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Game Theory

On the Complexity of Testing Membership in the Core of Min-Cost Spanning Tree Games

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Abstract: Let $N = \{1, \ldots, n\}$ be a finite set of players and K_N the complete graph on the node set $N \cup \{0\}$. Assume that the edges of K_N have nonnegative weights and associate with each coalition $S \subseteq N$ of players as cost $c(S)$ the weight of a minimal spanning tree on the node set $S \cup \{0\}$.

Using transformation from EXACT COVER BY 3-SETS, we exhibit the following problem to be NP-complete. Given the vector $x \in \Re^N$ with $x(N) = c(N)$, decide whether there exists a coalition S such that $x(S) > c(S)$.

Keywords: Cooperative game, core, spanning tree, NP-complete, X3C.

1 Introduction

In this note we investigate the computational complexity of finding optimal core allocations of a much studied cooperative N-person game. The *minimum cost spanning tree game*, MCST-game for short, is a game on the set $N = \{1, \ldots, n\}$ of players, the *grand coalition*, that is to be connected to the *supply node* 0. Establishing a direct link between any pair (i,j), where $i,j \in N \cup \{0\}$, is assumed to cost the nonnegative amount $w(i,j) = w(j,i)$. The objective is to create a connected graph on the node set $N \cup \{0\}$ and to divide the resulting total cost among the n players. The latter is represented by an *allocation vector* $x \in \Re^N$ whose components indicate the amount each individual player is to contribute to cover the total cost.

The vector x should be considered *fair* by the players and the coalitions they may form, *i.e.*, the amount $x(S)$ any coalition S has to pay cumulatively should never exceed the cost $c(S)$ of a minimum spanning tree on $S \cup \{0\}$, which is what S would have to invest in order to connect itself to 0 without using any other nodes. The *core* of the MCST-game consists of all vectors x that are fair in this sense.

Being defined by linear inequalities, the core of a MCST-game is a polyhedron in $\Re^N$. Applying a "greedy" allocation scheme, it is actually an easy algorithmic task to determine a core allocation. The scheme is already discussed in, *e.g.*, Claus

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and Kleitman [1973] and Bird [1976] and has been rigorously proved to yield a core allocation by Granot and Huberman [1981]: Find a minimum cost spanning tree T on $N \cup \{0\}$ and allocate to player i the weight of the first edge that i encounters on the (unique) path from i to 0 in T.

Notice, however, that this scheme seems to favor players that are at relatively far distance from 0 in T: the allocation to player i is just the connection cost of i to the rooted subtree of nodes reachable from 0 without passing through i. One way of overcoming such a bias would be to allocate a core vector that is optimal with respect to a linear function (which depends just on the players and not on their relative distance from the supply node). Linear optimization over a polyhedron implicitly necessitates a good description of the polyhedron.

Thus the problem of getting a complete overview of the core of MCST-games has received considerable attention. Positive answers were obtained for cases where the core (or a subset of the core) can be described by submodular (a.k.a. *convex*) constraints (cf. Shapley [1971]), which permits the determination of extremal core vectors by Edmonds' [1970] greedy algorithm (see, *e.g.*, Aarts [1994], Aarts and Driessen [1993], Granot and Huberman [1983], Kuipers [1994] or for related models Granot and Granot [1993], Meggido [1978] and Tamir [1991]).

The purpose of this note is to contribute a negative answer to the problem of determining the core of general MCST-games. We show that it is an NP-hard problem to decide whether a given vector is not a member of the core. Because of the polynomial equivalence of membership testing and optimizing linear functions relative to a polyhedron (see Grötschel et al. [1988]), it is therefore doubtful whether there exist efficient ways to obtain a complete overview of the core of a general MCST-game.

We remark that our result is not the first to indicate that core membership testing is generally NP-hard. Tamir [1991] points out that a result of Chvátal [1978] implies NP-hardness of membership testing for the class of network synthesis games, which includes the MCST-games. Deng and Papadimitriou [1994] have given the example of a game for which the Shapley value is easy to compute and the problem of deciding whether the Shapley value is not a core vector is NP-complete: every pair of players has a weight and the value of a coalition S is the sum of all weights of the pairs with both components in S.

It is curious to observe that a Deng-Papadimitriou game has a non-empty core (in which case it necessarily contains the Shapley value) if and only if the game is submodular. It should be interesting to have more genuine examples of cooperative games whose cores are *not* given by submodular constraints and, yet, core membership testing is provably polynomial.

2 Core Membership Testing is Co-NP-Complete

The remaining part of this note will be devoted to proving the headline of this section. Recall that a decision problem $\mathscr{P}$ is in the class NP if every instance of

$\mathscr{P}$ has a solution whose correctness can be checked in polynomial time (if an affirmative solution exists at all). $\mathscr{P}$ is *NP-complete* if $\mathscr{P}$ lies in NP and is *NP-hard*, i.e., every instance of any probem in NP can be transformed to an instance of $\mathscr{P}$ in polynomial time. A problem is *co-NP-complete* if its negative is NP-complete (see, *e.g.*, the well-known book of Garey and Johnson [1979]).

Theorem 2.1: The following problem is NP-complete:

Instance: An MCST-game (N, w) *and a vector* $x \in \Re^N$ *with* $x(N) = c(N)$.

Question: Is x not an element of the core of the game, i.e., does there exist a coalition $S \subset N$ *such that* $x(S) > c(S)$?

Proof: The problem is easily seen to be in NP by exhibiting a coalition S with $x(S) > c(S)$ (if such an S exists). To prove completeness we establish a polynomial transformation from EXACT COVER BY 3-SETS (X3C), which is one of the six basic NP-complete problems in Garey and Johnson [1979] and is given as follows:

EXACT COVER BY 3-SETS (X3C)

Instance: A finite set with $X = \{x_1, \ldots, x_{3q}\}$ and a collection $F = \{f_1, \ldots, f_{|F|}\}$ of 3-element subsets of X.

Question: Does F contain an *exact cover* for X, that is, a subcollection $F' \subseteq F$ such that every element of X occurs in exactly one member of F'?

Given an instance of X3C we construct the following MCST-game. It has 4 "layers" of players (see Figure 1). We have an element-player for each element of X and a set-player for each member from F. Furthermore we have a Steiner-point-player St and a guardian g. Thus, including the supply node 0, the graph of the game has $|X| + |F| + 3$ nodes

$$\{1, \ldots, 3q\} \cup \{3q + 1 \ldots, 3q + |F|\} \cup \{0, g, St\}.$$

The corresponding weight function is given as follows. First we define the weights on a subset E of the edges of the graph.

For each set $f_i = \{j, k, l\} (i = 1, \ldots, |F|)$ we set

- $w(3q + i, j) = w(3q + i, k) = w(3q + i, l) = q + 1,$
- $w(3q + i, St) = q,$
- $w(3q + i, g) = q + 1$

and additionally

- $w(g, 0) = 2q - 1,$
- $w(St, g) = q + 1.$

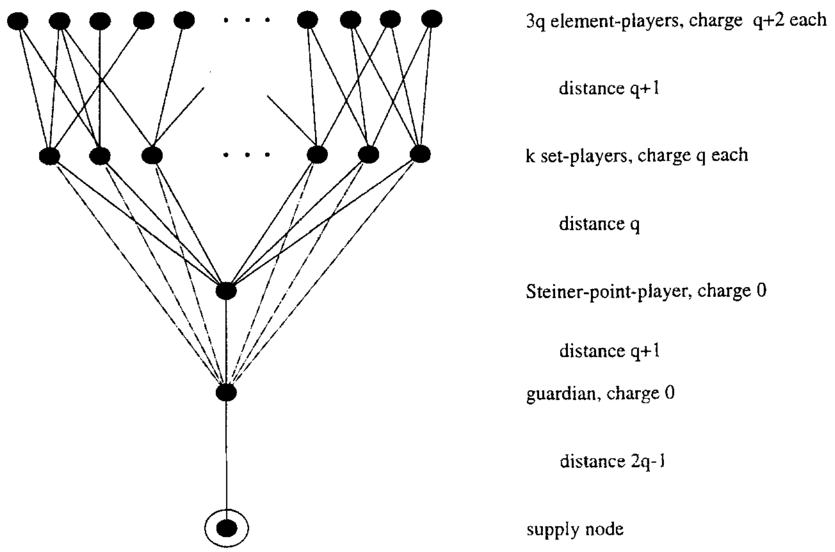

Fig. 1. The graph of the MCST-game

The weights of the other edges e are given by the distances induced from the graph $G(N, E)$ with weights as above. Note that our instance is Euclidean in the sense that it satisfies the triangle inequality.

Let $x \in \mathfrak{R}^{|X|+|F|+2}$ be given by

- $x_i = q + 2$ for $i = 1, \ldots, 3q$,
- $x_i = q$ for $i = 3q + 1, \ldots, 3q + k$,
- $x_{St} = x_g = 0$.

We claim that x is not in the core of the MCST-game if and only if F contains an exact cover.

To see this let us first make a preliminary consideration. Let $y \in \mathfrak{R}^{|X|+|F|+2}$ denote the vector

- $y_i = q + 1$ for $i = 1, \ldots, 3q$,
- $y_i = q$ for $i = 3q + 1, \ldots, 3q + k$,
- $y_{St} = q + 1$,
- $y_g = 2q - 1$.

This vector y is the core element of the MCST-game computed by the greedy algorithm in the last section. Considering $d := x - y$, we get $d(i) = 1$ for $i = 1, \ldots, 3q$, $d(i) = 0$ for $i = 3q + 1, \ldots, 3q + k$, $d(St) = -q - 1$ and $d(g) = -2q + 1$.

Let us comment on the role of g and St in our construction. If y is a fixed core vector and x a candidate for a core vector, we set $d = x - y$. Then $d(N) = 0$, and $x(S) > c(S)$ implies $d(S) > 0$. We would like such an 'unsatisfied' coalition S to exhibit an exact cover. So S should contain all element-players, which we try to achieve by giving the whole positive d-weight to the $3q$ element-players. If we could assume that S includes a further node g, then the (negative) d-weight of g could be used to "force" many element-players into S. Now $d(S) > 0$ implies $d(g) > -3q$. So $d(N) = 0$ necessitates the existence of a further point St. We show that our construction works with the choice of parameters as above.

Assume x is not in the core of the MCST-game and let S be a coalition satisfying $x(S) > c(S)$. Then $c(S) \geq y(S)$ implies $d(S) > 0$. We conclude that S must contain some element-player.

Any path from an element-player to 0 in $G(N, E)$ has to visit a covering set-player and g. Since the weights of edges not in E are induced we may assume w.l.o.g. that $g \in S$ and that S contains a covering set-player for each element-player in S. Because $\{g, St\} \nsubseteq S$ (otherwise $d \leq 0$), we then have $St \notin S$.

Under these assumptions we observe

$$c(S) = (q+1)|S \cap \{1, \ldots, 3q\}| + (q+1)|S \cap \{3q+1, \ldots 3q+k\}| + 2q - 1, \tag{1}$$

$$x(S) = (q+2)|S \cap \{1, \ldots, 3q\}| + q|S \cap \{3q+1, \ldots 3q+k\}|. \tag{2}$$

Hence

$$0 < x(S) - c(S) = |S \cap \{1, \ldots, 3q\}| - |S \cap \{3q+1, \ldots 3q+k\}| - 2q + 1 \tag{3}$$

implying $|S \cap \{3q+1, \ldots 3q+k\}| \leq q$.

Every set-player covers at most 3 element-players, thus we have

$$|S \cap \{1, \ldots, 3q\}| \leq 3|S \cap \{3q+1, \ldots 3q+k\}|. \tag{4}$$

Together with (3) this yields

$$2|S \cap \{3q+1, \ldots 3q+k\}| > 2q - 1 \tag{5}$$

implying $|S \cap \{3q+1, \ldots 3q+k\}| \geq q$ and hence $|S \cap \{3q+1, \ldots 3q+k\}| = q$. Again using (3) we get

$$0 < |S \cap \{1, \ldots, 3q\}| - 3q + 1 \Rightarrow S \supseteq \{1, \ldots, 3q\}. \tag{6}$$

By assumption, S contains a covering set-player for each of its element-players. Hence, our computations imply that the set-players in S must form an exact cover.

On the other hand, if F admits an exact 3-cover, the coalition S consisting of all element-players, an exact cover of set-players and the guardian satisfies

$$c(S) = (3q^2 + 3q) + (q^2 + q) + (2q - 1) = 4q^2 + 6q - 1, \tag{7}$$

$$x(S) = (3q^2 + 6q) + q^2 + 0 = 4q^2 + 6q \tag{8}$$

and hence $x(S) > c(S)$, which implies that x is not in the core of the MCST-game. □

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