

# LINEAR PROGRAMMING

[V. CH3]: DEGENERACY

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## DEFINITION/EXAMPLES OF DEGENERACY

PERTURBATION METHOD

FUNDAMENTAL THEOREM

GEOMETRY

OTHER PIVOT RULES

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As an example

$$\begin{array}{rcll}
 \zeta = & 5 + & x_3 - & 1x_1 \\
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 x_2 = & 5 + & 2x_3 - & 3x_1 \\
 x_4 = & 7 & & - 4x_1 \\
 x_5 = & \mathbf{0} & + & x_1
 \end{array}$$

It is clear that the problem is unbounded and therefore no more pivots were required.

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↪ A degenerate dictionary could cause difficulties for the simplex method. Problems arise, when a degenerate dictionary produces *degenerate pivots*.

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$$\begin{array}{rccccccc} \zeta = & 3 & - & 0.5x_1 & + & 2 & x_2 & - & 1.5w_1 \\ \hline x_3 = & 1 & - & 0.5x_1 & & & & - & 0.5w_1 \\ w_2 = & 0 & + & & x_1 & - & 1.0x_2 & + & w_1 \end{array}$$

For this dictionary, the entering variable is  $x_2$  and the ratio computed to determine the leaving variable is

$$\left\{ i \in \mathcal{B} : \frac{\bar{b}_i}{\bar{a}_{i2}} \text{ with } \bar{a}_{i2} > 0 \right\} = \left\{ \frac{0}{1.0} \right\}$$

Hence, the leaving variable is  $w_2$ .



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Hence, the leaving variable is  $w_2$ .

$\rightsquigarrow$  The fact that the ratio is zero means that *as soon as  $x_2$  is increased from zero to a positive value,  $w_2$  will go negative.*

Nonetheless, it can be *reclassified* from *nonbasic* to *basic* (with  $w_2$  going the other way). Look at the result of this degenerate pivot:

$$\begin{array}{rclclcl}
 \zeta = & 3 + & 1.5 x_1 - & 2w_2 + & 0.5 w_1 \\
 \hline
 x_3 = & 1 - & 0.5x_1 & - & 0.5w_1 \\
 x_2 = & & x_1 - & w_2 + & w_1
 \end{array}$$

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Note that

→  $\bar{\zeta}$  remains unchanged at 3. Hence, this degenerate pivot has not produced any increase in the objective function value.

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- $\bar{\zeta}$  remains unchanged at 3. Hence, this degenerate pivot has not produced any increase in the objective function value.
- Furthermore, the values of the variables have not even changed: both before and after this degenerate pivot, they are

$$(x_1, x_2, x_3, w_1, w_2) = (0, 0, 1, 0, 0)$$

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- Furthermore, the values of the variables have not even changed: both before and after this degenerate pivot, they are

$$(x_1, x_2, x_3, w_1, w_2) = (0, 0, 1, 0, 0)$$

But we are just representing the solution in a new way.

The entering variable for the next iteration is  $x_1$  and the leaving variable is  $x_3$ , producing a *nondegenerate* pivot that leads to

$$\begin{array}{rcl} \zeta = & 6 - & 3x_3 - & 2w_2 - & w_1 \\ \hline x_1 = & 2 - & 2x_3 & & - & w_1 \\ x_2 = & 2 - & 2x_3 - & & & w_2 \end{array}$$

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While it is typical for some pivot to “break away” from the degeneracy, the real danger is that

*the simplex method will make a sequence of degenerate pivots and eventually return to a dictionary that has appeared before, in which case the simplex method enters an infinite loop and never finds an optimal solution.*

This behavior is called *cycling*.

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While it is typical for some pivot to “break away” from the degeneracy, the real danger is that

*the simplex method will make a sequence of degenerate pivots and eventually return to a dictionary that has appeared before, in which case the simplex method enters an infinite loop and never finds an optimal solution.*

This behavior is called *cycling*.

↪ Cycling is rare! A program that generates random  $2 \times 4$  fully degenerate problems was run more than one billion times and did not find one example!

Unfortunately, under certain pivoting rules, cycling is possible.



In fact, cycling is possible even when using one of the most popular pivoting rules:

- Entering variable: Choose the entering variable as the one with the largest coefficient in the  $\zeta$ -row of the dictionary.
- Leaving variable: When two or more variables compete for leaving the basis, reading left to right, pick the first leaving-variable candidate from the list:

$$x_1, x_2, \dots, x_n, w_1, w_2, \dots, w_m$$

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$$x_1, x_2, \dots, x_n, w_1, w_2, \dots, w_m$$

Here is an example that cycles:

$$\begin{array}{rcccccc}
 \zeta = & + & x_1 & - & 2x_2 & & - & 2x_4 \\
 \hline
 w_1 = & - & 0.5x_1 & + & 3.5x_2 & + & 2x_3 & - & 4x_4 \\
 w_2 = & - & 0.5x_1 & + & x_2 & + & 0.5x_3 & - & 0.5x_4 \\
 w_3 = & 1 & - & x_1 & & & & & 
 \end{array}$$

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 \zeta = & + & 1 & x_1 - & 2x_2 & & - & 2x_4 \\
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 w_1 = & - & 0.5x_1 + & 3.5x_2 + & 2x_3 - & 4x_4 \\
 w_2 = & - & 0.5x_1 + & x_2 + & 0.5x_3 - & 0.5x_4 \\
 w_3 = & 1 - & x_1 & & & & & 
 \end{array}$$

For the first pivot,  $x_1$  enters and  $w_1$  leaves bringing us to:

$$\begin{array}{rcccccc}
 \zeta = & + & 1 x_1 - & 2x_2 & - & 2x_4 \\
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 w_1 = & - & 0.5x_1 + & 3.5x_2 + & 2x_3 - & 4x_4 \\
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 \end{array}$$

For the first pivot,  $x_1$  enters and  $w_1$  leaves bringing us to:

$$\begin{array}{rcccccc}
 \zeta = & - & 2w_1 + & 5 x_2 + & 4 x_3 - & 10x_4 \\
 \hline
 x_1 = & - & 2w_1 + & 7x_2 + & 4x_3 - & 8x_4 \\
 w_2 = & + & w_1 - & 2.5x_2 - & 1.5x_3 + & 3.5x_4 \\
 w_3 = & 1 + & 2w_1 - & 7x_2 - & 4x_3 + & 8x_4
 \end{array}$$

For the second iteration,  $x_2$  enters and  $w_2$  leaves bringing us to:

$$\begin{array}{rcccccc}
 \zeta = & + & 1x_1 & - & 2x_2 & & - & 2x_4 \\
 \hline
 w_1 = & - & 0.5x_1 & + & 3.5x_2 & + & 2x_3 & - & 4x_4 \\
 w_2 = & - & 0.5x_1 & + & x_2 & + & 0.5x_3 & - & 0.5x_4 \\
 w_3 = & 1 & - & x_1 & & & & & 
 \end{array}$$

For the first pivot,  $x_1$  enters and  $w_1$  leaves bringing us to:

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 x_1 = & - & 2w_1 & + & 7x_2 & + & 4x_3 & - & 8x_4 \\
 w_2 = & + & w_1 & - & 2.5x_2 & - & 1.5x_3 & + & 3.5x_4 \\
 w_3 = & 1 & + & 2w_1 & - & 7x_2 & - & 4x_3 & + & 8x_4
 \end{array}$$

For the second iteration,  $x_2$  enters and  $w_2$  leaves bringing us to:

$$\begin{array}{rcccccc}
 \zeta = & & - & 2w_2 & + & 1x_3 & - & 3x_4 \\
 \hline
 x_1 = & + & 0.8w_1 & - & 2.8w_2 & - & 0.2x_3 & + & 1.8x_4 \\
 x_2 = & + & 0.4w_1 & - & 0.4w_2 & - & 0.6x_3 & + & 1.4x_4 \\
 w_3 = & 1 & - & 0.8w_1 & + & 2.8w_2 & + & 0.2x_3 & - & 1.8x_4
 \end{array}$$

For the third iteration,  $x_3$  enters and  $x_1$  leaves:

$$\begin{array}{rcccccc}
 \zeta = & + & 4 & w_1 - & 16w_2 - & 5x_1 + & 6 & x_4 \\
 \hline
 x_3 = & + & & 4w_1 - & 14w_2 - & 5x_1 + & & 9x_4 \\
 x_2 = & - & & 2w_1 + & 8w_2 + & 3x_1 - & & 4x_4 \\
 w_3 = & 1 & & & & - & & x_1
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For the fourth iteration,  $x_4$  enters and  $x_2$  leaves:

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 \end{array}$$

For the fourth iteration,  $x_4$  enters and  $x_2$  leaves:

$$\begin{array}{r}
 \zeta = \quad + \quad 1 w_1 - \quad 4w_2 - \quad 0.5x_1 - \quad 1.5x_2 \\
 \hline
 x_3 = \quad - \quad 0.5w_1 + \quad 4w_2 + \quad 1.75x_1 - \quad 2.25x_2 \\
 x_4 = \quad - \quad 0.5w_1 + \quad 2w_2 + \quad 0.75x_1 - \quad 0.25x_2 \\
 w_3 = \quad 1 \qquad \qquad \qquad - \quad x_1
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In the fifth iteration,  $w_1$  enters and  $x_3$  leaves:

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 w_3 = \quad 1 \qquad \qquad \qquad - \quad x_1
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For the fourth iteration,  $x_4$  enters and  $x_2$  leaves:

$$\begin{array}{r}
 \zeta = \quad + \quad 1 w_1 - \quad 4w_2 - \quad 0.5x_1 - \quad 1.5x_2 \\
 \hline
 x_3 = \quad - \quad 0.5w_1 + \quad 4w_2 + \quad 1.75x_1 - \quad 2.25x_2 \\
 x_4 = \quad - \quad 0.5w_1 + \quad 2w_2 + \quad 0.75x_1 - \quad 0.25x_2 \\
 w_3 = \quad 1 \qquad \qquad \qquad - \quad x_1
 \end{array}$$

In the fifth iteration,  $w_1$  enters and  $x_3$  leaves:

$$\begin{array}{r}
 \zeta = \quad - \quad 2x_3 + \quad 4 w_2 + \quad 3 x_1 - \quad 6x_2 \\
 \hline
 w_1 = \quad - \quad 2x_3 + \quad 8w_2 + \quad 3.5x_1 - \quad 4.5x_2 \\
 x_4 = \qquad \quad x_3 - \quad 2w_2 - \quad x_1 + \quad 2x_2 \\
 w_3 = \quad 1 \qquad \qquad \qquad - \quad x_1
 \end{array}$$



Lastly, for the sixth iteration,  $w_2$  enters and  $x_4$  leaves:

$$\begin{array}{rcccccl}
 \zeta = & & & - & 2x_4 & + & 1x_1 & - & 2x_2 \\
 \hline
 w_1 = & + & 2x_3 & - & 4x_4 & - & 0.5x_1 & + & 3.5x_2 \\
 w_2 = & + & 0.5x_3 & - & 0.5x_4 & - & 0.5x_1 & + & x_2 \\
 w_3 = & 1 & & & & - & x_1 & & 
 \end{array}$$

Lastly, for the sixth iteration,  $w_2$  enters and  $x_4$  leaves:

$$\begin{array}{rcccccc}
 \zeta = & & & - & 2x_4 + & 1x_1 - & 2x_2 \\
 \hline
 w_1 = & + & 2x_3 - & 4x_4 - & 0.5x_1 + & 3.5x_2 \\
 w_2 = & + & 0.5x_3 - & 0.5x_4 - & 0.5x_1 + & x_2 \\
 w_3 = & 1 & & & - & x_1
 \end{array}$$

Note that we have come back to the original dictionary:

$$\begin{array}{rcccccc}
 \zeta = & + & x_1 - & 2x_2 & & - & 2x_4 \\
 \hline
 w_1 = & - & 0.5x_1 + & 3.5x_2 + & 2x_3 - & 4x_4 \\
 w_2 = & - & 0.5x_1 + & x_2 + & 0.5x_3 - & 0.5x_4 \\
 w_3 = & 1 - & x_1 & & & & 
 \end{array}$$

↪ from here on the simplex method simply cycles through these six dictionaries and never makes any further progress toward an optimal solution.

As bad as cycling is, the following theorem tells us that nothing worse can happen:

## THEOREM

*If the simplex method fails to terminate, then it must cycle.*

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### PROOF.

A dictionary is completely determined by specifying which variables are basic and which are nonbasic. There are only

$$\binom{n+m}{m} = \frac{(n+m)!}{n!m!}$$

different possibilities. This number is big, but it is finite. If the simplex method fails to terminate, it must visit some of these dictionaries more than once. Hence, the algorithm cycles.  $\square$

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Whenever a vanishing "rhs", (zero  $\bar{b}_i$ ), appears *perturb* it.

- set :  $\bar{b}_i = \bar{b}_i + \epsilon_i$

If there are lots of them, say  $k$ , perturb them all. Make the perturbations at different *scales*:

$$0 < \epsilon_k \ll \dots \ll \epsilon_2 \ll \epsilon_1 \ll \text{other nonzero data}$$

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$$0 < \epsilon_k \ll \dots \ll \epsilon_2 \ll \epsilon_1 \ll \text{other nonzero data}$$

Try to keep in mind that:

- no linear combination of the  $\epsilon_i$  's using coefficients that might arise in the course of the simplex method can ever produce a number whose size is of the same order as the data in the problem.
- each of the “lower down”  $\epsilon_i$  's can never “escalate” to a higher level.

Hence, *cancellations* can only occur on a given scale.

First example:

### EXAMPLE

$$\begin{array}{rclcl}
 \zeta & = & 0 & + & 6x_1 + 4x_2 \\
 \hline
 w_1 & = & 0 & + & 9x_1 + 4x_2 \\
 w_2 & = & 0 & - & 4x_1 - 2x_2 \\
 w_3 & = & 1 & & - x_2
 \end{array}$$



The first step is to introduce symbolic parameters

$$0 < \epsilon_3 \ll \epsilon_2 \ll \epsilon_1$$

to get a perturbed problem:

$$\begin{array}{rclclcl}
 \zeta = & 0 & & + & 6x_1 + & 4x_2 \\
 \hline
 w_1 = & 0 + \epsilon_1 & & + & 9x_1 + & 4x_2 \\
 w_2 = & 0 & + \epsilon_2 & - & 4x_1 - & 2x_2 \\
 w_3 = & 1 & & + \epsilon_3 & & - x_2
 \end{array}$$

This dictionary is *not degenerate*. The entering variable is  $x_1$  and the leaving variable is unambiguously  $w_2$ .

The next dictionary is

$$\begin{array}{rcccccc}
 \zeta = & 0 & + & 1.5\epsilon_2 & - & 1.5w_2 + & 1 x_2 \\
 \hline
 w_1 = & 0 + & \epsilon_1 + & 2.25\epsilon_2 & - & 2.25w_2 - & 0.5x_2 \\
 x_1 = & 0 & + & 0.25\epsilon_2 & - & 0.25w_2 - & 0.5x_2 \\
 w_3 = & 1 & & & + & \epsilon_3 & - & x_2
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 x_1 = & 0 & + & 0.25\epsilon_2 & - & 0.25w_2 - & 0.5x_2 \\
 w_3 = & 1 & & & + & \epsilon_3 & - & x_2
 \end{array}$$

For the next pivot, the entering variable is  $x_2$  and, using the fact that  $\epsilon_2 \ll \epsilon_1$ , we see that the leaving variable is  $x_1$ . The new dictionary is

$$\begin{array}{rcccccc}
 \zeta = & 0 & + & 2\epsilon_2 & - & 2w_2 - & 2x_1 \\
 \hline
 w_1 = & 0 + & \epsilon_1 + & 2\epsilon_2 & - & 2w_2 + & x_1 \\
 x_2 = & 0 & + & 0.5\epsilon_2 & - & 0.5w_2 - & 2x_1 \\
 w_3 = & 1 & - & 0.5\epsilon_2 + & \epsilon_3 + & 0.5w_2 + & 2x_1
 \end{array}$$

This last dictionary is optimal. At this point, we simply *drop* the symbolic  $\epsilon_i$  parameters and get an optimal dictionary for the *unperturbed* problem.

$$\begin{array}{rcll} \zeta = & 0 - & 2w_2 - & 2x_1 \\ \hline w_1 = & 0 - & 2w_2 + & x_1 \\ x_2 = & 0 - & 0.5w_2 - & 2x_1 \\ w_3 = & 1 + & 0.5w_2 + & 2x_1 \end{array}$$

When treating the  $\epsilon_i$ 's as symbols, the method is called the *lexicographic* method.

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↪ The lexicographic method does not affect the choice of entering variable but does amount to a precise prescription for the choice of leaving variable.

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$$\begin{array}{rcl} \zeta & = & 0 - 2w_2 - 2x_1 \\ w_1 & = & 0 - 2w_2 + x_1 \\ x_2 & = & 0 - 0.5w_2 - 2x_1 \\ w_3 & = & 1 + 0.5w_2 + 2x_1 \end{array}$$

When treating the  $\epsilon_i$ 's as symbols, the method is called the *lexicographic* method.

↪ The lexicographic method does not affect the choice of entering variable but does amount to a precise prescription for the choice of leaving variable.

↪ The lexicographic method produces a variant of the simplex method that never cycles:

## THEOREM

*The simplex method always terminates provided that the leaving variable is selected by the lexicographic rule.*

PROOF.

It suffices to show that no degenerate dictionary is ever produced.

As we have discussed before, the  $\epsilon_i$ 's operate on different scales and hence cannot cancel with each other. Therefore, can think of the  $\epsilon_i$ 's as a collection of independent variables.

PROOF.

It suffices to show that no degenerate dictionary is ever produced.

As we have discussed before, the  $\epsilon_i$ 's operate on different scales and hence cannot cancel with each other. Therefore, can think of the  $\epsilon_i$ 's as a collection of independent variables.

Extracting the  $\epsilon$  terms from the first dictionary, we see that we start with the following pattern:

$$\begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_m \end{array}$$

□



## PROOF CONT.

And, after several pivots, the  $\epsilon$  terms will form a system of linear combinations, say,

$$\begin{array}{cccc}
 r_{11}\epsilon_1 + & r_{12}\epsilon_2 + & \cdots & r_{1m}\epsilon_m \\
 r_{21}\epsilon_1 + & r_{22}\epsilon_2 + & \cdots & r_{2m}\epsilon_m \\
 \vdots & \vdots & \ddots & \vdots \\
 r_{m1}\epsilon_1 + & r_{m2}\epsilon_2 + & \cdots & r_{mm}\epsilon_m
 \end{array}$$

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 r_{21}\epsilon_1 + & r_{22}\epsilon_2 + & \cdots & r_{2m}\epsilon_m \\
 \vdots & \vdots & \ddots & \vdots \\
 r_{m1}\epsilon_1 + & r_{m2}\epsilon_2 + & \cdots & r_{mm}\epsilon_m
 \end{array}$$

This system of linear combinations is obtained from the original system by pivot operations and, since pivot operations are reversible, it follows that the rank of the two systems must be the same.

## PROOF CONT.

And, after several pivots, the  $\epsilon$  terms will form a system of linear combinations, say,

$$\begin{array}{cccc}
 r_{11}\epsilon_1 + & r_{12}\epsilon_2 + & \cdots & r_{1m}\epsilon_m \\
 r_{21}\epsilon_1 + & r_{22}\epsilon_2 + & \cdots & r_{2m}\epsilon_m \\
 \vdots & \vdots & \ddots & \vdots \\
 r_{m1}\epsilon_1 + & r_{m2}\epsilon_2 + & \cdots & r_{mm}\epsilon_m
 \end{array}$$

This system of linear combinations is obtained from the original system by pivot operations and, since pivot operations are reversible, it follows that the rank of the two systems must be the same.

Since the original system had rank  $m$ , we see that every subsequent system must have rank  $m$ .

→ There must be at least one nonzero  $r_{ij}$  in every row  $i$ , which of course implies that

→ None of the rows can be degenerate.

Hence, no dictionary can be degenerate. □

Second example:

### EXAMPLE

$$\begin{array}{rclcl}
 \zeta = & 0 & + & 2x_1 + & 4x_2 \\
 \hline
 w_1 = & 0 & + & x_1 - & x_2 \\
 w_2 = & 0 & + & 3x_1 - & x_2 \\
 w_3 = & 0 & - & 4x_1 + & x_2
 \end{array}$$

Perturb vanishing rhs

$$\begin{array}{rcllcl}
 \zeta = & 0.0 & & + & 2x_1 + & 4x_2 \\
 \hline
 w_1 = & 0.0 + & \epsilon_1 & & + & x_1 - & x_2 \\
 w_2 = & 0.0 & & + & \epsilon_2 & & + & 3x_1 - & x_2 \\
 w_3 = & 0.0 & & & & + & \epsilon_3 & - & 4x_1 + & x_2
 \end{array}$$

Perturb vanishing rhs

$$\begin{array}{rcllcl}
 \zeta = & 0.0 & & + & 2x_1 + & 4x_2 \\
 \hline
 w_1 = & 0.0 + & \epsilon_1 & & + & x_1 - & x_2 \\
 w_2 = & 0.0 & & + & \epsilon_2 & & + & 3x_1 - & x_2 \\
 w_3 = & 0.0 & & & & + & \epsilon_3 - & 4x_1 + & x_2
 \end{array}$$

or equivalently

$$\begin{array}{rcllclcl}
 \zeta = & 0.0 + & 0.0\epsilon_1 + & 0.0\epsilon_2 + & 0.0\epsilon_3 + & 2x_1 + & 4x_2 \\
 \hline
 w_1 = & 0.0 + & 1.0\epsilon_1 + & 0.0\epsilon_2 + & 0.0\epsilon_3 + & x_1 - & x_2 \\
 w_2 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 0.0\epsilon_3 + & 3x_1 - & x_2 \\
 w_3 = & 0.0 + & 0.0\epsilon_1 + & 0.0\epsilon_2 + & 1.0\epsilon_3 - & 4x_1 + & x_2
 \end{array}$$

$x_2$  enters and as  $\epsilon_2 \ll \epsilon_1$ ,  $w_2$  leaves.

Perturb vanishing rhs

$$\begin{array}{rcllcl}
 \zeta = & 0.0 & & + & 2x_1 + & 4x_2 \\
 \hline
 w_1 = & 0.0 + & \epsilon_1 & & + & x_1 - & x_2 \\
 w_2 = & 0.0 & & + & \epsilon_2 & & + & 3x_1 - & x_2 \\
 w_3 = & 0.0 & & & & + & \epsilon_3 - & 4x_1 + & x_2
 \end{array}$$

or equivalently

$$\begin{array}{rcllclcl}
 \zeta = & 0.0 + & 0.0\epsilon_1 + & 0.0\epsilon_2 + & 0.0\epsilon_3 + & 2x_1 + & 4x_2 \\
 \hline
 w_1 = & 0.0 + & 1.0\epsilon_1 + & 0.0\epsilon_2 + & 0.0\epsilon_3 + & x_1 - & x_2 \\
 w_2 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 0.0\epsilon_3 + & 3x_1 - & x_2 \\
 w_3 = & 0.0 + & 0.0\epsilon_1 + & 0.0\epsilon_2 + & 1.0\epsilon_3 - & 4x_1 + & x_2
 \end{array}$$

$x_2$  enters and as  $\epsilon_2 \ll \epsilon_1$ ,  $w_2$  leaves.

↪ Note that, as  $\epsilon_i$ 's have different scale, they will not cancel each other.

We get

$$\begin{array}{rcccccc}
 \zeta = & 0.0 + & 0.0\epsilon_1 + & 4.0\epsilon_2 + & 0.0\epsilon_3 + & 14 x_1 - & 4w_2 \\
 \hline
 w_1 = & 0.0 + & 1.0\epsilon_1 - & 1.0\epsilon_2 + & 0.0\epsilon_3 - & 2x_1 + & w_2 \\
 x_2 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 0.0\epsilon_3 + & 3x_1 - & w_2 \\
 w_3 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 1.0\epsilon_3 - & x_1 - & w_2
 \end{array}$$



We get

$$\begin{array}{rcccccc}
 \zeta = & 0.0 + & 0.0\epsilon_1 + & 4.0\epsilon_2 + & 0.0\epsilon_3 + & 14x_1 - & 4w_2 \\
 \hline
 w_1 = & 0.0 + & 1.0\epsilon_1 - & 1.0\epsilon_2 + & 0.0\epsilon_3 - & 2x_1 + & w_2 \\
 x_2 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 0.0\epsilon_3 + & 3x_1 - & w_2 \\
 w_3 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 1.0\epsilon_3 - & x_1 - & w_2
 \end{array}$$

$x_1$  enters and as  $\epsilon_2 + \epsilon_3 \ll \epsilon_1 - \epsilon_2$ ,  $w_3$  leaves. We get

$$\begin{array}{rcccccc}
 \zeta = & 0.0 + & 0.0\epsilon_1 + & 18\epsilon_2 + & 14\epsilon_3 - & 14w_3 - & 18w_2 \\
 \hline
 w_1 = & 0.0 + & 1.0\epsilon_1 - & 3.0\epsilon_2 - & 2.0\epsilon_3 + & 2w_3 + & 3w_2 \\
 x_2 = & 0.0 + & 0.0\epsilon_1 + & 4.0\epsilon_2 + & 3.0\epsilon_3 - & 3w_3 - & 4w_2 \\
 x_1 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 1.0\epsilon_3 - & w_3 - & w_2
 \end{array}$$

Done!

DEFINITION/EXAMPLES OF DEGENERACY

PERTURBATION METHOD

FUNDAMENTAL THEOREM

GEOMETRY

OTHER PIVOT RULES

Now that we have a Phase I algorithm and a variant of the simplex method that is guaranteed to terminate, we can summarize the main points of this chapter in the following theorem:

## THEOREM

*For an arbitrary linear program in standard form, the following statements are true:*

- *If there is no optimal solution, then the problem is either infeasible or unbounded.*
- *If a feasible solution exists, then a basic feasible solution exists.*
- *If an optimal solution exists, then a basic optimal solution exists.*

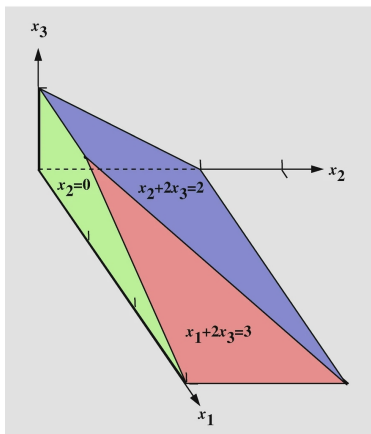
DEFINITION/EXAMPLES OF DEGENERACY

PERTURBATION METHOD

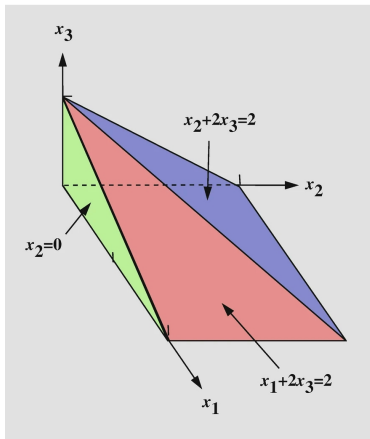
FUNDAMENTAL THEOREM

GEOMETRY

OTHER PIVOT RULES



$$\begin{array}{rcl}
 \max_x & x_1 + & 2x_2 + & 3x_3 \\
 \text{s.t.} & x_1 & + & 2x_3 \leq 3 \\
 & & x_2 + & 2x_3 \leq 2 \\
 & x_1, & x_2, & x_3 \geq 0
 \end{array}$$



$$\begin{array}{rcl}
 \max_x & x_1 + & 2x_2 + & 3x_3 \\
 \text{s.t.} & x_1 & + & 2x_3 \leq 2 \\
 & & x_2 + & 2x_3 \leq 2 \\
 & x_1, & x_2, & x_3 \geq 0
 \end{array}$$

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The second pivoting rule we consider is called **Bland's rule** or **smallest index rule**.

It stipulates that:

*Both the entering and the leaving variable be selected from their respective sets of choices by choosing the variable  $x_k$  with the smallest index  $k$ .*

## THEOREM

*The simplex method always terminates provided that both the entering and the leaving variable are chosen according to Bland's rule.*



Other rule called **random selection rule**.

*Select at random from the set of possibilities.*

Other rule called **greatest increase rule**.

*Pick the entering/leaving pair so as to maximize the increase of the objective function over all other possibilities.*

**Too much computations.**





