Computational Geometry

Tutorial #5 — Voronoi diagrams and enclosing disks

Institut für Betriebssysteme
und Rechnerverbund
Algorithmik

Peter Kramer | December 18th, 2025

Voronoi diagrams

Higher order

Farthest point

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Voronoi diagrams

Refresh

Voronoi diagrams
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A Voronoi diagram $\mathrm{Vor}(P)$ partitions a metric space based on which element of the discrete point set P is closest.

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How do the unbounded faces relate to the convex hull $\mathrm{conv}(P)$?

Voronoi diagrams
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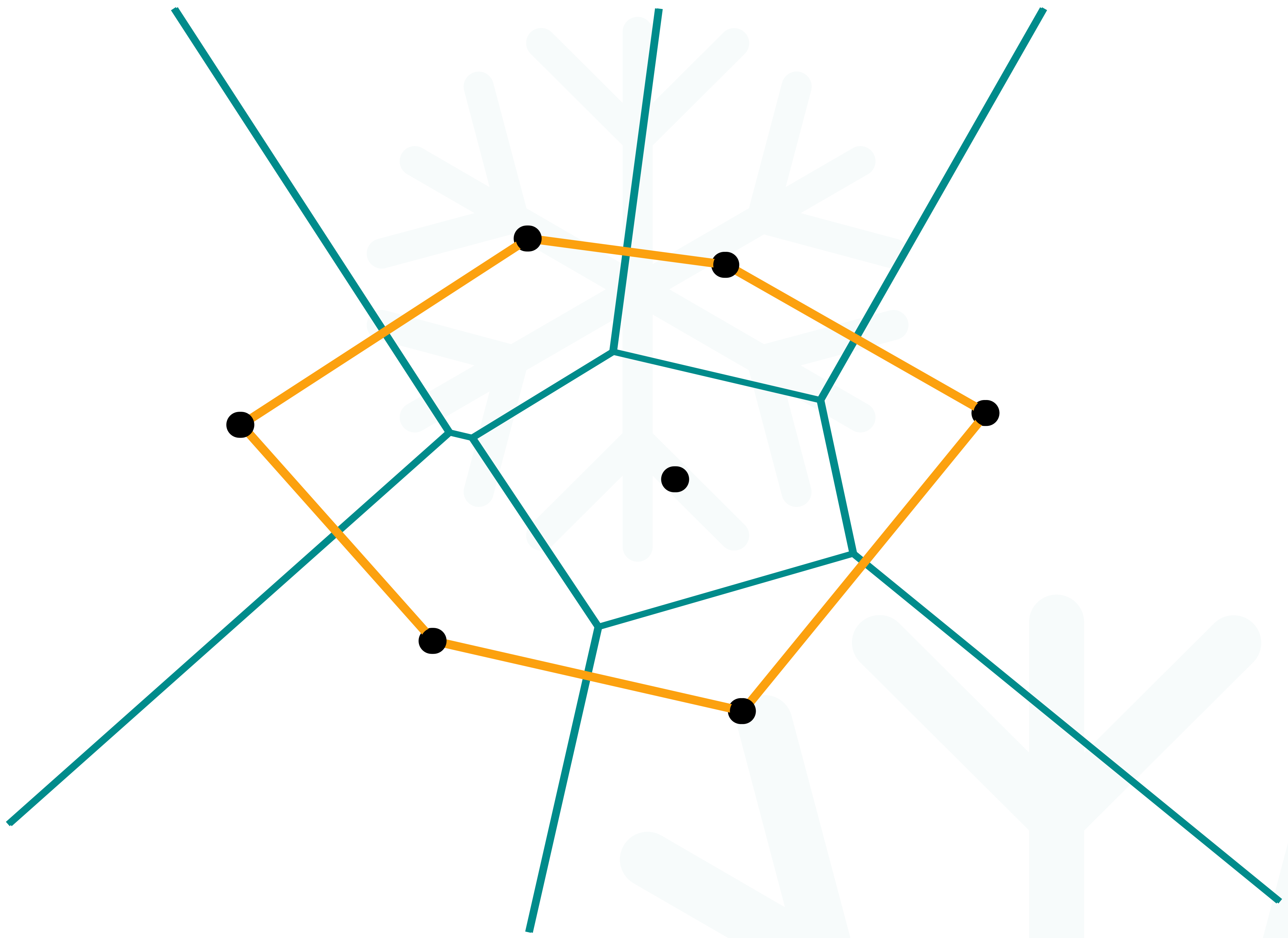
How do the unbounded faces relate to the convex hull $\mathrm{conv}(P)$*?*



Voronoi diagrams
Higher order

A Voronoi diagram $\mathrm{Vor}(P)$ partitions a metric space based on which element of the discrete point set P is closest.

*What if we wanted to divide based on which **two** points are closest?*

Voronoi diagrams

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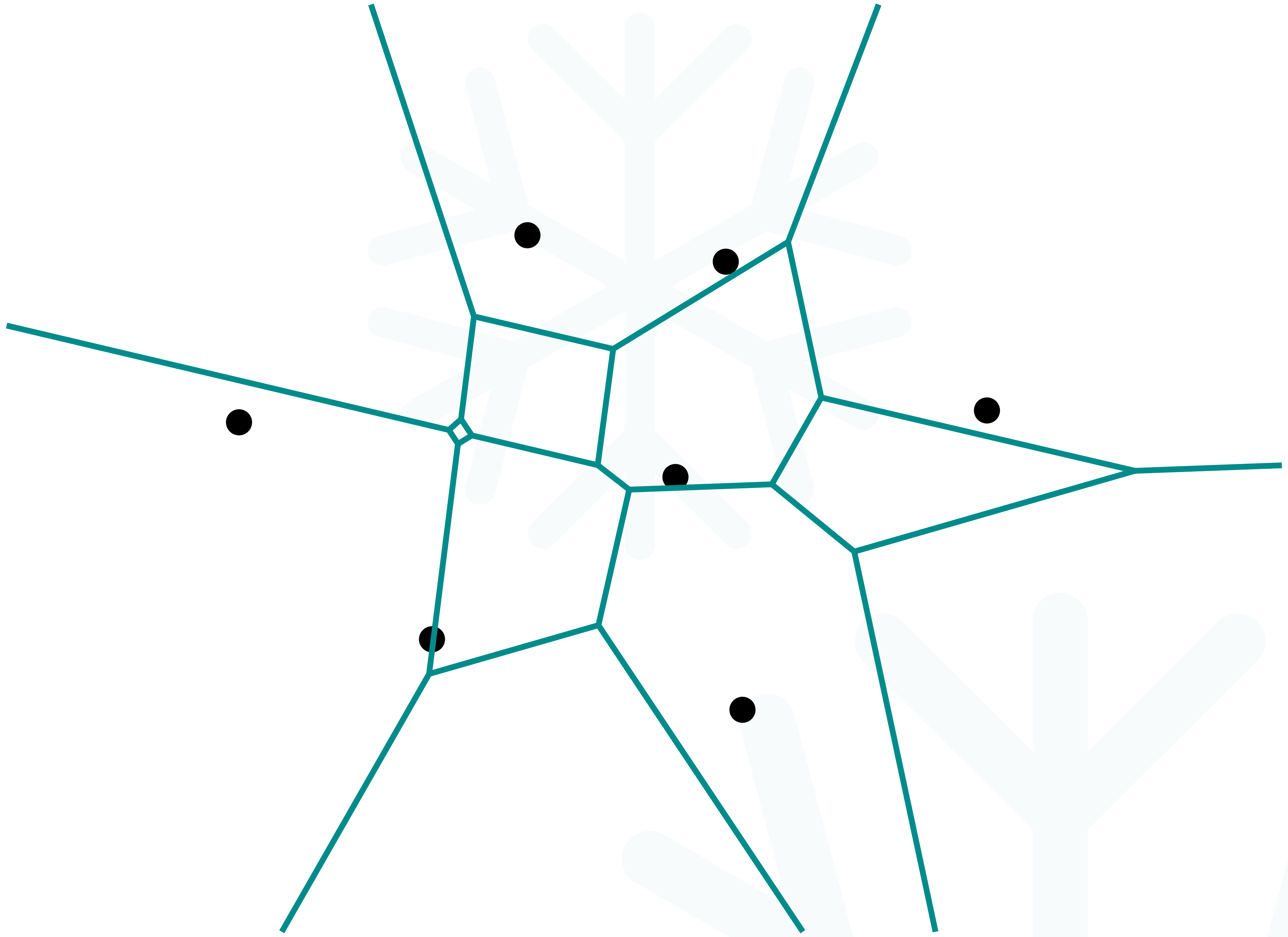
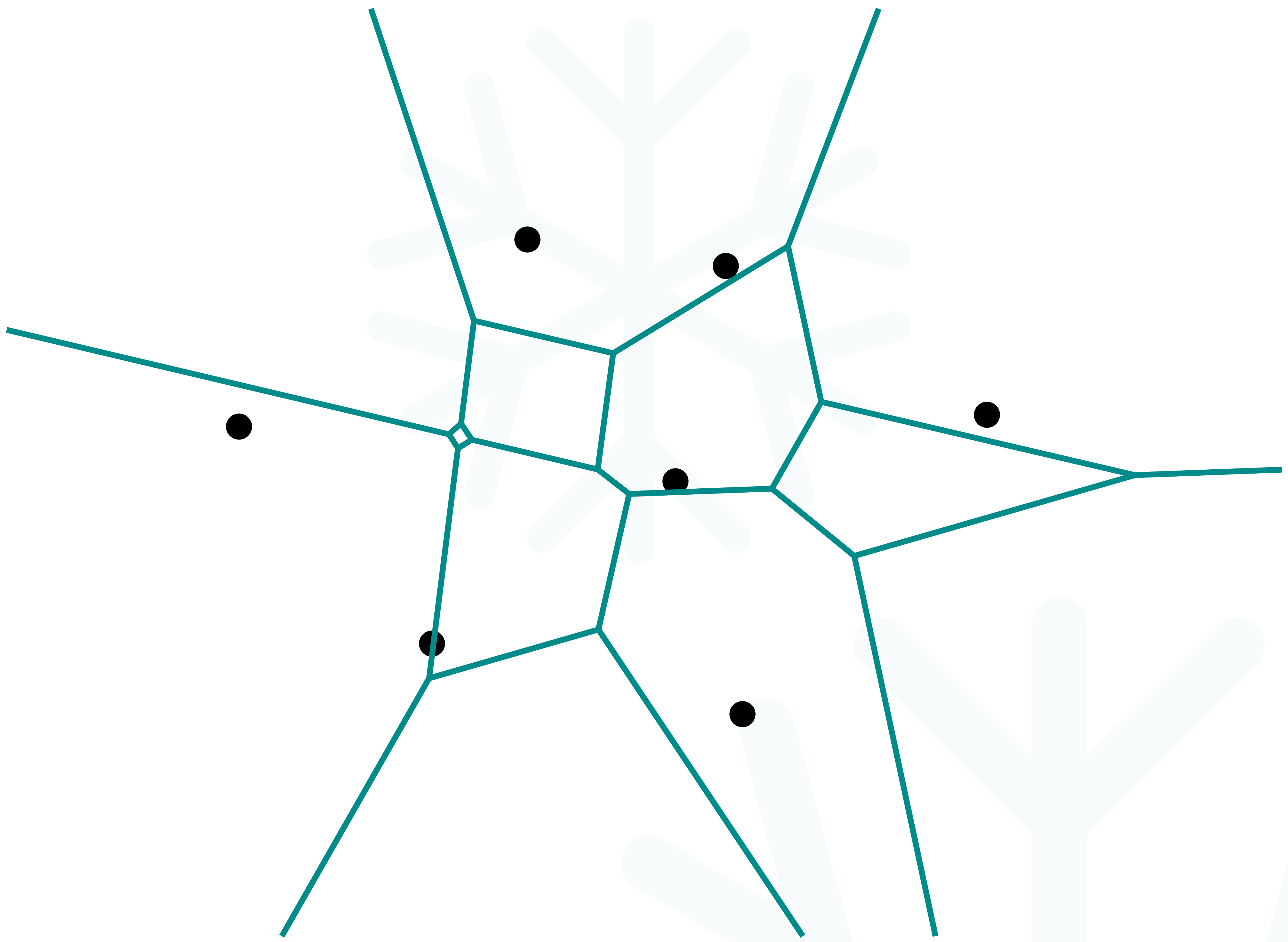


Voronoi diagrams
Higher order

An ith order Voronoi diagram of P divides a metric space based on **which i points** of a discrete set P are closest.

Here: Second order Voronoi diagram.

How can we derive this?



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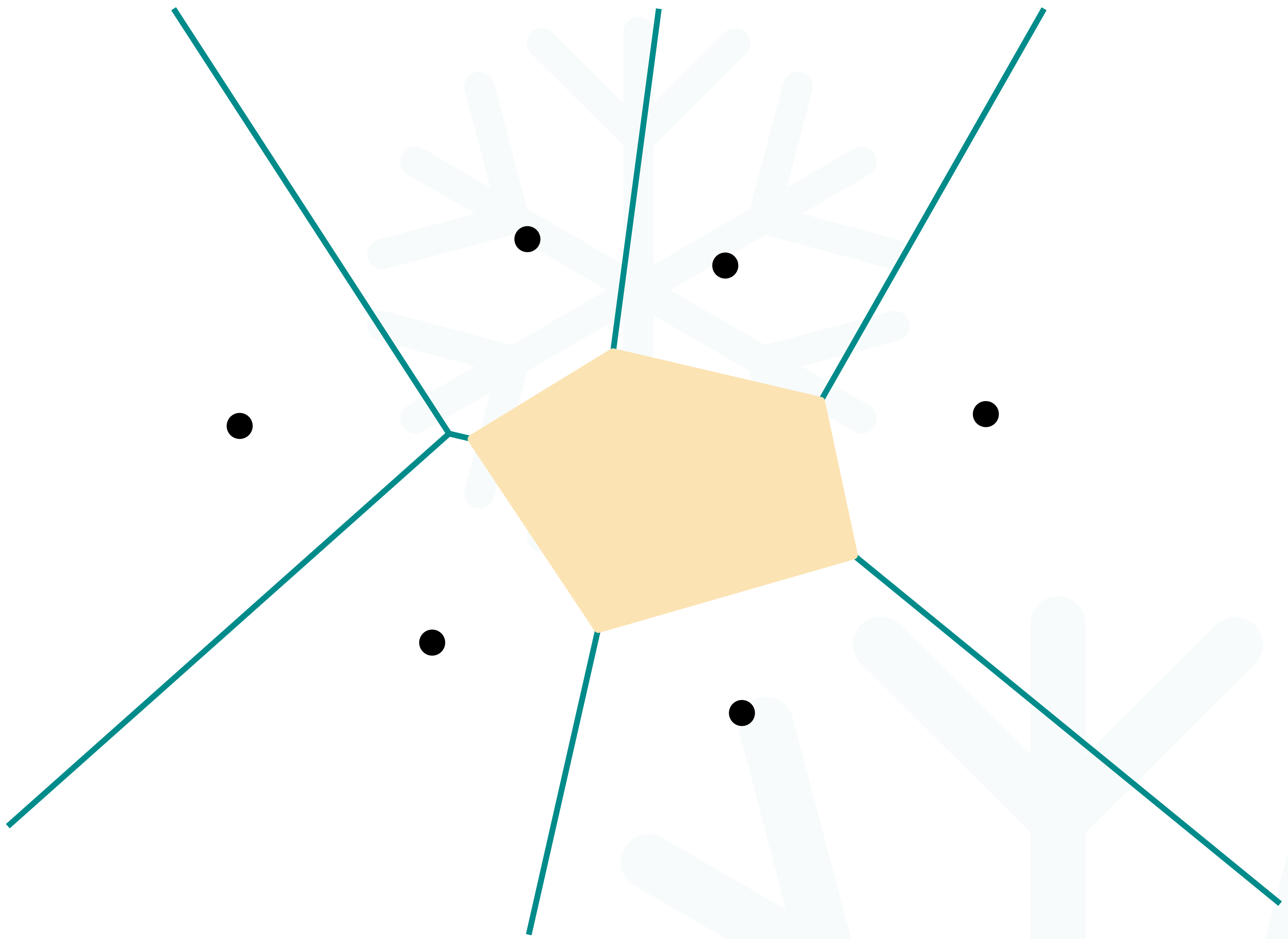

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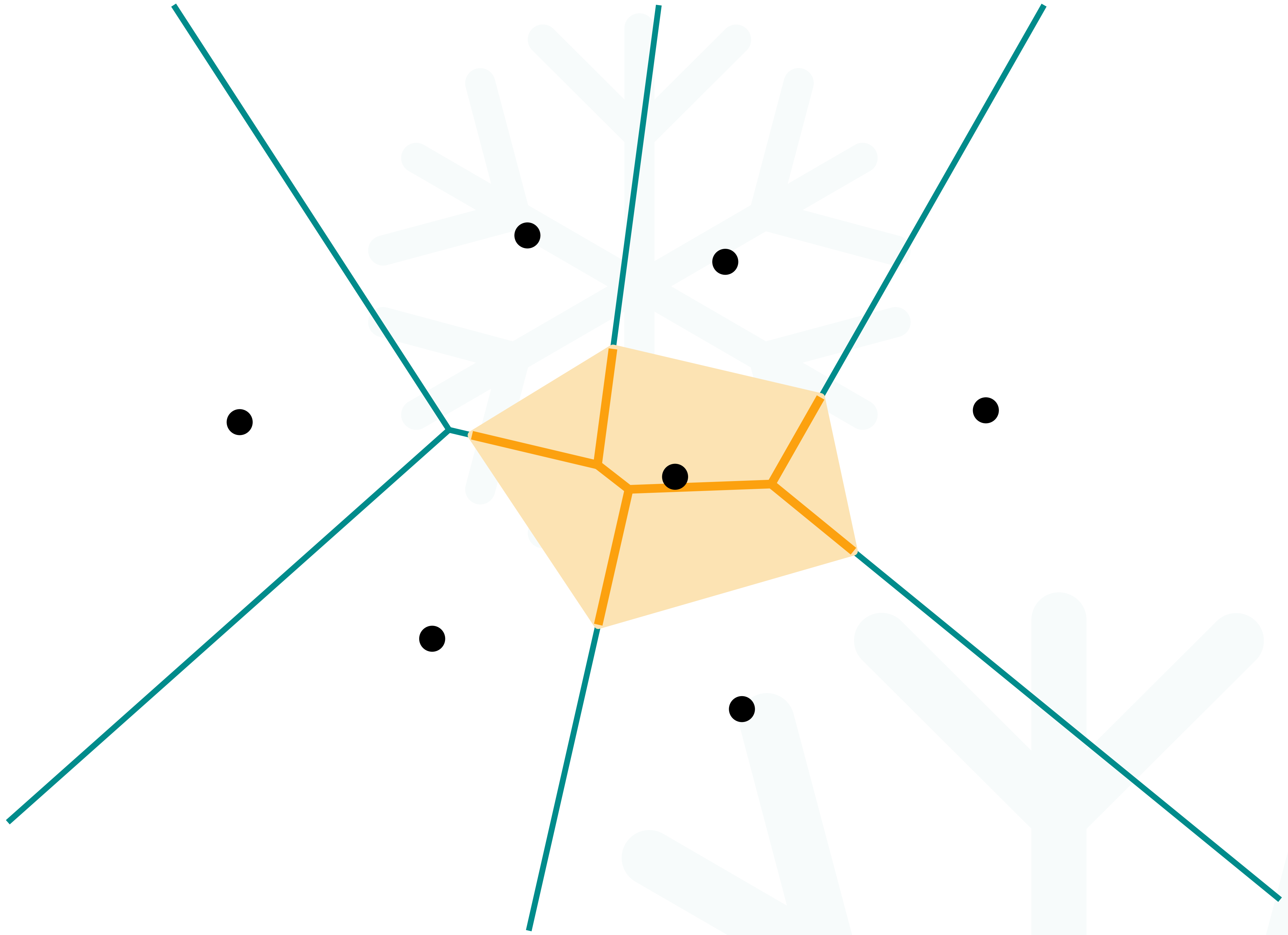
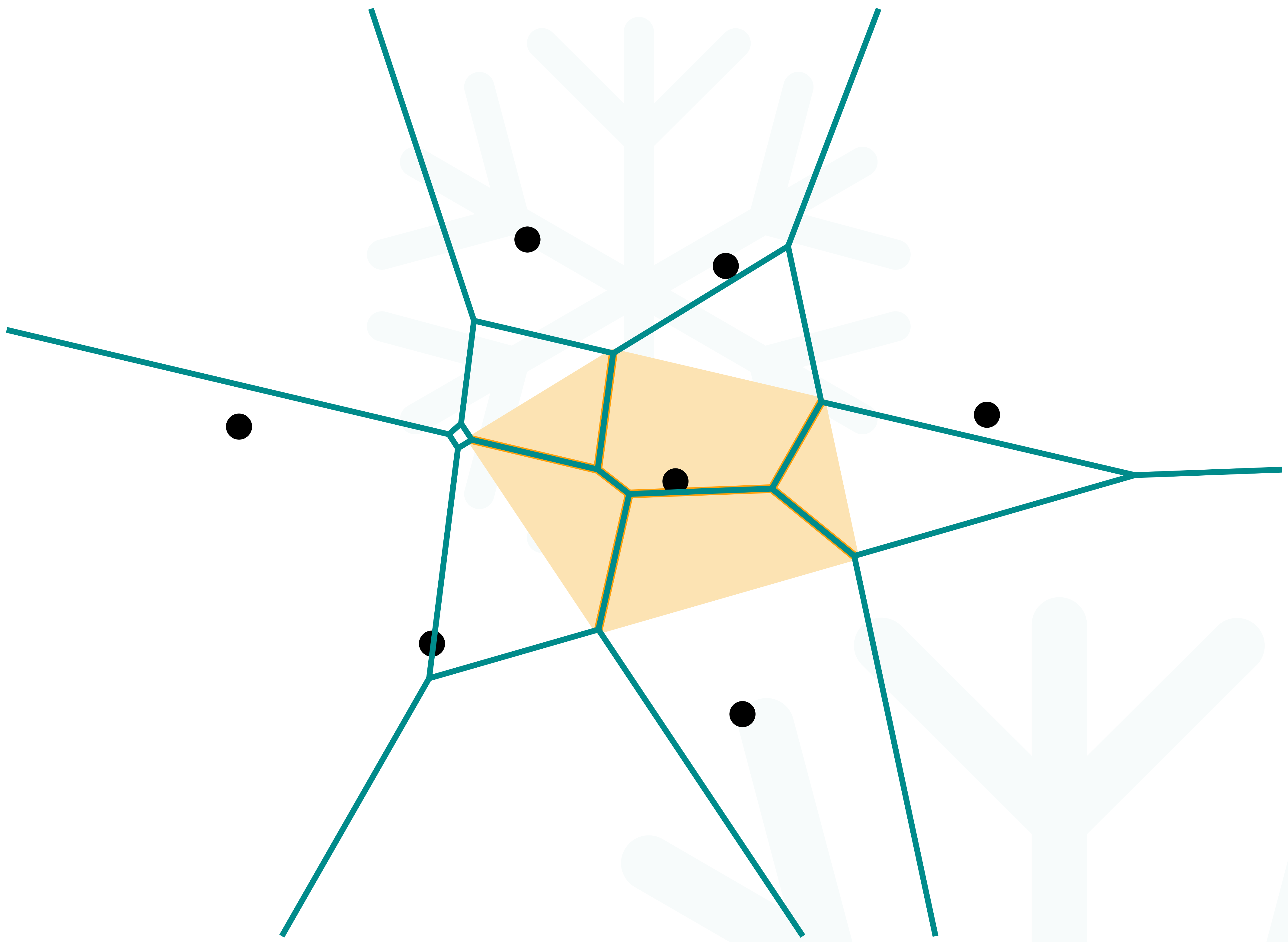


Voronoi diagrams

Higher order

An ith order Voronoi diagram $\mathrm{Vor}(P, i)$ divides a metric space based on **which i points** of the discrete set P are closest.

```
Basic idea for Vor(P, i+1):
  For region R in Vor(P, i) do:
    Let P_R = sites in P that define R
    R_{i+1} = Vor(P\P_R, i) ∩ R
    Replace R by R_{i+1}
```



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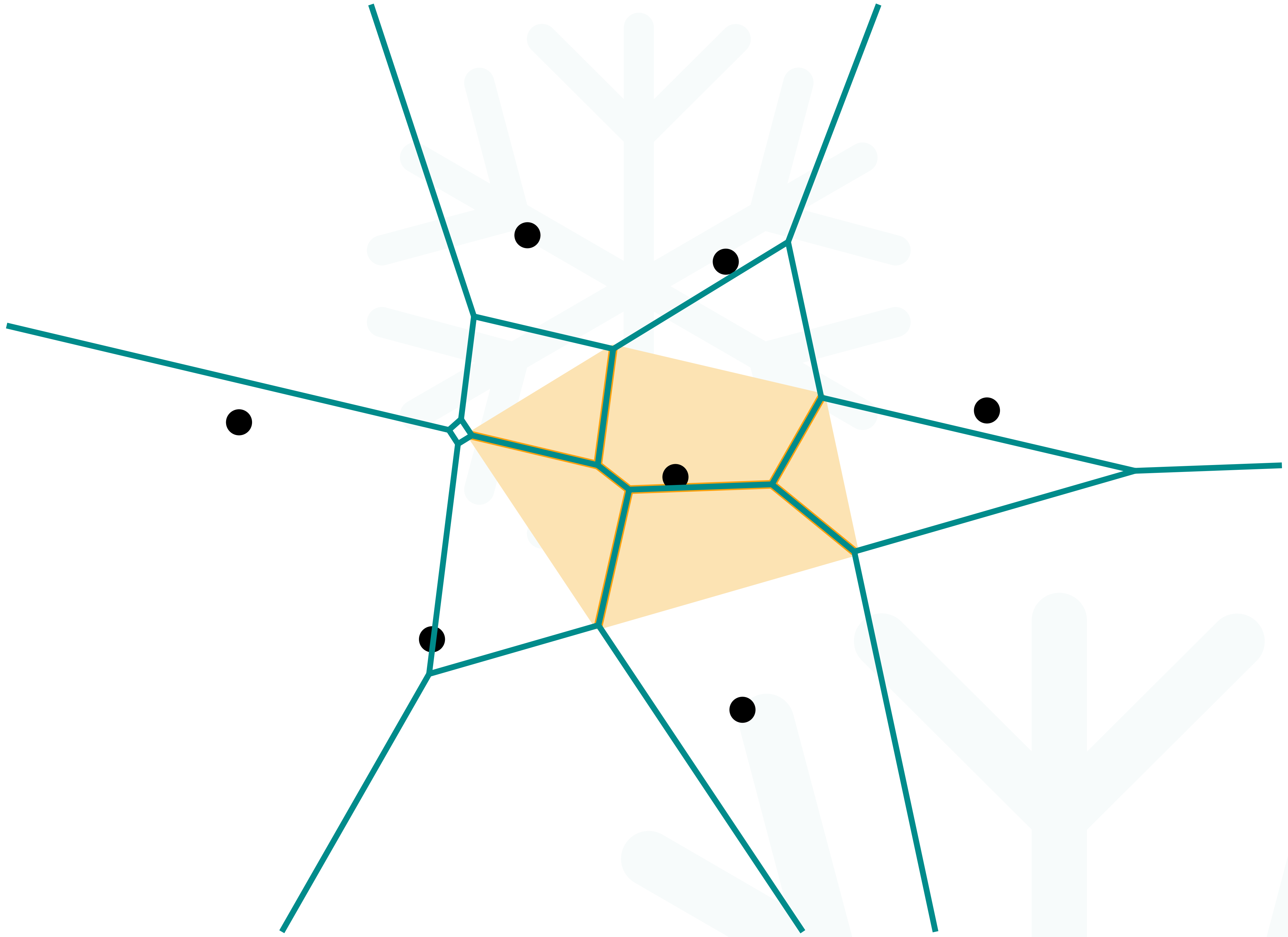


Voronoi diagrams

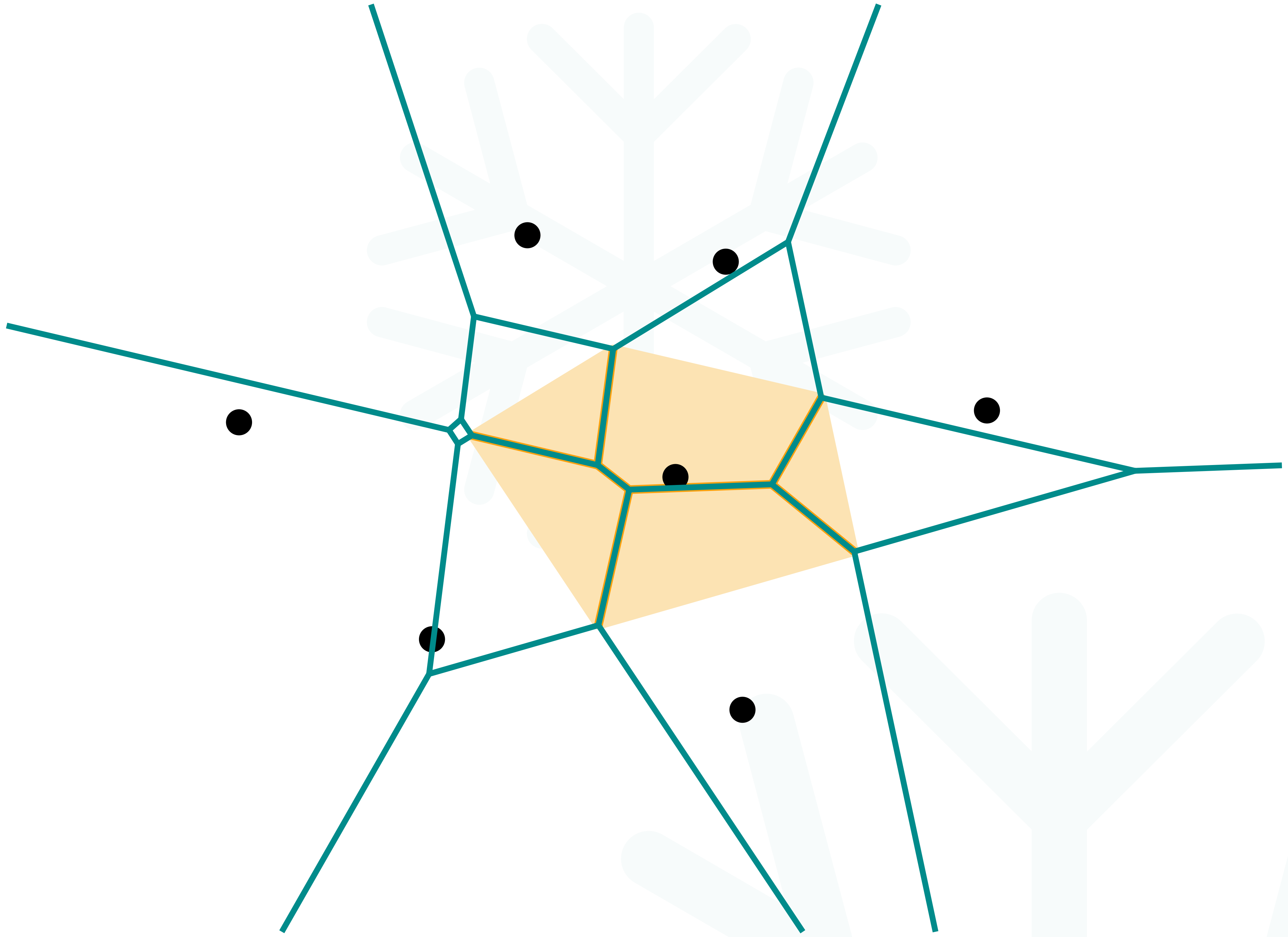
Higher order

An ith order Voronoi diagram $\mathrm{Vor}(P, i)$ divides a metric space based on **which i points** of the discrete set P are closest.

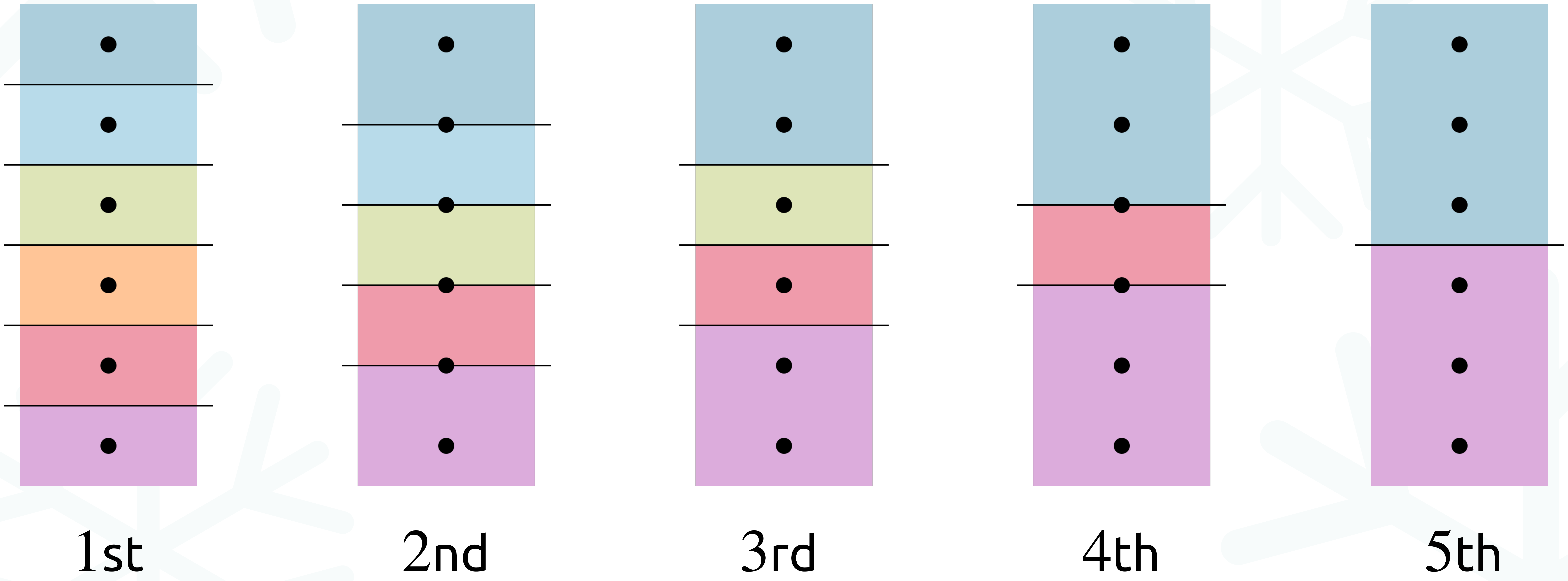
Using better methods:
Theorem E4.1 (Chan and Tsakalidis, 2015):
The ith order Voronoi diagram of n points in the plane can be computed in $\mathcal{O}(n \log n + ni \log i)$.

Theorem E4.2 (Chan et al, 2023):

[… or] in $\mathcal{O}(n \log n + ni)$ expected time.



Degenerate case: Collinearity



Voronoi diagrams

Farthest point

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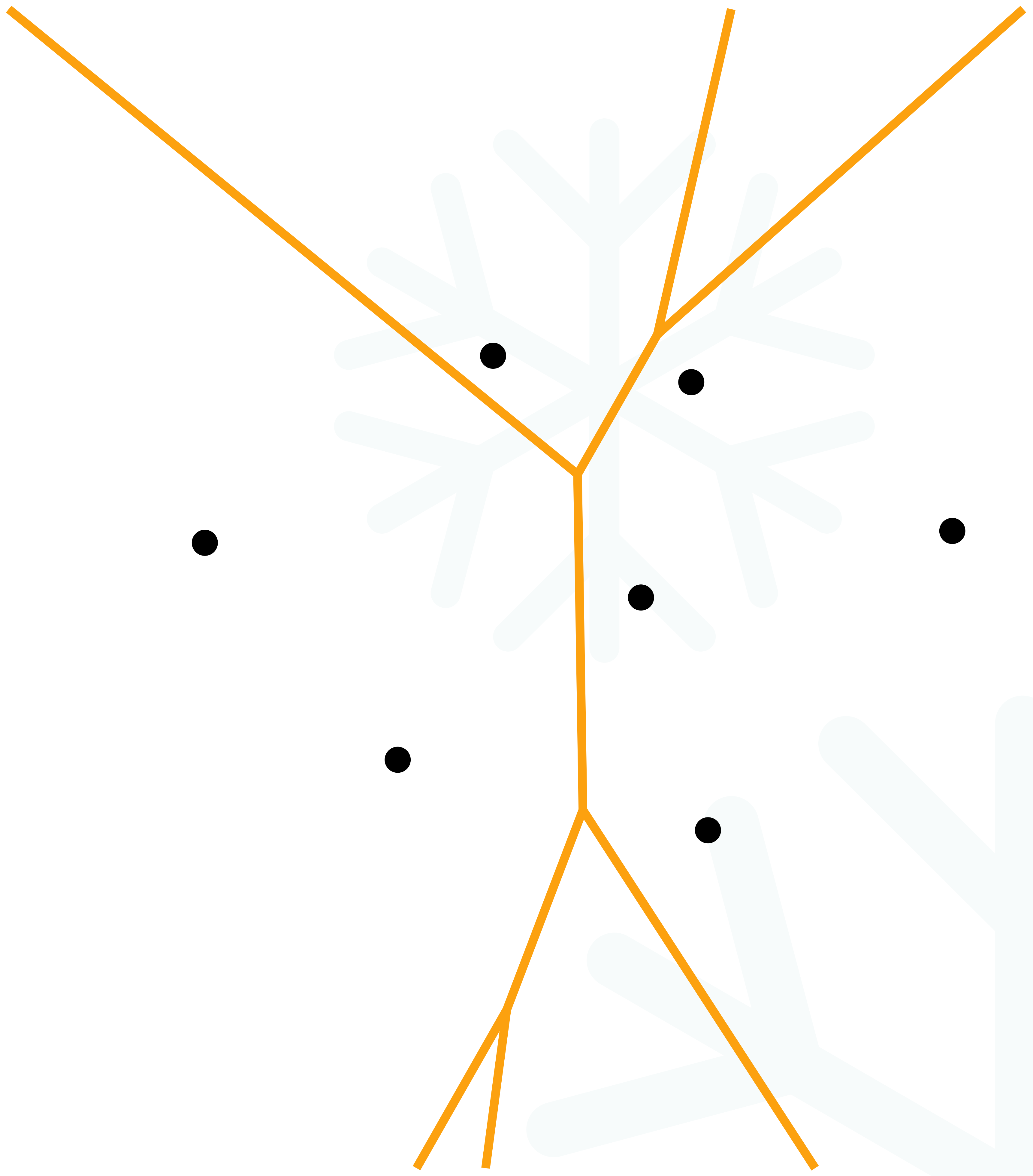
Voronoi diagrams
Farthest point

An $(n-1)$th order Voronoi diagram divides a metric space based on which element of a discrete point set P is **farthest**.

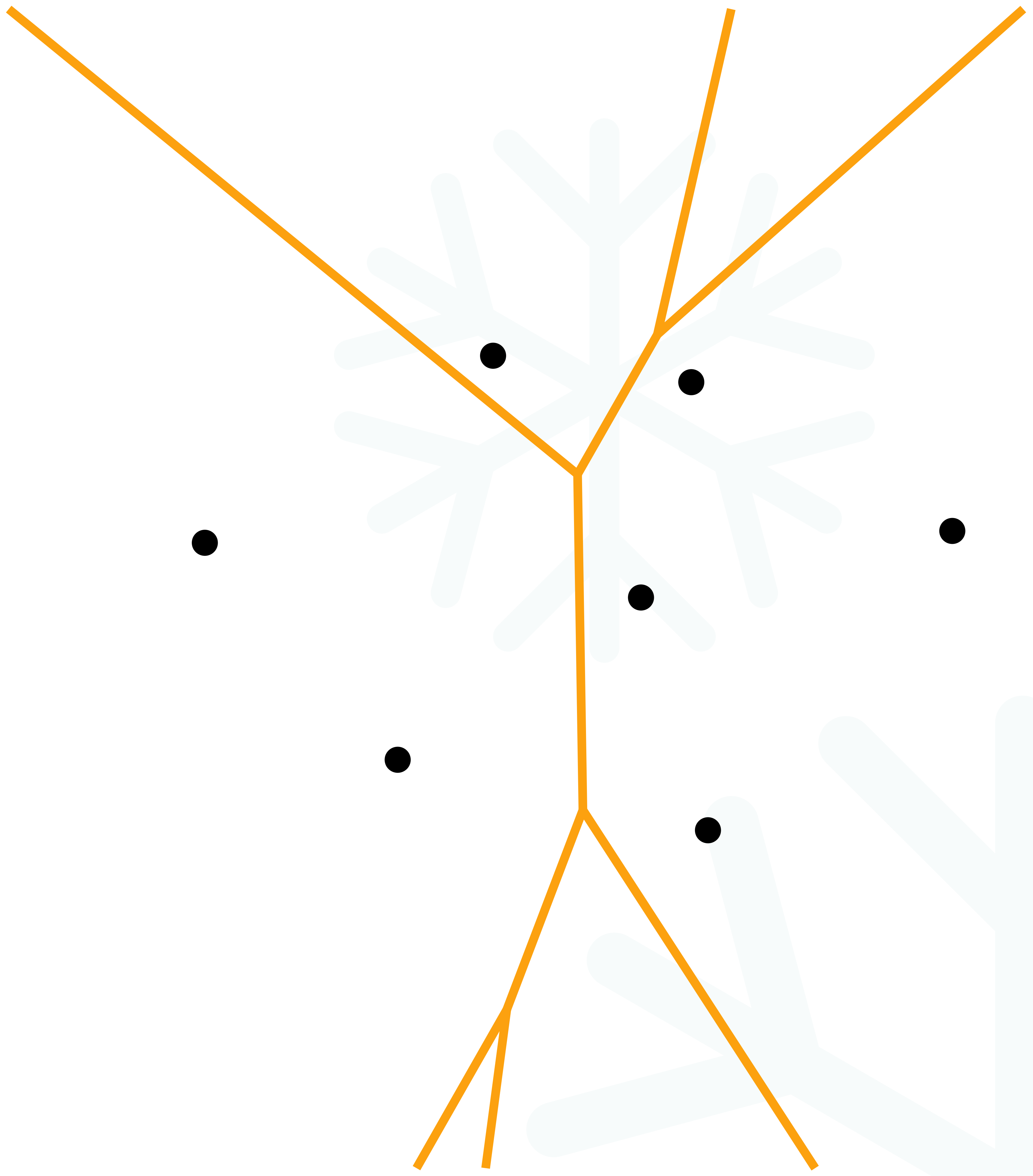


What can you say about this diagram?
How many regions?
What's the graph topology?

Voronoi diagrams

Farthest point

An $(n-1)$th order Voronoi diagram divides a metric space based on which element of a discrete point set P is **farthest**.



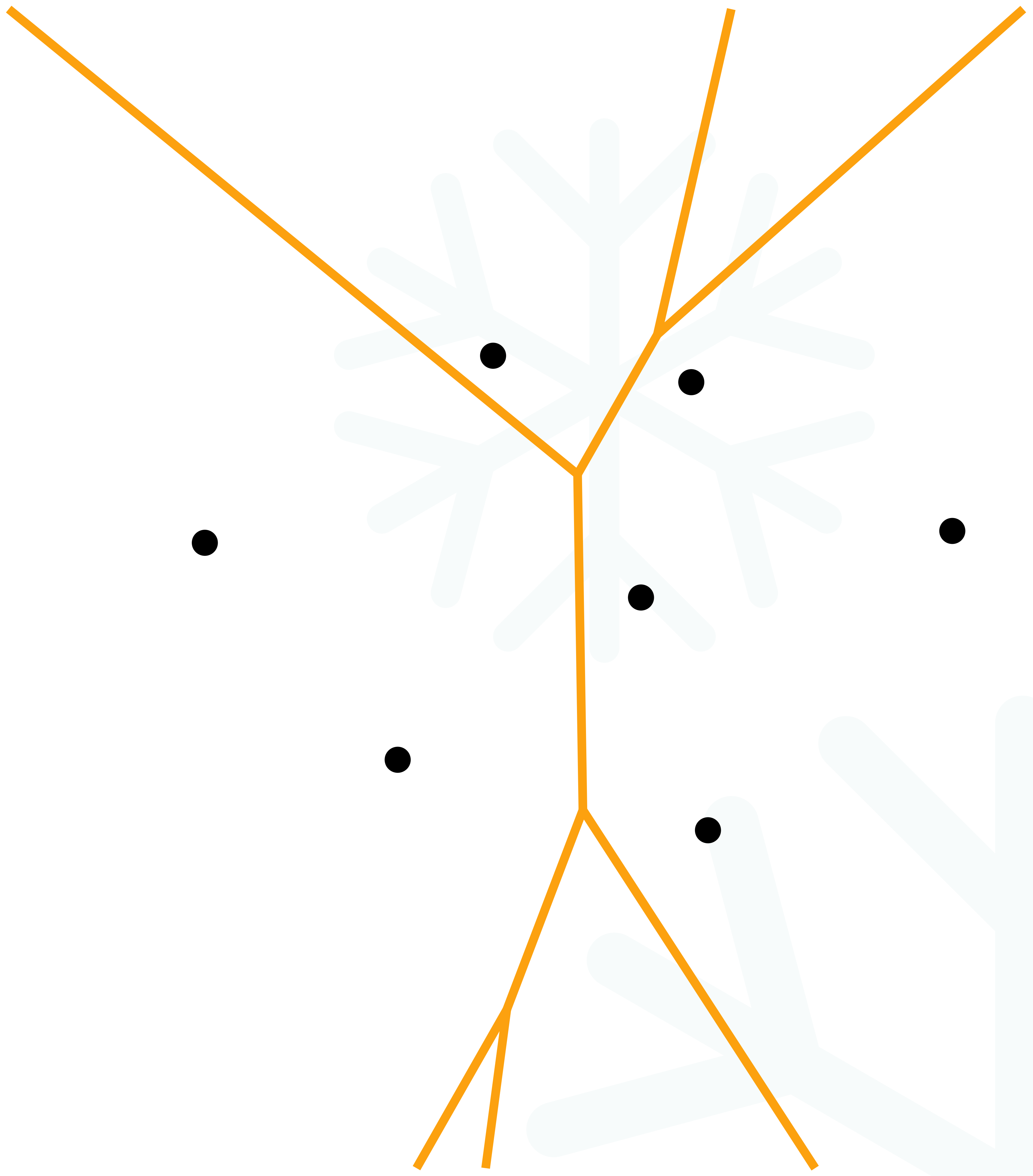
Can you think of some relation to the convex hull $\mathrm{conv}(P)$?

Voronoi diagrams

Farthest point

An $(n-1)$th order Voronoi diagram divides a metric space based on which element of a discrete point set P is **farthest**.

All cells are unbounded, i.e., the dual graph is a tree. A point $p \in P$ has a non-empty Voronoi region exactly if it lies on the boundary of the convex hull $\mathrm{conv}(P)$.



Voronoi diagrams

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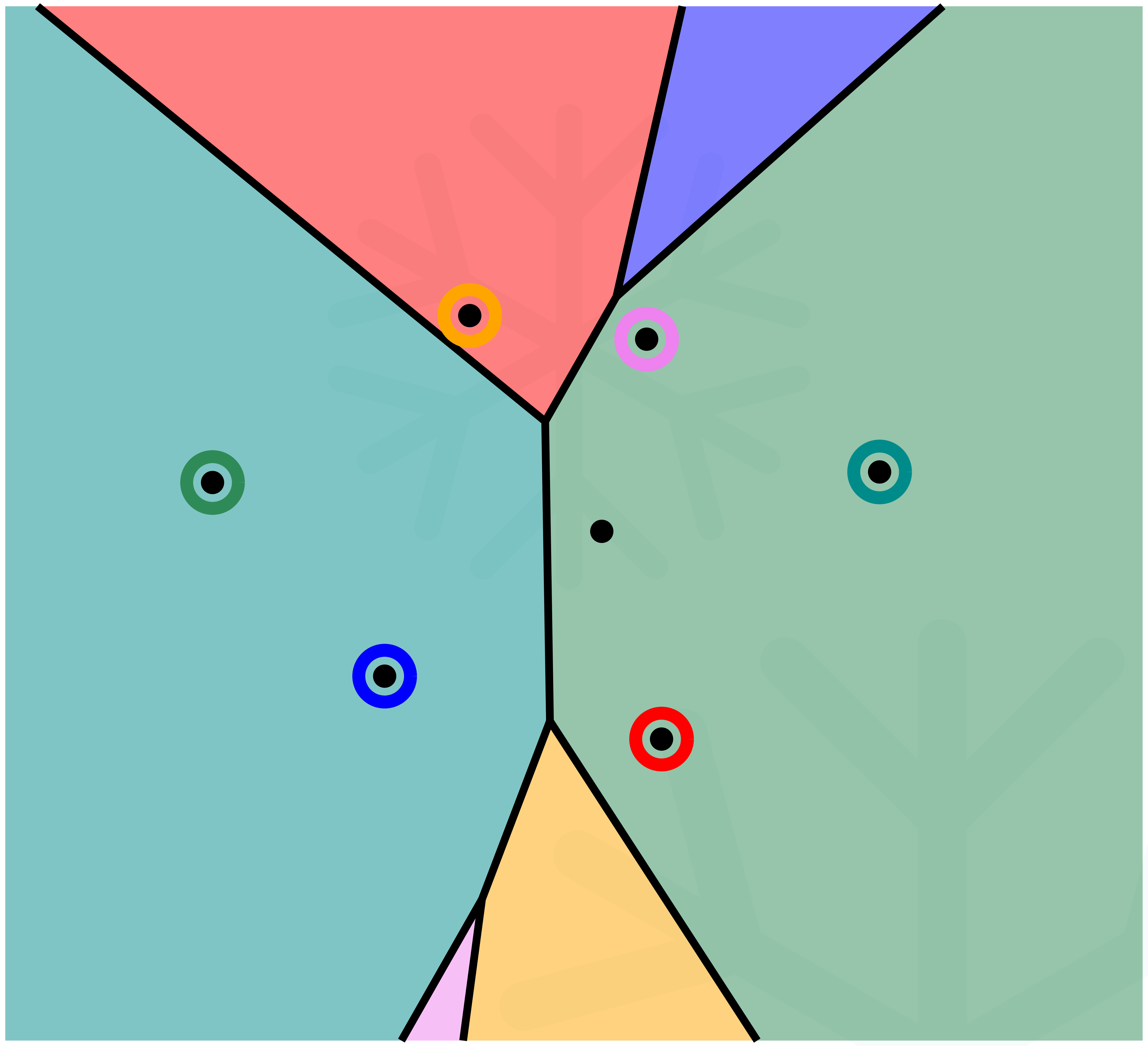
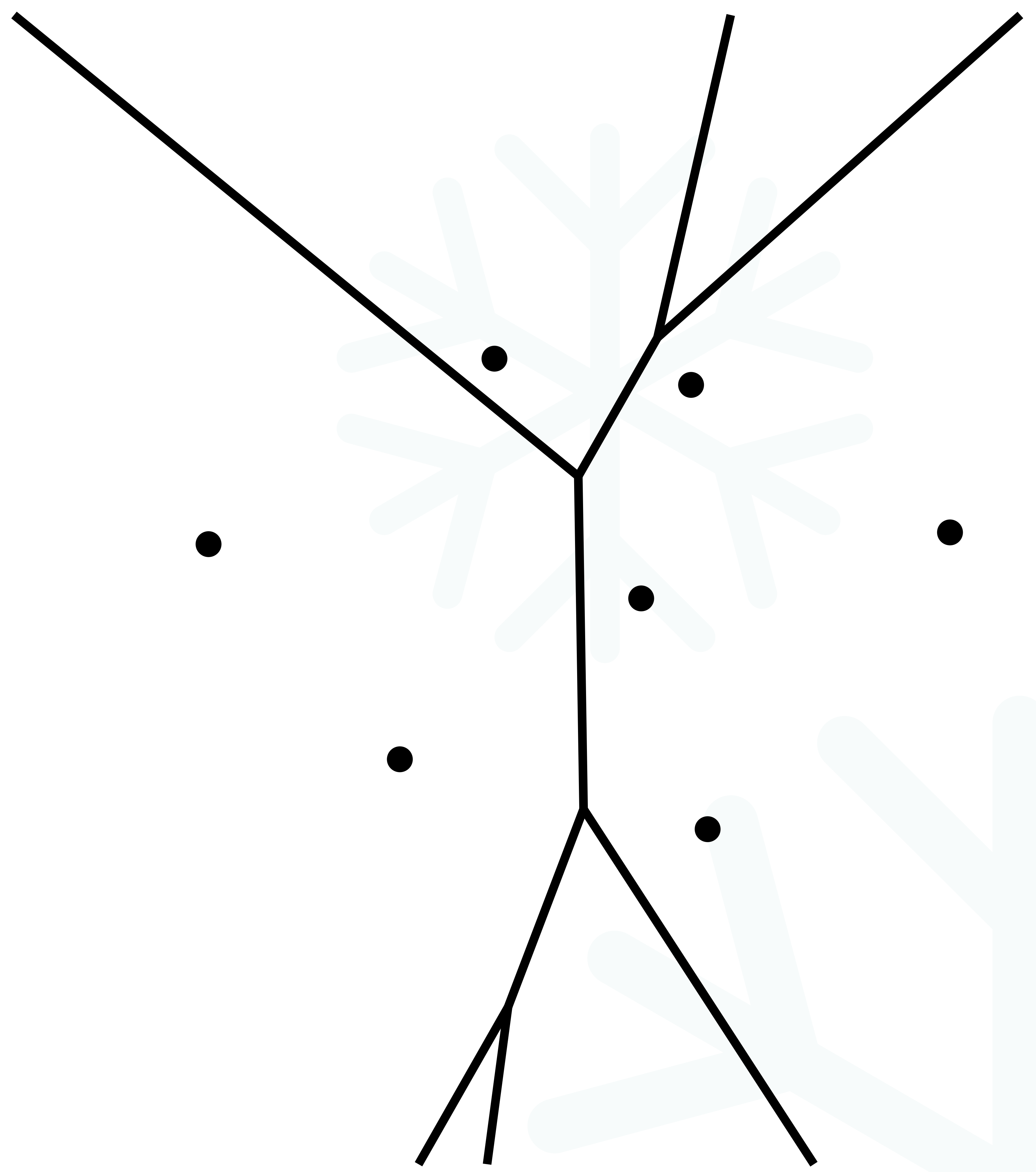


Voronoi diagrams
Farthest point

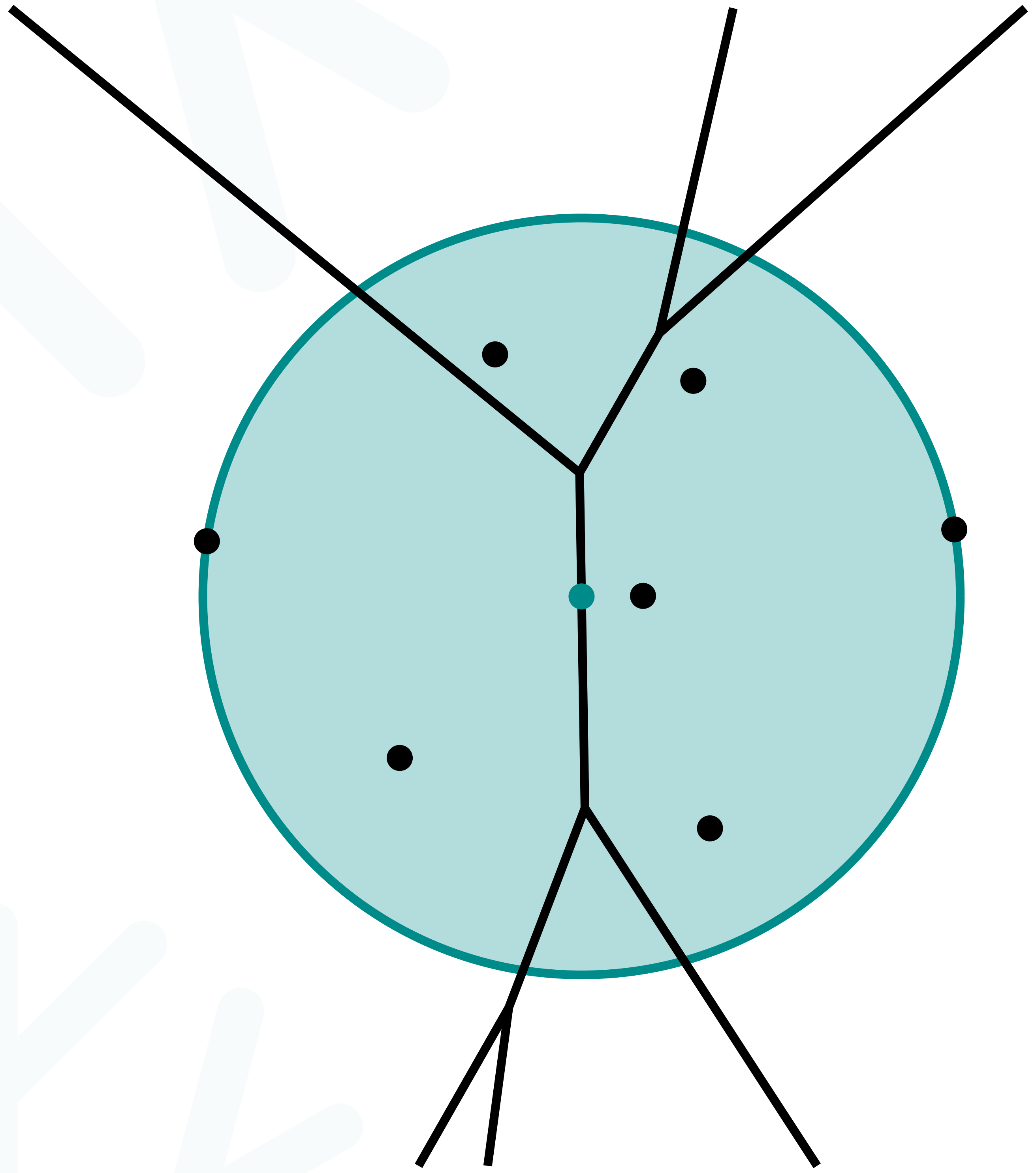
An $(n-1)$th order Voronoi diagram ***(farthest-point Voronoi diagram)*** divides a metric space based on which element of a discrete point set P is **farthest**.

For first order, we had the empty circumcircle property *(what's this?)*.

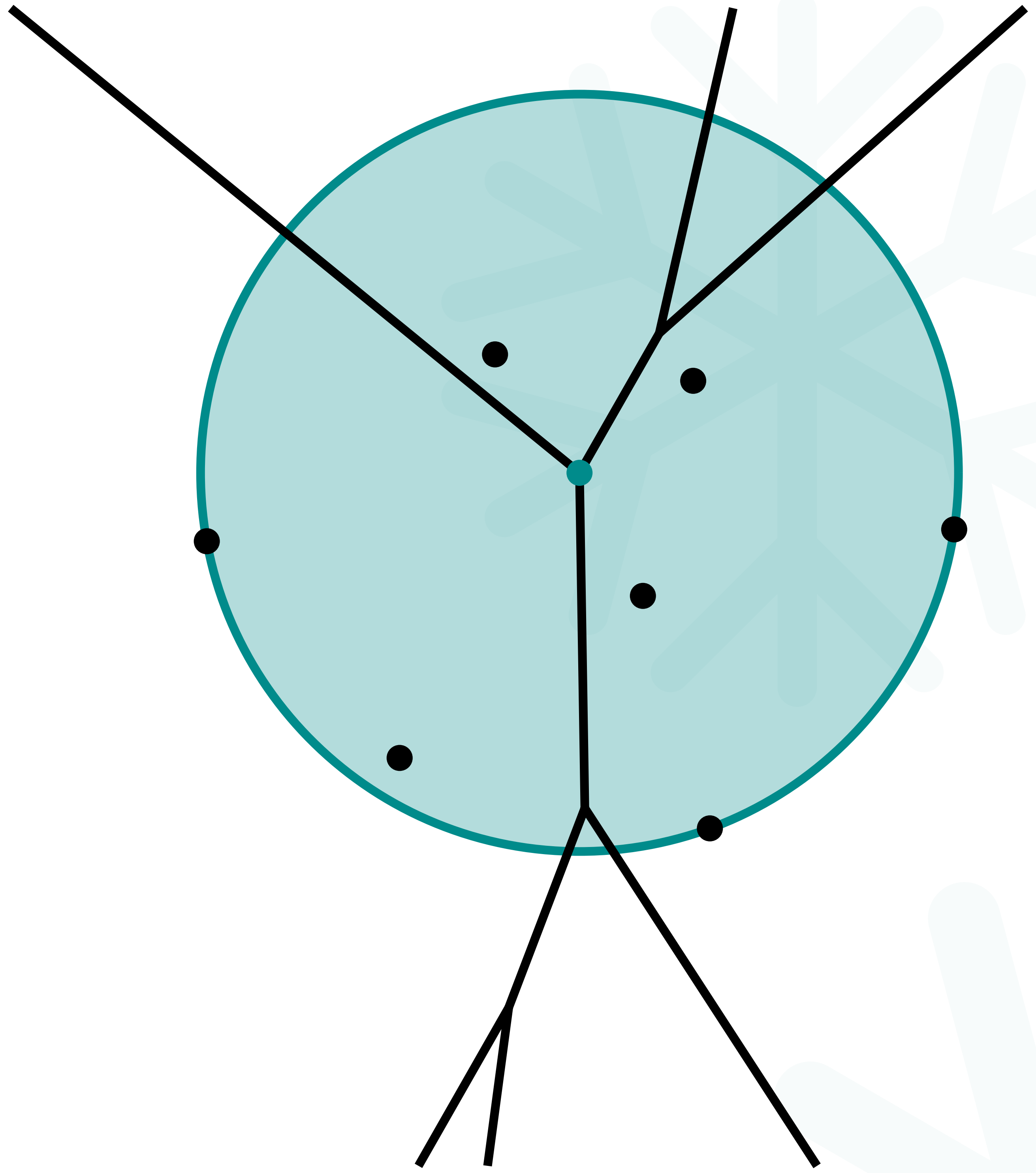
Does a similar property hold for every vertex and edge of the farthest point Voronoi diagram?



Voronoi diagrams
Farthest point



Edges are equidistant to **two** sites, closer to all other.

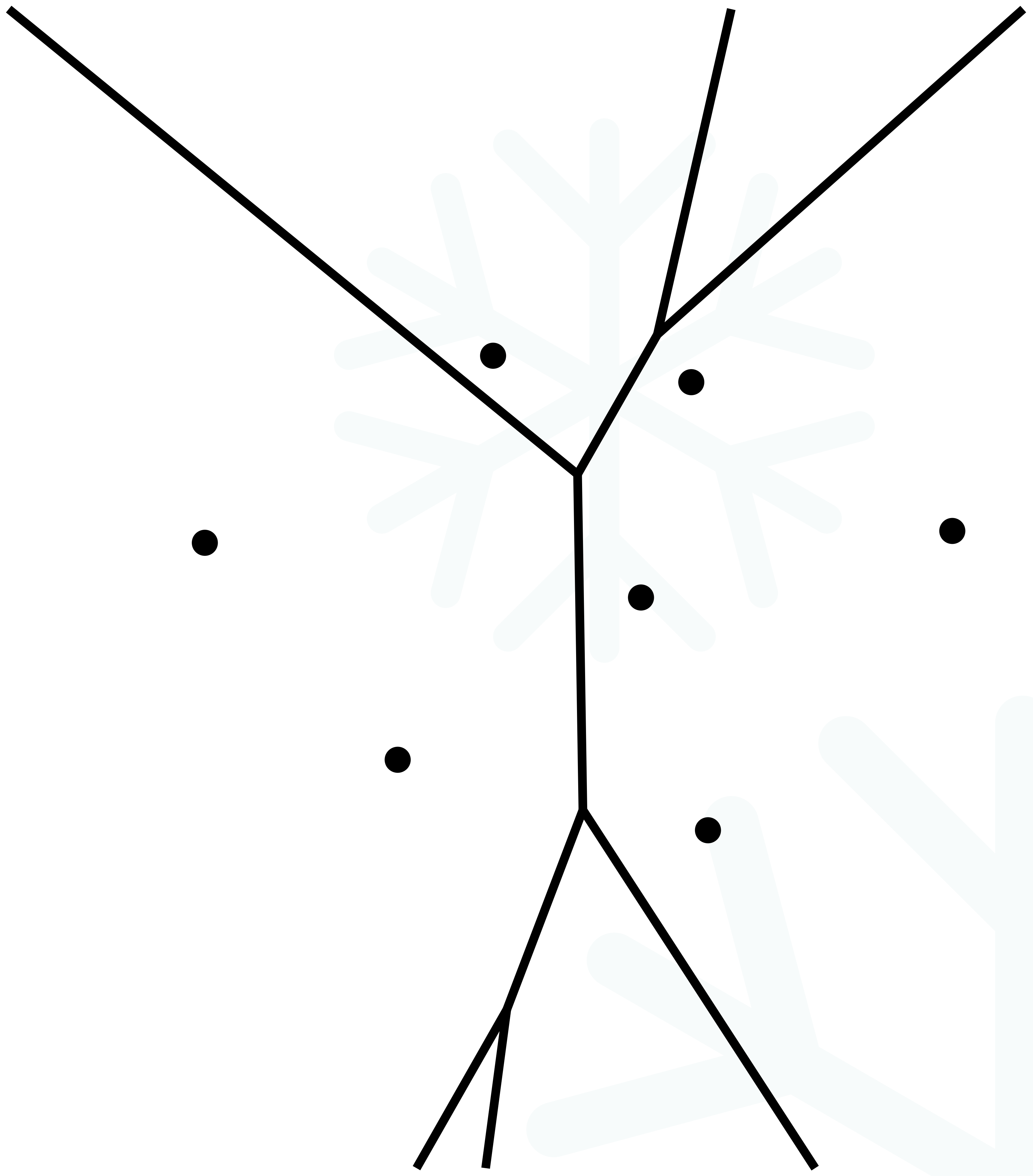


Vertices are equidistant to **three** sites, closer to all others.

Voronoi diagrams
Farthest point

An $(n-1)$th order Voronoi diagram divides a metric space based on which element of a discrete point set P is **farthest**.

Theorem E4.2 (Cheong et al., 2011):
The farthest point Voronoi diagram of n points in the plane can be computed in $\mathcal{O}(n \log^3 n)$ time.

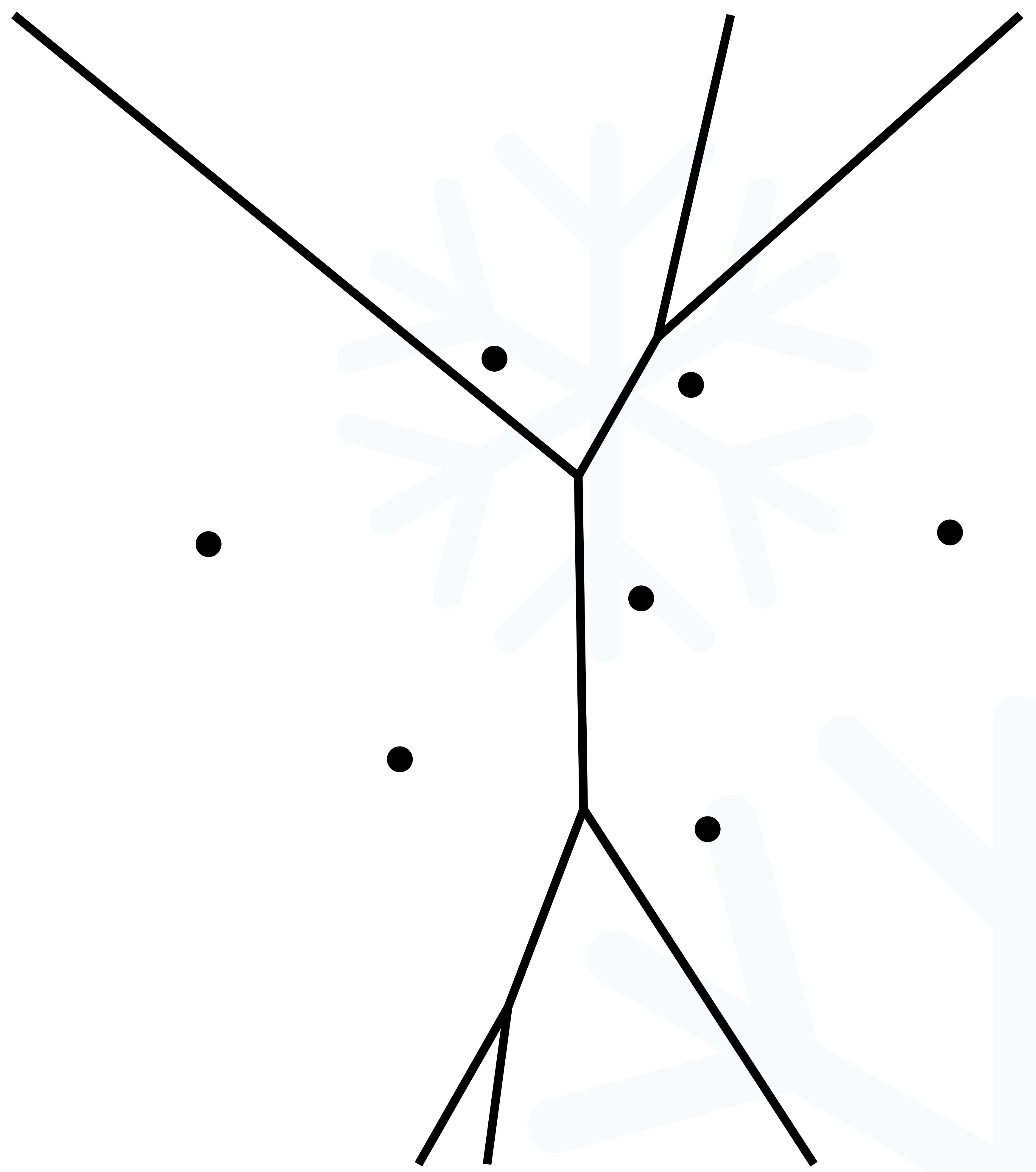


Voronoi diagrams

Farthest point

An $(n-1)$th order Voronoi diagram divides a metric space based on which element of a discrete point set P is **farthest**.

Using a DCEL, this graph structure can be stored such that the corresponding sites to each face, vertex, and edge can be accessed in $\mathcal{O}(1)$ time!



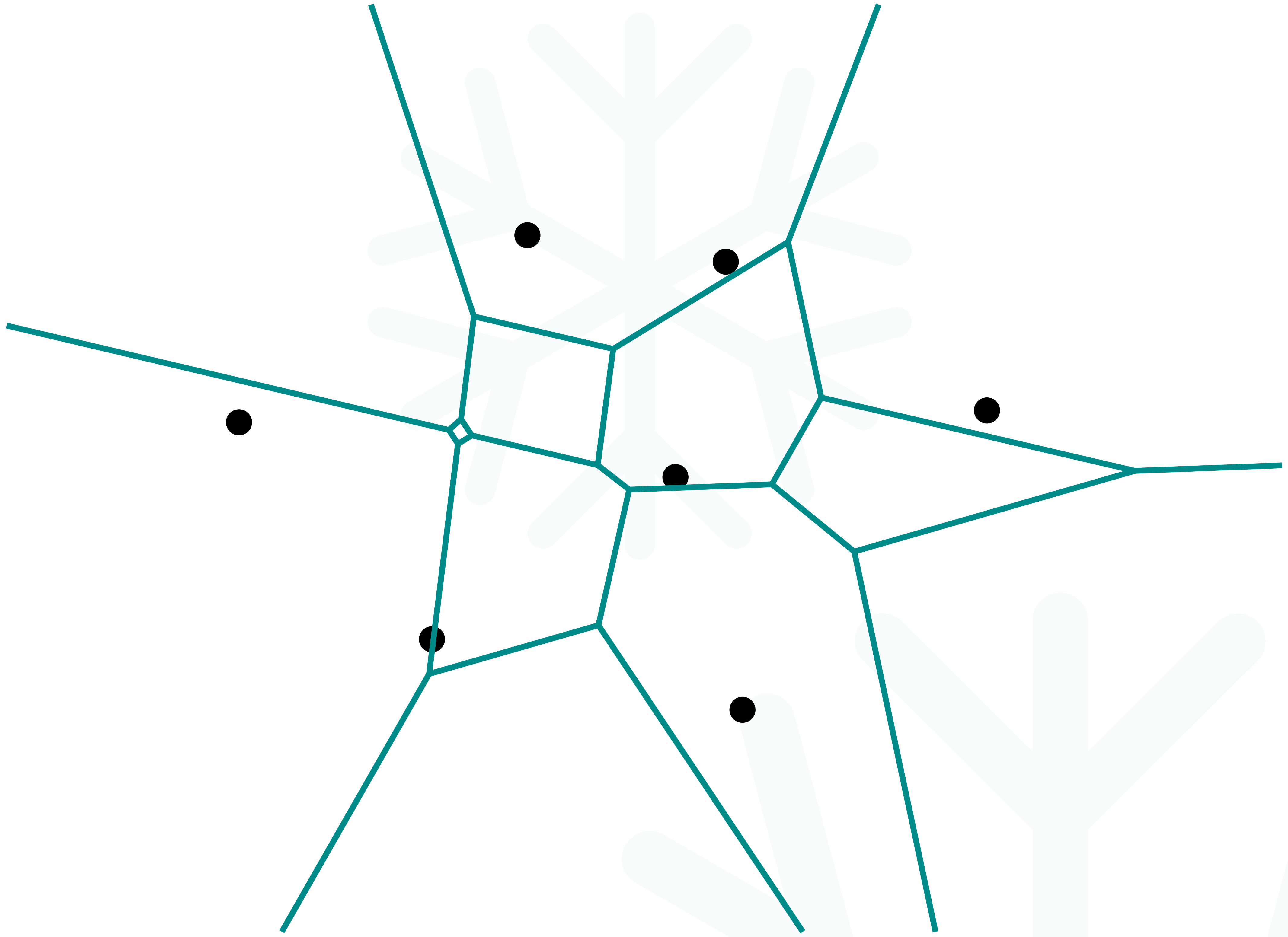
Voronoi diagrams

Circle properties

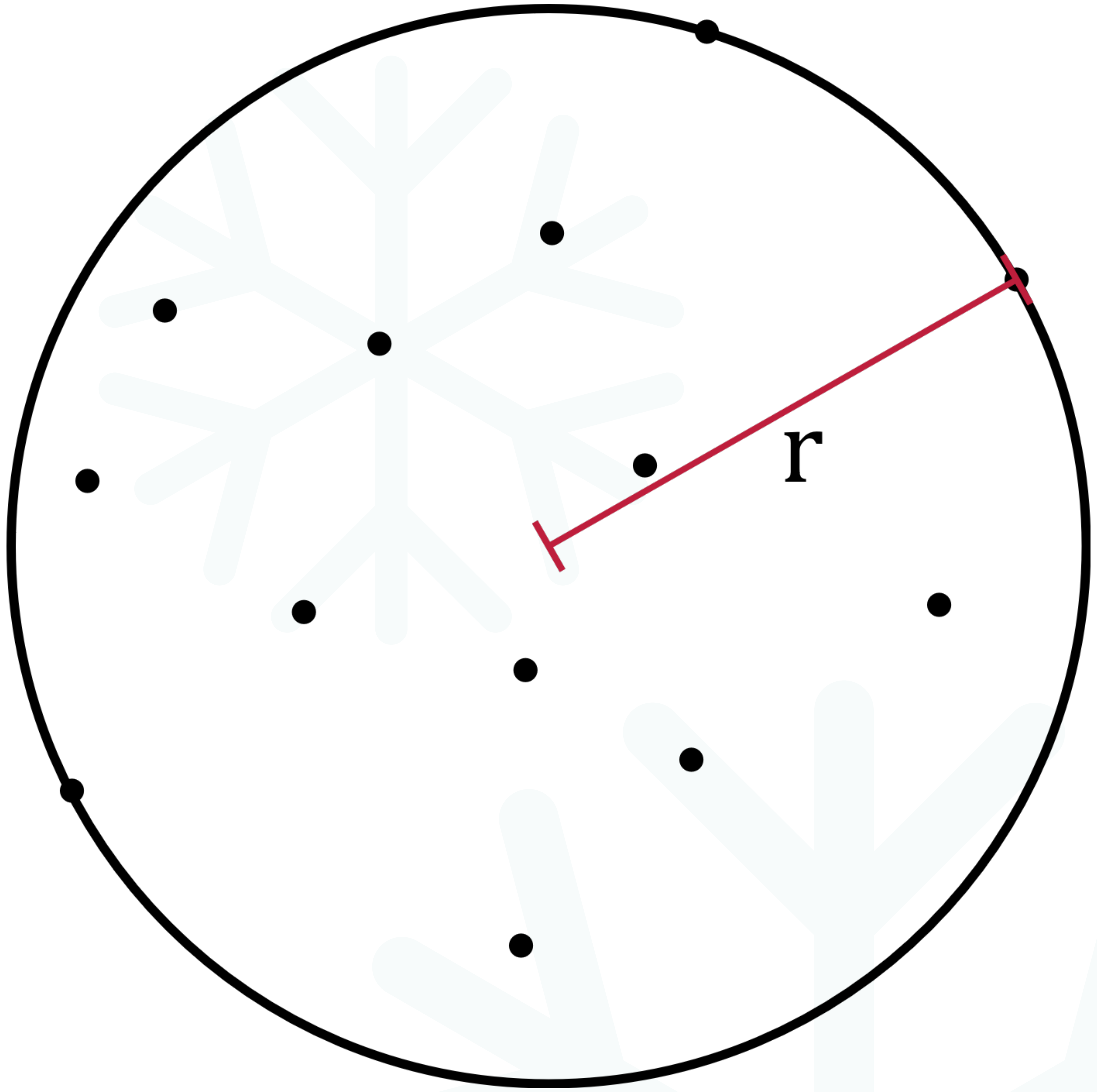
An ith order Voronoi diagram $\mathrm{Vor}(P, i)$ divides a metric space based on **which i points** of the discrete set P are closest.

For i points $M \subset P$, there is a **non-empty Voronoi region** in this diagram if there exists a **disk that encloses M but none of $P \setminus M$.**

The **Voronoi region of M is the set of all centres** of such circles: M **need not be contained in their region.**



Min enclosing disk



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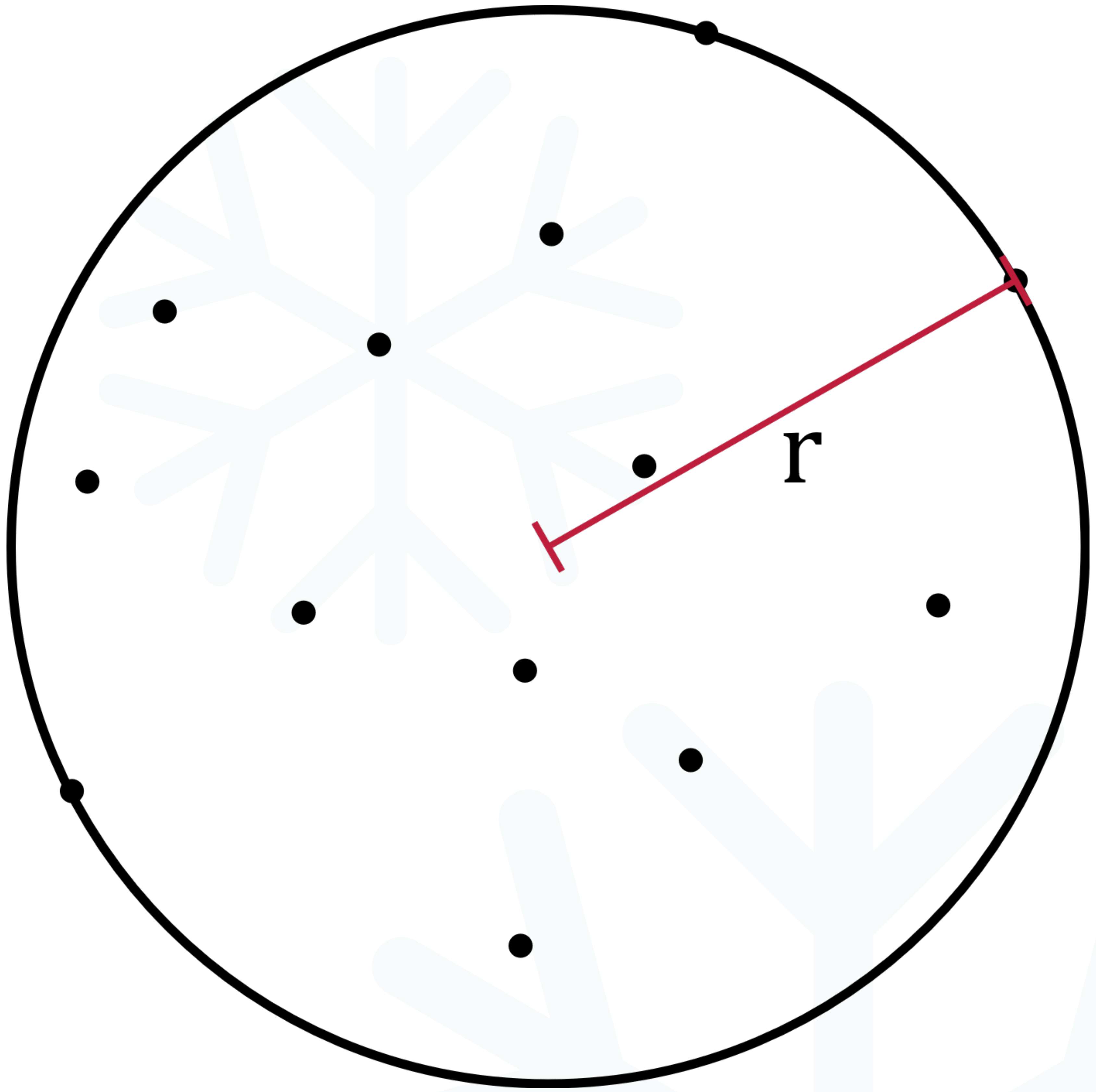
Given: Points $P := p_1, \ldots, p_n$ in the Euclidean plane, in general position (no four concyclic points).

Wanted: An enclosing disk $\mathrm{md}(P)$ of **minimal radius** r.



Can you think of a fast approximation method?
Which factor can you achieve?

Min enclosing disk

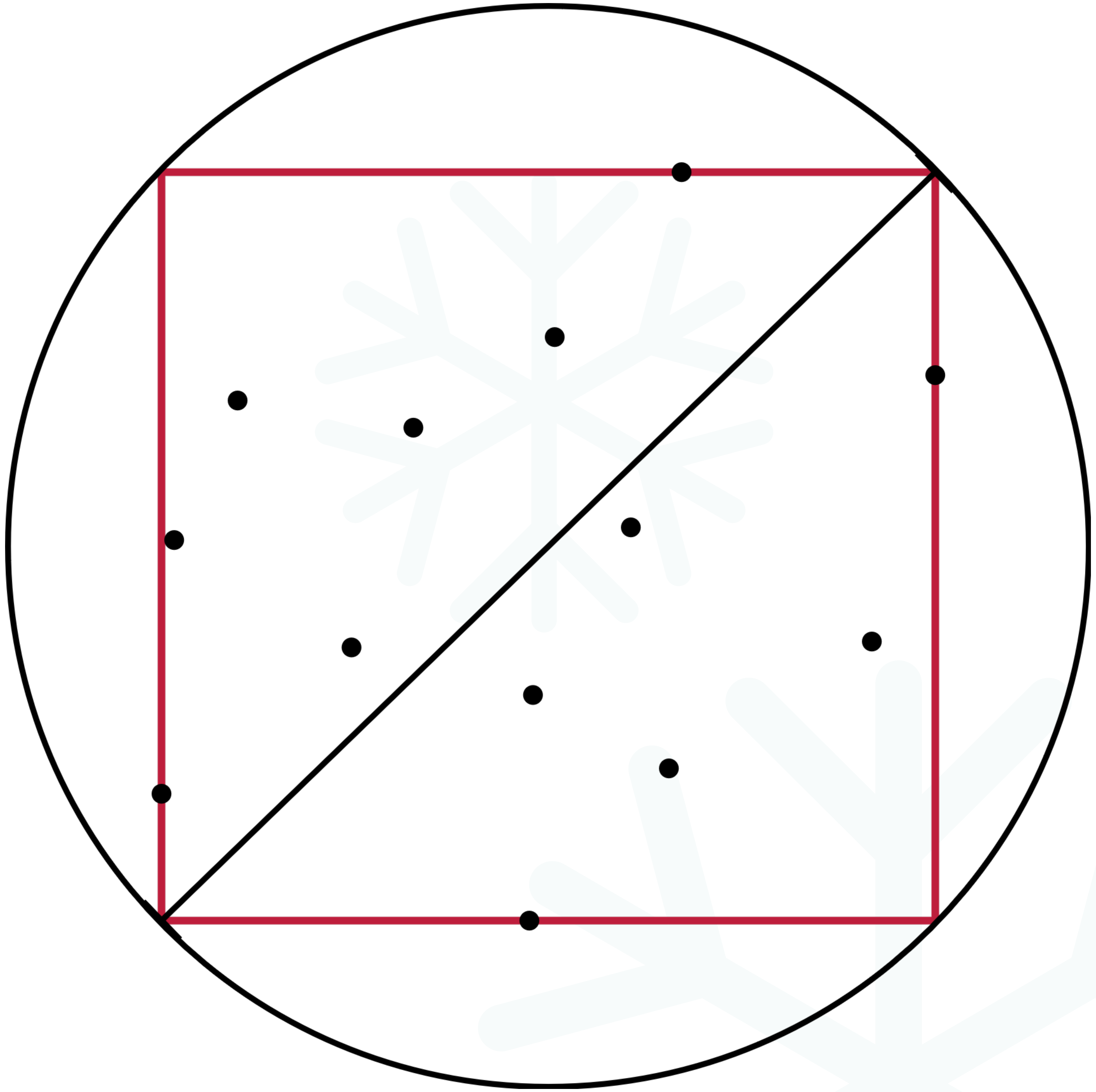
A $\sqrt{2}$-approximation

Given: Points $\mathscr{P} := p_1, \ldots, p_n$ in the plane, in general position.

Idea: Compute in $\mathcal{O}(n)$ an axis-aligned bounding box via $\min$ and $\max$ coordinates, use the smallest enclosing disk of those.

The diameter of this disk is at most $\sqrt{2}$ times larger than $\operatorname{diam}(\mathscr{P})$, which bounds the diameter of any enclosing disk from below.

Note: r is not necessarily equal to $\frac{1}{2}\operatorname{diam}(\mathscr{P})$.



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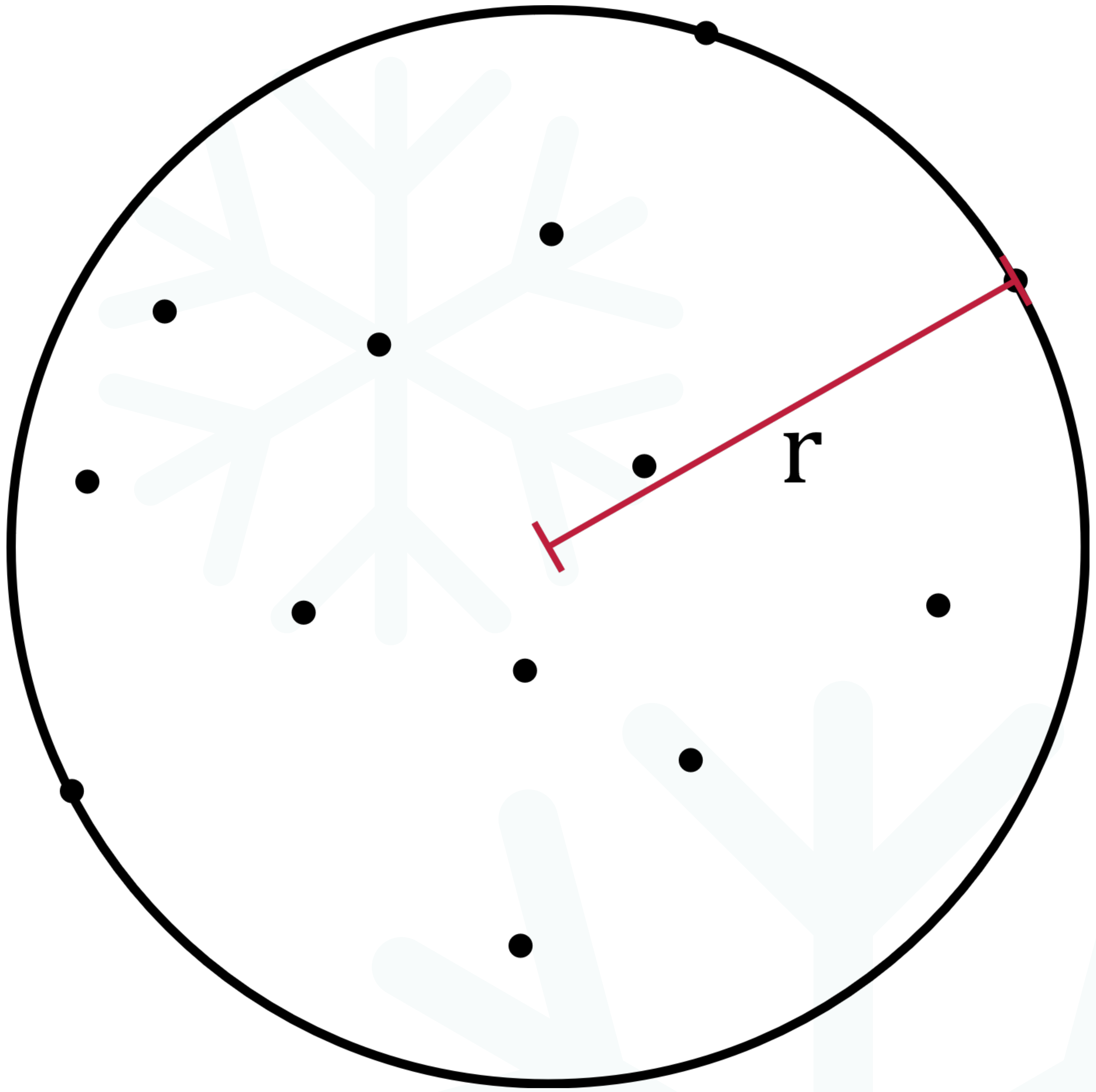
Let's try to solve it! :)



Thank you

… and see you next year :)