Asymmetric Rendezvous on the Plane

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Abstract

We consider rendezvous problems in which two players move on the plane and wish to cooperate in order to minimise their first meeting time. We begin by considering the case when they know that they are a distance d apart, but they do not know the direction in which they should travel. We also consider a situation in which player 1 knows the initial position of player 2, while player 2 is only given information on the initial distance of player 1. Finally we give some results for the case where one of the players is placed at an initial position chosen equiprobably from a finite set of points.

1 Introduction

Rendezvous problems are a type of search game in which two players cooperate in order to minimise their meeting time. Probably the first discussion of a rendezvous problem appears in Schelling [29] who discusses a situation in which two parachutists land in a field and wish to find each other as quickly as possible.

It is possible to formulate rendezvous problems in a wide variety of different settings. For example, rendezvous problems can involve two people meeting each other on discrete locations (Anderson and Weber [9], Alpern and Howard [6]), on an infinite line (Alpern and Beck [4], Anderson and Essegaier [8], Alpern and Gal [5], Alpern and Lim [24], Baston [14], Baston and Gal [15], Gal [19], Howard [23]), or on a circle [23]. Alpern [1] gives a comprehensive introduction to these problems. Other aspects of the problem that have been considered include rendezvous with more than two players (Lim, Alpern, and Beck [26], Lim and Alpern [25]), rendezvous with bounded resources (Alpern and Beck [2, 3]), and rendezvous evasion games (Alpern and Lim [7]). On a less serious note, see [22].

Despite the recent increase in interest in these problems, we know of no work that addresses the rendezvous problem in the plane, even though this was the original setting of Schelling.

In this paper we will consider a number of different scenarios in the plane. In each case two players are placed at different positions and each is given some information about the other's position. The aim is to choose a trajectory for each player in such a way as to minimise the expected meeting time or minimise the worst-case time, given maximum speeds v_1 and v_2 at which the players can move. Other aspects that we discuss are concerned with bounded resources, where player i can travel a total distance of f_i. We concentrate on *asymmetric* rendezvous search, where both players have agreed in advance on their role in a combined strategy. For example, they may agree on one player keeping her position, while the other player does all the searching. This is different to *symmetric* rendezvous search, where the players did not have a chance to agree on their roles in a combined strategy. Furthermore we can make different assumptions on the sense of direction: if each player has a compass that enables her to tell the direction of "north", we speak of an *oriented* problem, otherwise the problem is *unoriented*. In any case we assume that both players can tell a clockwise orientation from a counterclockwise one.

In a two-dimensional setting there are several different scenarios: The players may be required to meet at the same point at the same time ("zero visibility"), or they may be able to see each other when sufficiently close to each other (within a "visibility radius r"). They may also have some information on the other's initial position ("distance d"). We also consider situations with *asymmetric information* (as opposed to asymmetric strategies): For example player 1 may know the initial position of player 2, while player 2 does not know more than the initial distance of player 1.

The final part of the paper is concerned with the problem in which the players are constrained to move on a grid or network. This enables an optimal strategy to be derived for some problems when the version in which unconstrained

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movement is possible turns out to be frustratingly difficult. The problem on a grid also has interest because optimal policies for rendezvous on a grid may provide an approximation procedure for certain problems with a visibility radius.

The rest of the paper is organised as follows. In Section 2 we give an overview of some basics. Section 3 considers rendezvous search when the initial distance is known; we describe optimal strategies for scenarios at zero visibility, unbounded and bounded resources, as well as problems arising from limited visibility. In Section 4 we describe an asymptotically optimal strategy for a scenario in which the initial distance is unknown. In Sections 5 and 6 we discuss a variety of problems in which each player knows that the other player's initial location is taken from a finite set of candidate points.

2 Problem Setting and Fundamentals

The reader is referred to Alpern [1] for a discussion of the ways in which rendezvous problems can be formalised. We will depart from the formulation given by Alpern who assumes that both players are given initial positions which are independently chosen and uniform over the whole of a metric space X.

Since the rendezvous search takes place on the plane, it is only the relative position of the two players at the start which is important. Hence we may suppose, without loss of generality, that the initial position of player 1 is at the origin. The initial position of player 2 is one of a given set of points Y, which may be infinite. We shall assume that each point in Y is equally likely to be chosen as the initial position of player 2.

Care is needed to specify the information that is given to the two players. Following Alpern we can describe the information available to the players by specifying a subgroup of the isometry group on the plane. In fact we shall consider a group G of isometries with the property that $G(Y) = Y$ and also assume that the group G acts transitively on Y (i. e., any point in Y is taken into any other point in Y by some element of the group).

A solution to the problem will take the form of the specification of two policies, one for each player. Since we are considering the asymmetric rendezvous problem we can assume that each policy consists of a single trajectory. Because the players do not know their initial positions, the same trajectory is followed from each of the starting positions in Y. We write $s_i(t)$ for the trajectory for player i, $i = 1, 2$. We assume that $s_1(0)$ is the origin and $s_2(0)$ is in Y. Further we assume that player 1 can move with (maximal) speed v_1, while player 2 can move with speed v_2.

Every time the rendezvous search takes place one element $g \in G$ is chosen with an equal chance for each of the elements of G. Then the meeting takes place at the first time t for which $s_1(t) = g(s_2(t))$. The expected meeting time $T(s_1, s_2)$ is the average of this over G. Since G acts transitively on Y the averaging over G includes an average over different starting points in Y.

A key part of the problem formulation is the question of whether the two players share a common sense of orientation in the plane. We may assume that player 1 has a compass, so that she may travel North if she chooses. We will consider two different types of problem, which we call oriented and unoriented. In an oriented problem player 2 is also given a compass, in an unoriented problem she is not. In an unoriented problem the choice of a trajectory for player 2 leads to the player setting off with an equal chance of going in each of the possible directions from the starting point.

In terms of the subgroup G, an oriented problem has nothing other than translation moves included in G, while an unoriented problem also includes all rotation moves.

Alpern has pointed out that we also need to distinguish between an unoriented arrangement, in which the players share a common notion of "clockwise" on the surface, and one in which they do not. We will not consider this latter case, which corresponds to the inclusion of reflections in the group G.

The lemma below shows that, for zero visibility, we need only consider problems in which Y has zero measure in the plane. Y captures the uncertainty about the initial positions of the two players. If this uncertainty is too great, as happens when Y has non-zero measure, the players do not have sufficient information to guarantee a meeting.

Lemma 2.1 *If Y has non-zero measure then for any strategy the probability of the players meeting in finite time is zero.*

Proof: Observe that the probability of meeting before time t given a strategy pair (s_1, s_2) in an unoriented problem is the same as that which can be achieved in the oriented problem if player 2 randomises over all possible directions at the start. Thus it is enough to consider an oriented problem.

Consider the movement of player 1 relative to player 2. This is then equivalent to the problem in which player 2 has an initial uniform distribution over Y and player 1 searches for player 2 with a maximum speed of $v_1 + v_2$. The probability of a meeting before time t is the expected value (over player 1 trajectories) of the measure of the area traced out within Y by the player 1 trajectory up to time t, normalised by the area of Y. Since Y has non-zero measure and the measure of the trajectory is zero we are done. □

3 Rendezvous with a Known Distance

In this section we consider the case where both players know their original distance d, but do not know the direction of the other player. We assume that all directions are equally likely, meaning that from the point of view of either player, the location of the other player is chosen uniformly at random from the circle of radius d. The players meet when they occupy the same point at the same time. In addition to this basic scenario, we consider other scenarios involving resource constraints, an objective function of expected rendezvous time as well as worst-case rendezvous time, and a visibility radius r.

3.1 The Basic Scenario

Suppose the players start with the knowledge of their original distance d, and the visibility is zero. This situation has been considered by Alpern and Gal [5] for the one-dimensional case, with both players moving at unit speed; they showed that, if there is no common sense of direction for both players, then there is an optimal strategy that results in an expected rendezvous time of $\frac{13d}{8}$. In addition when there is a common sense of direction (i. e., a notion of "left" and "right"), it is easy to show that an optimal strategy yields an expected rendezvous time of d: player 1 moves a distance of $\frac{d}{2}$ to the right, while player 2 moves the same distance to the left. If the players do not meet (which happens with probability $\frac{1}{2}$), then both players reverse their directions and meet after moving a further distance of d.

In the following we assume, without loss of generality, that the maximal speed for player 1, v_1, is no smaller than the maximal speed for player 2, v_2. Both players have a common 2-dimensional sense of direction, i.e., a compass and a notion of "clockwise". We will denote by $t_d = \frac{d}{v_1+v_2}$ the minimum time for a meeting of the two players.

There is an easy asymmetric strategy that yields finite bounds: player 2 stays put, while player 1 does all the searching. (This strategy has been dubbed "waiting for momma" by Gal.) Clearly, this yields a worst case time of $\frac{d}{v_1}(1+2\pi)$ and an expected time of $\frac{d}{v_1}(1+\pi)$.

In the following lemma, we describe a better, "kissing circles" strategy:

Lemma 3.1 *There is a strategy that yields a worst case time of $t_d(1+2\pi)$ and an expected time of $t_d(1+\pi)$.*

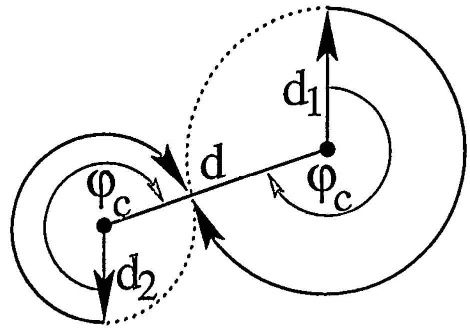

Figure 1: The "kissing circles" strategy

Proof: See Figure 1. Player 1 moves a distance of $d_1 = \frac{dv_1}{v_1+v_2}$ north, then at time $t_d = \frac{d_1}{v_1}$ she follows a clockwise circle around her original location. Similarly, player 2 moves a distance of $d_2 = \frac{dv_2}{v_1+v_2}$ south, then (at the same time $t_d = \frac{d_2}{v_2}$) she starts to follow a clockwise circle around her original location. At any given time $t \geq t_d$, player i has traveled an angle of $\varphi_i(t) = \frac{v_i}{d_i}(t - t_d)$. Let φ_c be the (clockwise) angle between the north direction and the directed line from player 1 to player 2 at time $t = 0$. Clearly, φ_c is also the clockwise angle between the south direction and the directed line from player 2 to player 1 at time $t = 0$. Since $\frac{v_i}{d_i} = \frac{v_1+v_2}{d}$, both players will have traveled the angle φ_c at the same time, and therefore meet (at $t = \frac{(1+\varphi_c)d}{v_1+v_2}$) at the point where their respective circles touch. This yields the values claimed for worst case and expected times. □

We can show that the kissing circles strategy is optimal:

Theorem 3.2 *The "kissing circles" strategy yields optimal worst case time and optimal expected time for a rendezvous.*

Proof: We have to show that no strategy can beat a worst case rendezvous time of $t_d(1+2\pi)$ or an expected rendezvous time of $t_d(1+\pi)$. Consider the position of player 2 in coordinates that are fixed at player 1, i.e., the vector $p(t) = s_2(t) - s_1(t)$. $p(0)$ is chosen uniformly at random from the circle C_p with radius d around the origin. By definition, the velocity $\dot{p}(t)$ has norm no greater than $v_1 + v_2$. When following $p(t)$, we have to search the circle C_p. Clearly, we have to travel a distance of at least d before encountering any points of C_p; since C_p has measure $2\pi d$ and the search speed is bounded by v_1+v_2, the worst case rendezvous time is bounded from below by $\frac{d(1+2\pi)}{v_1+v_2} = t_d(1+2\pi)$, and the expected rendezvous time is bounded from below by $\frac{d(1+\pi)}{v_1+v_2} = t_d(1+\pi)$. □

After the preceding discussion, it is straightforward to give a more general result for oriented rendezvous search: by using the orientation of the problem, we can reduce the rendezvous search problem with speeds v_1 and v_2 to a search problem with speed v_1+v_2. This is achieved in the following way:

Theorem 3.3 *Let $S(t)$ be an optimal trajectory for player 1, starting at the origin and searching with maximum speed $v_1 + v_2$ for a stationary player 2, positioned with a given probability distribution on a set Y. Then $S(t)$ induces an optimal pair of trajectories for the rendezvous search with mutual distribution Y.*

Proof: $S(t)$ can be converted into a pair of trajectories for the rendezvous search problem by taking

$$s_1(t) = \frac{v_1}{v_1+v_2}S(t)+s_1(0) \text{ and } s_2(t) = \frac{-v_2}{v_1+v_2}S(t)+s_2(0).$$

This guarantees that in a coordinate system centered at player 2, player 1 will follow the search path $S(t)$. □

3.2 Bounded Resources

Next consider the situation where both players have bounded resources: player i can travel at most a distance f_i before running out of fuel. In this scenario, the prime objective is to maximise the probability of a meeting. Problems of this type have been considered by Alpern and Beck [3] for the one-dimensional case. In the following, we describe a strategy that is probability-optimal; furthermore, among all strategies that are probability-optimal, it is also time-optimal: if there is a rendezvous, it will happen with smallest expected time. (See Alpern and Beck [2] for the one-dimensional case.)

Lemma 3.4 *There is a strategy that yields a probability of $\min\left\{\frac{f_1+f_2-d}{2\pi d}, 1\right\}$ for a rendezvous, provided that $f_1 + f_2 \geq d$. Conditional on there being a rendezvous, the worst case time is $\max\left\{\frac{f_1}{v_1}, \frac{f_2}{v_2}\right\}$; with $\frac{f_j}{v_j} \geq \frac{f_i}{v_i}$, the expected time is $\frac{\frac{f_i^2}{v_i}+\frac{f_j^2}{v_j}-dt_d}{2(f_1+f_2-d)}$ if $\frac{f_i}{v_i} \geq t_d$, and $\frac{f_j-f_i+d}{2v_j}$ otherwise.*

Proof: See Figure 2. Let $t_j = \frac{f_j}{v_j} \geq \frac{f_i}{v_i} = t_i$. Both players follow the kissing circles strategy until player i runs out of fuel, at time t_i, and has to stay at position $s_i(t_i)$. At time t_i, player j modifies her strategy: instead of a circle of radius d_j around her original position $s_j(0)$, she follows a circle of radius d around position $s^* = s_j(0) + s_i(t_i) - s_i(0)$. If $f_i \geq d_i$, i.e., $t_i \geq t_d$ and both players have already started along their kissing circles when player i runs out of fuel, then

$$\begin{aligned} & \|s_j(t_i) - s^*\| \\ = & \|(s_j(t_i) - s_j(0)) - (s_i(t_i) - s_i(0))\| \\ = & \|s_j(t_i) - s_j(0)\| + \|s_i(t_i) - s_i(0)\| \\ = & d_j + d_i \\ = & d \end{aligned}$$

by construction of the kissing circles. Furthermore, the new circle for player j will have a common tangent with her kissing circle, at position $s_j(t_i)$.

If $f_i \le d_i$, then player j continues along her straight path until both players have moved a combined total of d, before following a circle of radius d around s^*. Since player j has traveled $f_i \frac{v_j}{v_i}$ at time t_i, while player i has traveled f_i, this additional distance is $d - f_i(1 + \frac{v_j}{v_i})$.

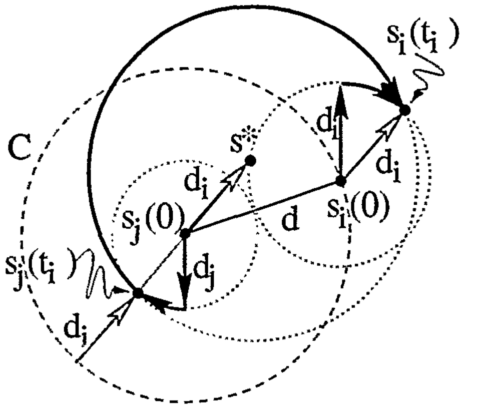

Figure 2: The strategy for bounded resources

Let C be the circle of radius d around $s_j(0)$. If the players have not met at time t_i, then player i is positioned on the circle $C + s_i(t_i) - s_i(0)$, i.e., on the circle of radius d around s^* that is searched by player j. If $2\pi d \ge f_1 + f_2 \ge d$, the described rendezvous strategy guarantees a rendezvous for a fraction of $\frac{f_1+f_2-d}{2\pi d}$ of the perimeter of C. For a uniform distribution along C, we get the claimed probability.

Now consider the set of positions where a rendezvous happens. In the worst case, this is at time $t_j = \frac{f_j}{v_j}$, when player j is just running out of fuel. For the expected time, first consider the case where $t_i \ge t_d$: for a time period of $t_i - t_d = \frac{f_i}{v_i} - t_d$, an arc of C is searched with speed $v_1 + v_2$, for a length of $\ell_1 = f_i \frac{v_1+v_2}{v_i} - d$; the rest of the search is done with speed v_j, covering an arc of length $\ell_2 = f_j - f_i \frac{v_j}{v_i}$. This yields an expected time of

$$\frac{\frac{t_d+t_i}{2}\ell_1 + \frac{t_i+t_j}{2}\ell_2}{\ell_1 + \ell_2} =$$

$$= \frac{\left(t_d + \frac{f_i}{v_i}\right)\left(f_i \frac{v_1+v_2}{v_i} - d\right) + \left(\frac{f_i}{v_i} + \frac{f_j}{v_j}\right)\left(f_j - f_i \frac{v_j}{v_i}\right)}{2(f_1 + f_2 - d)}$$

$$= \frac{\frac{f_i^2}{v_i} + \frac{f_j^2}{v_j} - dt_d}{2(f_1 + f_2 - d)}.$$

Now consider the case $t_d > t_i$. Player j reaches the search circle at time $t_i + \frac{d - t_i(v_1+v_2)}{v_j}$ and searches with speed v_j until time t_j, for an expected time of

$$\frac{t_i + \frac{d}{v_j} - f_i \frac{v_1+v_2}{v_i v_j} + t_j}{2} = \frac{\frac{f_i}{v_i} + \frac{d}{v_j} - \frac{f_i}{v_j} - \frac{f_i}{v_i} + \frac{f_j}{v_j}}{2} = \frac{f_j - f_i + d}{2v_j}.$$

□

Again, we can argue along the lines of Theorem 3.2 and Theorem 3.3 that this strategy is optimal:

Theorem 3.5 *The modified kissing circles strategy yields the maximum probability for a rendezvous. Among all strategies with maximum probability, it has smallest worst case time and optimal expected time for a rendezvous, assuming the players meet.*

Proof: It is impossible to search more than a portion of $f_1 + f_2 - d$ of the circle C, so the modified kissing circles strategy is probability-optimal. It is clear that the worst case time cannot be improved. To see that the expected time in case of a meeting is optimal, note that for any given time $t \ge t_d$, no search strategy can have searched a larger portion of C. □

3.3 Limited Visibility

Now let us consider the case of limited visibility, where the players meet when they come within distance r of each other. Theorem 3.3 allows us to focus on the search with a search radius r. Problems of this type have been dubbed "lawn mower problems" by Arkin, Fekete, and Mitchell [10]. If we consider the set Y to be a (possibly disconnected) piece of lawn, the problem of finding a shortest (closed) tour for a (circular) lawn mower of radius r is equivalent to finding a trajectory that minimises the worst case time for a rendezvous. Being a generalisation of the well-known Travelling Salesman Problem, the lawn mower problem is NP-hard, so one cannot hope to find an efficient algorithm that computes an optimal tour for any input Y. In [11], Arkin et al. give a number of efficient approximation algorithms for several versions of the problem.

However, the scenario of rendezvous search with a known distance d and search radius r is less general, so we do not have to deal with all the difficulties of the lawn mower problem. In fact, Gluss [20] has given a solution to this question when considering the min-max problem of "Searching for a circle in the plane". See Figure 3 for the case $d = 2r = 2$; the figure shows a trajectory which is optimal for the worst case. Note that a straightforward generalisation of the kissing circles strategy for zero visibility (yielding $(1+2\pi)(d-r)$) is not optimal, due to boundary effects.

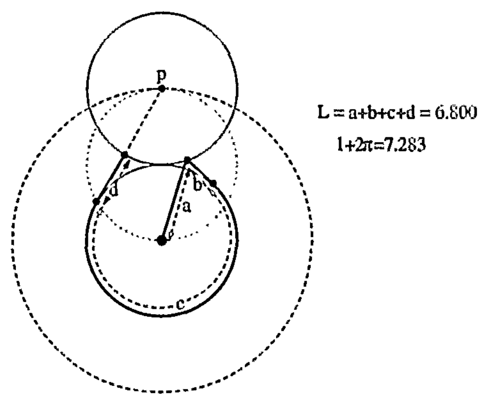

Figure 3: An optimal search trajectory with limited visibility

Things are much trickier if we want to minimise the *expected time* for a rendezvous. We think that an optimal solution will involve a spiral instead of the straight line labelled d in Figure 3, but an exact optimal solution is hard to determine.

4 Rendezvous with an Unknown Distance

Before we pursue the question of finding trajectories that minimise the expected time for a rendezvous, we will take a brief look at the situation where both players do *not* know each other's initial distance d. As before, it is possible to reduce the rendezvous search problem to a pure search problem. This search problem has been solved for the one-dimensional case by Beck and Newman [16]; a good overview can be found in [18]; see also [13].

In the two-dimensional scenario with an unknown distance, we cannot hope for a rendezvous if the visibility is zero – see Lemma 2.1. However, we can consider the two-dimensional case for a visibility radius of r by considering trajectories that will guarantee a rendezvous for any initial distance d. The rendezvous time (worst case or expected) can then be given as a function of d. Since we cannot impose any bounds on d, it makes sense to consider *asymptotically optimal* search trajectories: the value of a trajectory is $(1+\varepsilon_d)L_d$, where ε_d tends to zero as d grows to infinity, and L_d is a lower bound for the optimal value with distance d. This type of optimality has been discussed by Gal [18] for various types of search problems. He also uses ideas somewhat similar to the following.

In Figure 4 we show a "semi-circular spiral", a search trajectory which is asymptotically optimal for both worst case and expected time. Starting at the origin, the trajectory consists of semi-circles of radius $r, 2r, 3r, \ldots$ Each of the semi-circles "mows" an area of width $2r$. It is easy to see that (except for a set of points of measure zero at the "seams" of the semi-circles) no area is mown more than once. By travelling a distance of L, we can mow at most an area of $2rL + \pi r^2$. This means that we need at least a distance of $\frac{\pi d^2 - \pi r^2}{2r}$ to mow all points within d of the origin. It is straightforward to see that the semi-circular spiral trajectory mows the circle of radius d while travelling a distance of $\frac{\pi d^2 + O(d)}{2r}$. This implies that this trajectory is asymptotically optimal.

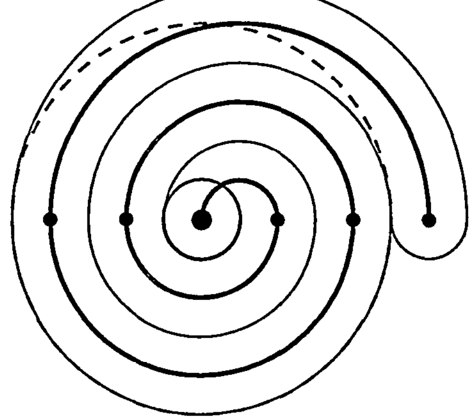

Figure 4: An asymptotically optimal trajectory when the distance is unknown

5 Rendezvous with a Fixed Point

In the following two sections we will discuss scenarios which are more complicated, owing to a combination of weaker assumptions on orientation, asymmetric information, or the more complex objective "expected time". We will show that, even for the special case of identical speeds, matters become very difficult.

We begin our discussions by considering the case where Y is a single point. Without loss of generality we take $Y = (d, 0)$. In this case the oriented problem is trivial, since the two players can move towards each other meeting at time $d/(v_1 + v_2)$. Thus we consider the unoriented problem in which player 2 does not know the direction of the starting point for player 1 (though she knows the distance). Notice that the problem is not the same for the two players: player 1 does know the initial position of player 2. For this problem G is simply the group of rotations about Y.

In practice rendezvous problems are usually solved by both players moving to some common focal point. This idea of coordination through some common point was the starting point for work in this area (see Schelling [29], Mosteller [28]). The problem in which Y is a single point also has the characteristic of having a unique point which both players can find: the initial position of player 2.

One easy solution is for player 2 to remain at her initial point and for player 1 to move on the trajectory $s_1(t) = (v_1 t, 0)$ so that the players meet at time d/v_1. As we shall see the players can do better than this.

Because of the special status of the initial point for player 2, it is better to make this the origin; so the initial position of player 1 becomes $(-d, 0)$. It is easier to use polar coordinates to describe the trajectories of the two players. We suppose that player i uses a trajectory

$$s_i(t) = (R_i(t), \Theta_i(t)),\ 0 < t,\ i = 1, 2.$$

The actual path traced out by player 2 is $g(s_2(t))$ which is

$$(R_2(t), \Theta_2(t) + \Theta_0),\ t > 0,$$

where Θ_0 is chosen with a uniform probability distribution over the interval $(0, 2\pi)$.

First we prove a lemma giving the form of the optimal policies for the two players.

Lemma 5.1 *If there is an optimal solution, then an optimal solution exists in which player 1 follows a trajectory of the form (in polar coordinates):*

$$\begin{aligned} s_1(t) &= (d - v_1 t, \pi),\ 0 \le t \le t_d, \\ s_1(t) &= (R(t), \Theta_1(t)),\ t > t_d, \end{aligned}$$

while player 2 uses the trajectory

$$\begin{aligned} s_2(t) &= (v_2 t, 0),\ 0 \le t \le t_d, \\ s_2(t) &= (R(t), \Theta_2(t)),\ t > t_d, \end{aligned}$$

where Θ_i are monotonic functions for $t \ge t_d$, with $\Theta_1(t_d) = \pi$ and $\Theta_2(t_d) = 0$. Moreover, the players move at their maximum speeds and in opposite rotational directions.

Proof: First we show that once the two players are at the same distance from the origin, they may as well remain at the same distance from the origin (since they can only meet when they have the same R value). Suppose that at times t_1 and t_2 the two players are at the same distance from the origin, but not for any $t \in (t_1, t_2)$. Then, since the players cannot meet at $t \in (t_1, t_2)$, we can replace both trajectories with straight line trajectories for this period. Suppose, without loss of generality, that $|s_1(t_1) - s_1(t_2)|/v_1 \geq |s_2(t_1) - s_2(t_2)|/v_2$. Then we generate a new solution which is no worse than the current one by setting the speed of s_1 to be v_1 over this section of trajectory: then the new trajectory reaches $s_2(t_2)$ at time $t_3 = t_1 + |s_1(t_1) - s_1(t_2)|/v_1 \leq t_2$. Then we choose the speed of s_2 over (t_1, t_3) so that s_1 and s_2 have the same R value throughout this interval. Finally we replace both trajectories from time t_3 onwards with the previous trajectories used from time t_2.

The next step is to show that both players are moving at maximum speed and with opposite rotational directions. This is achieved by observing that the meeting time is unaltered if the players use trajectories with an additional rotational component, provided that this is the same for both players. In other words if

$$s_i(t) = (R_i(t), \Theta_i(t) + \Phi(t)),\ i = 1, 2$$

for some fixed $\Phi(t)$, then all meeting times are unaltered. This transformation is only possible when the resulting trajectories do not break the speed limit.

We write $\phi(t)$ for the derivative of $\Phi(t)$ and $\theta_i(t)$ for the derivative of $\Theta_i(t)$. Note that if $\phi(t)$ is the same sign as $\theta_i(t)$ then the speed of player i is increased, and if they are of different signs then the speed is decreased. It is thus easy to see that the two players must move with opposite rotational directions, otherwise we may choose Φ to decrease both their speeds. This then makes it possible to speed up both players, keeping their paths the same, and decrease the meeting time.

Now suppose that player 1, say, is not moving at maximum speed. We may choose ϕ in such a way that the speed of player 2 is decreased and the speed of player 1 is increased, but remains less than v_1. Then we can improve the meeting times from this point on by increasing the speed of both trajectories. This contradicts the presumed optimality of the solution and hence we have established that each player moves at maximum speed.

The continuity of the trajectories implies that $\Theta_1(t_d) = \pi$ and $\Theta_2(t_d) = 0$, so it only remains to show that each Θ_i is monotonic. Observe that, with the form of optimal trajectories established, the rate at which a meeting occurs is either 0, or is governed entirely by $|\theta_1|$ and $|\theta_2|$. Consider replacing Θ_i by $\tilde{\Theta}_i$ where $d\tilde{\Theta}_1/dt = -|\theta_1|$ and $d\tilde{\Theta}_2/dt = |\theta_2|$. This ensures that the trajectory for player 1 takes a clockwise spiral around the origin rather than being reversed at any point. The rate at which meeting takes place is never lower than it was previously, until $\tilde{\Theta}_2$ reaches π when no more meetings take place. Thus the expected meeting time is no larger, and we have established a pair of optimal trajectories with the properties we claimed. □

Figure 5 shows the behaviour of the pair of optimal rendezvous trajectories in this case. Each executes a spiral towards the origin. To find the function describing that spiral turns out to be a problem in the calculus of variations.

We consider the case when $v_1 = v_2 = 1$ and $d = 2$. Then the fact that each player moves at the same speed and remains at the same radial distance, implies $\Theta_1(t) = \pi - \Theta_2(t)$ for $t > 1$. Thus we may take the trajectory for

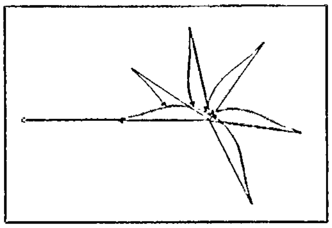

Figure 5: An optimal pair of trajectories

player 1 as given by the pair $(R(t), \pi - \Theta(t))$ for $t > 1$ with the conditions of the lemma.

It is convenient to express the trajectory in terms of the angle Θ, so we consider the function $y(\alpha), 0 \leq \alpha \leq 2\pi$ with $R(t) = y(\Theta(t))$ for $t > 1$.

Let $t(\alpha)$ be the time at which the trajectory reaches the angle α. The probability that a meeting takes place before time $1 + t(\alpha)$ is the probability that Θ_0 is in the range $(\pi - 2\alpha, \pi)$, and thus is α/π.

Hence the expected meeting time is given by the integral $\int_0^\pi (t(\alpha)/\pi) d\alpha$. Integrating by parts we see that this is

$$\int_0^\pi \left(1 - \frac{\alpha}{\pi}\right) \frac{dt}{d\alpha} d\alpha.$$

Since the trajectory moves at speed 1,

$$(dt/d\alpha)^2 = y(\alpha)^2 + (dy/d\alpha)^2.$$

So we need to minimise the integral

$$\int_0^\pi \left(1 - \frac{\alpha}{\pi}\right) \left(y(\alpha)^2 + (dy/d\alpha)^2\right)^{1/2} d\alpha$$

over choices of function y with $y(0) = 1$ and $y \geq 0$.

We can use the theory of the calculus of variations to carry out this minimisation. We wish to minimise an integral of the form $\int_0^\pi f(\alpha, y, dy/dt) d\alpha$, where

$$f(x, y, z) = \left(1 - \frac{x}{\pi}\right) (y^2 + z^2)^{1/2}$$

is convex on $[0, \pi] \times (0, \infty) \times \Re$. Hence a solution of a particular differential equation of the form $(d/dx) f_z = f_y$ with appropriate boundary conditions, will minimise the integral (see for example Troutman [30]).

The required differential equation is

$$\left(\frac{d}{d\alpha}\right) \left[\left(1 - \frac{\alpha}{\pi}\right) \left(y(\alpha)^2 + \left(\frac{dy}{d\alpha}\right)^2\right)^{-1/2} \frac{dy}{d\alpha}\right]$$

$$= \left(1 - \frac{\alpha}{\pi}\right) \left(y(\alpha)^2 + \left(\frac{dy}{d\alpha}\right)^2\right)^{-1/2} y.$$

After simplification this becomes

$$(\pi - \alpha) \left(y^3 + 2y \left(\frac{dy}{d\alpha}\right)^2 - y^2 \frac{d^2 y}{d\alpha^2}\right)$$

$$+\left(\frac{dy}{d\alpha}\right)^3 + \frac{dy}{d\alpha}y^2 = 0.$$

The only boundary condition we require in addition to setting $y(0) = 1$, is that $(dy/d\alpha)|_\pi = 0$. Any other value leads to an infinite value for $dy/d\alpha$ at π.

Figure 6 shows a numerical solution of this differential equation. Observe that $y(\pi)$ is strictly greater than zero (i. e., [illegible] will alwa[illegible]

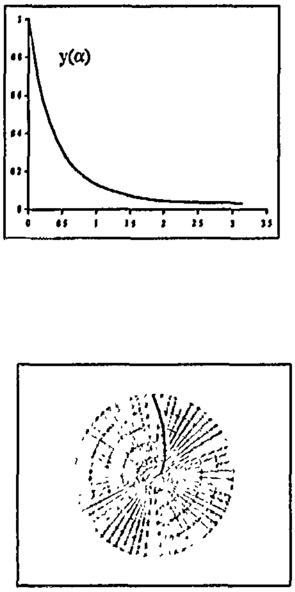

Figure 6: An optimal solution, shown in polar coordinates and Cartesian coordinates

6 Rendezvous with a Set of Points

In this section we consider the situation where Y is a finite set of points.

6.1 Oriented Rendezvous

First consider the oriented problem in which $Y = \{y_1, y_2, \ldots, y_k\}$, and we want to minimise the worst case time for a rendezvous. Suppose we already have a shortest route starting at the origin and visiting all the points in Y. Finding this is almost the same as solving the travelling salesman problem, except that we do not have to return to the origin. Since this is an NP-hard problem, we cannot hope to find such an optimal route in a short time. However, there are algorithms (see Arora [12], and Mitchell [27]) for the geometric TSP that can find a solution within $(1+\varepsilon)$ of the optimum in polynomial time for any fixed ε.

We suppose that the points in Y have been reordered so that there is an optimal solution to the problem of a shortest path in which the points are visited in the order: first y_1, then y_2, then y_3 and so on up to y_k. Let a_j be the vector $y_j - y_{j-1}$, $j = 2, 3, \ldots k$ and let $a_1 = y_1$.

The following lemma can be proved in exactly the same way as the optimality of the kissing circles solution, when Y is a circle. We consider the position of player 1 in a coordinate system which leaves player 2 fixed. Then player 1 has a maximum velocity of $v_1 + v_2$ and by construction visits the points of Y in order.

Lemma 6.1 *The solution in which player 1 moves in direction a_1, followed by direction a_2, etc. finishing by moving in direction a_k, with the direction a_j being followed for a time $|a_j|/(v_1+v_2)$, and in which player 2 moves at all times in the opposite direction to player 1, minimises the maximum meeting time.*

It should be noted that, when minimising the expected time instead of the worst case time, one has to consider a problem that is very similar to the so-called *minimum latency problem* (Blum et al. [17]) of finding a roundtrip that minimises the average travel time for the points. Like the travelling salesman problem, this problem is NP-hard; however, the best known approximation algorithms (see Goemans and Kleinberg [21]) have relatively high approximation factors, indicating that even the combinatorial structure of this problem is more involved than the structure of the TSP. There are additional difficulties when we consider the asymmetric case, since this brings into play the spirals that were needed in the previous section.

Now consider the problem in which the movement is restricted to a network forming a square grid, with each arc of length 1.

At this point we need to strengthen the assumptions that we make. We shall assume that $v_1 = v_2 = 1$ and that the points of Y are at the nodes of the network. Furthermore we assume that all distances between points in $Y \cup \{0\}$ are multiples of 2. Thus the points in Y are drawn from the set $\{(a,b)|\ a+b \text{ is even }\}$.

We suppose that, as before, a shortest route from the origin visiting all the points in Y exists in which the points are visited in the order $y_1, y_2, \ldots, y_k$.

Let $p(a,b)(t)$ be a particular arbitrary choice of shortest path on the square grid from position a to position b, where $t \in [0, d(a,b)]$ and $d(a,b)$ is the shortest distance from a to b on the network.

Set $x_1 = 0$. Now consider the strategy pair in which player 1 follows the path $p(x_1, y_1)$ up to time $t_1 = d(x_1, y_1)/2$ at which point player 1 is at position $x_2 = p(x_1,y_1)(t_1)$, and then the path $x_2 - y_1 + p(y_1, y_2)$ until time $t_2 = t_1 + d(y_1,y_2)/2$, and so on. Thus during stage i player 1 follows the path $x_i - y_{i-1} + p(y_{i-1}, y_i)$ until time $t_i = t_{i-1} + d(y_{i-1}, y_i)/2$ at which point she is at position $x_{i+1} = x_i - y_{i-1} + p(y_{i-1}, y_i)(t_i - t_{i-1})$. Meanwhile player 2 follows the other half of the $p(y_{i-1}, y_i)$ path during stage i. Thus player 2 moves in the direction $(d/dt)p(0,y_1)(d(0,y_1) - t)$ for $t \in (0, t_1)$, and at any time $t \in (t_j, t_{j+1})$ player 2 moves in the direction $(d/dt)p(y_{j-1}, y_j)(d(y_{j-1}, y_j) - t)$.

We need to establish that this pair of trajectories will stay on the square grid. Observe that, since all distances are even integers, all the times $t_1, t_2, \ldots t_k$ are integers. Thus if the trajectory followed up to time t_i remains on the grid, x_i will be at one of the nodes of the square grid, for $i = 1, 2, \ldots k$. But once it is established that x_i is at a node of the grid, it follows that the path $x_i - y_{i-1} + p(y_{i-1}, y_i)$ is on the grid. This shows what we require for player 1. In the same way, player 2 follows a path which at stage j is a translation of $p(y_{j-1}, y_j)(d(y_{j-1}, y_j) - t)$ and, because each t_i is even, player 2 starts each stage at a node of the grid. This is enough to show that player 2 also stays on the grid.

Lemma 6.2 *The strategy pair described above minimises the maximum time till meeting.*

Proof: Using the same technique as the lemma above, we observe that if we fix the frame of reference so that player 2 is stationary, a lower bound on the total distance travelled by player 1 is given by the TSP path which has total length $d(0, y_1) + \sum_{i=1}^{k-1} d(y_i, y_{i+1})$. In this frame of reference the maximum speed of player 1 is 2. Now observe that the total time taken under the strategy pair proposed is exactly half of the total length of the TSP path.

It only remains to be shown that this strategy pair guarantees a meeting before time t_k. If player 2 starts from position y_1 then at time t_1 she is at point x_1, by construction, and will meet player 1. Hence, if player 2 starts from position y_2 (a translation by $y_2 - y_1$), then at time t_1 she is at position $x_1 - y_1 + y_2$. Thus a meeting will take place at time t_2 at position x_2. Continuing in this way we find that if player 2 starts at y_j there will be a meeting at time t_j. □

6.2 Unoriented Rendezvous

Now we discuss an unoriented rendezvous problem on the grid. In this section we will concentrate on minimising the expected meeting time. Consider the problem where $Y = \{(1,1), (1,-1), (-1,-1), (-1,1)\}$. This is one of the simplest problems of this type which is symmetric, in the sense that the two players have the same information set at the start. Both players know that the other player is at the diagonally opposite corner of one of the 4 squares intersecting at the starting position, but have no information on which square. Notice that, even though the problem formulation is symmetric between player 1 and player 2, we are still considering asymmetric policies, in which the two players choose different strategies.

Since both players have to move along grid lines, we can show that, for an optimal policy, meetings only take palce at the nodes of the grid.

Lemma 6.3 *For the problem of minimising expected meeting time, there is an optimal strategy pair in which both players always move at speed 1, and all meetings happen at integer points and at integer times.*

Proof: We may restrict attention to strategy pairs which always move at speed 1, since other strategies can be improved by moving at speed 1 from one possible meeting point to the next, and replacing a waiting time by movement at speed 1 back and forth along one of the arcs in such a way as to reduce the time till meeting one of the trajectories for the other player. Now consider a trajectory which moves part way along an arc and then reverses direction. If this reversal happens without this also being a possible meeting time with one of the other player's trajectories, then we may replace this trajectory with another in which that section of arc is not traversed and the trajectory arives earlier at its next meeting point. But now observe that, at the first meeting that takes place part way along an arc, both players will have followed trajectories that consist of a sequence of unit length steps around the grid. This is a contradiction, since the initial distance between the players is even. Hence we may suppose that there are no reversals part way along an arc. The result follows. □

This allows us to indicate a trajectory by listing the sequence of directions in which a player travels, with each step being of unit length.

Theorem 6.4 *The expected meeting time is minimised by the strategy pair in which each player chooses an arbitrary direction to call North, and player 1 follows a trajectory NWSSEENN and player 2 follows a trajectory NSNSNSNS.*

Proof: Fix the orientation of the plane according to player 1's compass. Thus the direction that player 2 calls North may in fact be North, West, South or East. Observe that player 1 meets the 16 possible trajectories at times listed in the following Table 1.

Table 1: Meeting times along different trajectories

Player 2 initial direction	*Starting point* (1,1)	(-1,1)	(-1,-1)	(1,-1)
North	8	2	3	6
West	1	2	4	5
South	7	2	4	6
East	8	1	4	6

The meetings with these 16 trajectories occur according to the following pattern: 2 at time 1, 3 at time 2, 1 at time 3, 3 at time 4, 1 at time 5, 3 at time 6, 1 at time 7, and 2 at time 8. We represent this pattern of meetings by the sequence $(2, 3, 1, 3, 1, 3, 1, 2)$. The proof establishes that no other feasible pattern of meetings gives a lower expected meeting time.

First observe that for any strategy pair no more than three meetings can occur at the same time. This follows from the fact that two player 2 trajectories from different starting points that move in the same initial direction remain at the same distance apart. Hence if a meeting takes place with four different trajectories, they must all have had different initial directions. Thus they must approach the meeting with player 1 from four different directions. Now we can assume that player 1 is never stationary (since we can replace a stationary period in which no meetings take place with a back and forward motion). Hence the meeting with one of the four trajectories must have taken place earlier.

Now suppose that a meeting takes place at time t^* with three trajectories for player 2. We write $x(t)$ for the player 1 trajectory, and $y_i(t)$, $y_j(t)$ and $y_k(t)$ for the three player 2 trajectories. Notice that $x(t^*)$ is equidistant from $y_i(0)$, $y_j(0)$ and $y_k(0)$. Thus if each of these is a different starting point then $x(t^*) = 0$. On the other hand if two of the three player 2 trajectories have the same starting point then they can only meet again at that starting point. But in this case all the player 2 trajectories must be at their initial positions. Hence at most one meeting can take place at the next step.

Thus the meeting with three player 2 trajectories can only occur at a starting point, either for player 1 or player 2. Now we can use the symmetry of the rendezvous game. If the two player strategies are swapped exactly the same pattern of meetings takes place. But now a situation in which three meetings take place at the origin is transformed into one in which three meetings takes place at one of the initial positions for player 2. So the observation that if three meetings takes place at time t^* at most one meeting can take place at time $t^* + 1$ will apply in all circumstances.

Now consider any feasible pattern of meetings. We show that this can be transformed into the pattern of meetings $(2, 3, 1, 3, 1, 3, 1, 2)$ in a way which makes the expected meeting time no worse. First change any time at which there is no meeting into a time at which there is one meeting by moving forward the last meeting. Now change any time at which there is one meeting, provided it does not follow immediately after three meetings, into a time at which there are two meetings by moving forward the last meeting. Each of these changes will make the expected meeting time less. After this the pattern of meetings can be described by a sequence made up of from single elements '2' and pairs '3, 1'. The first element in the pattern will be '2' since it is impossible to meet three trajectories at time 1. Now move any '2', other than that at the start of the pattern, to the end of the pattern by repeatedly replacing the triple '2, 3, 1' by '3, 1, 2'. These changes make no difference to the expected meeting time. At the same time replace consecutive pairs '2, 2' with the pair '3, 1' except where this involves the first '2', each of these changes will improve the expected meeting time. Clearly the end result is the pattern we require. □

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