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Finding All Anchored Squares in a Convex Polygon in Subquadratic Time

by

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Abstract

We present an $O(n \log^2 n)$ method that finds all squares inscribed in a convex polygon with n vertices such that at least one corner lies on a vertex of the polygon.

Keywords: Computational geometry, polygons, convex sets, inscribed squares, pattern recognition

1 Introduction

Approximating a polygon with a simpler shape is a problem that has received a considerable amount of attention. Finding *inscribed* polygons has applications to pattern recognition, as well as being of theoretical interest in computational geometry. In [2], De Pano, Ke and O'Rourke have described an $O(n^2)$ algorithm for finding the largest inscribed square in a convex polygon $\mathcal{P}$ with n vertices.

The interest in inscribed squares has also been highlighted by Klee in his recent book [7].

Of particular interest are squares that are *anchored*: One corner of the square is located at a vertex of the polygon. While it is relatively easy to find anchored squares in quadratic time, it is nontrivial even to find all squares formed by the $O(n^2)$ diagonals of $\mathcal{P}$ in *subquadratic* time.

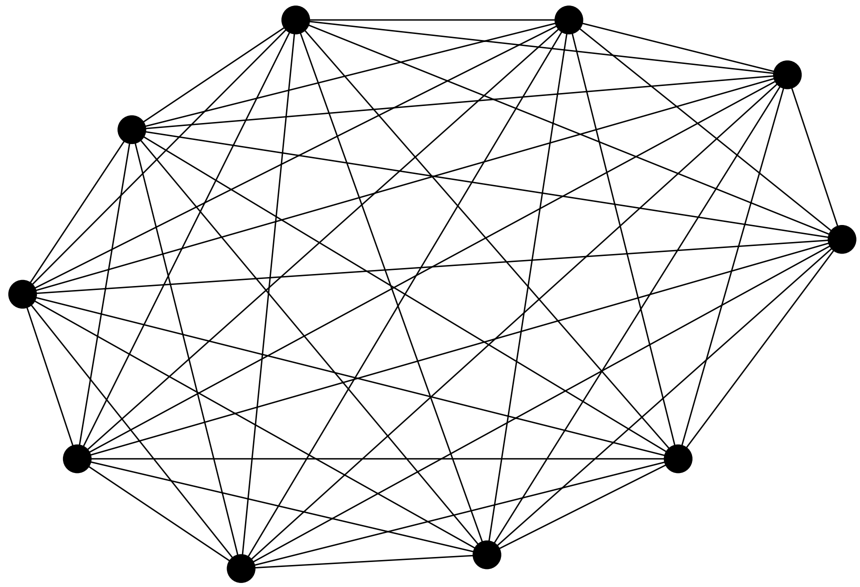

Figure 1: Pattern Recognition: Is there a square among the diagonals of $\mathcal{P}$?

2 Inscribed Squares and Dual Curves

In the following, we denote the corners of a square by s_1, s_2, s_3 and s_4 in counterclockwise order. The vertices of $\mathcal{P}$ are counterclockwise $v_1, \ldots, v_n$, while the edges are $e_1, \ldots, e_n$, where e_i has vertices v_i and v_{i+1}.

Let α be any point on a convex polygon $\mathcal{P}$. For any point p on $\mathcal{P}$, placing s_1 at α and s_2 at p positions s_3 at the point $R_\alpha(p)$. Obviously, $R_\alpha(p)$ is obtained by scaling the distance of p from α by a factor of $\sqrt{2}$ and rotating the resulting point by $\frac{\pi}{4}$ counterclockwise around α. Consequently, the locus $R_\alpha(\mathcal{P})$ of all possible positions of s_3 for s_1 at p and s_2 on $\mathcal{P}$ is a scaled and rotated copy of $\mathcal{P}$, called the *right dual curve* to $\mathcal{P}$.

Similarly, the *left dual curve* $L_\alpha(\mathcal{P})$ of $\mathcal{P}$ is the locus of all positions of s_3 with s_1 at α and s_4 on $\mathcal{P}$ and obtained by scaling $\mathcal{P}$ by $\sqrt{2}$ and a clockwise rotation of $\frac{\pi}{4}$ around α.

It is straightforward to verify the following lemma:

Lemma 2.1 *There is a one-to-one correspondence between squares inscribed in $\mathcal{P}$ anchored at α and points other than α where all three curves $\mathcal{P}$, $R_\alpha(\mathcal{P})$ and $L_\alpha(\mathcal{P})$ intersect.*

Before we describe how to use the dual curves for locating anchored squares, we note the following:

Theorem 2.2 *Let c be a closed convex curve in the plane and α be some extreme point on c. There is at most one square inscribed in c that is anchored at α.*

Proof:

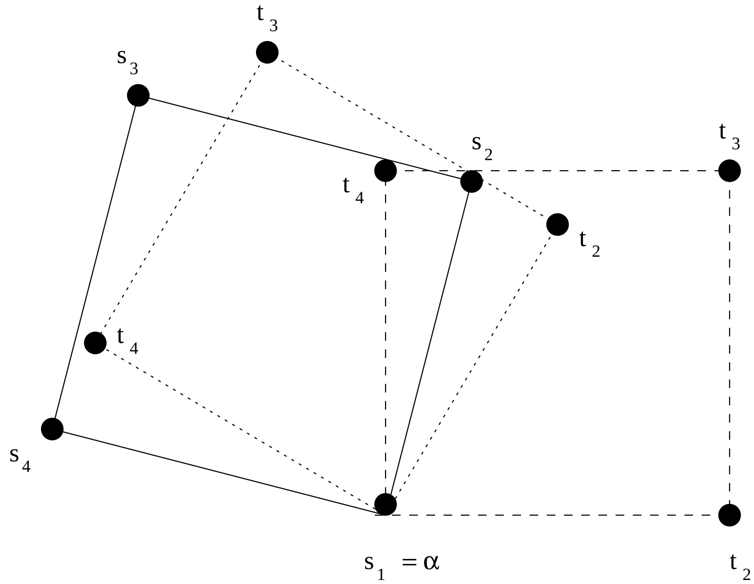

Figure 2: There is at most one inscribed square that is anchored at α

Assume there is an anchored square with corners $s_1 = \alpha$, s_2, s_3 and s_4 - see Figure 2. It is not hard to check that it is impossible to place another square with vertices $t_1 = \alpha$, t_2, t_3 and t_4, such that the seven points α, s_2, s_3, s_4, t_2, t_3 and t_4 form a convex arrangement.(One of the points s_2, s_4 will lie inside the square (α, t_2, t_3, t_4) or one of t_2, t_4 will lie inside the square (α, s_2, s_3, s_4).)

□

We distinguish two kinds of intersections between the dual curves: *Simple* intersections, where an intersection point can be separated from all other intersection points, and *nonsimple* intersections, which consist of a common segment of the polygons $R_\alpha(\mathcal{P})$ and $L_\alpha(\mathcal{P})$. Clearly, we get a nonsimple intersection only if there are two edges of $\mathcal{P}$ that enclose an angle of $\frac{\pi}{2}$ and have the same distance from α. This property enables us to check all nonsimple intersections in time $O(n \log n)$:

Algorithm Nonsimple
 for each edge e_i of $\mathcal{P}$ **do**
  if there is an edge e_j enclosing an angle of $\frac{\pi}{2}$ with e_i.
   Determine the unique point p_i on $\mathcal{P}$ that has the same

positive

   distance from e_i and e_j.
   Check whether $R_{p_i}(e_i)$ and $L_{p_i}(e_j)$ intersect on $\mathcal{P}$.
 return
End of Nonsimple.

Note that NONSIMPLE detects even those inscribed squares with corresponding nonsimple intersections that are not anchored at a vertex of the polygon $\mathcal{P}$.

3 Simple Intersections

We will now discuss the problem of detecting inscribed squares with corresponding simple intersection of the dual curves.

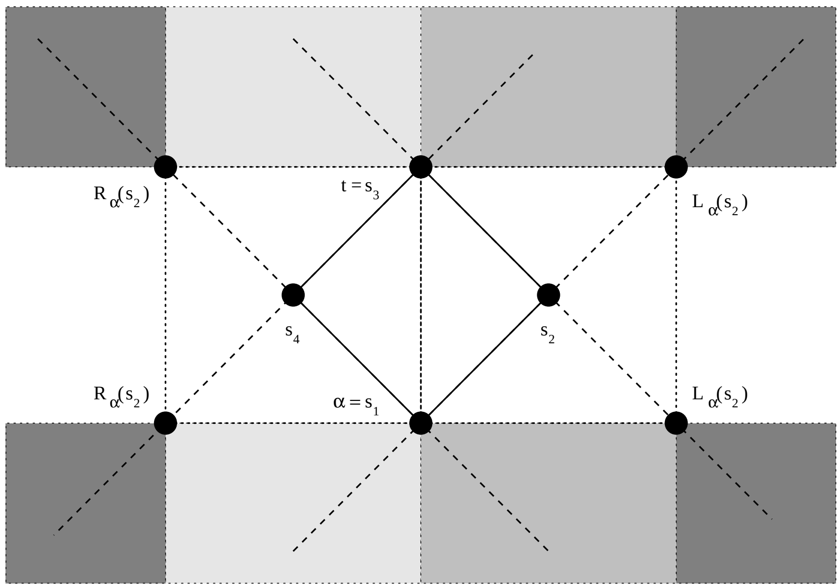

Figure 3: The situation for a square with a simple intersection

Assume α is an anchor point for which there exists an inscribed square with a simple intersection point t; see Figure 3. (The shaded areas indicate areas that cannot contain any part of $R_\alpha(\mathcal{P})$, or $L_\alpha(\mathcal{P})$ rsp., because of convexity.) We see that as a consequence of convexity of $\mathcal{P}$, $R_\alpha(\mathcal{P})$ and $L_\alpha(\mathcal{P})$, any other intersection point t' of the dual curves must satisfy $|\angle(t', \alpha, t)| > \frac{\pi}{4}$. Furthermore, for any two other such intersection points t' and t'', we get $|\angle(t', \alpha, t'')| < \frac{\pi}{4}$. Finally, we see that the two dual curves cross each other at t.

This implies the following algorithm:

Algorithm SQUARE

use NONSIMPLE to detect all nonsimple intersections.

for each vertex v_i of $\mathcal{P}$ **do**

if no nonsimple intersection t' for anchor point v_i,

use binary search to determine a simple intersection point t'.

if intersection point t' does not yield square,

Use binary search on $\{t \in L_\alpha(\mathcal{P}) \mid \frac{\pi}{4} < |\angle(t, \alpha, t')|\}$ to detect any simple intersection point t corresponding to an inscribed square.

return all squares Q_i.

End of SQUARE.

For the binary searches, we use the following idea:

Consider a ray from α through a vertex of $L_\alpha(\mathcal{P})$. In time $O(\log n)$, determine the (unique) intersection point $q \neq \alpha$ with $R_\alpha(\mathcal{P})$. If q lies outside $L_\alpha(\mathcal{P})$, an intersection must lie clockwise from q, as seen from α. If q lies inside $L_\alpha(\mathcal{P})$, an intersection must lie counterclockwise from q, as seen from α. When we are left with an edge as our search interval, we can calculate the intersection point.

Using this binary search on the vertices of $L_\alpha(\mathcal{P})$, we get an overall complexity of $O(n \log^2 n)$.

4 Conclusion

We have presented an $O(n \log^2 n)$ algorithm for determining all anchored squares inscribed in a convex polygon with n vertices. It is an open question whether there is a lower bound of $\Omega(n \log n)$; in that case, it would be particularly nice to improve our algorithm to $O(n \log n)$. This might be possible with a more sophisticated approach for locating simple intersections of the two dual curves.

Another interesting question is to give a subquadratic algorithm for finding maximal inscribed squares that are not anchored, i.e. that have no corners on vertices. This would improve the method of [2] for finding maximal inscribed squares to quadratic running time. It remains an open question whether there can be a superlinear number of maximal squares of this type.

Our method can be immediately generalized for finding inscribed rectangles with a given ratio of sides. Other quadrangles make it necessary to give some more specifications - we have omitted a detailed discussion at this point. It is not true for general convex quadrangles that there can only be one similar inscribed copy anchored at a vertex. (Theorem 2 cannot even be generalized to rhombi, i.e. quadrangles with four equal sides.)

We do conjecture, however, that the overall number of anchored quadrangles will still be linear.

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