On Minimum Stars, Minimum Steiner Stars, and Maximum Matchings

Sándor P. Fekete *
Center for Parallel Computing
Universität zu Köln
50923 Köln
GERMANY
sandor@zpr.uni-koeln.de

Henk Meijer †
Department of Computer Science
Queen's University
Kingston, Ont K7L 3N6
CANADA
henk@cs.queensu.ca

Abstract

We discuss properties and values of maximum matchings and minimum median problems for finite point sets. In particular, we consider "minimum stars", which are defined by a center chosen from the given point set, such that the total geometric distance $\min \|St\|$ to all the points in the set is minimized. If the center point is not required to be an element of the set (i. e., the center may be a Steiner point), we get a "minimum Steiner star", of total length $\min \|StSt\|$. As a consequence of triangle inequality, the total length $\max \|Mat\|$ of any maximum matching is a lower bound for the length $\min \|StSt\|$ of a minimum Steiner star, which makes the ratio $\frac{\min \|StSt\|}{\max \|Mat\|}$ interesting in the context of optimal communication networks. The ratio also appears as the duality gap in an integer programming formulation of a location problem by Tamir and Mitchell.

In this paper, we show that for an even set of points in the plane and Euclidean distances, the ratio $\frac{\min \|StSt\|}{\max \|Mat\|}$ cannot exceed $2/\sqrt{3}$. This proves a conjecture of Suri, who gave an example where this bound is achieved. For the case of Euclidean distances in two and three dimensions, we also prove upper and lower bounds for the maximal value of the ratios $\frac{\min \|St\|}{\min \|StSt\|}$ and $\frac{\min \|St\|}{\max \|Mat\|}$. We give tight upper bounds for the case where distances are measured according to the Manhattan metric: we show that in three-dimensional space, $\frac{\min \|StSt\|}{\max \|Mat\|}$ is bounded by 3/2, while in two-dimensional space $\min \|StSt\| = \max \|Mat\|$, extending some independent observations by Tamir and Mitchell. Finally, we show that $\frac{\min \|St\|}{\min \|StSt\|}$ is bounded by 3/2 in the two-dimensional case, and by 5/3 in the three-dimensional case.

*Parts of this work were done while visiting Queen's University, partially supported by the Deutsche Forschungsgemeinschaft, FE 407/3-1.

†Parts of this work were done while visiting Universität zu Köln, partially supported by NSERC.

SCG'99 Miami Beach Florida

1 Introduction

The problem of finding a maximum weight *matching* for a given set of vertices in a weighted graph is to find a set of disjoint edges, such that the total weight of all the edges is maximized. Determining an optimal matching is a classical algorithmic problem, and Edmonds' famous polynomial algorithm [6] is one of the milestones of combinatorial optimization.

On the other hand, it has been known for quite a while [9] that the task of finding a minimum weight *Steiner tree* is an NP-hard problem: find a network of smallest total length $\min \|StT\|$ that connects all given points, while allowing additional "Steiner" points for connecting edges. This algorithmic intractability differs drastically from the case where no Steiner points are allowed, so that the connected network has to be a minimum weight spanning tree (MST) of weight $\|MST\|$, which can be solved very efficiently. Many aspects of optimal Steiner trees have been considered, see the book [11] for an overview. One of the most famous problems related to geometric Steiner trees deals with the largest possible value of the ratio $\frac{\|MST\|}{\min \|StT\|}$. As Du and Hwang [5] managed to prove for the case of planar point sets with Euclidean distances, this ratio cannot exceed the value of $2/\sqrt{3}$, which is tight.

A special type of Steiner tree problems arises in the context of location theory: The so-called *Weber problem* asks for the location of a single center point, such that the sum of distances from the given points to the center is minimized. It was shown by Bajaj [1] that even for the simple case of 5 points in the Euclidean plane, a solution can in general not be expressed by radicals. (In particular, it is impossible to construct an optimal solution by means of ruler and compass). In the context of communication networks, the resulting tree has been called a *star* [8]. As in the case of general tree networks, we can distinguish the Steiner case (where the center point can be chosen anywhere) from the more restricted case, where the center point is required to be chosen from the given set. In the following, we will speak of "Minimum Steiner stars" (with a total edge length denoted by $\min \|StSt\|$) and "Minimum stars" (with a total edge length denoted by $\min \|St\|$).

When dealing with algorithmically hard problems like the task of designing optimal communication networks, it is of great importance to provide good upper and lower bounds for an optimal solution. It has been pointed out by Fingerhut, Suri, and Turner [8] that $\max \|Mat\|$ is a lower bound for $\min \|StSt\|$, which is an upper bound for $\min \|StT\|$.

This makes it interesting to consider the worst-case behavior of the ratio $\frac{\min \|StSt\|}{\max \|Mat\|}$. It was conjectured by Suri [14] that for the case of points in the plane with Euclidean distances, this ratio is bounded by $2/\sqrt{3}$ – the Steiner tree ratio. Proving this conjecture is one of the main results of this paper. In addition, we consider the worst-case behavior of the ratios $\frac{\min \|St\|}{\min \|StSt\|}$ and $\frac{\min \|St\|}{\max \|Mat\|}$. For the case of Euclidean distances in two and three dimensions, we prove upper and lower bounds for the largest possible values of these ratios.

The above problems are also of interest when distances are not measured according to the Euclidean metric. Of particular relevance is the case of rectilinear (or "Manhattan") distances, which arises in the context of VLSI layout. Tamir and Mitchell [15] have considered the ratio $\frac{\min \|StSt\|}{\max \|Mat\|}$ for the case of rectilinear distances, motivated by questions from cost allocation for matching games. They prove that in the case of points in the plane, $\frac{\min \|StSt\|}{\max \|Mat\|} = 1$, which implies that the core of a related matching game is nonempty. By establishing a tight upper bound on the ratio $\frac{\min \|StSt\|}{\max \|Mat\|}$, we can prove the largest possible value of the duality gap for their integer programming formulation for the case of Euclidean distances. In Section 5, we will prove that for the case of Manhattan distances, $\frac{\min \|St\|}{\max \|Mat\|} \leq \frac{3}{2}$, and $\frac{\min \|St\|}{\min \|StSt\|} \leq \frac{3}{2}$, which are both tight. Section 6 deals with Manhattan distances in three-dimensional space. We show that the ratio $\frac{\min \|StSt\|}{\max \|Mat\|}$ has the tight upper bound $\frac{3}{2}$, $\frac{\min \|St\|}{\min \|StSt\|}$ has the tight upper bound $\frac{5}{3}$, while the maximal value of $\frac{\min \|St\|}{\max \|Mat\|}$ lies between $\frac{5}{3}$ and 2. (See Table 1 at the end of this paper for an overview.)

Finally, we would like to note some further algorithmic implications: The result by Tamir and Mitchell [15] yields an $O(n)$ time algorithm for finding a maximum weight matching for a planar point set with Manhattan distances. With some extra work, some of the underlying properties of minimum Steiner stars have been used by Fekete [7] to construct an $O(n)$ time algorithm for finding a Traveling Salesman tour of maximum total length. (See the paper by Barvinok et al. [2] for more results on this problem.)

The rest of this paper is organized as follows. In Section 2 we introduce some basic notation and some general results. Section 3 deals with Euclidean distances in two-dimensional space, while Section 4 contains results for the case of Euclidean distances in three-dimensional space. In Sections 5 and 6, we consider Manhattan distances in two- and three-dimensional space. The concluding Section 7 contains a discussion of remaining open problems.

2 Preliminaries

Let $G = (V, E)$ be a graph with non-negative edge weights $w(e)$. Throughout this paper, the vertex set V of G will be represented by a point set $P = \{p_0, p_1, \ldots, p_{n-1}\}$ from Euclidean space, and edge weights correspond to geometric distances, according to some metric. A *star* of P is a set of $n-1$ edges (represented by line segments) connecting an element of P with all other elements of P. A *Steiner star* of P with center point c is a set of n edges (represented by line segments), connecting each point of P to c. A (perfect) matching of P is a set of $n/2$ edges that pair each point of P with another unique point of P. In the remainder of this paper, any star, Steiner star or matching is assumed to be a star, Steiner star or matching of P, denoted by the symbols St, $StSt$ and Mat. Their lengths are denoted by $\|St\|$, $\|StSt\|$, and $\|Mat\|$, for a specified metric $\|...\|$. Let $minSt$, $minStSt$ and $maxMat$ denote a star, Steiner star and matching of minimal, minimal and maximal length respectively.

Before we consider various geometric instances, we note a general bound on ratios that holds for all weight functions on the edges, even if we do not have triangle inequality.

Theorem 1 *For any weighted graph* G*, we have* $\min \|St\| \leq 2 \max \|Mat\|$.

Proof: Let matrix A be the distance matrix of the points in P, so $A(i,j)$ is the distance between p_i and p_j. Let S be the sum of all entries in A. Since $\|minSt\|$ is the minimal row sum of A, we have by the pigeonhole principle that $\|minSt\| \leq S/n$. The maximal matching consists of $n/2$ elements of A, so again by the pigeonhole principle we have $\max \|Mat\| \geq S/(2n)$. Hence,

$$\min \|St\| \leq S/n \leq 2 \max \|Mat\|. \quad \blacksquare$$

In a setting where distances are not induced by the geometry of points in a space of fixed dimension, this inequality is tight, even if we assume triangle inequality:

Theorem 2 *For each* $\epsilon > 0$*, there is a weighted graph for which* $\frac{\min \|St\|}{\max \|Mat\|} > 2 - \epsilon$.

Proof: Consider the complete graph on n vertices, with all edge weights being 1. Then $\min \|St\| = (n-1)$, and $\max \|Mat\| = \frac{n}{2}$. $\blacksquare$

Without assuming triangle inequality, the ratio $\frac{\min \|St\|}{\min \|StSt\|}$ may be unbounded, as the following example shows:

Theorem 3 *For each* $M > 0$*, there is a weighted graph for which* $\min \|St\| > M \min \|StSt\|$.

Proof: Let $G = (V, E)$ be the complete graph on n vertices, with each edge having weight $2M$. Let $\overline{G} = (V \cup \{c\}, \overline{E}$ be the complete graph on (n+1) vertices, with all edges adjacent to c having weight 1. Then $\min \|St\| = 2M(n-1) > Mn = M \min \|StSt\|$. $\blacksquare$

Assuming triangle inequality, we can give a bound for the ratio $\frac{\min \|St\|}{\min \|StSt\|}$:

Theorem 4 *For any graph* $\overline{G}$ *with edge weights satisfying triangle inequality, the inequality* $\min \|St\| \leq 2 \min \|StSt\|$ *holds.*

Proof: Let c be the center of an optimal Steiner star. Let $v_0 \in V$ be a vertex closest to c, and let $w(c, v_0) = d$. Then by triangle inequality, $w(v_0, v_i) \leq w(v_0, c) + w(c, v_i)$, so the star $St(v_0)$ with center v_0 satisfies $\min \|St\| \leq \|St(v_0)\| \leq (n-1)d + \min \|StSt\| \leq 2 \min \|StSt\|$. $\blacksquare$

Again, this estimate is tight:

Theorem 5 *For each* $\epsilon > 0$*, there is a graph with edge weights satisfying triangle inequality, such that* $\frac{\min \|St\|}{\min \|StSt\|} > 2 - \epsilon$.

Proof: Let G be the complete graph on n vertices, with all edge weights being 2. Let $\overline{G} = (V \cup \{c\}, \overline{E}$ be a complete graph on $(n+1)$ vertices, with all edges adjacent to c having weight 1. Then $\min \|St\| = 2(n-1)$, and $\min \|StSt\| = n$. ∎

In a geometric setting, distances in an arrangement of points are far more restricted, so the above ratios may no longer be best possible. It is the main purpose of this paper is to provide tight estimates for geometric scenarios.

3 Euclidean Distances in Two-dimensional Space

Throughout this and the following section, we will consider arrangements of points in two- and three-dimensional space, with distances measured according to the Euclidean metric. At several occasions, we make use of the following well-known theorem:

Proposition 6 (Cosine Theorem) *Given a triangle with edge lengths a, b, c. If γ is the angle opposite c, then*

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

3.1 Minimum Steiner Stars and Maximum Matchings

In this subsection, we give a proof of Suri's conjecture [14]. Throughout the section, distances are measured according to the Euclidean metric.

Lemma 7 *Given a triangle with edge lengths a, b and c, where the angle opposite c is $\geq 2\pi/3$, we have $a+b \leq 2/\sqrt{3}\cdot c$.*

Proof: Without loss of generality assume that $1 = a \leq b$. Then Proposition 6 implies that $c^2 \geq 1+b^2+b$. Now consider $\frac{(a+b)^2}{c^2} \leq \frac{(1+b)^2}{1+b^2+b} = 1 + \frac{b}{b^2+b+1}$. Since $b^2 - 2b + 1 \geq 0 \Leftrightarrow \frac{b}{b^2+b+1} \leq 1/3$, we get a minimal value of $4/3$, from which the claim follows. Equality holds for $\gamma = 2\pi/3$ and $b = 1$. ∎

Let l be a directed line in the plane. We say that l splits P if at most half of the points of P are to the right l and at most half of the points of P are to the left of l.

Lemma 8 *For any set of points P in two-dimensional space we can find three directed lines l_0, l_1 and l_2 such that the three lines intersect in a common point, all three lines split P and the smallest angle between any two lines is $\pi/3$.*

Proof:
The collection of splitting lines for a given direction α form a directed closed strip which we call S_α. Consider the strips S_0, $S_{\pi/3}$ and $S_{2\pi/3}$. If these three strips have a point in common we are done. Therefore assume without loss of generality that $S_0 \cap S_{\pi/3}$ lies to the left of $S_{2\pi/3}$. It follows that $S_\pi \cap S_{4\pi/3}$ lies to the right of $S_{5\pi/3}$.

We now consider the strips S_α, $S_{\alpha+\pi/3}$ and $S_{\alpha+2\pi/3}$, where α increases from $\alpha = 0$ to $\alpha = \pi$. The three strips move in a continuous manner. Suppose that for no value of α the three strips have a point in common. Then $S_\alpha \cap S_{\alpha+\pi/3}$ stays to the left of $S_{\alpha+2\pi/3}$, which contradicts the fact that $S_\pi \cap S_{4\pi/3}$ lies to the right of $S_{5\pi/3}$. Therefore there is a value of α for which the three strips S_α, $S_{\alpha+\pi/3}$ and $S_{\alpha+2\pi/3}$ have a point in common, which proves the lemma. ∎

Theorem 9 *For any set of points P in two-dimensional space with Euclidean distances and n even, we have the inequality $\min \|StSt\| \leq 2/\sqrt{3}\cdot \max \|Mat\|$.*

Proof: Find three lines l_0, l_1 and l_2 such that the three lines intersect in a common point, all three lines split P and the smallest angle between any two lines is $\pi/3$. These lines divide P into six sets A_0, A_1, A_2, B_0, B_1 and B_2, where A_i lies opposite B_i for all i, as shown in Figure 1. By assigning the points of P on the three lines to only one of the sets it belongs to, we can assume that $|A_0| + |A_1| + |A_2| = \frac{n}{2} = |B_0| + |B_1| + |B_2|$, as well as $|A_0| + |A_1| + |B_2| = \frac{n}{2} = |B_0| + |B_1| + |A_2|$, and $|A_0| + |B_1| + |B_2| = \frac{n}{2} = |B_0| + |A_1| + |A_2|$. This implies $|A_i| = |B_i|$ for all i.

Let O be the intersection of l_0, l_1 and l_2. Construct a Steiner Star $StSt$ with O as its center. Construct a matching Mat by matching points in A_i with points in B_i for all i.

We know from Lemma 7 that for each edge (p,q) in Mat, we have $\|p-O\| + \|O-q\| \leq 2/\sqrt{3}\cdot\|p-q\|$. So $\min \|StSt\| \leq \|StSt\| \leq 2/\sqrt{3}\cdot \|Mat\| \leq 2/\sqrt{3}\cdot \max\|Mat\|$. ∎

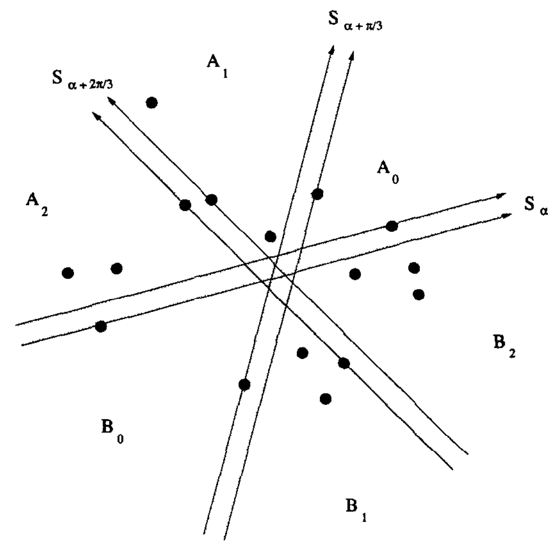

Figure 1: Finding a small Steiner Star and a large Matching

The inequality in Theorem 9 is tight:

Theorem 10 (Suri) *There is a set of points P in two-dimensional space with Euclidean distances for which the equality $\min \|StSt\| = 2/\sqrt{3}\cdot \max\|Mat\|$ holds.*

Proof: Suppose n is divisible by 6. Place $n/3$ points on each corner of an equilateral triangle with sides of length $2\sqrt{3}$. Then $\min \|StSt\| = 2n$ and $\max \|Mat\| = n\sqrt{3}$. ∎

3.2 Minimum Stars and Minimum Steiner Stars

In this subsection, we discuss the possible values of the ratio $\frac{\min \|St\|}{\min \|StSt\|}$. Clearly, this ratio is bounded from below by 1; in order to get upper bounds, we assume without loss of generality that the star center c of an optimal Steiner star $minStSt$ is not an element of P.

Let the line segments, or rays, of a Steiner star be r_i, and denote their lengths by a_i. Let α_i be the angle between the positive x-axis and ray r_i.

Lemma 11 *For the angles α_i of minStSt, we have*

$$\sum_{i=0}^{n-1} \cos\alpha_i = \sum_{i=0}^{n-1} \sin\alpha_i = 0.$$

Proof: Consider ray r starting at the origin and ending in (p_x, p_y). Let α be the angle between the positive x-axis and the line segment from $(x,0)$ to (p_x,p_y). Let $f(x)$ be the distance from the point $(x,0)$ to (p_x,p_y). So $f(x) = \sqrt{(p_x-x)^2+p_y^2}$ and

$$\frac{df(x)}{dx} = \frac{-(p_x-x)}{\sqrt{(p_x-x)^2+p_y^2}} = -\cos\alpha$$

If c is not an element of the set P, then at c the derivative of the sum of the lengths of all rays should be zero in all directions, so for all θ we have

$$\sum_{i=0}^{n-1} \cos(\alpha_i+\theta) = 0,$$

from which the lemma follows. ∎

Lemma 12 *Let StSt be a Steiner star of P and let a_i, r_i and α_i be defined as above. Let b_i be the distance between p_0 and p_i. If $a_0 = a_i$ for all i and if*

$$\sum_{i=0}^{n-1} \cos\alpha_i = \sum_{i=0}^{n-1} \sin\alpha_i = 0 \quad \textit{then} \quad \sum_{i=0}^{n-1} b_i \le a_0 n\sqrt{2}.$$

Proof: We have $\sum_{i=0}^{n-1}\cos(\alpha_i+\theta) = 0$ for all θ, so without loss of generality we can assume that P is rotated around the origin so that $p_0 = (a_0, 0)$. We have

$$b_i = 2a_0\sin(\alpha_i/2) = a_0\sqrt{2(1-\cos\alpha_i)}$$

(see also Figure 2). The function $f(\alpha) = \sqrt{1-\cos\alpha}$, where $0 \le \alpha < 2\pi$, is a concave function of $\cos\alpha$. By Jensen's inequality [13] we have

$$\sum_{i=0}^{n-1}\frac{1}{n}\sqrt{1-\cos\alpha_i} \le \sqrt{1-(\sum_{i=0}^{n-1}\frac{1}{n}\cos\alpha_i)} = 1$$

from which the result follows. ∎

Now we can prove the following upper bound:

Theorem 13 *For any set of points P in two-dimensional space with Euclidean distances we have* $\min\|St\| \le \sqrt{2}\cdot\min\|StSt\|$.

Proof: If the center of *minStSt* is an element of P, the lemma holds, so assume that the center of the *minStSt* is not an element of P. Without loss of generality assume that the center of *minStSt* is the origin, that r_0 is a shortest ray and that r_0 runs along the positive x-axis, as shown in Figure 2.

Consider the star St centered around the end-point of r_0. Denote the rays of St by r_i' and their length by a_i'. Using triangle inequality we have

$$a_i' \le (a_i - a_0) + 2a_0\sin(\alpha_i/2).$$

So from Lemma 12 we have

$$\begin{aligned}\min\|St\| \le \|St\| &= \sum_{i=0}^{n-1} a_i' \le \sum_{i=0}^{n-1} a_i - na_0 + na_0\sqrt{2} \\ &\le \min\|StSt\| + \min\|StSt\|(\sqrt{2}-1) \\ &= \sqrt{2}\cdot\min\|StSt\|. \quad ∎\end{aligned}$$

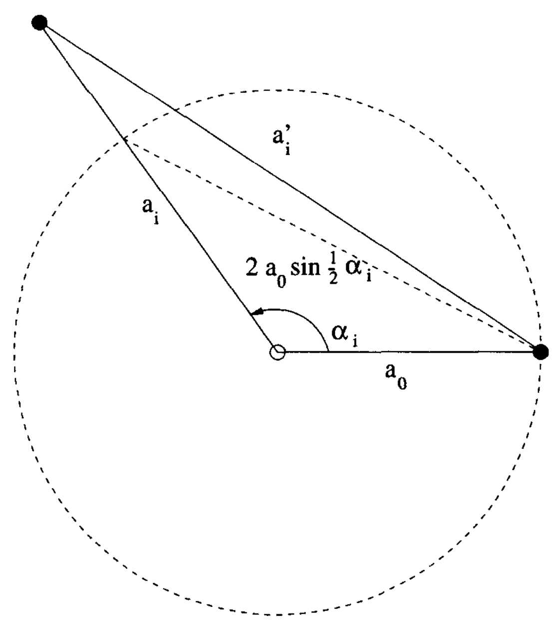

Figure 2: Length of a chord in a circle

As a lower bound, we get the following:

Theorem 14 *For any $\epsilon > 0$, there is a set of points P in two-dimensional space with Euclidean distances for which* $\min\|St\| > 4/\pi\cdot\min\|StSt\| - \epsilon$.

Proof: Let P be a set of points spaced evenly on the unit circle. Assume that P includes the point $(x,y) = (0,-1)$. The center of the *minStSt* is the center of the circle, so $\min\|StSt\| = n$. Consider the star St centered around $(0,-1)$. Denote the rays of St by r_i and their lengths by a_i. Let α_i be the angle between the positive x-axis and ray r_i. We have

$$a_i = 2\sin\alpha_i$$

Therefore the average ray length in the limit is

$$\frac{1}{\pi}\int_0^{\pi/2} 4\sin\alpha\, d\alpha = 4/\pi,$$

from which the theorem follows. ∎

3.3 Minimum Stars and Maximum Matchings

It is not hard to derive upper and lower bounds for the largest possible value of $\frac{\min\|St\|}{\max\|Mat\|}$ by using the previous results of this section:

Theorem 15 *For any set of points P in two-dimensional space with Euclidean distances and n even, we have the inequality* $\min \|St\| \leq 2\sqrt{2}/\sqrt{3} \cdot \max \|Mat\|$.

Proof: This follows immediately from Theorems 9 and 13. ∎

Theorem 16 *There is a set of points P in two-dimensional space with Euclidean distances for which* $\min \|St\| = 4/3 \cdot \max \|Mat\|$.

Proof: Suppose n is divisible by 6. Place $n/3$ points on each corner of an equilateral triangle with sides of length 6. Then $\min \|St\| = 4n$ and $\max \|Mat\| = 3n$. ∎

4 Euclidean Distances in Three-dimensional Space

4.1 Minimum Steiner Stars and Maximum Matchings

Following an idea similar to the one from Section 3.1, we show that there always exist three orthogonal planes that partition P into 8 octants such that opposite octants contain the same number of points.

Let p be a plane. We say that p splits P if at most half of the points of P are on one one side of p and at most half of the points of P are on the other side of p.

Let p_0, p_1 and p_2 be three orthogonal planes, each of which splits P. Each plane p_i divides P into points above p_i, on p_i and below p_i. Let Q_0^i be the points in P below p_i and Q_1^i be the points in P above p_i. We assign the points on p_i to either Q_0^i and Q_1^i in such a way that $|Q_0^i| = |Q_1^i|$. We will call $|Q_0^i|$ and $|Q_1^i|$ the set of points below and above p_i respectively, even though some of these points may in fact lie on p_i. We define for $i, j, k \in \{0, 1\}$:

$$Q_{ijk} = Q_i^2 \cap Q_j^1 \cap Q_k^0.$$

For example, Q_{110} is the set of points in P above p_2, above p_1 and below p_0. Since $|Q_0^i| = |Q_1^i|$, we can derive the following equalities:

$$\begin{aligned}
|Q_{000}| + |Q_{001}| &= |Q_{110}| + |Q_{111}| \\
|Q_{100}| + |Q_{101}| &= |Q_{010}| + |Q_{011}| \\
|Q_{000}| + |Q_{010}| &= |Q_{101}| + |Q_{111}| \\
|Q_{001}| + |Q_{011}| &= |Q_{100}| + |Q_{110}| \\
|Q_{000}| + |Q_{100}| &= |Q_{011}| + |Q_{111}| \\
|Q_{001}| + |Q_{101}| &= |Q_{010}| + |Q_{110}|
\end{aligned}$$

We show that we can always find three orthogonal planes such that opposite octants, i.e. Q_{ijk} and $Q_{(1-i)(1-j)(1-k)}$ have the same cardinality.

Lemma 17 *For any set of points P in three-dimensional space, we can find three orthogonal planes such that the balancedness condition* $|Q_{ijk}| = |Q_{(1-i)(1-j)(1-k)}|$ *holds for all* i, j, k.

Proof: Notice that it suffices to find three orthogonal planes such that $|Q_{000}| = |Q_{111}|$, since this implies the equality $|Q_{ijk}| = |Q_{(1-i)(1-j)(1-k)}|$ for all i, j, k. We first assume that the points are in general position, in the sense that if we project all points in P onto the $(z = 0)$ plane, then no three points are collinear and no line through two points is perpendicular to another line through two points. Let p_2 be a splitting plane parallel to the $(z = 0)$ plane. We map all points from P onto the $(z = 0)$ plane, and call the projected points from Q_1^2 and Q_0^2 the black and white points respectively.

The problem is now a two-dimensional one. Orthogonal splitting planes p_0 and p_1 will become splitting lines l_0 and l_1 in the projection. Let a directed line l be a splitting line if at most half of the black and white points are to the left of l and at most half lie to the right of l. Because of the non-degeneracy assumption there are at most two points on a splitting line l. The collection of splitting lines for a given direction α form a directed closed strip which we call S_α. Let l_α be the splitting line in the middle of S_α. Consider the strips S_α and $S_{\alpha+\pi/2}$. The corresponding lines l_α and $l_{\alpha+\pi/2}$ divide the points into 4 subsets Q_{ij} for $i, j \in \{0, 1\}$, where Q_{ij} is the projection of $Q_{0ij} \cup Q_{1ij}$. Therefore $|Q_{00}| = |Q_{11}|$ and $|Q_{01}| = |Q_{10}|$, as illustrated in Figure 3.

If α increases at most one point is added to and at most one point is removed from Q_{ij} at any one time. Changes only occur when S_α or $S_{\alpha+\pi/2}$ is a line. Because of the non-degeneracy assumption it is not possible that both S_α and $S_{\alpha+\pi/2}$ are lines. Either

- the number of points in two opposite quadrants such as Q_{00} and Q_{11} both increase or both decrease, or
- two neighboring quadrants exchange a point.

In the first case, $|Q_{000}|$ and $|Q_{111}|$ either both increase, both decrease, or only one changes. It is not possible for $|Q_{000}|$ to decrease and for $|Q_{111}|$ to increase or vice-versa. In the latter case, at most one of $|Q_{000}|$ and $|Q_{111}|$ changes. Therefore if for some value of α_0 we have $|Q_{000}| < |Q_{111}|$, while for α_1 we have $|Q_{000}| > |Q_{111}|$, then there is an α with $\alpha_0 < \alpha < \alpha_1$ for which $|Q_{000}| = |Q_{111}|$.

Consider first the strips S_0 and $S_{\pi/2}$. If $|Q_{000}| = |Q_{111}|$ then we are done. Therefore suppose that $|Q_{000}| < |Q_{111}|$. It follows that for strips S_π and $S_{3\pi/2}$ we have the inequality $|Q_{000}| > |Q_{111}|$. Therefore there is a value of α with $0 < \alpha < \pi$ for which $|Q_{000}| = |Q_{111}|$.

If the non-degeneracy assumption does not hold, we can move all points by an infinitesimal small distance, in such a way that the assumption does hold. The construction shown above gives three orthogonal splitting planes of the perturbed set. The same planes partition P in the correct way, whereby the perturbation of a point that lies on a splitting plane determines to which sides of the plane this point should be assigned. ∎

Lemma 18 *Given a triangle with side lengths a, b and c, where the angle between a and b is* $\geq \pi/2$, *we have* $a + b \leq \sqrt{2}c$.

Proof: We proceed similar to the proof of Lemma 7. Without loss of generality assume that $1 = a \leq b$. Then Proposition 6 implies that $c^2 \geq 1 + b^2$. Now consider $\frac{(a+b)^2}{c^2} \leq \frac{(1+b)^2}{1+b^2} = 1 + \frac{2b}{1+b^2}$. Since $b^2 - 2b + 1 \geq 0 \Leftrightarrow \frac{2b}{1+b^2} \leq 1$, this has a maximal value of 2, from which the claim follows. Equality holds for $\gamma = \pi/2$ and $b = 1$. ∎

Theorem 19 *For any set of points P in three-dimensional space with Euclidean distances, we have* $\min \|StSt\| \leq \sqrt{2} \cdot \max \|Mat\|$.

Proof: Find three orthogonal splitting planes such that $\|Q_{ijk}\| = \|Q_{(1-i)(1-j)(1-k)}\|$ for all i, j, k. Let O be the intersection of these three planes. Construct a Steiner Star $StSt$ with O as its center. Construct a matching Mat by

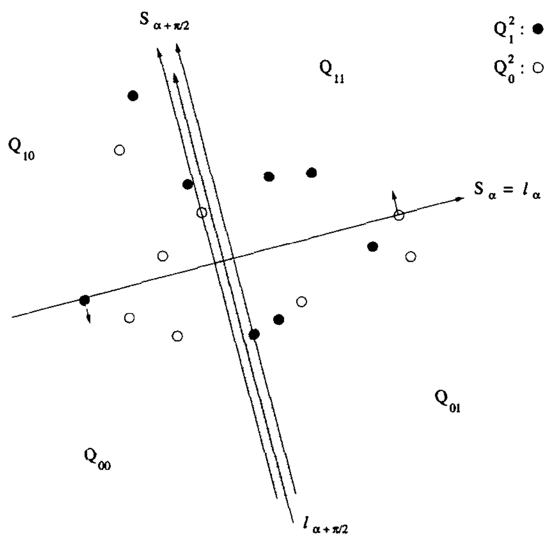

Figure 3: Orthogonal splitting strips

connecting points in each octant to points in the opposite octant.

For each edge (p, q) in Mat, the angle between the ray from O to p and the ray from O to q is $\geq \pi/2$, from which we derive by Lemma 18 that $\|p-O\| + \|O-q\| \leq \sqrt{2} \cdot \|p-q\|$. So $\min \|StSt\| \leq \|StSt\| \leq \sqrt{2} \cdot \|Mat\| \leq \sqrt{2} \cdot \max \|Mat\|$. ∎

In the following, we give the best lower bound that we know of:

Theorem 20 *There is a set of points P in three-dimensional space with Euclidean distances for which we have the equality* $\min \|StSt\| = \sqrt{3}/\sqrt{2} \cdot \max \|Mat\|$.

Proof: Suppose n is divisible by 4. Place $n/4$ points on each corner of a tetrahedron with sides of length 2. Then $\min \|StSt\| = n\sqrt{3}/\sqrt{2}$ and $\max \|Mat\| = n$. ∎

4.2 Minimum Stars and Minimum Steiner Stars

In this subsection, we discuss the possible values of the ratio $\frac{\min \|St\|}{\min \|StSt\|}$ for points in three-dimensional space. Our upper bound carries over from the two-dimensional case:

Theorem 21 *For any set of points P in three-dimensional space with Euclidean distances we have* $\min \|St\| \leq \sqrt{2} \cdot \min \|StSt\|$.

Proof:
The proof is virtually the same as the proof for Theorem 13. Without loss of generality assume that the center of $minStSt$ is the origin, that r_0 is a shortest ray and that r_0 runs along the positive x-axis. Let α_i be the angle between r_i and the positive x-axis. Then we can show that $\sum(\cos \alpha_i) = 0$ similarly to the way this was shown in the proof of Lemma 11. Therefore the result of Theorem 13 applies here as well. ∎

The best lower bound is not too far off:

Theorem 22 *For any* $\epsilon > 0$, *there is a set of points P in three-dimensional space with Euclidean distances for which* $\min \|St\| > 4/3 \cdot \min \|StSt\| - \epsilon$.

Proof: Let P be a set of points evenly distributed over the unit sphere. Assume P includes the point $(x, y, z) = (0, -1, 0)$. The center of $minStSt$ is the center of the sphere, so $\min \|StSt\| = n$. Consider the star St centered around the points $(0, -1, 0)$. Denote the rays of St by r_i and their lengths by a_i. Let α_i be the angle between the $(y = 0)$-plane and ray r_i. We have

$$a_i = 2 \sin \alpha_i$$

The average ray length can be computed as follows. Let θ be the angle between a ray and the $(y = 0)$ plane and ϕ the angle between the projection of the ray onto the $(y = 0)$ plane and the positive x-axis. If $\Delta\theta = \theta_1 - \theta_0$ and $\Delta\phi = \phi_1 - \phi_0$ are small, then the surface area covered by all rays with angles in this range is approximately equal to $4 \sin \theta \cos \theta \Delta\phi \Delta\theta$, so the area of the surface of the sphere is

$$\int_0^{\pi/2} \int_0^{2\pi} 4 \sin \theta \cos \theta \, d\phi d\theta = 4\pi$$

and the average ray length is

$$\frac{1}{4\pi} \int_0^{\pi/2} \int_0^{2\pi} (4 \sin \theta \cos \theta) \, (2 \sin \theta) \, d\phi d\theta = 4/3$$

from which the lemma follows. ∎

4.3 Minimum Stars and Maximum Matchings

We know from Theorem 1 that the ratio $\frac{\min \|St\|}{\max \|Mat\|}$ cannot exceed 2. We conclude our discussion on Euclidean distances by giving a lower bound on the largest possible value of this ratio.

Theorem 23 *There is a set of points P for which the equality* $\min \|St\| = 3/2 \cdot \max \|Mat\|$ *holds.*

Proof: Suppose n is divisible by 8. Place $n/4$ points on each corner of a tetrahedron with sides of length 4. Then $\min \|St\| = 3n$ and $\max \|Mat\| = 2n$. ∎

5 Manhattan Distances in Two-dimensional Space

5.1 Minimum Steiner Stars and Maximum Matchings

Independent from our work, the following proposition was also noted by Tamir and Mitchell [15]. Since some of the steps are of importance for our further results, we give a sketch of the proof.

Proposition 24 *For any set of point P in two-dimensional space with Manhattan distances and n even, we have the equality* $\max \|Mat\| = \min \|StSt\|$.

Sketch: It is not hard to see that the center for an optimal Steiner star must both be a median of the x-coordinates and the y-coordinates of the points in P. Assume without loss of generality that an optimal center is located at $(0, 0)$ and consider the numbers n_1, n_2, n_3, and n_4 of points in each of the four quadrants, with points on the boundary of two quadrants assigned in a suitable way. Using $n_1 +$

$n_2 = n_3 + n_4$ and $n_2 + n_3 = n_4 + n_1$, we get $n_1 = n_3$ and $n_2 = n_4$, i.e., diagonally opposite quadrants must contain the same number of points. This allows us to match points from opposite quadrants. It is straightforward to see that to each edge of the matching, we have a corresponding pair of edges of the Steiner star with the same total length, implying that the total length of the matching is equal to the total length of the Steiner star. ∎

5.2 Minimum Stars and Minimum Steiner Stars

In the following, we will consider the ratio $\frac{\min \|St\|}{\min \|StSt\|}$. For any point $p_i = (x_i, y_i) \in P$, $\|St(p_i)\|$ is the total length of the star centered at p_i. We denote by α_n the supremum of the values $\frac{\min \|St\|}{\min \|StSt\|}$ for point sets of cardinality n. Without loss of generality, we may assume that $\min \|StSt\| > 0$, and thus $\min \|StSt\| = 1$. Furthermore, we may assume that the origin is an optimal Steiner center.

We will make use of the following lemma:

Lemma 25 *For any n, there are point sets for which the ratio $\frac{\min \|St\|}{\min \|StSt\|}$ attains the value α_n.*

Proof: For any fixed n, the set of point arrangements with $\min \|StSt\| = 1$ and optimal Steiner center O is a compact subset of R^{2n}. Since $\min \|St\|$ is a continuous function on R^{2n}, the claim holds. ∎

Lemma 26 *Let P be a set of n points with $\frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$. Then for all $p_i \in P$, $\|St(p_i)\| = \alpha_n$.*

Proof: Suppose there is a point p_i that satisfies $\|St(p_i)\| > \min \|St\|$. For sufficiently small ε, replacing p_i by the point $p_i' = (1-\varepsilon)p_i$ does not turn p_i' into an optimal star center. Thus, the replacement reduces $\min \|StSt\|$ by some small ε', but $\min \|St\|$ by not more than ε'. Therefore, the new arrangement has a ratio of at least $\frac{\min \|St\| - \varepsilon'}{\min \|StSt\| - \varepsilon'} > \frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$, a contradiction. ∎

It is straightforward to see that this implies the following:

Corollary 27 *For an arrangement with $\frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$, we cannot move a vertex such that $\min \|StSt\|$ decreases by ε and $\min \|St\|$ by not more than ε.*

Corollary 28 *For an arrangement with $\frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$, we cannot move a vertex such that $\min \|StSt\|$ remains the same, one or more of the $\|St(p_i)\|$ increase, and none of them decrease.*

Furthermore, we get:

Corollary 29 *For any arrangement of points in two-dimensional space with Manhattan distances and with $\frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$, there cannot be two points $p_1 = (x_1, y_1) \neq p_2 = (x_2, y_2)$, such that*

$$\begin{aligned} 0 &\leq (x_1, y_1) \leq (x_2, y_2), \text{ or} \\ 0 &\leq (-x_1, y_1) \leq (-x_2, y_2), \text{ or} \\ 0 &\leq (x_1, -y_1) \leq (x_2, -y_2), \text{ or} \\ 0 &\leq (-x_1, -y_1) \leq (-x_2, -y_2). \end{aligned}$$

Proof: In any of the four cases, it is straightforward to see that $\|St(p_1)\| < \|St(p_2)\|$. ∎

The next lemma shows that we may restrict our attention to arrangements with extreme points on the coordinate axes:

Lemma 30 *For any arrangement in two-dimensional space with Manhattan distances and with $\frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$, any point $p_i = (x_i, y_i)$ with minimal or maximal x_i among the points in P must have $y_i = 0$. Conversely, minimal or maximal y_i implies $x_i = 0$.*

Proof: Without loss of generality consider a point p_i with maximal y_i and assume $x_i > 0$. Let $x_k = \max\{x_j | j \neq i\}$. If $x_i - x_k = \delta > 0$, replace p_i by $(x_k, y_i + \delta)$; this does not change $\min \|St\|$ or $\min \|StSt\|$, and allows us to consider without loss of generality the case $x_i - x_k \leq 0$.

For sufficiently small ε, replace p_i by $p_i' = p_i + (-\varepsilon, \varepsilon)$. This does not change $\min \|StSt\|$, increases $\|St(p_k)\|$, and does not decrease $\|St(p_j)\|$ for any $j \neq k$, contradicting Corollary 28. ∎

Lemma 31 *For any n, we have $\alpha_n \leq \alpha_{kn}$.*

Proof: Suppose we have an arrangement of n points with $\frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$. Replace each point by k copies; this yields an arrangement of kn points with $\frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$. ∎

Lemma 32 *For any arrangement in two-dimensional space with Manhattan distances and with $\frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$, there can be at most four points p_i with $x_i, y_i \neq 0$.*

Proof: Assume that there are at least five points not on coordinate axes. Then we may assume without loss of generality that two of them (say, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$) are in the positive quadrant. Because of Corollary 29, we may assume that $0 < x_2 < x_1$ and $0 < y_1 < y_2$. Define

n_0 = the number of vertices i with $x_1 < x_i$
n_1 = the number of vertices $i \neq 1, 2$ with $x_1 \leq x_i \leq x_2$
n_2 = the number of vertices i with $x_i < x_2$
m_0 = the number of vertices i with $y_2 < y_i$
m_1 = the number of vertices $i \neq 1, 2$ with $y_1 \leq y_i \leq y_2$
m_2 = the number of vertices i with $y_i < y_1$

So $n_2 > n_0$ and $m_2 > m_0$. Let ε_x and ε_y be such that $\varepsilon_x/\varepsilon_y = (m_2 - m_0)/(n_2 - n_0)$. Also let ε_x and ε_y be positive but smaller than the smallest non- zero difference between the x-coordinates and y-coordinates of any two points respectively. Now replace p_1 by $p_1' = p_1 + (\varepsilon_x, -\varepsilon_y)$ and p_2 by $p_2' = p_2 + (-\varepsilon_x, +\varepsilon_y)$ It is not hard to see that these replacements do not decrease $\|St(p_i)\|$ for any $i \neq 1, 2$. Furthermore, $\min \|StSt\|$ does not change. The value $\|St(p_1)\|$ changes by $\Delta_1 = (n_2 + n_1 - n_0)\varepsilon_x + (-m_2 + m_1 + m_0)\varepsilon_y + 2\varepsilon_x + 2\varepsilon_y$. So $\Delta_1 > (n_2 - n_0)\varepsilon_x + (-m_2 + m_0)\varepsilon_y = (m_2 - m_0)\varepsilon_y + (-m_2 + m_0)\varepsilon_y = 0$. Similarly, the value $\|St(p_2)\|$ changes by $(-n_2 + n_1 + n_0)\varepsilon_x + (m_2 + m_1 - m_0)\varepsilon_y + 2\varepsilon_x + 2\varepsilon_y$, which is also positive. This contradicts Corollary 28. ∎

In order to analyze the limit of the sequence α_n, we define a sequence β_n. This is the supremum of the values $\frac{\min \|St\|}{\min \|StSt\|}$ for all arrangements of n points, such that any point lies on a coordinate axis.

With the help of Corollary 29, Lemma 31, and Lemma 32, it is not hard to prove that for arrangements with many points, the bounded number of points not on coordinate axes becomes negligible for the worst case ratio:

Lemma 33 $\limsup_{n\to\infty}\alpha_n = \limsup_{n\to\infty}\beta_n$.

Proof: For any n, consider a point arrangement P_n with $\min\|StSt\| = 1$ and $\min\|St\| = \alpha_n$. By Lemma 32, for any P_n, there can be at most 4 points not on coordinate axes; by Corollary 29, we conclude that the points on the axes are positioned at $p_1 = (d_1, 0)$, $p_2 = (0, d_2)$, $p_3 = (-d_3, 0)$, $p_4 = (0, -d_4)$, with multiplicities n_1, n_2, n_3, n_4 and $\sum_{i=1}^4 n_i \geq n-4$.

Because of Lemma 31, we are done if there are only finitely many P_n with a point not on an axis. So assume there are infinitely many P_n with a point $p_0 = (x_0, y_0) > (0,0)$. By Corollary 29, we conclude that for any such P_n, $d_1 > x_0$ and $d_2 > y_0$. If d_1 and d_2 tend to zero as n becomes large, the contribution of p_0 to $\min\|StSt\|$ and $\min\|St\|$ becomes arbitrarily small, and we are done.

So suppose without loss of generality that the distance $d_1 = \max\{d_j | j = 1, \ldots, 4\}$ and that d_1 remains bounded from below. Since $\min\|StSt\| = 1$, this means that n_1 remains bounded. As n becomes large, it follows that some n_i becomes arbitrarily large. Then $\min\|StSt\| = 1$ implies that d_i tends to zero. Let St_j be the star centered at p_j. Using 0 as a lower and $3d_1$ as an upper bound for the distance of points not on axes to p_1 and p_i, respectively, we get

$$\begin{aligned}\|St(p_1)\| &\geq n_2d_2 + n_3d_3 + n_4d_4 + (n_2+n_3+n_4)d_1 \\ &= \sum_{j=1}^4 n_jd_j + (\sum_{j=1}^4 n_j - 2n_1)d_1,\end{aligned}$$

whereas

$$\|St(p_i)\| \leq 12d_1 + \sum_{j=1}^4 n_jd_j + (\sum_{j=1}^4 n_j - 2n_i)d_i.$$

This means that for sufficiently large n, we have

$$\|St(p_1)\| - \|St(p_i)\| \geq (d_1 - d_i)\sum_{j=1}^4 n_j - 2n_1d_1 + 2n_id_i - 12d_1,$$

which is positive, since $(d_1 - d_i)\sum_{j=1}^4 n_j$ gets arbitrarily large for increasing n, while all other terms are bounded from below. This contradicts Lemma 26, and we are done. ∎

In order to establish an upper bound of the sequence β_n, we need the following lemma:

Lemma 34 *Let* $0 \leq \lambda_1, \ldots, \lambda_{2d} < \frac{1}{2}$, *such that* $\sum_{i=1}^{2d}\lambda_i = 1$. *Then*

$$\sum_{i=1}^{2d}\frac{\lambda_i}{1-2\lambda_i} \geq \frac{d}{d-1}.$$

Proof: Since $f(x) = \frac{x}{1-2x}$ is a convex function on the interval $[0, \frac{1}{2})$, we have from Jensen's inequality [13]

$$f(\sum_{i=1}^{2d}\frac{1}{2d}\lambda_i) \leq \frac{1}{2d}\sum_{i=1}^{2d} f(\lambda_i).$$

So

$$\frac{1}{2d-2} \leq \frac{1}{2d}\sum_{i=1}^{2d}\frac{\lambda_i}{1-2\lambda_i}$$

from which the lemma follows. ∎

Now we can proceed to prove the following:

Theorem 35 *For any set of points P in two-dimensional space with Manhattan distances,* $\min\|St\| \leq \frac{3}{2}\min\|StSt\|$ *holds.*

Proof: By Lemma 33, we only have to show that $\beta_n \leq \frac{3}{2}$. Similar to Lemma 25, we can assume that there are sets of n points for which $\frac{\min\|St\|}{\min\|StSt\|} = \beta_n$. By Lemma 29, these sets consist of n_1 points p_1 at position $(d_1, 0)$, of n_2 points p_2 at position $(0, d_2)$, of n_3 points p_3 at position $(-d_3, 0)$, and of n_4 points p_4 at position $(0, -d_4)$, with $d_i \geq 0$. We assume that

$$\min\|StSt\| = \sum_{i=1}^4 n_id_i = 1. \tag{1}$$

Furthermore, observe that $\|St(p_i)\| = \sum_{j\neq i} n_j(d_i + d_j) = (\sum_{j=1}^4 n_jd_j) + (\sum_{j=1}^4 n_jd_i) - 2n_id_i = 1 + (n - 2n_i)d_i$. By Lemma 26, we have $\|St(p_i)\| = \beta_n$, which implies

$$d_i = \frac{\beta_n - 1}{(n-2n_i)}. \tag{2}$$

Equations 1 and 2 yield

$$\frac{1}{\beta_n - 1} = \sum_{i=1}^4\frac{n_i}{(n-2n_i)}. \tag{3}$$

With $\frac{n_i}{n} = \lambda_i$, Lemma 34 implies $\frac{1}{\beta_n - 1} \geq 2$, so $\beta_n \leq \frac{3}{2}$, and we are done. ∎

Again, we can show that this bound is best possible:

Theorem 36 *There is a set of points P in two-dimensional space with Manhattan distances for which we have the equality* $\min\|St\| = \frac{3}{2}\min\|StSt\|$.

Proof: Suppose n is divisible by 4. Place $n/4$ points on each corner of the points $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$. Then $\min\|St\| = 3n/2$ and $\min\|StSt\| = n$. ∎

5.3 Minimum Stars and Maximum Matchings

Together with Theorem 24, the above tight bound implies the following:

Corollary 37 *For any set of points P in two-dimensional space with Manhattan distances and with n even, we have the inequality* $\min\|St\| \leq \frac{3}{2}\max\|Mat\|$, *which is tight.*

6 Manhattan Distances in Three-dimensional Space

6.1 Minimum Steiner Stars and Maximum Matchings

It was noted by Tamir and Mitchell [15] that $\max\|Mat\| = \min\|StSt\|$ may no longer be valid in three-dimensional rectilinear space:

Proposition 38 (Tamir, Mitchell)
For the point set $P = \{(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\}$, *we have* $\max\|Mat\| < \min\|StSt\|$.

In fact, Proposition 38 provides an example with the ratio $\max\|Mat\|/\min\|StSt\| = \frac{3}{2}$. The following result shows that this is a worst case example:

Theorem 39 *For any set of point P in three-dimensional space with Manhattan distances and with n even, we have the inequality* $\min \|StSt\| \leq \frac{3}{2} \max \|Mat\|$.

Proof: Assume that $O = (0,0,0)$ is the center of an optimal Steiner tree, so $|\{p_i | x_i < 0\}| \leq \frac{n}{2}$, $|\{p_i | x_i > 0\}| \leq \frac{n}{2}$, $|\{p_i | y_i < 0\}| \leq \frac{n}{2}$, etc. Then $\min \|StSt\| = \sum_i |x_i| + |y_i| + |z_i|$. Without loss of generality, assume that $\sum_i |z_i| \leq \sum_i |x_i| \leq \sum_i |y_i|$. Now consider the point set $P' = \{p_i' || i = 1, \ldots, n\}$, where $p_i = (x_i, y_i, 0)$. Because of the above conditions, O is the center of an optimal Steiner star $minStSt'$ for P'. We have $\|minStSt'\| = \sum_i |x_i| + |y_i| \geq \frac{2}{3} \min \|StSt\|$. By Proposition 24, P' has a matching Mat' of total weight $\|minStSt'\|$, and the claim follows. ∎

6.2 Minimum Stars and Minimum Steiner Stars

Using the same tools as in the two-dimensional case, we can show the following upper bound:

Theorem 40 *For any set of points P in three-dimensional rectilinear space we have* $\min \|St\| \leq \frac{5}{3} \min \|StSt\|$.

Proof: We proceed similarly to the proof of Theorem 35. Note that Lemmas 25, 26, and Corollaries 27 and 28 stay valid without any change, as well as Lemmas 31 and 34. It is straightforward to modify Corollary 29 and Lemma 30 to higher dimensions. Lemma 32 is replaced by the following three-dimensional version:

For any arrangement with $\frac{\min \|St\|}{\min \|StSt\|} = \alpha_n$, there can be at most eight points p_i not on coordinate axes.

This is shown as follows: Suppose there are nine points not on coordinate axes, then there must be at least two points $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ in the same octant, say the positive one. By the analogue of Corollary 29, we cannot have $0 \leq (x_1, y_1, z_1) \leq (x_2, y_2, z_2)$ or $0 \leq (x_2, y_2, z_2) \leq (x_1, y_1, z_1)$ with $p_1 \neq p_2$. This allows us to consider without loss of generality $0 < x_1 < x_2$, $0 < y_2 < y_1$ and apply the same modification to the x- and y-coordinates of p_1 and p_2 as in the proof of Lemma 32.

With the help of these lemmas, the claim of Lemma 33 still holds. Using Lemma 34 for $d = 3$, we get

$$\frac{1}{\beta_n - 1} = \sum_{i=1}^{6} \frac{n_i}{(n - 2n_i)} \geq \frac{d}{d-1} = \frac{3}{2},$$

implying $\beta_n \leq \frac{5}{3}$. This concludes the proof. ∎

A matching lower bound exists:

Theorem 41 *There is a point set P in three-dimensional rectilinear space for which we have the equality* $\min \|St\| = \frac{5}{3} \min \|StSt\|$.

Proof: Suppose n is divisible by 6. Place $n/6$ points on each of the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$. Then $\min \|St\| = 5n/3$ and $\min \|StSt\| = n$. ∎

6.3 Minimum Stars and Maximum Matchings

For this ratio, we have the same lower bound as in the previous subsection:

Theorem 42 *There is a set of points P for which the equality* $\min \|St\| = \frac{5}{3} \max \|Mat\|$ *holds.*

Proof: Suppose n is divisible by 6. Place $n/6$ points on each of the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$. Then $\min \|St\| = 5n/3$ and $\max \|Mat\| = n$. ∎

At this point, our best upper bound is 2, which holds in general, as shown in Theorem 1.

7 Conclusion

We have derived a number of upper and lower bounds for the largest possible value of the ratios between the size of Minimum stars, Minimum Steiner stars, and Maximum matchings. An overview of our bounds is given in Table 1. Some of these bounds are not tight; in all cases, we believe that the lower bounds are more likely to be correct. This belief is strengthened by the fact that some of the tools we used for the case of Manhattan distances (in particular, Lemma 26) are true for Euclidean distances as well. We note the following conjectures:

Conjecture 43 *For any point set P in two-dimensional Euclidean space, we have* $\min \|St\| \leq 4/\pi \cdot \min \|StSt\|$.

Conjecture 44 *For any point set P in three-dimensional Euclidean space, we have* $\min \|St\| \leq 4/3 \cdot \min \|StSt\|$.

Table 1: Lower and upper bounds for maximal values of ratios

Dist	Dim	max ratio	Lower bd.	Upper bd.
L_2	2D	$\frac{\min \|StSt\|}{\max \|Mat\|}$	$\frac{2}{\sqrt{3}} = 1.15\ldots$	$\frac{2}{\sqrt{3}} = 1.15\ldots$
		$\frac{\min \|St\|}{\min \|StSt\|}$	$\frac{4}{\pi} = 1.27\ldots$	$\sqrt{2} = 1.41\ldots$
		$\frac{\min \|St\|}{\max \|Mat\|}$	$\frac{4}{3} = 1.33\ldots$	$\frac{2\sqrt{2}}{\sqrt{3}} = 1.63\ldots$
	3D	$\frac{\min \|StSt\|}{\max \|Mat\|}$	$\frac{\sqrt{3}}{\sqrt{2}} = 1.22\ldots$	$\sqrt{2} = 1.41\ldots$
		$\frac{\min \|St\|}{\min \|StSt\|}$	$\frac{4}{3} = 1.33\ldots$	$\sqrt{2} = 1.41\ldots$
		$\frac{\min \|St\|}{\max \|Mat\|}$	$\frac{3}{2} = 1.5$	2
L_1	2D	$\frac{\min \|StSt\|}{\max \|Mat\|}$	1	1
		$\frac{\min \|St\|}{\min \|StSt\|}$	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$
		$\frac{\min \|St\|}{\max \|Mat\|}$	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$
	3D	$\frac{\min \|StSt\|}{\max \|Mat\|}$	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$
		$\frac{\min \|St\|}{\min \|StSt\|}$	$\frac{5}{3} = 1.66\ldots$	$\frac{5}{3} = 1.66\ldots$
		$\frac{\min \|St\|}{\max \|Mat\|}$	$\frac{5}{3} = 1.66\ldots$	2

Acknowledgments

We would like to thank Gordon Wilfong for first bringing this problem to our attention in 1996 and Arie Tamir for pointing out his related previous work.

References

[1] C. Bajaj. The algebraic degree of geometric optimization problems. *Discrete and Computational Geometry*, **3** (1988), 177-191.

[2] A. I. Barvinok, D. S. Johnson, G. J. Woeginger, and R. Woodroofe. The maximum Traveling Salesman Problem under polyhedral norms. *Proceedings of the 6th*

International Integer Programming and Combinatorial Optimization Conference (IPCO VI), Springer LNCS 1412, 1998, 195–201.

[3] M. Ben-Or. Lower bounds for algebraic computation trees. *Proceedings of the 15th ACM Symposium on the Theory of Computing*, 1983, 80–86.

[4] M. Blum, R. W. Floyd, V. R. Pratt, R. L. Rivest, and R. E. Tarjan. Linear time bounds for median computations. *Proceedings of the 4th ACM Symposium on the Theory of Computing*, 1972, 119–124.

[5] D.-Z. Du and F. K. Hwang. A proof of the Gilbert-Pollak conjecture on the Steiner ratio. *Algorithmica*, **7** (1992), 121–135.

[6] J. Edmonds. Maximum matching and a polyhedron with 0, 1-vertices. *Journal of Research of the National Bureau of Standards (B)*, **69** (1965), 69–87.

[7] S. P. Fekete. Simplicity and hardness of the Maximum Traveling Salesman Problem under geometric distances. To appear in *Proceedings of the ACM/SIAM Symposium on Discrete Algorithms*, 1999. Available at `http://www.zpr.Uni-Koeln.DE/~papers`.

[8] J. A. Fingerhut, S. Suri, and J. S. Turner. Designing least-cost nonblocking broadband networks. *Journal of Algorithms*, **24** (1997), 287–309.

[9] M. R. Garey and D. S. Johnson. *Computers and intractability. A guide to the theory of NP-completeness.* Freeman, New York, 1979.

[10] A. J. Goldmann. Optimal center location in simple networks. *Transportation Science*, **5** (1971), 212-221.

[11] F. K. Hwang, D. S. Richards, and P. Winter. *The Steiner Tree Problem.* Elsevier Science, Amsterdam 1992.

[12] O. Kariv and L. S. Hakimi. An algorithmic approach to network location problems. II: The p-medians. *SIAM Journal on Applied Mathematics*, **37** (1979), 539–560.

[13] M. Spivak. Calculus. W.A. Benjamin, Menlo Park, California, 1967.

[14] S. Suri. Problem # 5. Problem session of the *14th ACM Symposium on Computational Geometry*, 1998. `http://www.cs.duke.edu/~pankaj/scg98-openprobs /open-probs.html`.

[15] A. Tamir and J. S. B. Mitchell. A maximum b-matching problem arising from median location models with applications to the roommates problem. *Mathematical Programming*, **80**, 2 (1998), 171–194.