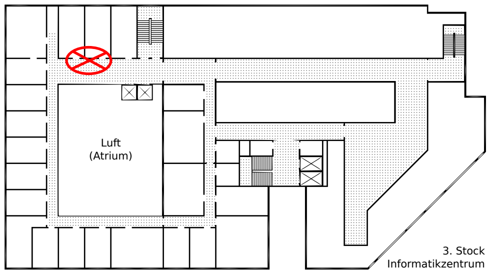
Mathematical Methods of Algorithms — Winter 2023/24

Dr. Phillip Keldenich
Dr. Ahmad Moradi

Due: 13.12.2023
Discussion: 13.12.2023

Homework 2

Solutions are to be left in the dedicated cupboard (see the pic) until 15:00 on the due date. Please put your name on all pages.



Exercise 1 (Primal-Dual computations): 1 ✓

Consider the following linear programming problem:

$$
\begin{array}{rrrrrl}
\max & +2x_1 & +8x_2 & -1x_3 & -2x_4 & \\
 & +2x_1 & +3x_2 & & +6x_4 & \leq 6 \\
 & -2x_1 & +4x_2 & +3x_3 & & \leq 1.5 \\
 & +3x_1 & +2x_2 & -2x_3 & -4x_4 & \leq 4
\end{array}
$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Suppose that, in solving this problem, you have arrived at the following dictionary:

$$
\begin{array}{rrrrrr}
\zeta = & 3.5 - & 0.25w_1 + & 6.25x_2 - & 0.5w_3 - & 1.5x_4 \\
\hline
x_1 = & 3 - & 0.5w_1 - & 1.5x_2 & - & 3x_4 \\
w_2 = & 0 + & 1.25w_1 - & 3.25x_2 - & 1.5w_3 + & 13.5x_4 \\
x_3 = & 2.5 - & 0.75w_1 - & 1.25x_2 + & 0.5w_3 - & 6.5x_4
\end{array}
$$

(a) Write down the dual problem.

(b) In the dictionary shown above, which variables are basic? Which are nonbasic?

(c) Write down the primal solution corresponding to the given dictionary. Is it feasible? Is it degenerate?

(d) Write down the corresponding dual dictionary.

(e) Write down the dual solution. Is it feasible?

(f) Do the primal/dual solutions you wrote above satisfy the complementary slackness property?

(g) Is the current primal solution optimal?

(h) For the next (primal) pivot, which variable will enter if the largest-coefficient rule is used? Which will leave? Will the pivot be degenerate?

Exercise 2 (Primal-Dual relationship): 1 ✓
Consider the following LP problem:

$$\begin{array}{lll} \max & +2x_1 & +3x_2 \\ & +2x_1 & +3x_2 \leq 30 \\ & +x_1 & +2x_2 \geq 10 \\ & +x_1 & -x_2 \leq 1 \\ & -x_1 & +x_2 \leq 1 \\ & x_1 \geq 0 & \end{array}$$

a) Write the dual problem.

b) Using the optimality conditions derived from the theory of duality, and without using the simplex method, find the optimal solution of the dual knowing that the optimal solution of the primal is $(\frac{27}{5}, \frac{32}{5})$.

Exercise 3 (Primal-Dual relationship): 1 ✓
The CEO of a large corporation needs the solution to the following linear programming problem:

$$\begin{array}{lll} \max & -2x_1 & -2x_2 - 6x_3 \\ & +x_1 & -2x_2 - 2x_3 \leq -15 \\ & -2x_1 & +x_2 - x_3 \leq -30 \\ & x_1, x_2, x_3 \geq 0 & \end{array}$$

Because the numbers represent millions of dollars, an error would be very costly. Hence, he's independently delegated the task of computing the solution to two subordinates: Alice and Brad. Alice came back with the following values for the primal and, by the way, for the associated dual values:

$$x_1^* = 25, x_2^* = 20, x_3^* = 0 \text{ and } y_1^* = y_2^* = 2$$

Brad provided the following values, which are different from Alice's:

$$x_1^* = 9, x_2^* = 0, x_3^* = 12 \text{ and } y_1^* = y_2^* = 2$$

Since the two solutions disagree, you're being asked to determine who, if anyone, is correct. So, who is? Explain.

Exercise 4 (Write dual): 1 ✓
write the dual of the following problem:

$$\begin{array}{lll} \max & \sum_{j \in J} \sum_{i \in I} r_j x_{ij} & \\ & \sum_{j \in J} x_{ij} \leq b_i & , \quad \forall i \in I \\ & \sum_{i \in I} x_{ij} \leq d_j & , \quad \forall j \in J \\ & \sum_{i \in I} p_i x_{ij} = p_j \sum_{i \in I} x_{ij} & , \quad \forall j \in J \\ & x_{ij} \geq 0 & , \quad \forall i \in I \ \forall j \in J \end{array}$$

where p is a given constant vector.

Exercise 5 (Circulation Problem): 1 ✓
Consider the linear programming problems whose right-hand sides are identically zero:.

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j & & \\
& \sum_{j=1}^{n} a_{ij} x_j & \leq 0, \qquad & i = 1, 2, \ldots, m \\
& x_j & \geq 0, \qquad & j = 1, 2, \ldots, m
\end{aligned}
$$

Show that either $x_j = 0$ for all j is optimal or else the problem is unbounded.