Linear Programming

[V. ch8]: Problems in General Form

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Very often, in practice, we have problems as follows (allowing for infinite bounds).

$$\max c^T x \text{ s.t.}$$

$$a \leq Ax \leq b$$

$$\ell \leq x \leq u$$

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Can we extend Simplex to handle such problems directly?

Example

$$
\begin{array}{lrrrrr}
\text{maximize} & & 3x_1 - & x_2 & & \\
\text{subject to} & 1 \leq & -x_1 + & x_2 \leq & 5 \\
& 2 \leq & -3x_1 + & 2x_2 \leq & 10 \\
& -\infty \leq & 2x_1 - & x_2 \leq & 0 \\
& -2 \leq & x_1 & \leq & \infty \\
& 0 \leq & & x_2 \leq & 6
\end{array}
$$

Some notes:

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- This is often the model professional LP-solvers handle; some parts of their interfaces refer to this type of model.
- We do not have a general $x \geq 0$ constraint; 0 is no longer special. Instead of a fixed lower bound of zero, we have different lower and upper bounds.

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Some notes:

- This is often the model professional LP-solvers handle; some parts of their interfaces refer to this type of model.
- We do not have a general $x \geq 0$ constraint; 0 is no longer special. Instead of a fixed lower bound of zero, we have different lower and upper bounds.
- In the general case, we will have infinities as some lower or upper bounds. We let $\infty \cdot x = \infty$ for $x > 0$, $\infty \cdot x = 0$ for $x = 0$ and $\infty \cdot x = -\infty$ for $x < 0$.

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The slack w_1 for $1 \leq -x_1 + x_2 \leq 5$ is now simply $w_1 = -x_1 + x_2$ with bounds $1 \leq w_1 \leq 5$.

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The slack w_1 for $1 \leq -x_1 + x_2 \leq 5$ is now simply $w_1 = -x_1 + x_2$ with bounds $1 \leq w_1 \leq 5$.

Result has only equality constraints and variables with upper and lower bounds.

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\begin{array}{lrrrrl}
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\text{subject to} & w_1 = & -x_1 + & x_2 & & \\
 & w_2 = & -3x_1 + & 2x_2 & & \\
 & w_3 = & 2x_1 - & x_2 & & \\
 & -2 \leq & x_1 & & \leq & \infty \\
 & 0 \leq & & x_2 & \leq & 6 \\
 & 1 \leq & w_1 & & \leq & 5 \\
 & -\infty \leq & w_2 & & \leq & 10 \\
 & 0 \leq & w_3 & & \leq & 0
\end{array}
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GENERAL DICTIONARIES

General dictionaries look different because we need more information. The overall idea is still to describe basic variables and the objective in terms of non-basic ones.
Can we still simply set non-basic variables to zero to obtain a basic (dictionary) solution?

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ℓ	u						
				-2^*		0^*	
				∞		6	
		ζ	$=$	$3x_1$	$-$	x_2	$=-6$
1	5	w_1	$=$	$-x_1$	$+$	x_2	$=2$
2	10	w_2	$=$	$-3x_1$	$+$	$2x_2$	$=6$
$-\infty$	0	w_3	$=$	$2x_1$	$-$	x_2	$=-4$

Is this dictionary feasible?

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How far can we increase x_1? $w_1 \geq 1 \Rightarrow x_1 \leq -1$, $w_2 \geq 2 \Rightarrow x_1 \leq -\frac{2}{3}$, $w_3 \leq 0 \Rightarrow x_1 \leq 0$.
When x_1 is increased to -1, w_1 hits its lower bound (becomes non-basic); w_1 is leaving variable!

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$-\infty$	0	w_3	$=$	$-2w_1$	$+$	x_2	$= -2$

Is this now optimal? No! We could increase x_2 to improve ζ! How far?
x_1: no limit, $w_2 \geq 2 \Rightarrow x_2 \leq 1$, $w_3 \leq 0 \Rightarrow x_2 \leq 2$; w_2 is leaving variable!

Another General Pivot

We say that w_2 becomes *non-basic at its lower bound*.
Result of pivoting out w_2 in favor of x_2:

ℓ	u						
				1^*		2^*	
				5		10	
		ζ	$=$	$3w_1$	$-$	$2w_2$	$=-1$
-2	∞	x_1	$=$	$2w_1$	$-$	w_2	$=0$
0	6	x_2	$=$	$3w_1$	$-$	w_2	$=1$
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No! 0 is not special anymore; degeneracy now means a basic variable is at one of its bounds.

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ℓ	u						
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				5		10	
		ζ	$=$	$3w_1$	$-$	$2w_2$	$= -1$
-2	∞	x_1	$=$	$2w_1$	$-$	w_2	$= 0$
0	6	x_2	$=$	$3w_1$	$-$	w_2	$= 1$
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Is this now optimal? No! We can increase w_1 from its lower bound! How far?

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ℓ							
	u			1^* / 5		2^* / 10	
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Is this now optimal? No! We can increase w_1 from its lower bound! How far?
The increase is limited to at most 1 unit due to w_3 hitting its upper bound. w_3 becomes non-basic at its upper bound.

Next General Pivot

Result of pivoting out w_3 in favor of w_1:

ℓ	u						
				$-\infty$		2^*	
				0^*		10	
		ζ	$=$	$3w_3$	$+$	w_2	$= 2$
-2	∞	x_1	$=$	$2w_3$	$+$	w_2	$= 2$
0	6	x_2	$=$	$3w_3$	$+$	$2w_2$	$= 4$
1	5	w_1	$=$	w_3	$+$	w_2	$= 2$

Next General Pivot

Result of pivoting out w_3 in favor of w_1:

ℓ	u						
				$-\infty$		2^*	
				0^*		10	
		ζ	$=$	$3w_3$	$+$	w_2	$= 2$
-2	∞	x_1	$=$	$2w_3$	$+$	w_2	$= 2$
0	6	x_2	$=$	$3w_3$	$+$	$2w_2$	$= 4$
1	5	w_1	$=$	w_3	$+$	w_2	$= 2$

Is this now optimal?

Next General Pivot

Result of pivoting out w_3 in favor of w_1:

ℓ	u						
				$-\infty$		2^*	
				0^*		10	
		ζ	$=$	$3w_3$	$+$	w_2	$= 2$
-2	∞	x_1	$=$	$2w_3$	$+$	w_2	$= 2$
0	6	x_2	$=$	$3w_3$	$+$	$2w_2$	$= 4$
1	5	w_1	$=$	w_3	$+$	w_2	$= 2$

Is this now optimal? No! We cannot increase w_3, but we can increase w_2 from its lower bound!

NEXT GENERAL PIVOT

Result of pivoting out w_3 in favor of w_1:

ℓ	u						
ℓ				$-\infty$		2^*	
	u			0^*		10	
		ζ	$=$	$3w_3$	$+$	w_2	$= 2$
-2	∞	x_1	$=$	$2w_3$	$+$	w_2	$= 2$
0	6	x_2	$=$	$3w_3$	$+$	$2w_2$	$= 4$
1	5	w_1	$=$	w_3	$+$	w_2	$= 2$

Is this now optimal? No! We cannot increase w_3, but we can increase w_2 from its lower bound!
Result of pivoting out x_2 in favor of w_2:

ℓ				$-\infty$		0	
	u			0^*		6^*	
		ζ	$=$	$1.5w_3$	$+$	$0.5x_2$	$= 3$
-2	∞	x_1	$=$	$0.5w_3$	$+$	$0.5x_2$	$= 3$
2	10	w_2	$=$	$-1.5w_3$	$+$	$0.5x_2$	$= 3$
1	5	w_1	$=$	$-0.5w_3$	$+$	$0.5x_2$	$= 3$

NEXT GENERAL PIVOT

Result of pivoting out w_3 in favor of w_1:

ℓ				$-\infty$		2^*	
	u			0^*		10	
		ζ	$=$	$3w_3$	$+$	w_2	$=2$
-2	∞	x_1	$=$	$2w_3$	$+$	w_2	$=2$
0	6	x_2	$=$	$3w_3$	$+$	$2w_2$	$=4$
1	5	w_1	$=$	w_3	$+$	w_2	$=2$

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Result of pivoting out x_2 in favor of w_2:

ℓ				$-\infty$		0	
	u			0^*		6^*	
		ζ	$=$	$1.5w_3$	$+$	$0.5x_2$	$=3$
-2	∞	x_1	$=$	$0.5w_3$	$+$	$0.5x_2$	$=3$
2	10	w_2	$=$	$-1.5w_3$	$+$	$0.5x_2$	$=3$
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Next General Pivot

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ℓ	u						
				$-\infty$ / 0^*		2^* / 10	
		ζ	$=$	$3w_3$	$+$	w_2	$= 2$
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Is this now optimal? Yes! Objective coefficients positive, both variables at their upper bound!

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As stated before, from an interface standpoint, most professional solvers implement this type of interface, where any linear expression can be given a lower and upper bound simultaneously without needing two matrix rows.
If one can query which variables are basic, one will notice that basic variables need not be 0, but can be at one of their bounds.

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Furthermore, a basis usually consists of a mixture of variables and constraints (we now have a more direct correspondence between constraints and their “slack” variables).

What about Phase I/Dual Simplex?

We will present both at the same time (with a modified objective, dual feasibility is easy to obtain and we can use dual Simplex to find a feasible solution).
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W.l.o.g. also complementary: $v_i q_i = 0, s_j h_j = 0$!
Note: Very similar to making a free variable from two non-negative ones, but with different objective coefficients!

Preliminaries for Dual Simplex

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Our Dual Simplex for general problems will solve this type of problem.

GENERAL DUAL SIMPLEX EXAMPLE

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\begin{array}{lrrrrl}
\text{maximize} & & 2x_1 - & x_2 & & \\
\text{subject to} & 0 \leq & x_1 + & x_2 \leq & 6 \\
& 2 \leq & -x_1 + & 2x_2 \leq & 10 \\
& -\infty \leq & x_1 - & x_2 \leq & 0 \\
& -2 \leq & x_1 & \leq & \infty \\
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Note: Infinities in the objective! We use our conventions. $-\infty$ indicates infeasibility!

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Also, we cannot use row operations on the objective. But we can use them on the constraints!

$$-\xi = -6y_1^+ - 10y_2^+ - 2z_1^+ + z_2^+ + 2y_2^- - \infty y_3^- - \infty z_1^- - 5z_2^-$$

We have the following dictionary (with no objective):

$$\begin{aligned} z_1 &= -2 + y_1 - y_2 + y_3 \\ z_2 &= 1 + y_1 + 2y_2 - y_3 \end{aligned}$$

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If we change the primal objective to $\eta = -2x_1 - x_2$, this will not happen! We then start with $z_1 = 2, z_2 = 1$, which is feasible.

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At the solution $z_1 = 2, z_2 = 1$, we have $-\xi = -6y_1^+ - 10y_2^+ - 2z_1 + z_2 + 2y_2^- - \infty y_3^-$. We can take left and right partial derivatives of $-\xi$ to look for improvements; note that z_1, z_2 are functions of y_1, y_2, y_3 here!

INITIAL PRIMAL DICTIONARY

$$\begin{aligned} z_1 &= 2 + y_1 - y_2 + y_3 \\ z_2 &= 1 + y_1 + 2y_2 - y_3 \end{aligned}$$

How does our initial primal dictionary look?

- Original problem gives matrix and bounds.
- How do we know which non-basic variable is at its upper, and which at its lower bounds?

The last question is the only difficult part, but complementarity helps here, as well.
$z_1 > 0 \Rightarrow z_1^+ > 0 \Rightarrow h_1 > 0 \Rightarrow g_1 = 0 \Rightarrow x_1 = \ell_1$ (at lower bound),
$z_2 > 0 \Rightarrow z_2^+ > 0 \Rightarrow h_2 > 0 \Rightarrow g_2 = 0 \Rightarrow x_2 = \ell_2$ (at lower bound).

ℓ	u						
				-2^*		1^*	
				∞		5	
		η	$=$	$-2x_1$	$-$	x_2	$= 3$
1	5	w_1	$=$	x_1	$+$	x_2	$= -1$
2	10	w_2	$=$	$-x_1$	$+$	$2x_2$	$= 4$
$-\infty$	0	w_3	$=$	x_1	$-$	x_2	$= 3$

Dual Pivot

$$-\xi = -6y_1^+ - 10y_2^+ - 2z_1 + z_2 + 2y_2^- - \infty y_3^-$$

$$\begin{aligned} z_1 &= 2 + y_1 - y_2 + y_3 \\ z_2 &= 1 + y_1 + 2y_2 - y_3 \end{aligned}$$

Derivative for increasing y_1:
Derivative for decreasing y_1:
Derivative for increasing y_2:
Derivative for decreasing y_2:
Derivative for increasing y_3:
Derivative for decreasing y_3:

Dual Pivot

$$-\xi = -6y_1^+ - 10y_2^+ - 2z_1 + z_2 + 2y_2^- - \infty y_3^-$$

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Derivative for increasing y_1: $-6 - 2 \cdot 1 + 1 \cdot 1 = -7 < 0$.
Derivative for decreasing y_1:
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Derivative for decreasing y_2:
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DUAL PIVOT

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Derivative for decreasing y_2: $2 - (-2 \cdot (-1) + 1 \cdot 2) = -2 < 0$
Derivative for increasing y_3:
Derivative for decreasing y_3:

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Which variable hits 0 first?

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Which variable hits 0 first? Only z_2 moves towards 0, and hits 0 for $y_1 = -1$.

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Which variable hits 0 first? Only z_2 moves towards 0, and hits 0 for $y_1 = -1$.

$$\begin{aligned} z_1 &= 1 + z_2 - 3y_2 + 2y_3 \\ y_1 &= -1 + z_2 - 2y_2 - y_3 \end{aligned}$$

DUAL PIVOT

$$-\xi = -6y_1^+ - 10y_2^+ - 2z_1 + z_2 + 2y_2^- - \infty y_3^-$$

$$\begin{aligned} z_1 &= \quad 2 + \quad y_1 - \quad y_2 + \quad y_3 \\ z_2 &= \quad 1 + \quad y_1 + \quad 2y_2 - \quad y_3 \end{aligned}$$

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Which variable hits 0 first? Only z_2 moves towards 0, and hits 0 for $y_1 = -1$.

$$\begin{aligned} z_1 &= \quad 1 + \quad z_2 - \quad 3y_2 + \quad 2y_3 \\ y_1 &= \quad -1 + \quad z_2 - \quad 2y_2 - \quad y_3 \end{aligned}$$

Analyzing derivatives shows that this is actually optimal. The primal dictionary is updated as follows: w_1 leaves, x_2 enters. $y_1^- > 0 \Rightarrow q_1 > 0 \Rightarrow p_1 = 0 \Rightarrow w_1 = a_1$.