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On approximately fair cost allocation in Euclidean TSP games

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Abstract. We consider the problem of allocating the cost of an optimal traveling salesman tour in a fair way among the nodes visited; in particular, we focus on the case where the distance matrix of the underlying TSP problem satisfies the triangle inequality. We thereby use the model of TSP games in the sense of cooperative game theory. We give examples showing that the core of such games may be empty, even for the case of Euclidean distances. On the positive, we develop an LP-based allocation rule guaranteeing that no coalition pays more than α times its own cost, where α is the ratio between the optimal TSP-tour and the optimal value of its Held-Karp relaxation, which is also known as the solution over the "subtour polytope". A well-known conjecture states that $\alpha \leq \frac{4}{3}$. We also exhibit examples showing that this ratio cannot be improved below $\frac{4}{3}$.

Summary. Wir betrachten die Aufgabe, die Kosten einer optimalen Traveling-Salesman-Tour fair unter den besuchten Knoten zu verteilen; insbesondere untersuchen wir den Fall, daß die Kostenmatrix des zugrundeliegenden TSP-Problems die Dreiecksungleichung erfüllt. Dazu wird das Modell von TSP-Spielen im Sinne der kooperativen Spieltheorie benutzt. Wir zeigen anhand eines Beispiels, daß der Core eines solchen Spiels leer sein kann, selbst im Falle euklidischer Distanzen. Andererseits geben wir eine LP-basierte Verteilungsregel an, die garantiert, daß keine Koalition mehr als das α-fache ihrer eigenen Kosten bezahlen muß, wobei α das Verhältnis zwischen den Kosten einer optimalen TSP-Tour und dem Optimum der Held-Karp-Relaxation ist, die auch als Lösung über dem "subtour polytope" bekannt ist. Es wird allgemein vermutet, daß $\alpha \leq \frac{4}{3}$. Abschließend geben wir eine Klasse von Beispielen an, die beweist, daß keine allgemeine Verteilungsregel für das TSP-game ein generell besseres Verhältnis als $\frac{4}{3}$ zwischen der Belastung einer Koalition und ihren Kosten garantieren kann.

Key words: Cooperative game, core, cost allocation, traveling salesman, Held-Karp bound

Schlüsselwörter: Kooperative Spiele, Core, Kostenallokation, Traveling Salesman, Held-Karp Schranke.

1 Introduction

Traveling salesman games ("TSP-games") are examples of games in the sense of cooperative game theory, i.e., mathematical models for cost allocation among individuals in a set N of so-called *players* that may form coalitions $S \subset N$ in order to achieve a common benefit or to provide a service, thereby incurring a *cost* $c(S)$.

In this setting, we define a *cooperative game* to be a pair (N, c), where N is a (finite) set of *players* and

$$c : 2^N \to \Re$$

the *characteristic* function of the game that assigns to every *coalition* $S \subseteq N$ its *cost* $c(S)$. N is often called the *grand coalition* of the game.

Example 1.1 (*Minimum Cost Spanning Tree Game*). Let $N = \{1, 2, \ldots, n\}$ and $N_0 := N \cup \{0\}$, and consider the complete undirected graph K_{N_0} with N_0 as its set of nodes. Each edge ij, $i, j \in N_0$ of K_{N_0} is weighted with a non-negative real number d_{ij}. For every $S \subseteq N$, let

$c(S)$ = minimum cost of a tree spanning all nodes in $S \cup \{0\}$. □

Minimum cost spanning tree games ("MCST-games") have received much attention in the literature (see, e.g., [1], [2], or [13]). They arise, for example, in the following modeling context: node 0 represents a *supply* node for electricity. The other nodes ("players") $i \in N$ are sites that are to be supplied with electricity. The weight d_{ij} reflects the cost of

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establishing a direct link between nodes i and j. So $c(S)$ is the minimal cost the coalition S would incur upon linking itself to 0 without using links to $N \setminus S$ that may already be established.

The game theoretic model of central interest here was introduced by Potters et al. [25] and is based on the same underlying combinatorial structure as the model of a MCST-game. The difference lies in the way the cost of a coalition is assessed.

Example 1.2 (*Traveling Salesman Game*). Let N, N_0, and d_{ij} be given as in Example 1.1 and define for every coalition $S \subseteq N$,

$$c(S) = \text{minimum length of a traveling salesman tour visiting all nodes in } S \cup \{0\}\ . \qquad \Box$$

A modeling context for Example 1.2 can be the following: node 0 is a server for a new local area network in a larger company. The players in N stand for different departments within the company, acting on independent budgets. $c(S)$ is the cost of connecting all departments in S to the server in a ring structure (say, token ring or FDDI).

Given a cooperative game (N, c), one faces the computational problem of finding an *allocation* vector $x \in \Re^N$ that is *fair* in a mathematically well-defined way. A set of mathematical criteria for fairness of an allocation is a *solution concept* for the game. There have been many solution concepts proposed in cooperative game theory (see, e.g., [28]).

The notion of fairness we use here goes back to von Neumann and Morgenstern [20] and tries to allocate the cost in such a way that no coalition S would be better off playing the game without $N \setminus S$ (condition (ii) below). We also want the total cost to be allocated (condition (i)). This leads us to the definition of the *core* of a game (N, c) to be the set $core(c)$ of all vectors $x \in \Re^N$ such that

(i) $\sum_{i \in N} x_i \geq c(N)$.

(ii) $\sum_{i \in S} x_i \leq c(S)$ for all $S \subseteq N$.

Note that $core(c)$ is a polyhedron in $\Re^N$. So a core vector can, in principle, be found by linear programming.

Already in the case of MCST-games, however, such an approach is unrealistic because of the exponentially many linear inequalities defining $core(c)$. On the other hand, there is a well-known efficient procedure for computing a core vector x relative to a MCST-game (see [2]): construct a minimum cost spanning tree T in K_{N_0} and direct the edges of T away from 0 so that the supply node 0 becomes the root of T. Allocate now to each $i \in N$ as x_i the weight of the (unique) edge in T that enters node i. It is straightforward to check that this x is indeed a core vector. (Without going into details, we remark that the core of a MCST-game in its entirety is less straightforward to analyze than this procedure might suggest: it is NP-hard to decide whether a given $x \in \Re^n$ lies in the core of a given MCST-game (see Faigle et al. [8]).)

In the case of TSP-games, further complications arise. The computation of the value $c(S)$ alone is an NP-hard problem (see Garey and Johnson [11]). It has also been observed that the core of a TSP-game may be empty (see Tamir [30]). (Tamir's model actually is slightly more general in that it does not exclude possibly non-hamiltonian graphs).

In the present paper, we consider the special class of *Euclidean* TSP-games, where the complete graph K_{N_0} can be represented by points in the Euclidean plane so that the weights d_{ij} are just the Euclidean distances between the points i, j. This class of TSP-games appears quite natural but so far, it has not received much attention in the literature.

Kuipers [18] has shown that TSP-games with 5 or less players have a non-empty core. We show in Sect. 2 that this result cannot be improved even for the class of Euclidean TSP-games. Therefore, the question arises whether allocation vectors x for TSP-games can be computed that are "approximately" core vectors.

Our notion of an ϵ-approximation of the core stipulates that a coalition S should be charged with an allocation that does not exceed the cost $c(S)$ by more than a fraction ϵ (see, e.g., Faigle and Kern [7] for references for other notions of "ϵ-approximation"). In other words, for $\epsilon \geq 0$, we define the *ϵ-core* of the cooperative game (N, c) to be the set ϵ-$core(c)$ of all vectors $x \in \Re^N$ satisfying

(i) $\sum_{i \in N} x_i \geq c(N)$.

(ii) $\sum_{i \in S} x_i \leq (1 + \epsilon)c(S)$ for all $S \subseteq N$.

Ideally, we would like to find a vector in the ϵ-*core* where ϵ is as small as possible. In Sect. 3, we prove that TSP-games whose weights satisfy the triangle inequality (as is the case in Euclidean TSP-games) always have ϵ-approximately fair (core) allocations for $\epsilon = \frac{1}{2}$. If a well-known conjecture concerning the Held-Karp relaxation of the Euclidean TSP is true, our allocation rule presented in Sect. 3 achieves $\epsilon = \frac{1}{3}$. Moreover, these allocations can be computed in polynomial time even when the value of the optimal tour is not known.

Note that $\epsilon = \frac{1}{3}$ would be best possible as we show in Sect. 4 that no allocation rule can guarantee an $\epsilon < \frac{1}{3}$. One of these results was already briefly summarized in Faigle and Kern [7]. Here we give full details and proofs. We end with some open problems and concluding remarks in Sect. 5.

2 A minimal example with empty core

Kuipers [18] showed that under quite general assumption (including the Euclidean case), every TSP-game with up to 5 players has a nonempty core. The following example shows that for $n = 6$ players, there exist Euclidean instances with empty core.

Example 2.1 (see Fig. 1). Consider an equilateral triangle of side length $l = \sqrt{3}$, centered at 0. Label the vertices by 1, 2, 3. Place three more nodes 4, 5, 6 at equal distance d from the center, such that node i lies on the line segment $\overline{0, i-3}$.

The distance d will be chosen appropriately. Let f denote the distance $d_{56} = d_{46} = d_{45}$; clearly, $f = \sqrt{3}d$. Let h denote the distance $d_{15} = d_{16} = d_{24} = d_{26} = d_{34} = d_{35}$. Applying Pythagoras' Theorem to the triangle $\Delta(5P3)$, we get

$$h = \sqrt{\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}d\right)^2 + \left(\frac{(1-d)}{2}\right)^2},$$

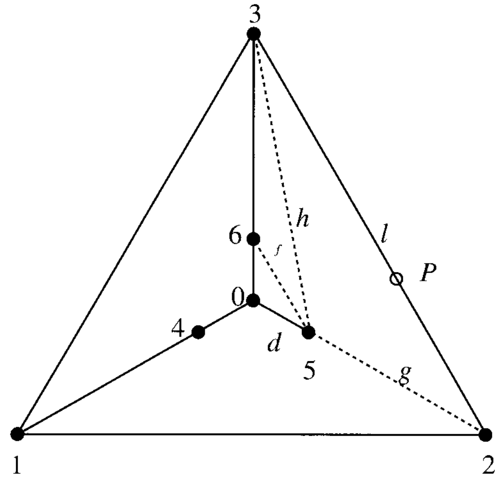

Fig. 1. A minimum Euclidean example with empty core

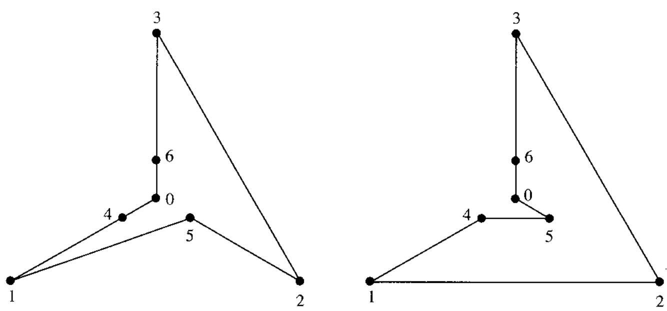

Fig. 2. The two candidates for optimal TSP tours

we conclude that $h = \sqrt{1+d+d^2}$. Finally, let $g := d_{36} = d_{25} = d_{14}$, i.e., $g = 1 - d$. As a consequence, the two obvious candidates for an optimal TSP tour of length L have the following lengths L_1 and L_2 (see Fig. 2):

$$L_1 = l + 3g + 2d + h = 3 - d + \sqrt{3} + \sqrt{1+d+d^2},$$

$$L_2 = 2l + 2g + 2d + f = 2 + 2\sqrt{3} + \sqrt{3}d.$$

Now suppose $x \in \mathbb{R}^6$ were a core allocation. In particular, we must have

1. $\sum_{i=1}^{6} x_i = L$,
2. $x_i + x_{i+1} + x_{i+3} + x_{i+4} \leq 2d + 2g + l$ for $i = 1, 2$ and $x_3 + x_1 + x_6 + x_4 \leq 2d + 2g + l$.

Adding up the three inequalities, we get

$$L = \sum_{i=1}^{6} x_i \leq \frac{3}{2}(2d + 2g + l) = 3 + \frac{3}{2}\sqrt{3}.$$

If the optimal tour has length L_1, we get

$$L_1 = 3 - d + \sqrt{3} + \sqrt{1+d+d^2} \leq 3 + \frac{3}{2}\sqrt{3}.$$

The last inequality holds if and only if

$$d \geq \frac{1}{4(\sqrt{3}-1)} > \frac{1}{3}.$$

If the optimal tour has length L_2, we get

$$L_2 = 2 + 2\sqrt{3} + \sqrt{3}d \leq 3 + \frac{3}{2}\sqrt{3},$$

imposing the condition

$$d \leq \frac{1}{\sqrt{3}} - \frac{1}{2} < \frac{1}{10}.$$

We conclude that there is no fair cost allocation if neither of these two conditions is satisfied, e.g., for $d = \frac{1}{4}$.

3 Approximately fair allocations for Euclidean instances

We have seen in the previous section that not all TSP instances allow a fair cost allocation. This makes it desirable to examine *approximately fair cost allocations*, where the customers can be overcharged by a certain percentage, or the supplier is allowed to run a certain deficit.

Interesting allocation rules (different from core allocations) have been studied in the context of TSP games (see [25]). In this section, we will approach the question from a geometric point of view and make use of linear programming duality.

There is another good reason for considering approximately fair cost allocations: Since computing the length L of an optimal TSP tour is NP-hard, we cannot expect to find an efficient way of computing cost allocations, whether they are fair or not. These computational difficulties make it desirable to consider performance bounds on cost allocations which can be computed in polynomial time. In Sect. 3.2 below, we will introduce a modified game with cost function $c_{(HK)} \leq c$, for which we can efficiently compute core allocation vectors z. It is known that $c_{(HK)} \geq \frac{2}{3}c_{Chr}$, where c_{Chr} denotes the cost function corresponding to the well-known Christofides heuristic. Hence, by scaling the vector z by a factor $\alpha = \frac{c_{Chr}(N)}{c_{HK}(N)} \leq \frac{3}{2}$, we get an ε-approximately fair allocation for the original TSP game with $\varepsilon \leq \frac{1}{2}$ – see Theorem 3.6 below for details. In case an optimal global tour of length L is known, we may of course scale the vector z by a factor of $\overline{\alpha} = \frac{L}{c_{HK}(N)} \leq \alpha$ to obtain an $\overline{\varepsilon} \leq \varepsilon \leq \frac{1}{2}$.

3.1 Geometric cost allocation

We encounter methods of allocating the cost of transportation in many instances in everyday life. Two of the easiest allocation rules are also the most common ones: A taxi charges by the distance that is traveled by an individual customer. On the other hand, public mass transportation typically charges a flat fee for anyone who uses it, regardless of the distance. The practicality of these two allocation rules relies on the fact that the number of players in a game is either extremely small, i.e., 1, or arbitrarily large. (Insufficiencies of the latter assumption are reflected by the deficits of most public transportation.)

In many cities, there are attempts to refine the fares by using a "zone structure": The region is subdivided into traffic zones, and a customer is charged a certain amount for crossing from one zone into the other (see Fig. 3 for a practical example).

For our purposes, a subdivision of the plane into fixed zones (e.g., concentric circles around the depot) is much

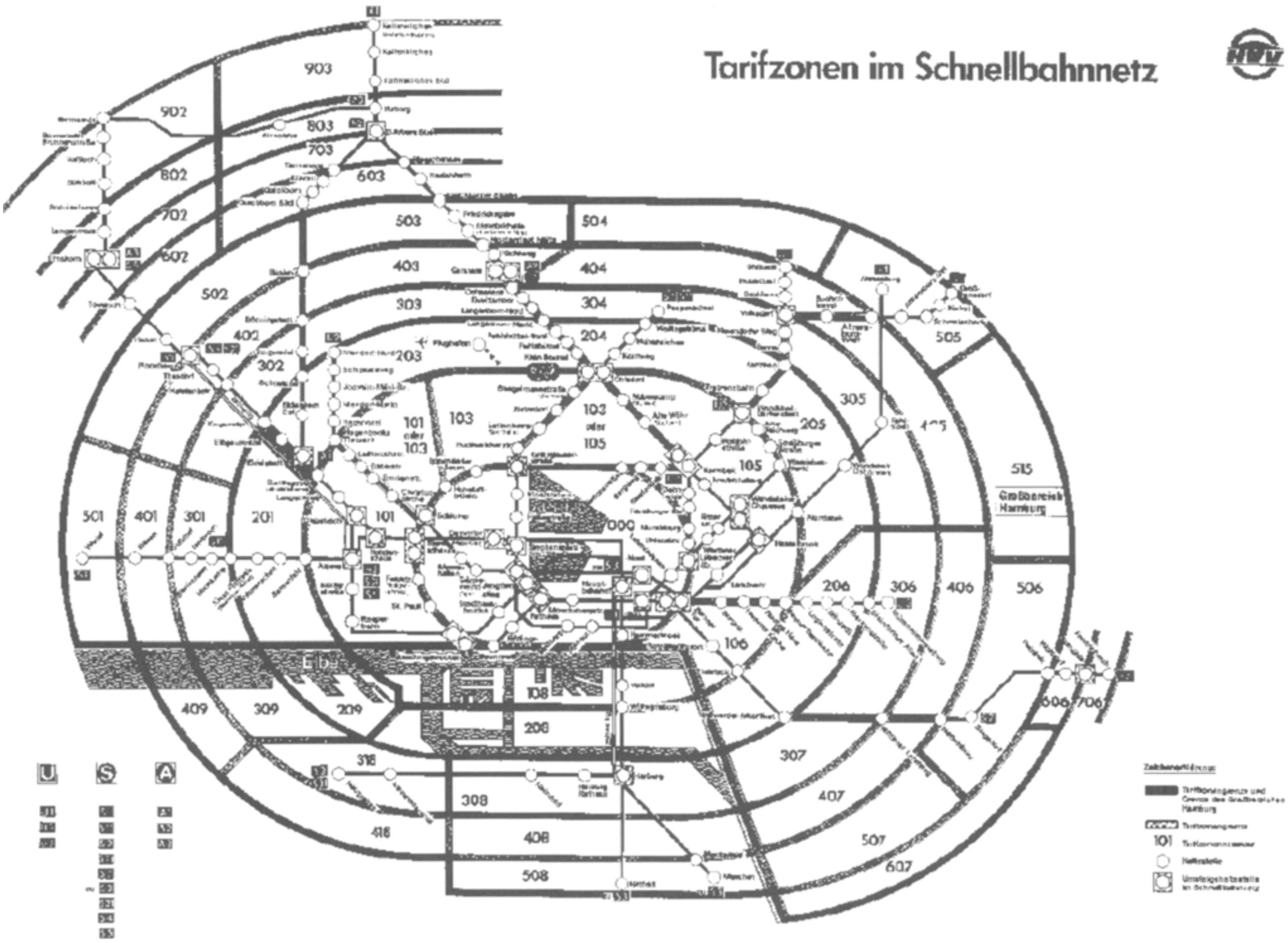

Fig. 3. The fare zones of Hamburg, reproduced with kind permission of the Hamburger Verkehrsverbund HVV

too crude. Instead, we have to take into account the relative position of the customers. In the following, we will describe a geometric cost allocation method that follows this idea.

Definition 3.1. *For a given set of vertices in the plane, a* moat *is a simply closed strip of constant width that separates two nonempty complementary subsets of the vertices. The* inside *of the moat is the region containing the vertex* 0, *the other region we call the* outside. *A* moat packing *is a collection of moats with pairwise disjoint interior. The* cost *of a moat packing is twice the sum over all widths.*

Note that any tour has to cross every moat twice, hence the cost of a TSP tour is greater than or equal to the cost of a moat packing. Fig. 4 shows a moat packing for an instance of cities in the American midwest, distances are taken from Nemhauser and Wolsey ([19], p. 530).

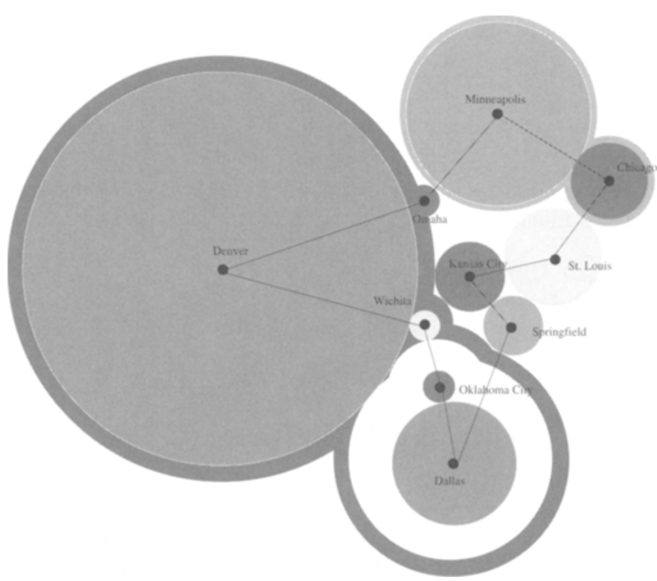

Fig. 4. A moat packing for an instance of 10 cities in the midwestern United States

3.2 Moats and the subtour polytope

The cost of any moat packing can be allocated as follows: Distribute twice the width of any moat among the vertices on the outside (in an arbitrary way). It is intuitively clear that if the total cost of the moat packing is distributed this way, the resulting distribution is such that no coalition pays more than its TSP cost. (We will prove that formally later on.) Note, however, that by distributing the cost of a moat packing, we will in general not get an allocation vector of the TSP game, since the cost of a moat packing is in general strictly less than the cost of an optimal TSP tour. The situation where the two costs (maximum moat packing and minimum TSP tour) coincide occurs precisely if there is a moat packing and a tour such that:

1. no part of the given tour is left uncovered, and
2. each moat is crossed by the tour exactly twice.

In the general situation, i.e., when the maximum moat packing has a cost strictly less than the optimum TSP tour, we can still get an allocation by simply scaling the cost distribution above appropriately. This yields an approximately fair allocation for the TSP game, which will be discussed below in detail.

The allocated cost for any moat packing is twice the sum of the moat widths. Moat widths are required to be nonnegative and the sum of the widths of moats separating two vertices must not exceed their distance. This motivates the following linear program:

$$\max\left(2\sum_{\{S,\overline{S}\}\in\mathcal{M}} w_{\{S\overline{S}\}}\right)$$

(M) subject to the constraints

$$\begin{aligned} w_{\{S\overline{S}\}} &\geq 0 \quad \text{for all} \quad \{S,\overline{S}\}\in\mathcal{M},\\ \sum_{ij\in\delta(S)} w_{\{S\overline{S}\}} &\leq d_{ij} \quad \text{for all} \quad i,j\in N_0. \end{aligned}$$

Here $\mathcal{M}$ denotes the set of all nontrivial partitions $\{S,\overline{S}\}$ of N_0 (assuming that the vertex 0 is contained in $\overline{S}$) and $\delta(S)$ is the set of all edges that join a vertex from S to a vertex from $\overline{S}$. Note that there are exponentially many variables. Furthermore, there may be solutions to the linear program which do not correspond to a moat packing, since there may be positive $w_{\{S_1\overline{S_1}\}}, w_{\{S_2\overline{S_2}\}}$ with $S_1\cap S_2\neq\emptyset$ and $S_1\not\subseteq S_2$ and $S_2\not\subseteq S_1$, in which case the two corresponding moats are forced to intersect. However, for any instance of (M), there is an optimal solution for which the sets with $w_{\{S\overline{S}\}}>0$ have the special structure of a *nested family*:

Definition 3.2. *A family of partitions* $\{S_1,\overline{S_1}\},\ldots,\{S_k,\overline{S_k}\}$ *is called* nested, *if for any two partitions* $\{S_i,\overline{S_i}\}$ *and* $\{S_j,\overline{S_j}\}$, *we have* $S_i\cap S_j=\emptyset$ *or* $S_i\subseteq S_j$ *or* $S_j\subseteq S_i$.

For details on nested families, see Pulleyblank [26]; a proof for the above claim can be found in Cornuéjols, Fonlupt and Naddef [5].

It is not hard to see that a solution with the structure of a nested family allows a moat packing. The details of using a moat packing for allocating the optimal value of (M) will be discussed in the following section. We will assume that triangle inequality and (for the sake of simplicity) symmetry hold for the distance function. We assume the distance function to be defined for all $ij, i,j\in N_0$.

How can we solve (M) in polynomial time? Consider its dual, i.e., the following linear problem:

$$\min\sum_{ij\in\binom{N_0}{2}} d_{ij}x_{ij}$$

(D) subject to the constraints

$$\begin{aligned} x_{ij} &\geq 0 \quad \text{for all} \quad i,j,\\ \sum_{ij\in\delta(S)} x_{ij} &\geq 2 \quad \text{for all} \quad \{S,\overline{S}\}\in\mathcal{M}, \end{aligned}$$

Proposition 3.3. *Let* $d: N_0^2\to\mathbb{R}_+$ *be an instance of the symmetric TSP satisfying the triangle inequality* $d_{ij}+d_{jk}\geq d_{ik}$ *for all* i,j,k. *Then the corresponding program* (D) *has an optimal solution* $x\in\mathbb{R}^{\binom{N_0}{2}}$ *satisfying*

1. $x_{ij}\leq 1$ *for all* $ij\in\binom{N_0}{2}$,
2. $\sum_{\substack{j=0\\ j\neq i}}^{N} x_{ij}=2$ *for all* $i\in N_0$.

Proof. Consider an optimal solution to (D) and suppose there was a vertex i with $\sum_{j\neq i}x_{ij}>2$. If there was only one vertex $k\neq i$ with $x_{ki}>0$, we would have $x_{ik}>2$ and we could lower x_{ik} and stay feasible. Since $d_{ik}\geq 0$, this would not increase the objective value. So assume there are two vertices k_1 and k_2 with $x_{ik_1}>0$ and $x_{ik_2}>0$. In that case we can lower both x_{ik_1} and x_{ik_2} and raise $x_{k_1k_2}$ by the same value and maintain feasibility. Since by triangle inequality $d_{k_1i}+d_{ik_2}\geq d_{k_1k_2}$, this does not increase the objective value. So we may assume that for any vertex i, we have $\sum_{j\neq i}x_{ij}=2$. Furthermore, there can be no edge k_1k_2 with $x_{k_1k_2}>1$, since otherwise the total weight of the edges leaving the set of vertices k_1,k_2 would be deficient, i.e., we would get the violated constraint

$$\begin{aligned} \sum_{ij\in\delta(\{k_1,k_2\})} x_{ij} &= \sum_{i\neq k_1}x_{ik_1}+\sum_{i\neq k_2}x_{ik_2}-2x_{k_1k_2}\\ &= 4-2x_{k_1k_2}<2.\square \end{aligned}$$

This means we may consider the following linear program (HK) instead of (D):

$$\min\sum_{ij\in\binom{N_0}{2}} d_{ij}x_{ij}$$

(HK) subject to the constraints

$$\begin{aligned} x_{ij} &\geq 0 \quad \text{for all} \quad i,j,\\ x_{ij} &\leq 1 \quad \text{for all} \quad i,j,\\ \sum_{ij\in\delta(S)} x_{ij} &\geq 2 \quad \text{for all} \quad \{S,\overline{S}\}\in\mathcal{M},\\ \sum_{j\neq i} x_{ij} &= 2 \quad \text{for all} \quad i\in N_0. \end{aligned}$$

The feasible region for this second linear program is known as the *subtour polytope* S^n, the program itself as *Held-Karp relaxation*: Any of the constraints corresponding to a moat variable is a so-called *subtour elimination constraint*.

These *subtour elimination constraints* were first introduced by Dantzig, Fulkerson and Johnson [6]. Grötschel and Padberg showed that they are facet-inducing for $n \geq 4$.

Grötschel, Lovász and Schrijver [14], and Karp and Papadimitriou [17], showed that a polynomial method for solving the separation problem for a polytope yields a polynomial method for optimization by means of the ellipsoid method. Padberg and Hong [22] (see also Padberg and Wolsey [24] and Padberg and Rao [23]) demonstrated how to solve separation for the subtour polytope in polynomial time by using the method of Gomory and Hu [13] for finding the minimum cost cut in a graph. Thus we know that optimization over the subtour polytope is possible in polynomial time by means of the ellipsoid algorithm.

For a comprehensive study of optimizing over the subtour polytope, see Boyd [3] and also Boyd and Pulleyblank [4].

Summarizing, we state:

Theorem 3.4. *We can determine an optimal moat packing in polynomial time.* □

Since the feasible region Q^n of the Traveling Salesman Polytope is contained in S^n, any optimal solution to minimizing over S^n is a lower bound for the optimal value of TSP. It was proved by Wolsey [32] and by Shmoys and Williamson [27] that for any distance function d satisfying the triangle inequality, this bound can be at worst 2/3 of the optimum:

Theorem 3.5 ([32],[27]). *If the distances satisfy the triangle inequality then the optimum value of (HK) is at least $\frac{2}{3}$ of the length of a shortest tour.* □

It is a well-known open conjecture that the factor of 2/3 can be replaced by 3/4. This is known as the *Held-Karp Conjecture.*

It should be noted that Shmoys and Williamson have shown an even stronger version of Theorem 3.5:

Theorem 3.6 ([27]). *If the distances satisfy the triangle inequality then the optimum value of (HK) is at least $\frac{2}{3}$ of the length of a tour obtained by the method of Christofides.* □

This means we can guarantee allocation of 2/3 of the cost of an *approximate tour* where both the allocation and the tour have been computed in polynomial time.

3.3 Tours and moats

We now proceed to describe how to obtain allocation vectors from moat packings. The idea is to distribute twice the width of every moat in an arbitrary way (e.g., uniformly) among the vertices on its outside. Let w^* be an optimal solution of the linear program (M) and define

$$x_i := 2 \cdot \sum_{\substack{\{S,\bar{S}\} \in \mathcal{M} \\ i \in S, 0 \in \bar{S}}} \frac{w^*_{S,\bar{S}}}{|S|}.$$

By linear programming duality this vector is in the core of the game associated with the linear program (HK) in a natural way:

Define the cost of a coalition $S \subseteq N$ by

$$c_{(HK)}(S) := \min \sum_{i,j} d_{ij} z_{ij}$$

subject to the constraints

$$\begin{aligned} z_{ij} &\geq 0 \quad \text{for all} \quad i,j, \\ \textstyle\sum_{ij \in \delta_S(T)} z_{ij} &\geq 2 \quad \text{for all} \quad 0 \in T \subset S \cup \{0\}. \end{aligned}$$

(Here $\delta_S(T)$ denotes the set of edges joining T to $(S \cup \{0\} \setminus T)$.) The general idea to apply linear programming duality to combinatorial games goes back to [21].

The vector $x \in \mathbb{R}^N$ is easily seen to be a core vector of the game defined by the cost function $c_{(HK)}$: Indeed, by linear programming duality we have

$$\begin{aligned} \sum_{i \in N} x_i &= 2 \cdot \sum_{i \in N} \sum_{\substack{\{S,\bar{S}\} \in \mathcal{M} \\ i \in S, 0 \in \bar{S}}} \frac{w^*_{S,\bar{S}}}{|S|} = 2 \cdot \sum_{\{S,\bar{S}\} \in \mathcal{M}} \sum_{i \in S} \frac{1}{|S|} w^*_{S,\bar{S}} \\ &= 2 \cdot \sum_{\{S,\bar{S}\} \in \mathcal{M}} w^*_{S,\bar{S}} = c_{(HK)}(N). \end{aligned}$$

To see that this is in fact a fair allocation vector, recall that for a coalition $S \subseteq N$, its cost $c_{(HK)}(S)$ is defined as the optimum of an LP. Its dual is given by

$$c_{(HK)}(S) = \max \quad 2 \cdot \sum_{0 \notin T \subset S} w_{T,(S \cup \{0\}) \setminus T}$$

subject to the constraints

$$\begin{aligned} \sum_{\substack{0 \notin T \subset S \\ \delta_S(T) \ni ij}} w_{T,(S \cup \{0\}) \setminus T} &\leq d_{ij} \; \forall i,j \\ w_{T,(S \cup \{0\}) \setminus T} &\geq 0. \end{aligned}$$

Now an optimum solution w^* of (M) induces a feasible solution of the LP above by $\tilde{w}_T := \sum_{S \cap U = T} w^*_U$. Hence we get

$$\begin{aligned} c_{(HK)}(S) &\geq 2 \cdot \sum_{0 \notin T \subset S} \tilde{w}_T \\ &= 2 \cdot \sum_{\substack{(i,T) \\ i \in T \subseteq S}} \frac{1}{|T|} \tilde{w}_T \\ &= \sum_{i \in S} 2 \cdot \sum_{i \in U} \frac{w^*_U}{|U \cap S|} \\ &\geq \sum_{i \in S} 2 \cdot \sum_{i \in U} \frac{w^*_U}{|U|} \\ &= \sum_{i \in S} x_i. \end{aligned}$$

Note that we can choose any distribution of the cost of a moat among the outside vertices, i.e., instead of taking

$$x_i := 2 \cdot \sum_{\substack{\{S,\bar{S}\} \in \mathcal{M} \\ i \in S, 0 \in \bar{S}}} \frac{w_{S,\bar{S}}}{|S|},$$

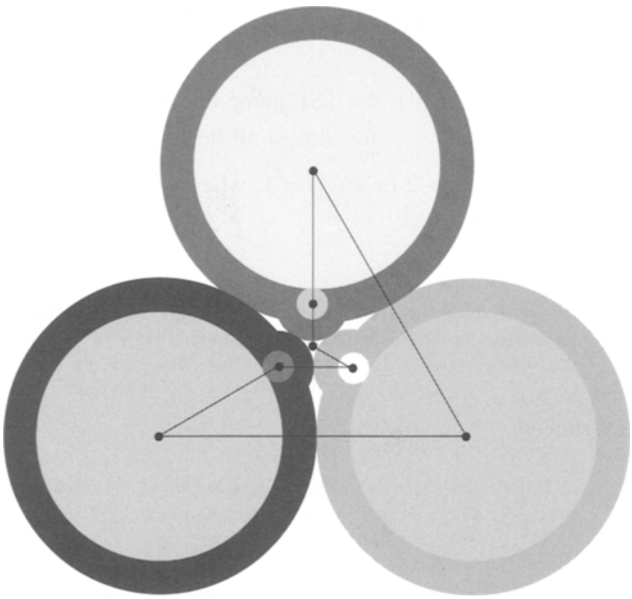

Fig. 5. A moat packing

we could choose any

$$x_i := 2 \cdot \sum_{\substack{i \in S \\ 0 \in \bar{S}}} \lambda_{Si} w_{S,\bar{S}}$$

with $\sum_{i \in S} \lambda_{Si} = 1, 0 \leq \lambda_{Si} \leq 1$ without changing the validity of any of the above statements.

If the optimal tour is not an optimal solution of (HK) then for any moat packing either at least one moat is traversed more than two times (see Fig. 5) or the optimal tour runs through territory which is not covered by any moat.

Proposition 3.7. We would like to note that even if $c(N) > c_{(HK)}(N)$, the core may be nonempty. The easiest example for this situation arises from Example 2.1 (see Figs. 1 and 5): It can be shown that an optimal moat packing has cost $3(\frac{f}{2} + g + \frac{l}{2}) + 2(d - \frac{f}{2}) = (3 - d) + (3 + d)\frac{\sqrt{3}}{2} =: L_3$. Now consider the case where d satisfies one of the bounds given in Example 2.1 with equality; in that case, we have a tour of length $L = 3 + 3\frac{\sqrt{3}}{2} > L_3$, meaning that we cannot distribute the cost of the tour by the Held-Karp allocation rule. However, one can check that allocating $\sqrt{3}$ to each of the players on the outside vertices and $1 - \frac{\sqrt{3}}{2}$ to each of the players on the inside vertices is a fair allocation.

3.4 Approximately fair allocations

Summarizing the results of the previous two sections, we state:

Theorem 3.8. *For TSP games with triangle inequality, there are vectors $x_1, x_2 \in \mathbb{R}^N$, which can be computed in polynomial time and satisfy the following conditions:*

(i) x_1 is an ε-approximative core allocation for $\varepsilon = \frac{1}{2}$.

(ii) If the "$\frac{3}{4}$-conjecture" on the Held-Karp bound is true, x_2 is an ε-approximative core allocation for $\varepsilon = \frac{1}{3}$.

(iii) x_1 ε-approximately allocates the cost of an approximative TSP tour obtained by the Christofides heuristic for $\varepsilon = \frac{1}{2}$.

Proof. Let x as above denote the fair allocation of the associated LP-game. Let $c^*_{(HK)} := c_{(HK)}(N)$ denote the optimum value of (HK) and let L be the length of a shortest tour. Obviously,

$$\hat{x} := \frac{L}{c^*} x$$

is a vector in $\mathbb{R}^N$ satisfying

1. $\hat{x}(N) = L$ and
2. $\hat{x}(S) = \frac{L}{c^*} x(S) \leq \frac{L}{c^*} L(S) \leq \frac{3}{2} L(S)$.

Set $x_1 = \frac{3}{2}x$, $x_2 = \frac{4}{3}x$ and the claim follows. □

We will see in the following Sect. 4 that it is impossible to achieve a better general bound than $\varepsilon = \frac{1}{3}$, even in the case of Euclidean distances.

Remark 3.9. Potters, Curiel and Tijs [25] have shown that in the case of a distance function induced by the Euclidean metric for a planar arrangement of points, any convex arrangement of players, i.e., a set of points which lie on the boundary of their convex hull, guarantees a fair cost allocation. As it was shown by Fekete and Pulleyblank [9], any such arrangement has a moat packing with cost equal to the length of the optimal tour, implying that this special case is covered by our above approach.

Remark 3.10. We would like to point out that the above worst case estimate for the ratio $\frac{L}{c^*_{(HK)}}$ is far from the average case ratio. It can be shown (cf. [29],[12])) that if $p_1, \ldots, p_n, \ldots$ are independently uniformly distributed in the unit square, and $L(p_1, \ldots, p_n)$ denotes the length of a shortest tour through $p_1, \ldots, p_n$ and $c^*_{(HK)}(p_1, \ldots, p_n)$ denotes the optimum value of the Held-Karp relaxation, then there exist constants β_{TSP} and $\beta_{(HK)}$, such that

$$\frac{L(p_1, \ldots, p_n)}{\sqrt{n}} \overset{n \to \infty}{\longrightarrow} \beta_{TSP} \text{ a.s.}$$

and

$$\frac{c^*_{(HK)}(p_1, \ldots, p_n)}{\sqrt{n}} \overset{n \to \infty}{\longrightarrow} \beta_{(HK)} \text{ a.s.}$$

Numerical experiments are reported [12] saying that $\beta_{TSP} \approx 0.709$ and $\beta_{(HK)} \approx 0.7$. This means that in the average one can expect ε-approximately fair allocations for $\varepsilon \approx 0.013$.

4 A lower bound on ε

We have seen in the previous section that (subject to the conjecture on the Held-Karp bound) there is a way to guarantee $\varepsilon \leq \frac{1}{3}$. In the following, we present an example with ε arbitrarily close to this bound of $\frac{1}{3}$.

Theorem 4.1. *The bound $\varepsilon \leq \frac{1}{3}$ resulting from the conjecture on the Held-Karp bound is best possible.*

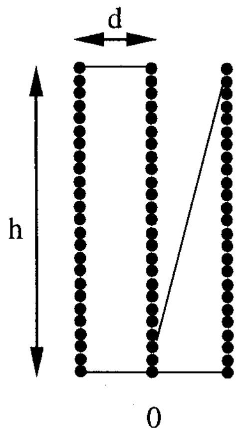

Fig. 6. An example with $\varepsilon \longrightarrow \frac{1}{3}$

Proof. Consider a set of n points, evenly distributed on 3 columns, as shown in Fig. 6. Let h denote the height of the columns. Since the distance d between two columns can be made arbitrarily small (provided we place a sufficient number of points on each column), a shortest tour has length approximately equal to $4h$. On the other hand, any coalition consisting of two of the three columns is willing to pay at most $2h + 4d$. Hence any potential core vector must satisfy $x(N) \leq \frac{3}{2}(2h + 4d) = 3h + 6d$. This shows that there are no ε-approximately fair allocations for $\varepsilon \leq \dfrac{4h}{3h + 6d} - 1$. If we let d tend to 0, we get the desired result. □

Combined with our results in Sect. 3, this bound implies a previous result due to Goemans [12], stating that there are TSP instances for which the ratio between the length of a shortest tour and the optimal value of the Held-Karp relaxation is arbitrarily close to $\frac{4}{3}$.

5 Concluding remarks

A proof of the "$\frac{3}{4}$-conjecture" on the Held-Karp relaxation would completely settle the question about the worst case analysis of the ε-approximability problem. On the other hand, it may be possible to find completely different methods of constructing approximately fair allocations with guaranteed worst case approximation error $\varepsilon = \frac{1}{3}$. For example, Faigle and Kern [7] propose some allocation rules based on the minimum spanning tree allocation and the distance from the supply node. An empirical study of randomly generated small instances ($n \leq 10$) seems to indicate that the Held-Karp allocation rule is by far preferable to other heuristic rules based on minimum spanning trees and relative distance functions, cf. Hunting [15].

Another interesting question concerns the average case behavior. We conjecture that if the points are independently and uniformly generated, say, in the unit square, then the probability of the core being empty tends to zero as $n \longrightarrow \infty$, where n is the number of players. One way of approaching this problem may be using the so-called *zero-or-one law* from probability theory (see Feller [10], Vol. II, Theorem 3, Chapter IV.6). This is easily seen to imply the following: Let $X_0, X_1, X_2, \ldots$ be independent random variables that are uniformly distributed in the unit square. Let $A \subseteq \left([0,1]^2\right)^\infty$ be defined as

$$A := \{(a_0, a_1, a_2, \ldots) \mid \text{the TSP game has nonempty core for almost all begin sequences}\}.$$

Then either $P(A) = 0$ or $P(A) = 1$, where P is the infinite product measure on $\left([0,1]^2\right)^\infty$.

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