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Traveling the boundary of Minkowski sums

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Abstract

We consider the problem of traveling the contour of the set of all points that are within distance 1 of a connected planar curve arrangement $\mathcal{P}$, forming an embedding of the graph G. We show that if the overall length of $\mathcal{P}$ is L, there is a closed roundtrip that visits all points of the contour and has length no longer than $2L + 2\pi$. This result carries over in a more general setting: if R is a compact convex shape with interior points and boundary length ℓ, we can travel the boundary of the Minkowski sum $\mathcal{P} \oplus R$ on a closed roundtrip no longer than $2L + \ell$. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

When planning the motion of a robot, we are confronted with the problem of moving an object while avoiding collision with a set of obstacles. For the case of translational motion, this means considering Minkowski sums of the form $\mathcal{P} \oplus R$, where $\mathcal{P}$ is the set of positions of a reference point and R is the shape of the robot itself. $\mathcal{P}$ is a set of feasible positions if and only if $\mathcal{P} \oplus R$ does not contain any part of an obstacle. In this context, Minkowski sums have been considered in [2,6–8,10–13,15,16], sometimes under the name of "configuration space obstacles". Computing the unbounded face of the Minkowski sum of two simple polygons was considered in [15].

In many cases, it is of particular importance to protect the boundary of a Minkowski sum: if a mobile obstacle (e.g., a person) enters the set of points that can be met by an industrial robot, we may have to stop the robot in order to avoid a serious accident. (A special case arises when a minimal distance r from a set of stationary machinery has to be maintained. In this situation, R is a ball of radius r.) This makes it interesting to find short closed roundtrips that visit all points of the boundary of a Minkowski sum; such a tour could then be followed by a laser or other scanning device.

In this paper, we establish that all boundary points of the Minkowski sum $\mathcal{P} \oplus R$ of a connected arrangement $\mathcal{P}$ of planar curves with total length L and a convex planar shape R of perimeter ℓ can be traveled along a closed roundtrip of length no more than $2L + \ell$, even if the boundary of $\mathcal{P} \oplus R$ is disconnected.

This result is also of importance in the more general situation where we have to search a polygonal area A with a scanning device of shape R. This so-called "lawn mowing problem" is NP-hard and was considered in [9,1]. As it was shown in [1], our

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inequality allows the construction of approximation algorithms with small approximation factors.

2. The inequality

We will make use of the following proposition, following from Cauchy's formula—see [3–5,14].

Proposition 2.1 (Cauchy, 1841). *Let C_1 and C_2 be closed convex curves of length ℓ_1 and ℓ_2, such that C_1 is contained in the interior of C_2. Then $\ell_2 \geqslant \ell_1$.*

Now we proceed to the main theorem. We start with the special case where $\mathcal{P}$ consists of a finite number of line segments, i.e., where $\mathcal{P}$ can be considered a plane drawing of a graph G. For easier notation we write $S^{(R)} := S \oplus R$ for any set S, and $\delta S^{(R)}$ for the boundary points of this set. (See Fig. 1.)

Theorem 2.2. *Let $\mathcal{P}$ be a connected planar arrangement of edges and let R be a closed convex curve of length ℓ. If L is the overall length of the edges in $\mathcal{P}$, there is a closed roundtrip of length at most $2L + \ell$ that visits all points of $\delta\mathcal{P}^{(R)}$.*

Proof. As a first step, replace G by a tree G_t in the following way: As long as there are cycles in the graph, pick a vertex v contained in a cycle where it has neighbors v_a and v_b. Replace v by the vertices v_1 and v_2, where v_1 is adjacent to v_a and v_b is adjacent to all other neighbors of v. Note that *geometrically*, v_1 and v_2 will be identical.

Now we proceed by induction over the number of edges of G_t. The claim is trivial if G (and hence G_t) is a point. Let e be an edge of length $l(e)$ in G_t that is adjacent to a leaf of G_t. Let $\mathcal{Q}$ be the edge arrangement of length $l(\mathcal{Q})$ left after removing e, let F_t be the tree obtained from G_t by removing e, and let $T_{\mathcal{Q}}$ be a tour of length at most $2l(\mathcal{Q}) + \ell$ that visits all points in $\delta\mathcal{Q}^{(R)}$. We assume as part of the induction hypothesis that the set of points where $T_{\mathcal{Q}}$ does not self-intersect can be decomposed into a finite number of (open) intervals $\mathcal{I}_i$. This implies that there is only a finite number of boundary points of these intervals, called the set P of *critical self-intersection points*. These points subdivide $\mathcal{Q}$ into a finite number of *paths*. (Note that a path is counted with a multiplicity according to the number of times that it is traveled by $T_{\mathcal{Q}}$.) Clearly, each point in P has an even number of sections adjacent to it. Furthermore, we assume that the family $\mathcal{C}$ of connected components of $\mathbb{R}^2 \setminus T_{\mathcal{Q}}$ has a finite number of members, and that for each of them, $T_{\mathcal{Q}}$ has a well-defined finite (counterclockwise) winding number. Let $\mathcal{C}_e$ ($\mathcal{C}_o$) be the set of components for which $T_{\mathcal{Q}}$ has an even (odd) winding number. We say that an orientation of a path P separating a component in $\mathcal{C}_o$ from a component in $\mathcal{C}_e$ is *positive*, if the odd component lies to the left when traveling the oriented path; otherwise we follow a *negative orientation*.

Next assign orientations to all paths: for paths with even multiplicity, orient half of these identical paths in each direction; for paths with odd multiplicity, orient one more in a positive than in a negative way. As a result, when going around a point in P, incoming and outgoing paths will alternate. It is straightfor-

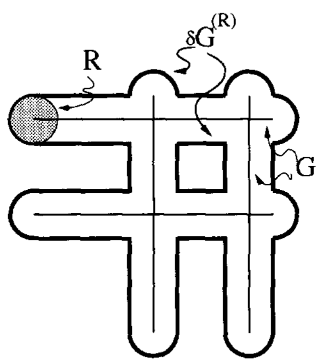

Fig. 1. The set $\delta G^{(R)}$.

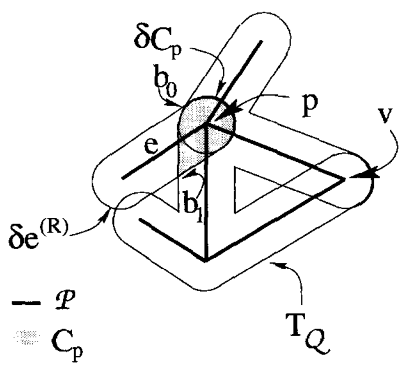

Fig. 2. Constructing the tour $T_{\mathcal{P}}$.

ward to see that following a "left-first" strategy (i.e., at each point in P, choose the unused path that forms the smallest counterclockwise angle with the previous path), we get a set of closed (sub)tours that are neither self-crossing nor do they cross each other. Furthermore, these subtours can be merged into one noncrossing tour that visits all points in $\mathbb{R}^2$ the same number of times as $T_{\mathcal{Q}}$. In the following assume without loss of generality that $T_{\mathcal{Q}}$ is such a noncrossing tour.

If $\delta\mathcal{Q}^{(R)} = \delta\mathcal{P}^{(R)}$, we are done, so assume that there is a point of $e^{(R)}$ which is not contained in $\mathcal{Q}^{(R)}$, implying that $\delta\mathcal{Q}^{(R)}$ and $\delta e^{(R)}$ intersect.

$\delta e^{(R)}$ has a simple tour T_e of length precisely $2l(e) + \ell$. Let p be the endpoint of e that is also a point of $\mathcal{Q}$; in particular, the set $p \oplus R$ is contained in $e^{(R)} \cap \mathcal{Q}^{(R)} = C$. Let C_p be the connected component of C that contains p. Since the length $l(\delta conv(C_p))$ of the boundary of the convex hull of C_p is not longer than $l(\delta(C_p))$, the length of the boundary of C_p, we conclude by Proposition 2.1 that we have $l(\delta C_p) \geqslant \ell$. Now consider the union $T_{\mathcal{Q}\cup e}$ of the tours $T_{\mathcal{Q}}$ and T_e: By assumption, $\delta\mathcal{Q}^{(R)}$ and $\delta e^{(R)}$ intersect, so $T_{\mathcal{Q}\cup e}$ can be traversed as one connected tour that visits each point of $\delta\mathcal{P}^{(R)}$ and each point of δC_p. Furthermore, points of δC_p that are also points of $\delta\mathcal{Q}_t^{(R)}$ are visited by both $T_{\mathcal{Q}}$ and T_e, hence visited twice by $T_{\mathcal{Q}\cup e}$.

Now consider the (finite) set B of critical intersection points of T_e and $T_{\mathcal{Q}}$, i.e., boundary points of intervals where T_e is disjoint from $T_{\mathcal{Q}}$. Let $\{b_0, b_1, \ldots, b_{2k} = b_0\} = B' \subseteq B$ be a set of critical intersection points on the boundary of C_p, such that for any $i \in \{0, \ldots, k-1\}$, the piece of δC_p from b_{2i} to b_{2i+1} belongs to $T_{\mathcal{Q}}$, while the piece of δC_p from b_{2i+1} to b_{2i+2} belongs to T_e. For any b_i with i odd, let $b_{\pi(i)}$ be the first b_j encountered by $T_{\mathcal{Q}}$ after running through the piece of C_p from b_{i-1} to b_i. Since T_e is simple and $T_{\mathcal{Q}}$ noncrossing, the tour formed by the section of $T_{\mathcal{Q}}$ from b_i to $b_{\pi(i)}$ and of the section of δC_p from $b_{\pi(i)}$ to b_i is noncrossing and the section of δC_p from b_i to $b_{\pi(i)}$ is contained in it. Therefore, the section of δC_p from b_i to $b_{\pi(i)}$ cannot contain any other points in B', so $b_{\pi(i)} = b_{i+1}$. This means that we can decompose $T_{\mathcal{Q}\cup e}$ into δC_p and a remaining tour $T_{\mathcal{P}}$, consisting of the section of T_e from b_{2i} to b_{2i+1} and of the section of $T_{\mathcal{Q}}$ from b_{2i+1} to b_{2i+2}. By construction, $T_{\mathcal{P}}$ visits all points of $\delta\mathcal{P}^{(R)}$ at least once and has length at most $2l(\mathcal{Q}) + \ell + 2l(e) + \ell - l(\delta C_p) \leqslant 2l(\mathcal{Q}) + 2l(e) + \ell = 2L + \ell$. From its construction it follows that $T_{\mathcal{P}}$ satisfies the additional induction hypotheses on self-intersection, subdivision of the plane, and well-defined finite winding number. □

It should be noted that for arrangements $\mathcal{P}$ of straight edges, the given bound holds with equality if and only if $\mathcal{P}$ consists of one straight edge of length L; it follows from the above proof that there will be some slack as soon as δC_p is not a unit circle, which is forced in any other case.

Similarly, we get the same result for more general arrangements of curves:

Corollary 2.3. *Let $\mathcal{P}$ be a connected planar arrangement of piecewise differentiable curves. Let C be a compact convex shape with interior points. If L is the overall length of the curves, and l is the length of the boundary of C, there is a closed roundtrip of length at most $2L + l$ that visits all points of $\partial(\mathcal{P} \oplus C)$.*

The proof follows from considering series of edge arrangements that converges towards the individual curves. The upper bound on the length of a tour of $\delta\mathcal{P}^{(R)}$ carries over in the limit.

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References

[1] E. Arkin, S.P. Fekete, J.S.B. Mitchell, Approximation algorithms for lawnmowing and milling, ZPR Report 97-255, available at ftp://ftp.zpr.uni-koeln.de/pub/paper/zpr97-255.ps.gz, 1997.

[2] J. Basch, L.J. Guibas, G.D. Ramkumar, L. Ramshaw, Polyhedral tracings and their convolution, Manuscript, http://robotics.Stanford.EDU/~ramkumar/papers/3d-conv.ps.gz, 1996.

[3] W. Blaschke, Vorlesungen über Integralgeometrie, Hamburger Mathematische Einzelschriften, 1935.

[4] A. Cauchy, Œuvres 6 (1) 369–375; 2 (1) 167–177.

[5] A. Cauchy, Note sur divers théorèmes à la rectification…, C.R. Paris 13 (1841) 1060–1065.

[6] L.J. Guibas, L. Ramshaw, J. Stolfi, A kinetic framework for computational geometry, in: Proc. 24th Ann. IEEE Symp. Foundations of Computer Science, 1983, pp. 100–111.

[7] L.J. Guibas, R. Seidel, Computing convolutions by reciprocal search, Discrete Comput. Geom. 2 (1985) 175–193, 1987.

[8] S. Har-Peled, T.M. Chan, B. Aronov, D. Halperin, J. Snoeyink, The complexity of a single face of a Minkowski sum, in: Proc. 7th Canadian Conf. Computational Geometry, 1995, pp. 91–96.

[9] K. Iwano, P. Raghavan, H. Tamaki, The traveling cameraman problem, with applications to automatic optical inspection, in: Proc. 5th Ann. Internat. Symp. Algorithms Comput. (ISAAC-94), Lecture Notes in Comput. Sci., Springer, Berlin, 1994.

[10] A. Kaul, M.A. O'Connor, Computing Minkowski sums of regular polyhedra, Report RC 18891 (82557) 5/12/93, IBM T.J. Watson Research Center, Yorktown Heights, NY, 1993.

[11] A. Kaul, M.A. O'Connor, V. Srinivasan, Computing Minkowski sums of regular polygons, in: Proc. 3rd Canadian Conf. Computational Geometry, 1991, pp. 74–77.

[12] J.-C. Latombe, Robot Motion Planning, Kluwer Academic Publishers, Boston, 1991.

[13] T. Lozano-Pérez, Spatial planning: A configuration space approach, IEEE Trans. Comput. C-32 (1983) 108–120.

[14] J. Pach, P.K. Agarwal, Combinatorial Geometry, Wiley, New York, 1995, pp. 283–284.

[15] G.D. Ramkumar, An algorithm to compute the Minkowski sum outer-face of two simple polygons, in: Proc. ACM Symp. Computational Geometry, 1996, pp. 234–242.

[16] G.D. Ramkumar, Algorithms to compute the convolution and Minkowski sum outer-face of two simple polygons, in: Proc. 28th Ann. ACM Symp. Theory of Computing, 1996.