $D_a \prec_x C_b$. Hence, $\{C_b \mid b > i\}$ and $\{D_a \mid a > j\}$, and $\{C_b \mid b < i\}$ and $\{D_a \mid a < j\}$ must be matched, which is impossible. □

Lemma 10 *In a box visibility representation of a complete graph, let C be a maximum length x-chain. There cannot be an x-chain D of length ≥ 3, such that $C \cap D = \emptyset$ and $C \cup D$ forms a z-antichain.*

Proof. Suppose there are chains C and D as described in the lemma. By Lemma 8, there is a y-matching between C and D. Assume this y-matching exists between the boxes C_0, C_1 and C_2 in C and D_0, D_1 and D_2 in D. Assume without loss of generality that $C_0 \prec_x C_1 \prec_x C_2$ and $D_0 \prec_x D_1 \prec_x D_2$. From Lemma 9 we know that this matching consists of the edges (C_0, D_0), (C_1, D_1) and (C_2, D_2). This is shown to be impossible in Lemma 7. □

Lemma 11 *In a box visibility representation of a complete graph, let C be a maximum length x-chain. There cannot be an x-chain D of length ≥ 5 such that $C \cap D = \emptyset$.*

Proof. Suppose there are chains C and D as described in the lemma, where D has size at least 5. By Lemma 8, there is a yz-matching between C and D. Since this yz-matching has size ≥ 5, it contains either a y- or a z-matching of size 3 by the pigeonhole principle. From Lemmas 9 and 7 we derive that such a configuration cannot exist. □

Lemma 12 *In a box visibility representation of a complete graph, let C be a maximal z-antichain, consisting of the m boxes $C_0, \ldots, C_{m-1}$. Let $a^* = \max\{\min_z(C_i) \mid i = 0, \ldots, m-1\}$ and $b^* = \min\{\max_z(C_i) \mid i = 0, \ldots, m-1\}$. Then the set of all boxes can be partitioned into $C^{(-z)} = \{D \mid \max_z(D) < a^*\}$, C and $C^{(+z)} = \{D \mid \min_z(D) > b^*\}$. Moreover, for any two boxes $D_i \in C^{(-z)}$ and $D_j \in C^{(+z)}$, we have $D_i \prec_z D_j$.*

Proof. Since C is a maximal z-antichain, every element D not in C sees an element C_i in the z-direction. $D \prec_z C_i$ implies $\max_z(D) < a^*$, and $C_i \prec_z D$ implies $b^* < \min_z(D)$. Since C represents a complete graph, we must have $a^* < b^*$ and hence $C^{(-z)}$, C and $C^{(+z)}$ form a partition. $D_i \prec_z D_j$ for all $D_i \in C^{(-z)}$ and $D_j \in C^{(+z)}$ follows from the definition of $C^{(-z)}$ and $C^{(+z)}$. □

Lemma 13 *In a box visibility representation of a complete graph, consider a maximal z-antichain C and the related partition $\{C^{(-z)}, C, C^{(+z)}\}$. If $C^{(-z)}$ contains two elements D_0 and D_1 such that $D_0 \prec_x D_1$, then $C^{(+z)}$ is an x-antichain.*

Proof. Suppose there are $D_0, D_1 \in C^{(-z)}$ with $D_0 \prec_x D_1$ and $D_2, D_3 \in C^{(+z)}$ with $D_2 \prec_x D_3$. Since $\max_z(D_i) < a^* < b^* < \min_z(D_j)$ for $i = 0, 1$, $j = 2, 3$, we get $D_i \prec_z D_j$, contradicting Lemma 6. □

Lemma 14 *In a box visibility representation of a complete graph with no chain longer than 4, there cannot be an antichain of size larger than 8.*

Proof. Let C be a maximum size antichain. Without loss of generality let it be a z-antichain. Since there are no chains longer than 4, there is a decomposition of C into at most 4 y-antichains, i.e., x-chains. Let C_0, C_1, C_2 and C_3 be 4 x-chains in such a decomposition, where some of these chains may have length 0. Without loss of generality assume that the lines parallel to the x-axis passing through all boxes in C_i have y-coordinates larger than the y-coordinates of the lines passing through all boxes in C_{i+1} for $0 \le i < 3$.

If the longest chain C_i has size 4 then from Lemma 10 we derive that C_j has size < 3 for $i \neq j$. So the two longest chains in a decomposition contain a combined total of at most 6 elements. Without loss of generality we assume that C_1 is a maximal chain, otherwise add boxes to C_1 until this is no longer possible. We now have to consider the following two cases:

1. If $|C_0| > 1$, then $|C_0 \cup C_1| \le 6$, $|C_2| \le 1$ because of Lemma 13 and $|C_3| \le 1$ because of Lemma 13, so C has at most 8 elements.

2. If $|C_0| \le 1$, then if necessary, we add boxes to C_2 until we get a maximal chain C_2', possibly reducing C_0, C_1, C_3 to C_0', C_1', C_3'. If $|C_3'| > 1$, then we have the same situation as in case 1. If $|C_3'| \le 1$, then $|C_0'| \le 1$ and $|C_1' \cup C_2'| \le 6$, so C has at most 8 elements.

□

Lemma 15 *Suppose there is no chain longer than 4 in a box visibility representation of a complete graph G. Then G can have at most 18 vertices.*

Proof. Let C be the arrangement of boxes. Since there are no chains longer than 4, there is a decomposition of C into at most 4 z-antichains. So C can be decomposed into z-antichains C_0, C_1, C_2, and C_3, where some of these antichains may have size 0. Without loss of generality, we may assume that C_1 is maximal and for the sets $C_1^{(+z)}$ and $C_1^{(-z)}$ as defined in Lemma 12, $C_1^{(+z)} = C_0$ and $C_1^{(-z)} = C_2 \cup C_3$. Now consider the following cases:

1. C_0 contains a nontrivial x-chain and a nontrivial y-chain. By Lemma 13, neither C_2 nor C_3 can contain a nontrivial x- or y-chain, implying $|C_2| = |C_3| = 1$. By Lemma 14, C_0 and C_1 cannot contain more than 8 elements, meaning that C contains at most $8 + 8 + 1 + 1 = 18$ elements.

2. C_0 contains a nontrivial x-chain, but no nontrivial y-chain. By assumption, C_0 contains at most 4 elements; by Lemma 14, C_1 contains at most 8 elements. By Lemma 13, $C_2 \cup C_3$ cannot contain a nontrivial x-chain, so the z-antichains C_2 and C_3 must be y-chains and $C_2 \cup C_3$ is an x-antichain. Suppose $C_2 \cup C_3$ contains more than 6 elements, then one of these two y-chains has size 4 and the other has size ≥ 3. Lemma 10 shows that this is not possible. Therefore, $C_2 \cup C_3$ can contain at most 6 elements, implying that C contains at most $4 + 8 + 6 = 18$ boxes.

3. C_0 contains no nontrivial x- or y-chains, i. e., contains at most one element. We add boxes to C_2 until it is a maximal z-antichain C_2'; possibly, we reduce C_0, C_1, C_3 to C_0', C_1', C_3'. If C_3' contains more than one element, then cases 1 and 2 show that the size of C is at most 18. If C_3' has size ≤ 1, than C contains at most $1 + 8 + 8 + 1 = 18$ boxes.

□

4.3. An Upper Bound on BR's

With the help of the Lemmas proved in the previous section, we can now establish an upper bound for the size of a complete graph that has a BR by general boxes:

Theorem 16 *There is no BR for K_{184}.*

Proof. Let C be a set of boxes whose visibility graph is complete. As was shown in Ref. [9], a largest x-chain C_x in C cannot exceed length 55. By Lemma 11, there cannot be an x-chain of length 5 that is disjoint from C_x. This means that after removing a maximum size x-chain, a maximum size y-chain, and a maximum size z-chain from C we are left with a BR such that there are no chains longer than 4. As shown in Lemma 15, this BR contains at most 18 boxes, implying that the size of C is at most $3 \times 55 + 18 = 183$. □

5. Upper Bounds on BR's with Limited Numbers of Shapes and Sizes

In this section, we prove that there cannot be a BR by unit cubes for a complete graph with ten vertices. This implies upper bounds for limited numbers of shapes and sizes.

5.1. Forbidden Configurations

Lemma 17 *In a unit cube box visibility representation of a complete graph, there cannot be boxes C_0, C_1, C_2, D_0, D_1 and D_2, such that $C_0 \prec_x C_1 \prec_x C_2$, $D_0 \prec_y D_1 \prec_y D_2$, and ($C_0 = D_0$ or $C_0 \prec_z\succ D_0$ or $C_0 = D_1$).*

Proof. The three forbidden configurations are shown in Figure 13.

If $C_0 = D_0$, then $\max_x(D_2) = \min_x(D_2) + 1 < \max_x(C_0) + 1 < \max_x(C_1) < \min_x(C_2)$, so $D_2 \prec_x C_2$. However, $\max_y(C_2) = \min_y(C_2) + 1 < \max_y(D_0) + 1 < \max_y(D_1) < \min_y(D_2)$, which implies $C_2 \prec_y D_2$.

If $C_0 \prec_z\succ D_0$, then $\max_y(C_0) = \min_y(C_0) + 1 < \max_y(D_0) + 1 < \min_y(D_2)$, so $C_0 \prec_y D_2$. Similarly $D_0 \prec_x C_2$. The previous argument for the case $C_0 = D_0$ now applies here as well.

If $C_0 = D_1$, then $\max_x(D_2) = \min_x(D_2) + 1 < \max_x(C_0) + 1 < \min_x(C_2)$, so $D_2 \prec_x C_2$ and $\max_x(D_0) = \min_x(D_0) + 1 < \max_x(C_0) + 1 < \min_x(C_2)$, so $D_0 \prec_x C_2$. However, since $\min_y(D_2) - \max_y(D_0) > 1$, this is impossible. □

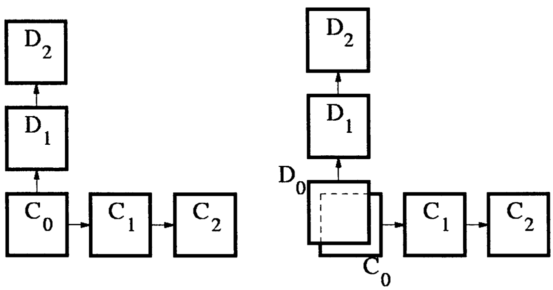


Fig. 13. Three forbidden configurations.

Lemma 18 *If in a unit cube box visibility representation of a complete graph, there are boxes C_0, C_1, C_2, D_0, D_1 and D_2, such that $C_0 \prec_x C_1 \prec_x C_2$ and $D_0 \prec_y D_1 \prec_y D_2$, then $C_1 = D_1$ or $C_1 \prec_z\succ D_1$.*

Proof. It is an immediate consequence of Lemma 17 that C_1 cannot see D_1 in the x- or y-direction. Figure 14 shows that the two chains of cubes can form a complete graph of size 5 or 6. □

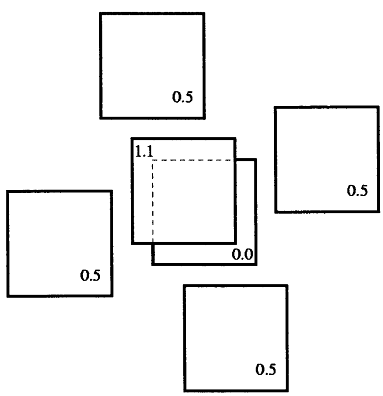


Fig. 14. Two chains of size 3 forming K_6.

Lemma 19 *In a unit cube box visibility representation of a complete graph, there cannot be an x-chain of size 4 and a y-chain of size 3.*

Proof. Suppose the two chains do exist. Let the x-chain consist of the boxes C_0, C_1, C_2 and C_3 with $C_0 \prec_x C_1 \prec_x C_2 \prec_x C_3$. Let the y-chain be $D_0 \prec_y D_1 \prec_y D_2$. If $C_3 \prec_y\succ D_2$ or $C_3 \prec_z\succ D_2$, then $\max_x(C_0) < \min_x(C_3) - 2 < \max_x(D_2) - 2 = \min_x(D_2) - 1 < \max_x(D_0) - 1 = \min_x(D_0)$, so $C_0 \prec_x D_2$ and $C_0 \prec_x D_0$, which is impossible. Therefore $C_3 \prec_x\succ D_2$. Similarly we can derive that $C_3 \prec_x\succ D_0$, which is impossible. □

Lemma 20 *If in a unit cube box visibility representation of a complete graph, there are boxes C_0, C_1, C_2, D_0, D_1 and D_2, such that $C_0 \prec_x C_1 \prec_x C_2$ and $D_0 \prec_y D_1 \prec_y D_2$, then ($C_0 \prec_x D_0 \prec_y C_2$ and $C_0 \prec_y D_2 \prec_x C_2$) or ($D_0 \prec_y C_0 \prec_x D_2$ and $D_0 \prec_x C_2 \prec_y D_2$).*

Proof. We know from Lemmas 17 and 18 that $C_1 = D_1$ or $C_1 \prec_z\succ D_1$ and that for all $i \in \{0, 2\}$ and $j \in \{0, 2\}$ it is not the case that $C_i \prec_z\succ D_j$.

If $C_0 \prec_x\succ D_0$ it follows that $C_0 \prec_x D_0$. It cannot be the case that $D_0 \prec_x C_2$, since we would have $C_0 \prec_x D_0 \prec_x C_2$ and $D_0 \prec_y D_1 \prec_y D_2$, which is not possible by Lemma 17. So $D_0 \prec_y C_2$. From $C_0 \prec_x D_0$ we can conclude that C_0 does not see D_2 in the x-direction, so $C_0 \prec_y D_2$. The latter in its turn implies that $D_2 \prec_x C_2$. Figure 14 shows a configuration satisfying these visibilities.

Similarly, if $C_0 \prec_y\succ D_0$, we derive that $D_0 \prec_y C_0 \prec_x D_2$ and $D_0 \prec_x C_2 \prec_y D_2$, which proves the lemma. □

Lemma 21 *In a unit cube box visibility representation of a complete graph, there cannot be chains of size 3 in the x-, y- and z-direction.*

Proof. Suppose such chains exist. Let these chains be defined by $C_0 \prec_x C_1 \prec_x C_2$, $D_0 \prec_y D_1 \prec_y D_2$ and $E_0 \prec_z E_1 \prec_z E_2$. From Lemma 20 we know that the x- and y-chains look as in Figure 14, with or without one of the middle cubes. Because the distance between C_0 and C_2 as well as between D_0 and D_2 is less than 2, we know that $E_1 \in \{C_1, D_1\}$. Because of symmetry we conclude that $C_1 \in \{E_1, D_1\}$ and $D_1 \in \{C_1, E_1\}$, which implies that $C_1 = D_1 = E_1$.

If the boxes C_0 and C_2 are above the middle cube C_1, i.e., if $\min_z(C_i) > \min_z(C_1)$ for $i = 0, 2$, then E_0 can see one, but not both of these two boxes. In the same way, we exclude the situation where C_1 lies above both C_0 and C_2. Therefore without loss of generality we may assume that $\min_z(C_0) < \min_z(C_1) < \min_z(C_2)$ and $\min_z(D_0) < \min_z(C_1) < \min_z(D_2)$. So $C_0 \prec_z E_2$, $D_0 \prec_z E_2$, $E_0 \prec_z C_2$ and $E_0 \prec_z D_2$. Therefore, $\min_x(E_2) < \min_x(C_1) < \min_x(E_0)$ and $\min_y(E_2) < \min_y(C_1) < \min_y(E_0)$, which means that C_1 blocks the lines of sight between E_0 and E_2, so the visibility graph is not complete. □

Lemma 22 *In a unit cube box visibility representation of a complete graph, there cannot be x-chains C and D of length 4 with a yz-matching between C and D.*

Proof. Let the chains C and D be defined by $C_0 \prec_x C_1 \prec_x C_2 \prec_x C_3$ and $D_0 \prec_x D_1 \prec_x D_2 \prec_x D_3$. We know from Lemma 9 that the yz-matching between C and D consists of edges (C_i, D_i) for $0 \le i \le 3$. From Lemma 7 we derive that two edges correspond to y- and two to z-visibilities.

Without loss of generality assume that $C_0 \prec_y D_0$. We first show that it is impossible that $C_2 \prec_y\succ D_2$ or $C_3 \prec_y\succ D_3$. Suppose that $C_2 \prec_y\succ D_2$. From Lemmas 5 and 6 we derive that we may assume that $C_2 \prec_y D_2$ and $C_0 \prec_y D_2$. So

$\max_x(C_0) > \min_x(D_2)$. However, $\max_x(C_0) = \min_x(C_0) + 1 < \max_x(D_0) + 1 < \min_x(D_2)$, which is a contradiction.

Therefore we may assume that the yz-matching between C and D consists of $C_0 \prec_y D_0$, $C_1 \prec_y D_1$, $C_2 \prec_z D_2$ and $C_3 \prec_z D_3$. Because of Lemma 6, assume that $D_0 \prec_x C_1$ and $C_0 \prec_y D_1$, as shown in Figure 15. Since $\max_x(C_1) < \min_x(C_2) < \max_x(D_2) < \min_x(D_3)$, we know that $C_1 \prec_x D_3$, so $C_0 \prec_x D_3$ as well. Similarly, we derive that $D_1 \prec_x C_3$.

We show that in the projection into the plane $z = 0$, the lines of sight between C_0 and D_3 and between C_0 and C_3 intersect C_1. Since $D_0 \prec_x C_1$, we have $\max_y(C_0) < \max_y(C_1) < \min_y(D_1)$. Now

$$D_1 \prec_x C_3 \Longrightarrow \min_y(D_1) < \max_y(C_3) \Longrightarrow \min_y(C_1) < \min_y(C_3),$$

$$D_1 \prec_x D_3 \Longrightarrow \min_y(D_1) < \max_y(D_3) \Longrightarrow \min_y(C_1) < \min_y(D_3).$$

This implies that the z-coordinates of all lines of sight between C_3 and C_0 and between D_3 and C_0 are larger than $\max_z(C_1)$, or all are smaller than $\min_z(C_1)$. Assume without loss of generality that they are all larger. Then $\max_z(C_1) < \max_z(C_3) < \min_z(D_3)$, so $C_1 \prec_z D_3$, which contradicts the fact that $C_1 \prec_x D_3$. □

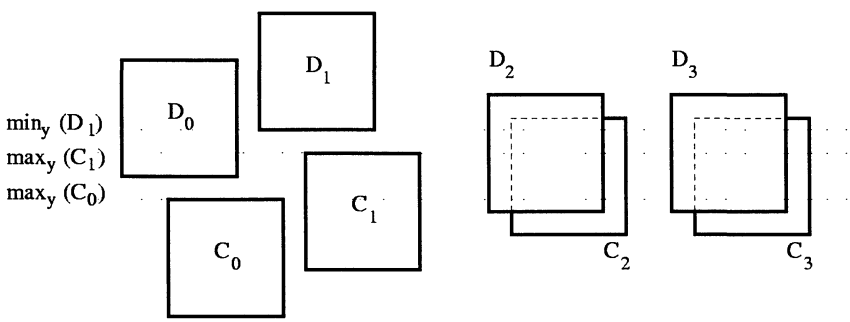


Fig. 15. Two x-chains of size 4.

Figure 16 shows that the above lemma does not hold if the chains C and D have length 3.

Lemma 23 *In a unit cube box visibility representation of a complete graph, with z-antichain $C = \{C_0, C_1, C_2, C_3\}$, there can be at most one box D with $C_i \prec_z D$ for all i.*

Proof. If C contains a chain of length 3, no box can see all elements in C in the z-direction, so C has one of the configurations shown in Figure 11. It is not hard to see that at most one box can be placed on top of C. □

Lemma 24 *In a unit cube box visibility representation of a complete graph, with z-antichain $C = \{C_0, C_1, C_2\}$, there can be at most two boxes D_j with $C_i \prec_z D_j$ for all i and j.*

Proof. If C contains no chain of length 3, we may assume that its configuration satisfies $C_0 \prec_x C_1$, $C_0 \prec_y C_2$ and $C_1 \prec_y C_2$. We first observe that

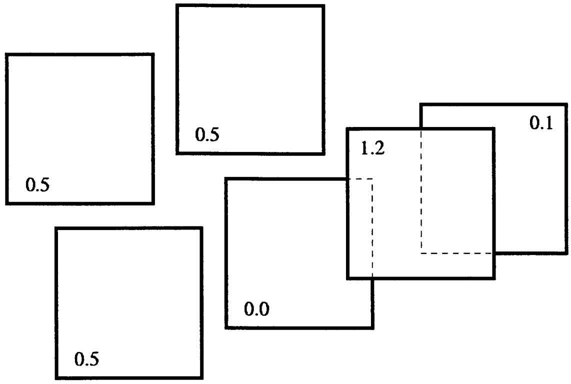


Fig. 16. Two x-chains of size 3 forming K_6.

any box placed above C covers the rectangle defined by $\min(\max_y(C_0), \max_y(C_1))$, $\min_y(C_2)$, $\max_x(C_0)$ and $\min_x(C_1)$, so we may assume that $C_i \prec_z D_0 \prec_z D_1 \prec_z D_2 \prec_z \ldots$ for all i and that the configuration looks similar to the one shown in Figure 17. In the projection onto the $z = 0$ plane as shown in Figure 17, the right-bottom corner of D_0 is in C_1, so D_1 cannot lie left above D_0. The left-bottom corner of D_0 lies in C_0, so D_1 cannot lie right above D_0. Therefore, the chain $D_0 \prec_z D_1 \prec_z \ldots$ is such that each box is placed below the previous one. Since these boxes also have to see C_2 in the z-direction, the chain $D_0 \prec_z D_1 \prec_z \ldots$ consists of at most 2 boxes. □

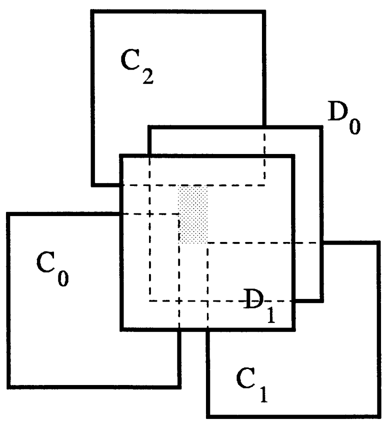


Fig. 17. Size 3 z-antichain plus size 2 z-chain.

5.2. *An Upper Bound on BR's with Limited Numbers of Shapes and Sizes*

Theorem 25 *There is no BR for K_{10} using only unit cubes.*

Proof. By Lemma 21, we may assume that in at most two directions the longest chain has length ≥ 3. If there is no direction in which there is a chain of length 3, then it follows straightforwardly from Dilworth's Theorem 4 that the number of boxes is at most 8. So we assume that there is at least one chain of length 3. We first deal with the case that there is only one direction in which there is a chain of length 3; assume that this is the z-direction. Let C be a maximum size z-antichain. If there is a second maximal z-antichain D, it follows from Lemma 12 that either C is "higher" than D (i. e., there are no $C_i \in C$ and $D_j \in D$ such that $C_i \prec_z D_j$), or D is higher than C. Without loss of generality let C be a highest maximal z-antichain.

Let C consist of the boxes $C_0, C_1, \ldots, C_{m-1}$ with $\max_z(C_i) > \max_z(C_{i+1})$ for $0 \le i < m-1$. Since there are no x- or y-chains of size 3, we have $m \le 4$.

Define $A^* = \max_z(C_0)$, $a^* = \max_z(C_{m-1})$, $b^* = \min_z(C_0)$, $B^* = \min_z(C_{m-1})$, so $B^* \le b^* < a^* \le A^*$, where equality only holds when $m = 1$. Consider the partition $\{C^{(-z)}, C, C^{(+z)}\}$, introduced in Lemma 12. We further partition $C^{(-z)}$ into the sets $C^{(-z)}_{far} = \{C_i \mid \max_z(C_i) < B^*\}$ and $C^{(-z)}_{near} = C^{(-z)} \setminus C^{(-z)}_{far}$. Also $C^{(+z)}_{far} = \{C_i \mid \min_z(C_i) > A^*\}$ and $C^{(+z)}_{near} = C^{(+z)} \setminus C^{(+z)}_{far}$. So we have a partition $\{C^{(-z)}_{far}, C^{(-z)}_{near}, C, C^{(+z)}_{near}, C^{(+z)}_{far}\}$. Notice that $C^{(+z)}_{near} \cup \{C_0\}$ and $C^{(-z)}_{near} \cup \{C_{m-1}\}$ are z-antichains, so $|C^{(-z)}_{near}| \le m-1$. Since C is the highest z-antichain of size m, we derive that $|C^{(+z)}_{near}| \le m-2$. Moreover, $C_i \prec_z C_0$ for all $C_i \in C^{(-z)}$ and $C_{m-1} \prec_z C_i$ for all $C_i \in C^{(+z)}$. We proceed by examining the different possible values of m.

1. $|C| = 1$. The set of all boxes forms a z-chain, so the largest graph with a BR has size 7 (Ref. [9]).

2. $|C| = 2$. Suppose there are 10 cubes forming a complete visibility graph. Partition the set of boxes D into k sets $D_0, D_1, \ldots,$ by taking $D_0 =$ set of minimal elements with respect to $\prec_z$, $D_1 =$ set of minimal elements in $D \setminus D_0$, $D_2 =$ set of minimal elements in $D \setminus (D_0 \cup D_1)$ etc. Since $|C| = 2$, we have $|D_i| \le 2$.

   The boxes with the lowest z-coordinate in each set D_i form a z-chain of length k, so $k \le 7$ following Ref. [9]. Therefore at least 3 sets D_i have size 2. Let D_a, D_b and D_c be three sets of size 2 with $a < b < c$. Using an argument similar to the one used in the proof of Lemma 8, we know that there must be z-matching between D_a and D_b and between D_b and D_c. So the boxes in D_a, D_b and D_c can be partitioned into z-chains $A_a \prec_z A_b \prec_z A_c$ and $B_a \prec_z B_b \prec_z B_c$, with $D_i = \{A_i, B_i\}$ for $i = a, b, c$.

   Assume without loss of generality that $A_a \prec_x B_a$. From Lemma 6 we derive that $A_c \prec_y\succ B_c$. Without loss of generality assume that $A_b \prec_x\succ B_b$. It is not hard to see that this implies that $A_b \prec_x B_b$. So boxes A_a, A_b, B_a and B_b are positioned as shown in Figure 18(a) with $A_b \prec_x B_a$ or they are such that $A_a \prec_x B_b$. We only consider the first case, since the proof for the latter proceeds similarly.

   We derive from $A_b \prec_x B_a$ that $a+1 = b$. We show that there are at most two boxes A_i with $A_i \prec_z A_a$ and at most two boxes C_i with $B_b \prec_z C_i$. Consider the boxes C_i. We first assume without loss of generality that $\max_y(B_a) > \max_y(B_b)$. The box C_i has to see A_b and B_a in the z-direction, so seen from a point at $y = +\infty$, the line of sight between C_i and B_a passes over B_b, or more precisely, $\max_y(B_b) < \max_y(C_i)$. If $C_0 \prec_x\succ C_1$ then C_0 and C_1 cannot both see A_b and B_a and if $C_0 \prec_y\succ C_1$ they cannot both see B_a and B_b in the z-direction. From this it follows that the boxes C_i form a z-chain $C_0 \prec_z C_1 \prec_z \ldots$.

In the projection into the plane $z = 0$, the right-bottom corner of C_0 (see Figure 18(b)) is in B_b, so C_1 cannot lie left above C_0. The right-top corner of C_0 lies in the shaded area shown which contains B_a, so C_1 cannot lie left below C_0. So the chain $C_0 \prec_z C_1 \prec_z \ldots$ is such that each box is placed to the right of the previous one. Since these boxes also have to see A_b in the z-direction, the chain $C_0 \prec_z C_1 \prec_z \ldots$ consists of at most 2 boxes.

Similarly, we show that there at most two boxes A_i with $A_i \prec_z A_a$. Finally, since the maximal z-antichain has size 2, there can be at most one more box which sees B_b in the x or y direction (either A_c or B_c), so the number of boxes cannot exceed $4 + 2 + 2 + 1 = 9$.

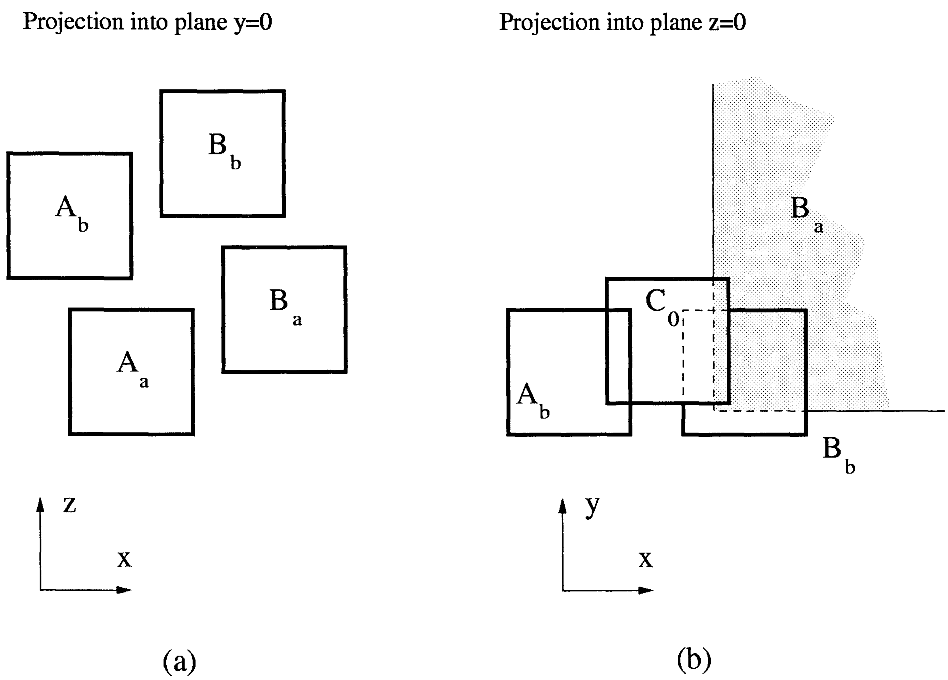


Fig. 18. Maximal z-antichain of size 2

3. $|C| = 3$. From Lemma 24 we know that $|C^{(+z)}_{far}| \leq 2$ and $|C^{(-z)}_{far}| \leq 2$. If $|C^{(-z)}_{near}| = 2$, then for the antichain $D := C^{(-z)}_{near} \cup \{C_{m-1}\}$, we have $|D| = 3$; by definition, we have $B_i \prec_z B_j$ for any $B_i \in D$ and $B_j \in C^{(+z)}$, so by

Lemma 24 $|C^{(+z)}| \leq 2$, and the number of boxes is $|C^{(-z)}_{far}| + |C^{(-z)}_{near}| + |C| + |C^{(+z)}| \leq 2 + 2 + 3 + 2 = 9$. If $|C^{(-z)}_{near}| \leq 1$, then the number of boxes is $|C^{(-z)}_{far}| + |C^{(-z)}_{near}| + |C| + |C^{(+z)}_{near}| + |C^{(+z)}_{far}| \leq 2 + 1 + 3 + 1 + 2 = 9$.

4. $|C| = 4$. From Lemma 23 we know that $|C^{(+z)}_{far}| \leq 1$ and $|C^{(-z)}_{far}| \leq 1$. If $|C^{(-z)}_{near}| = 3$, then $|C^{(-z)}_{near} \cup \{C_{m-1}\}| = 4$. If $|C^{(-z)}_{near}| = 3$, then for the antichain $D := C^{(-z)}_{near} \cup \{C_{m-1}\}$, we have $|D| = 4$; by definition, we have $B_i \prec_z B_j$ for any $B_i \in D$ and $B_j \in C^{(+z)}$, so by Lemma 24 $|C^{(+z)}| \leq 1$, and the number of boxes is $|C^{(-z)}_{far}|+|C^{(-z)}_{near}|+|C|+|C^{(+z)}| \leq 1+3+4+1 = 9$. If $|C^{(-z)}_{near}| = 2$, then for $D := C^{(-z)}_{near} \cup \{C_{m-1}\}$, we have $|D| = 3$; by definition, we have $B_i \prec_z B_j$ for any $B_i \in D$ and $B_j \in C^{(+z)}$, so by Lemma 24 $|C^{(+z)}| \leq 2$, and the number of boxes is $|C^{(-z)}_{far}| + |C^{(-z)}_{near}| + |C| + |C^{(+z)}| \leq 1+2+4+2 = 9$. If $|C^{(-z)}_{near}| \leq 1$, then the number of boxes is $|C^{(-z)}_{far}| + |C^{(-z)}_{near}| + |C| + |C^{(+z)}_{near}| + |C^{(+z)}_{far}| \leq 1+1+4+2+1 = 9$.

We now deal with the case that there are two directions with chains of length at least 3. Suppose that the longest chains in the x- and y-direction have length at least 3. From Lemma 19 we know that both chains have length 3. Lemma 20 describes the configuration of the two chains, which is demonstrated in Figure 14. It can easily be seen that in this case any z-antichain has size at most 5.

Suppose there are 10 cubes forming a complete visibility graph. Let C be the set of all boxes that are minimal with respect to $\prec_z$ and let D be the set of all boxes that are maximal with respect to $\prec_z$. Clearly, C and D are both z-antichains. Since there is no z-chain of length 3, we have $|C \cup D| = 10$; since $|C| \leq 5$ and $|D| \leq 5$, we get $C \cap D = \emptyset$. Using an argument similar to the one used in the proof of Lemma 8, we know that there is a z-matching between C and D. Let C_4 be the highest cube in C and let D_4 be the cube in D that is matched to C_4. Then D_4 must see all cubes of C in the z-direction, including all three elements of an x- or y-chain of length 3, which is impossible. Therefore in this case the number of boxes is at most 9.

□

This leaves a gap of size 1 between the best known lower bound and the best proven upper bound. The preceding arguments give a large amount of structure for any potential BR of K_9 by unit cubes. We expect that lowering the upper bound to 8 by going through all the cases is possible but very tedious. In the interest of readability of this paper, we leave this question open for the time being. However, we do believe the following:

Conjecture 26 *K_9 does not have a BR by unit cubes.*

Using the result of Theorem 25, we can now find upper bounds on BR's with a limited number of shapes and size. These bounds follow immediately from the above theorem and the fact that a rectangle of a certain shape can have 6 different sizes.

Theorem 27 *A BR of a complete graph using boxes of k different shapes has size at most equal to $54k$.*

Theorem 28 *A BR of a complete graph using boxes of k different sizes has size at most equal to* $9k$.

The results for complete graphs that have a BR are summarized in the following Table 3; again, "min" indicates the largest known representation of a complete graph using only the given number of shapes or sizes, while "max" is the best known upper bound.

Table 3. Lower and upper bounds for BR's

shapes	min	max	sizes	min	max
1	30	54	1	8	9
2	44	108	2	14	18
3	50	162	3	20	27
4	52	183	4	25	36
5	52	183	5	31	45
6	56	183	6	36	54
..	..	...	7	42	63
			8	46	72
			9	48	81
			10	50	90
			11	51	99
			12	52	108
			..	..	...
			17	52	153
			18	53	162
			19	54	171
			..	..	...
			24	54	183
			25	55	183
			26	56	183
			..	..	...

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