LINEAR PROGRAMMING

[V. CH10]: APPLICATION: TSP

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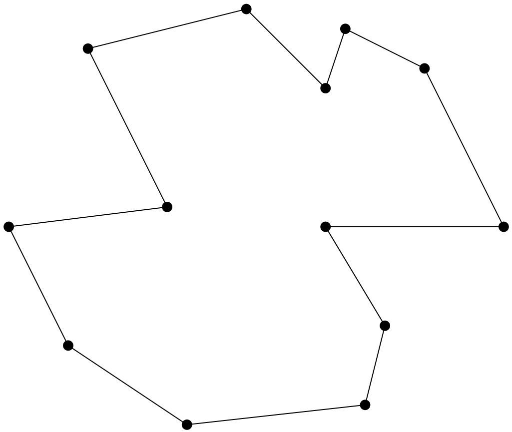
DEFINITION & MODEL

CUTTING PLANES

BRANCH, CUT & PRICE

Traveling Salesman Problem

For a given set V of n cities, (sometimes also called vertices) with given costs $c_{ab} = c_{ba}$ for going from any city a to any city b, compute the shortest round trip through all cities, visiting each city exactly once.



Integer Programming Model

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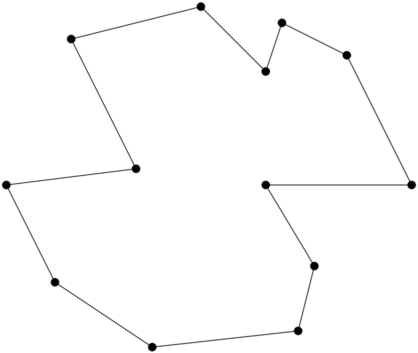

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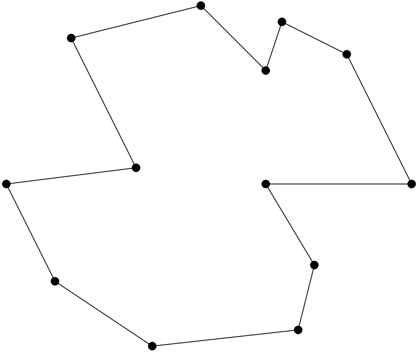
- $\min \sum c_e x_e$



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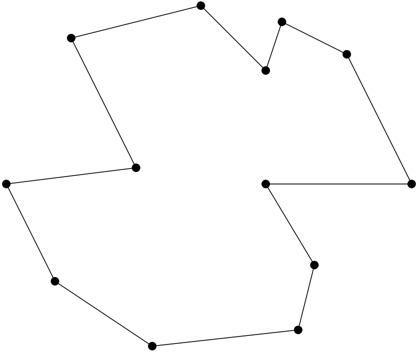

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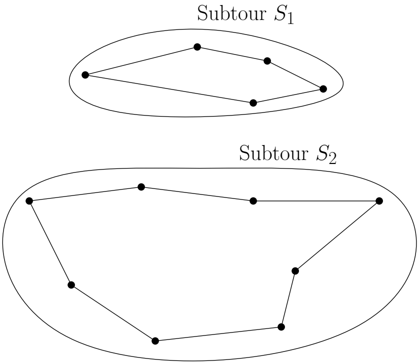

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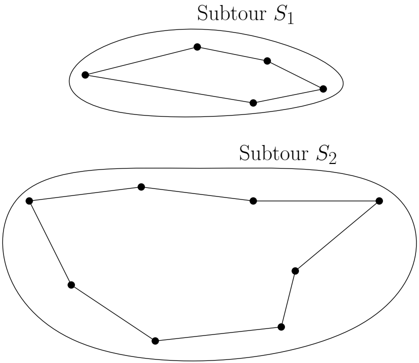
- One constraint per city $v \in V$: $\sum_{e \in \delta(\{v\})} x_e = 2$.
- *Subtour elimination* constraints: $\forall S \subsetneq V, S \neq \emptyset : \sum_{e \in \delta(S)} x_e \geq 2$.



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- This is very similar to cutting plane generation (which is part of the algorithm anyways).
- How hard is it to *separate* subtours?

Separating Subtour Constraints

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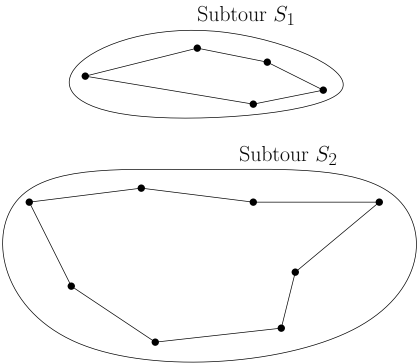
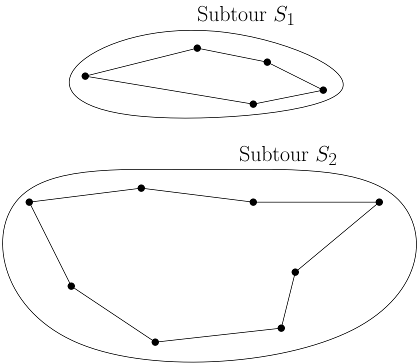


SEPARATING SUBTOUR CONSTRAINTS

Integral: Easy (BFS/DFS): $\sum_{e\in\delta(S_1)} x_e = 0 < 2 \rightarrow$ add (violated) constraint $\sum_{e\in\delta(S_1)} x_e \geq 2$



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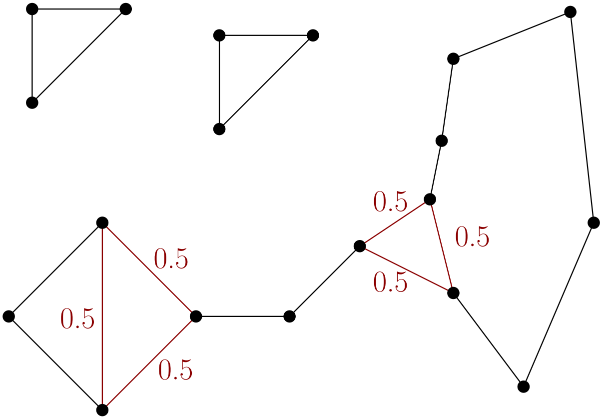

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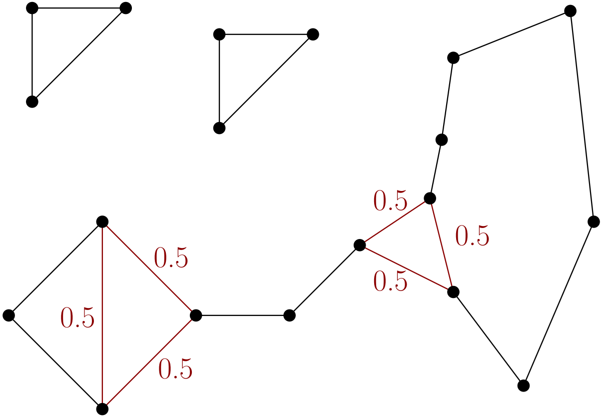

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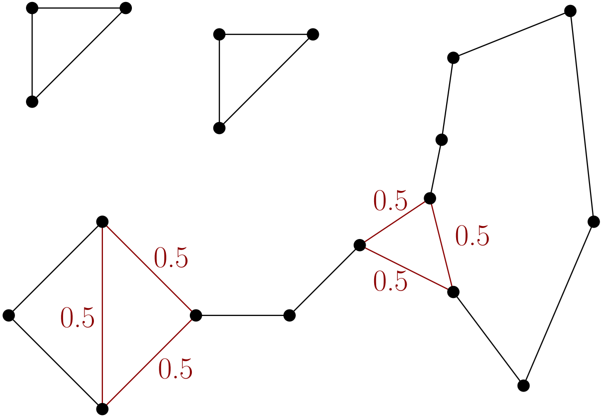

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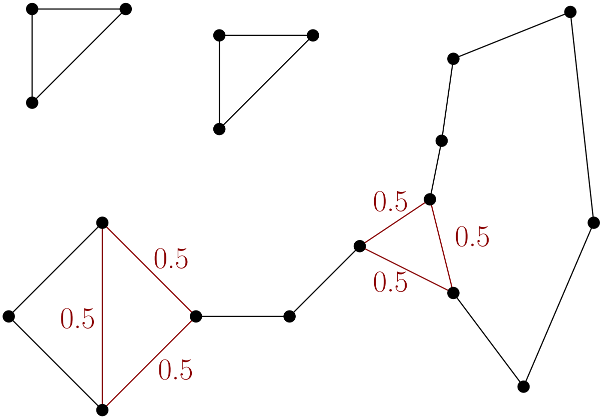

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Exact separation? Minimum graph cut (Stoer-Wagner algorithm).

Minimum Graph Cut

Given a graph $G = (V, E)$ with weighted undirected edges $w(e) \geq 0$, find a partition

$V = S \cup T, S \cap T = \emptyset, S, T \neq \emptyset$, which minimizes

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- If the minimum cut is strictly below 2, S and T are vertex sets of violated subtour constraints.
- The high running time may not be worth it — usually, at the very least, one should run the cheaper methods first.

Example Time

Interactive example at https://www.math.uwaterloo.ca/tsp/app/diy.html.

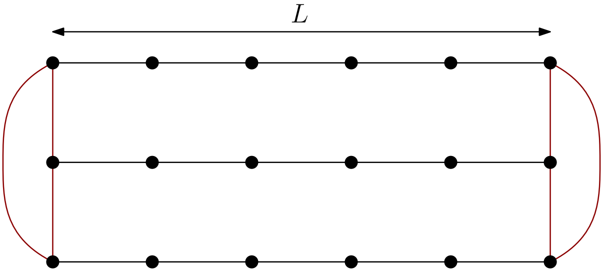
Definition & Model

Cutting Planes

Branch, Cut & Price

Cutting Planes for Tours

Besides subtours, what else are good cutting planes for the TSP?



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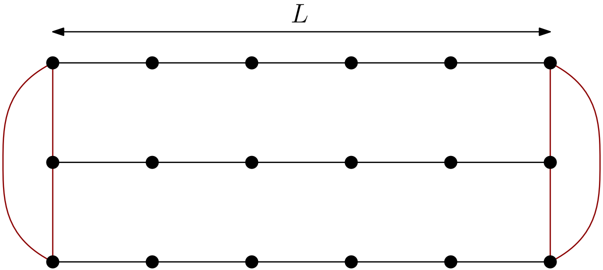

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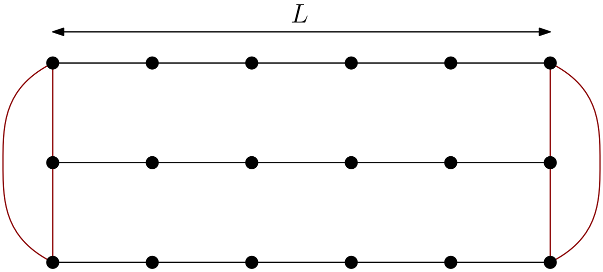

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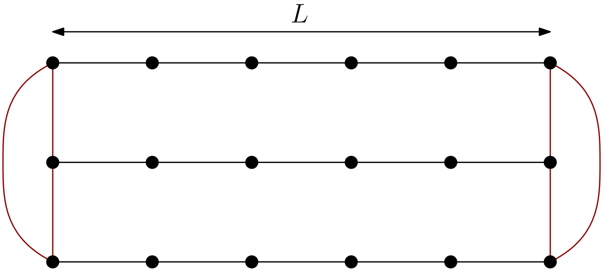

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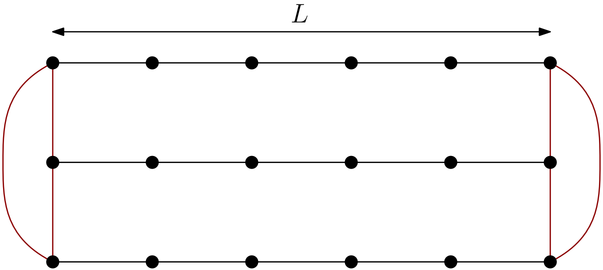

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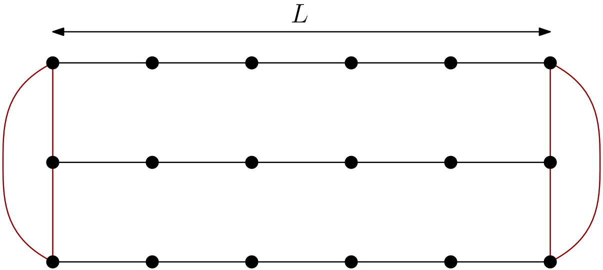

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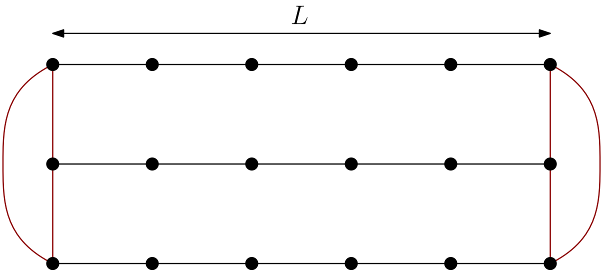
- The *(asymptotic) integrality gap* of this TSP formulation (with all subtours) is at least 4/3.
- 4/3-conjecture: This is actually the integrality gap, i.e., there are no worse instances than this.

Comb Inequalities

How do we find cuts for this solution?



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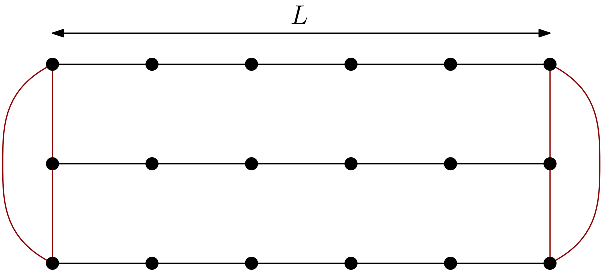

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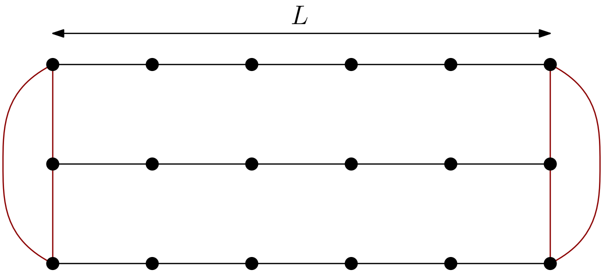

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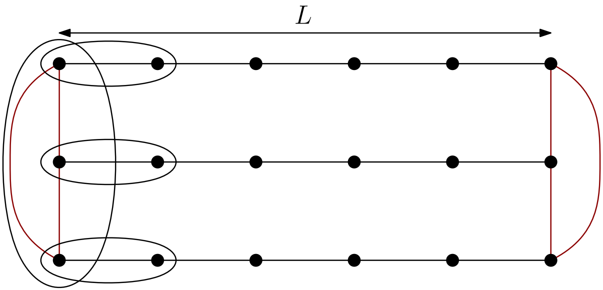

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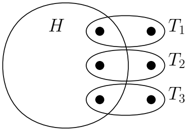


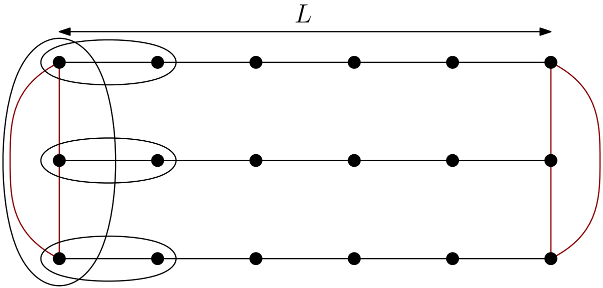
Suppose we have $H, T_1, \ldots, T_k \subset V$:

- $\forall i \in \{1, \ldots, k\} : H \cap T_i \neq \emptyset$ (handle meets each tooth),
- $\forall i \in \{1, \ldots, k\} : T_i \setminus H \neq \emptyset$ (teeth have vertex outside handle),
- $\forall i \neq j \in \{1, \ldots, k\} : T_i \cap T_j = \emptyset$ (teeth are disjoint),
- k is odd,

then every valid tour has

$$\sum_{e \in \delta(H)} x_e + \sum_{i=1}^{k} \sum_{e \in \delta(T_i)} x_e \geq 3k + 1.$$

Comb Inequalities: Our Example



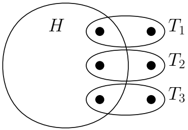
Here, we have:

$$\sum_{e\in\delta(H)} x_e = 3, \sum_{e\in\delta(T_i)} x_e = 2,$$

$$\sum_{e\in\delta(H)} x_e + \sum_{i=1}^{k} \sum_{e\in\delta(T_i)} x_e = 3 + 3\cdot 2 = 9 < 10 = 3k+1,$$

thus this comb inequality is a violated cutting plane!

Comb Inequalities: Correctness Proof



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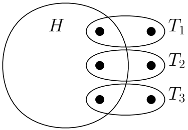

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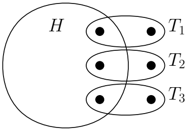

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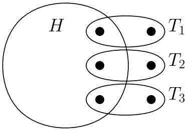

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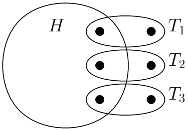

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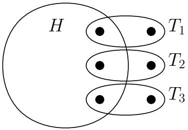

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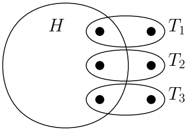

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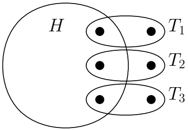

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  - T_i contributes at least 3 to the sum $\mathcal{S}$.

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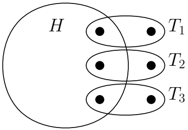

Let $H, T_1, \ldots, T_k$ be a comb, and let $\mathcal{S} = \sum_{e \in \delta(H)} x_e + \sum_{i=1}^{k} \sum_{e \in \delta(T_i)} x_e$.

First, consider a single T_i and some tour R through all cities.

- Both $H \cap T_i$ and $T_i \setminus H$ are nonempty!
- Therefore, the tour has to enter and exit each such set at least once.
- If R contains an edge e_i from $H \cap T_i$ to $H \setminus T_i$:
  - Edge e_i contributes 1 to the sum $\mathcal{S}$ (crosses H).
  - Furthermore, e_i does not cross T_i, so there must be two more edges in R crossing T_i.
  - T_i contributes at least 3 to the sum $\mathcal{S}$.
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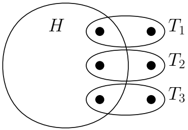

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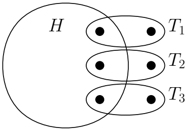

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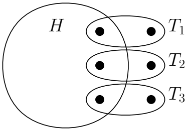

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$\Rightarrow$ $\mathcal{S} \geq 3k + 1$.

Example Time: Combs

Interactive example at https://www.math.uwaterloo.ca/tsp/app/diy.html.
Random seed: 1234, 50 cities.
Optimal solution (through comb and subtour cuts only): 51991.

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For more, see
David L. Applegate, Robert E. Bixby, Vašek Chvatál and William J. Cook.
The Traveling Salesman Problem: A Computational Study.
Princeton Series in Applied Mathematics (2006), Princeton University Press.

Definition & Model

Cutting Planes

Branch, Cut & Price

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Solution: Do not consider all edges all the time — most are, after all, never useful!

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Of course, we have to integrate cutting planes into the dual — as dual variables.

Dual of the TSP

$$\min c^T x \text{ s.t.}$$

$$\forall v \in V : \sum_{e \in \delta(\{v\})} x_e = 2,$$

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Intuition/Geometry: Zone & Moat packing — see https://www.math.uwaterloo.ca/tsp/app/diy.html.

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We know how to add new edges to cuts in this representation!

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With cuts (including subtours) represented as set families $\mathcal{F}$ with associated $\mu_{\mathcal{F}}$:

- let $e(\mathcal{F}) = \bigcup_{S \in \mathcal{F}} \delta(S)$ be the edges crossing any set in $\mathcal{F}$,
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So we need to find all edges with violated constraints, i.e., cases with

$$\alpha_e = c_e - y_v - y_w + y_e - \sum_{\mathcal{F} : vw \in e(\mathcal{F})} \chi(e, \mathcal{F}) z_{\mathcal{F}} < 0.$$

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We have everything we need to compute α_e. But it's too slow because of the sum of cuts.
We need a way to (quickly) underestimate $\overline{\alpha}_e \leq \alpha_e$; then check α_e only if $\overline{\alpha}_e < 0$.

Overestimating Dual Cut Sums

Observe that

$$\sum_{\mathcal{F}:vw\in e(\mathcal{F})} \chi(e,\mathcal{F}) z_{\mathcal{F}} = \sum_{\mathcal{F}} \sum_{S\in\mathcal{F}:vw\in\delta(S)} z_{\mathcal{F}}$$

$$= \sum_{\mathcal{F}} \sum_{S\in\mathcal{F}:v\in S} z_{\mathcal{F}} + \sum_{\mathcal{F}} \sum_{S\in\mathcal{F}:w\in S} z_{\mathcal{F}} - 2\sum_{\mathcal{F}} \sum_{S\in\mathcal{F}:\{v,w\}\subseteq S} z_{\mathcal{F}}.$$

$$\leq \sum_{\mathcal{F}} \sum_{S\in\mathcal{F}:v\in S} z_{\mathcal{F}} + \sum_{\mathcal{F}} \sum_{S\in\mathcal{F}:w\in S} z_{\mathcal{F}}.$$

We set

$$\overline{y}_v = y_v + \sum_{\mathcal{F}} \sum_{S\in\mathcal{F}:v\in S} z_{\mathcal{F}} \text{ and } \overline{\alpha}_e = c_e - \overline{y}_v - \overline{y}_w \leq \alpha_e.$$

This check can be done reasonably quickly, apparently even for about 10^6 cities.
Geometry can be used to speed this up even further.

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- Overall approach in Concorde is much more cut-y and much less branch-y.