Computational Geometry
Chapter 7: Location Problems

Sándor P. Fekete

Algorithms Division
Department of Computer Science
TU Braunschweig

1. Introduction
2. Manhattan distances
3. Euclidean distances
4. Galois and Bajaj
5. Continuous sets

Superhero! (Supervillain?)

Superhero! (Supervillain?)

Francis Sciabbala

Superhero! (Supervillain?)

Superhero! (Supervillain?)

THE DELIVERATOR

Motivation

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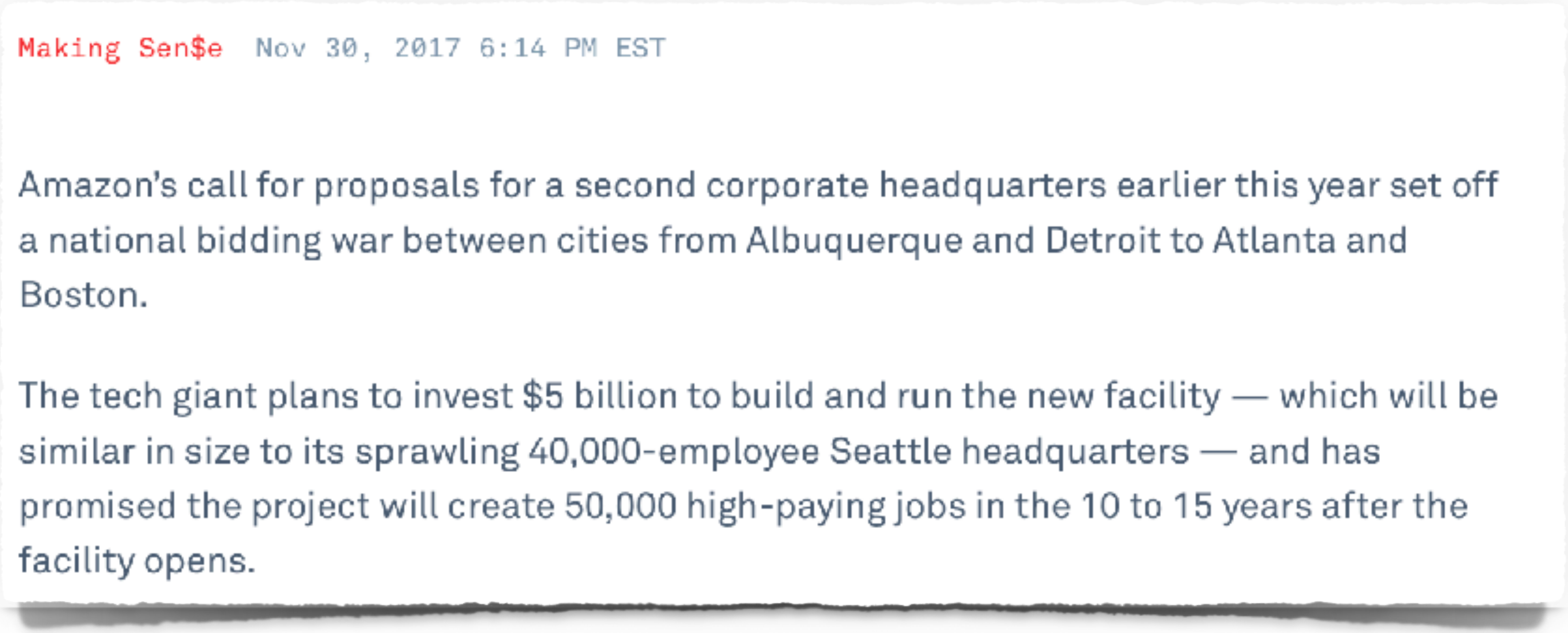
Amazon HQ2

Amazon HQ2

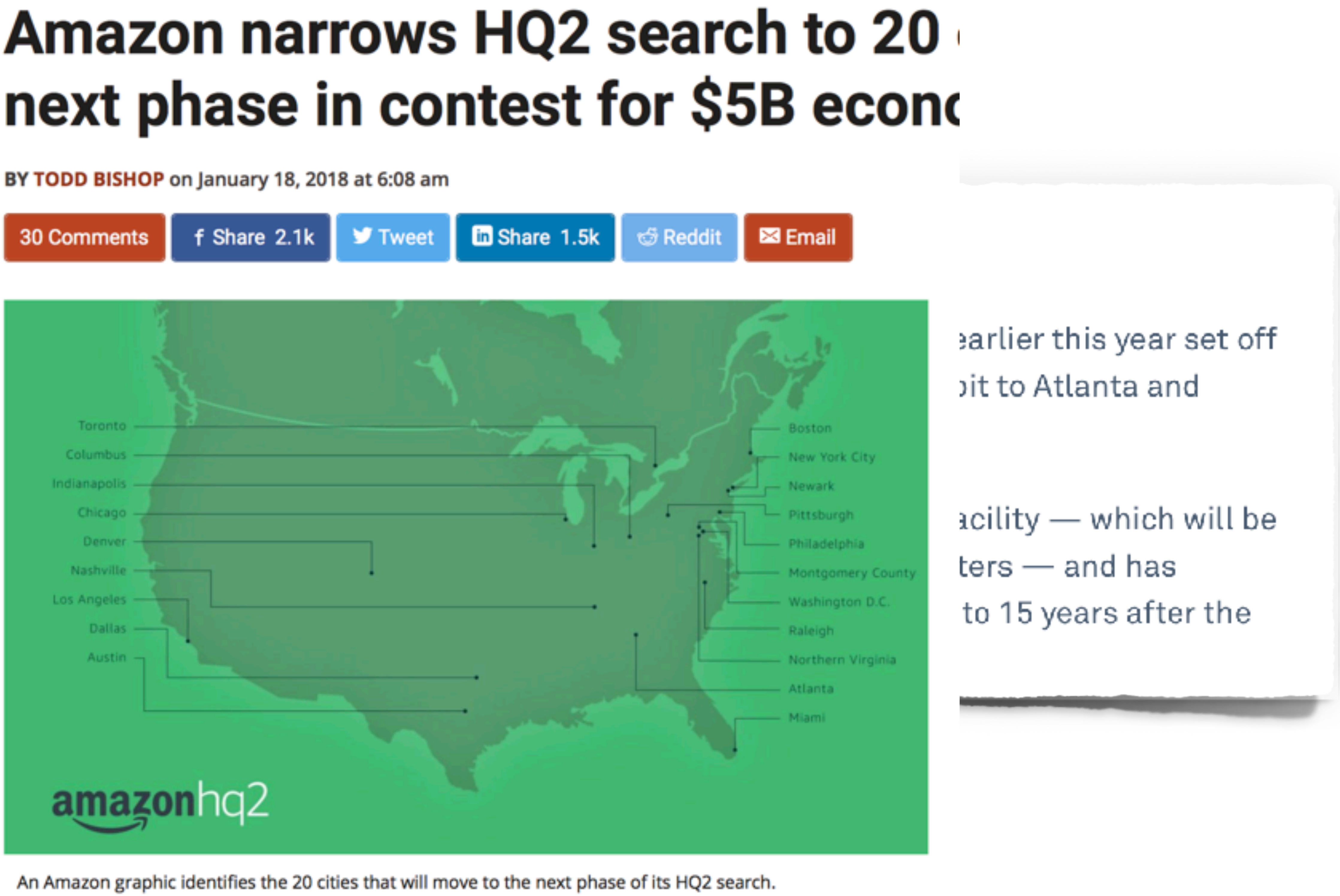


Making Sen$e Nov 30, 2017 6:14 PM EST

Amazon's call for proposals for a second corporate headquarters earlier this year set off a national bidding war between cities from Albuquerque and Detroit to Atlanta and Boston.

The tech giant plans to invest $5 billion to build and run the new facility — which will be similar in size to its sprawling 40,000-employee Seattle headquarters — and has promised the project will create 50,000 high-paying jobs in the 10 to 15 years after the facility opens.

Motivation



Amazon narrows HQ2 search to 20
next phase in contest for $5B econ

BY **TODD BISHOP** on January 18, 2018 at 6:08 am

An Amazon graphic identifies the 20 cities that will move to the next phase of its HQ2 search.

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Motivation

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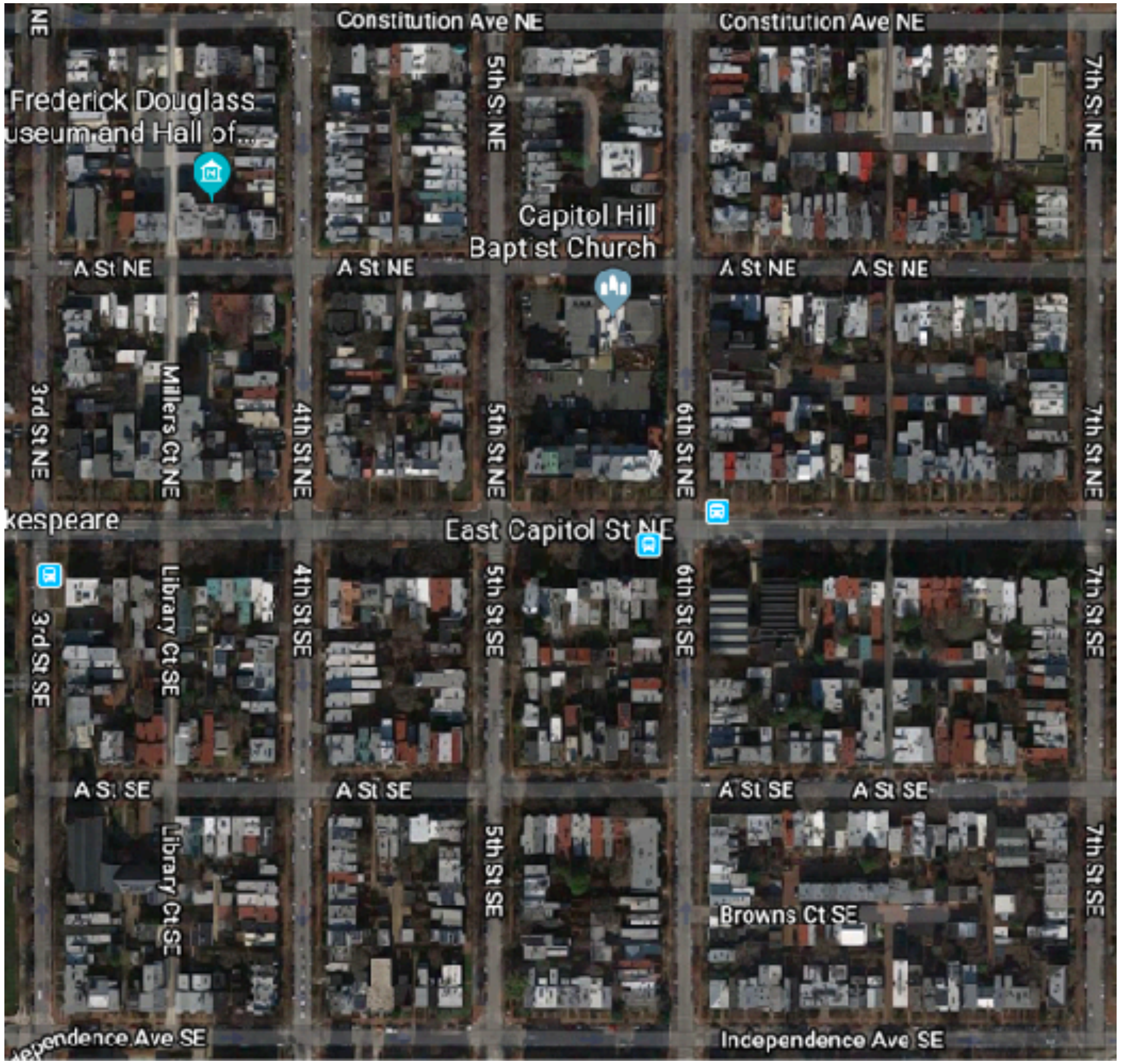

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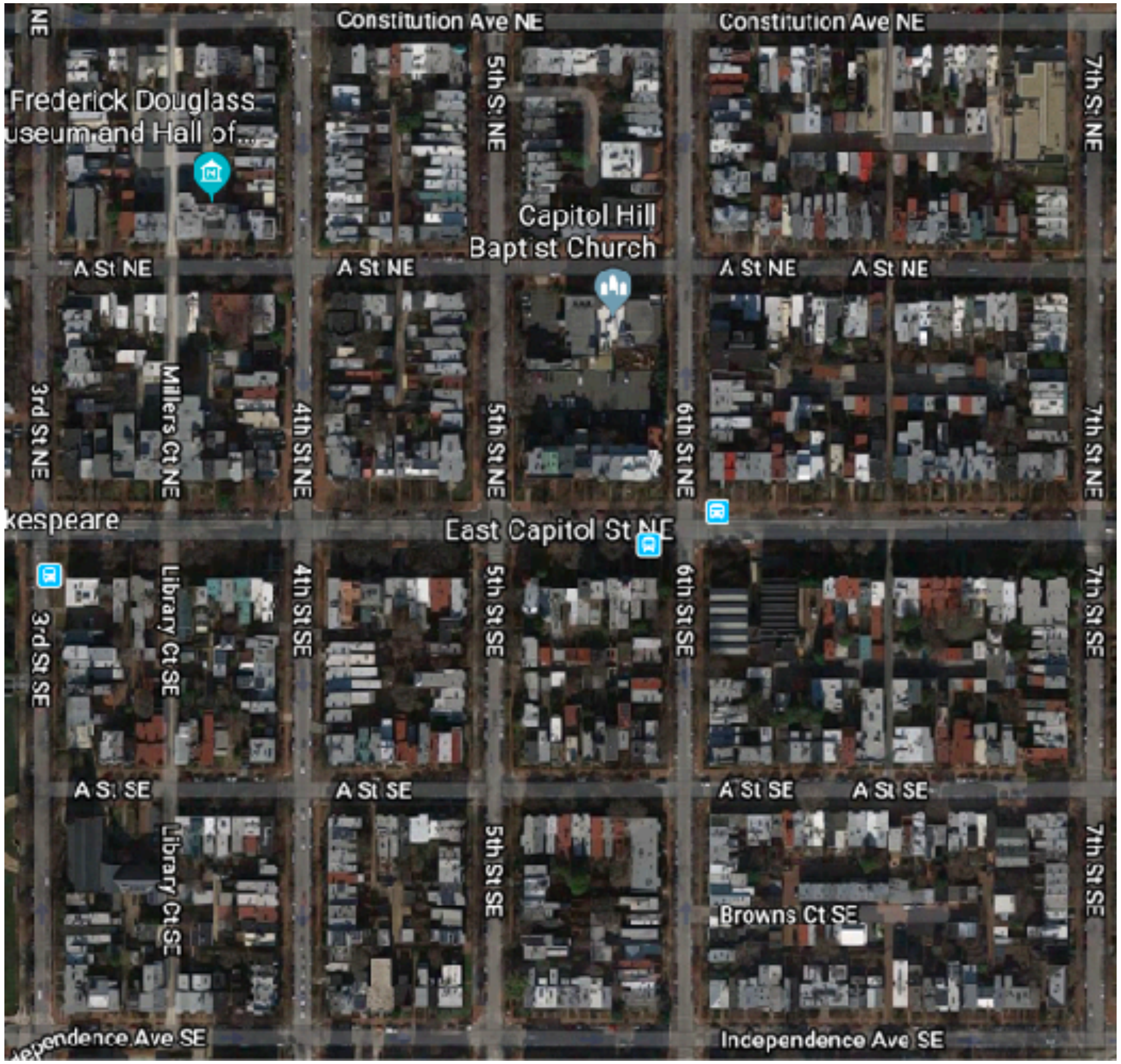
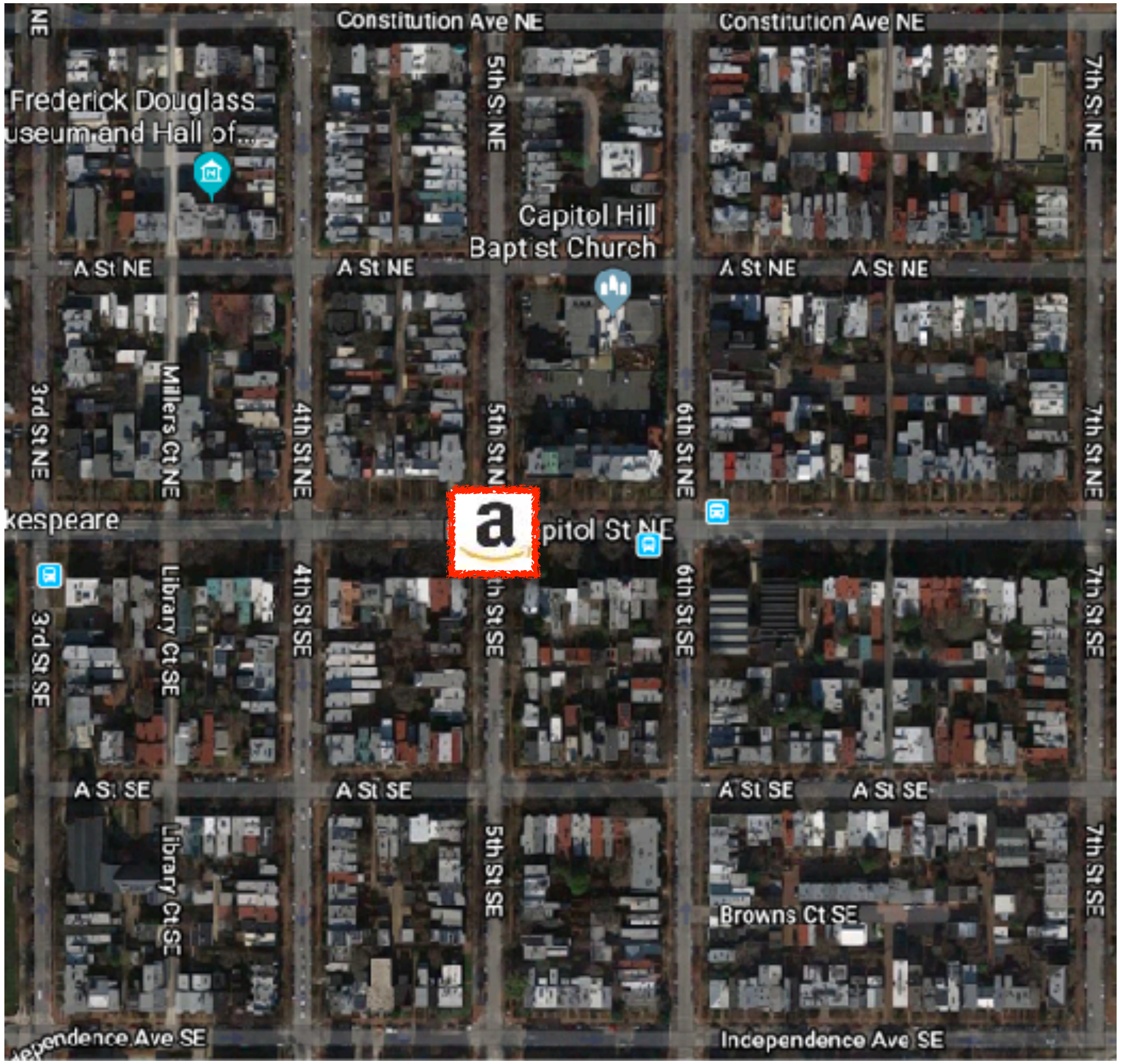
Motivation



Optimal location?



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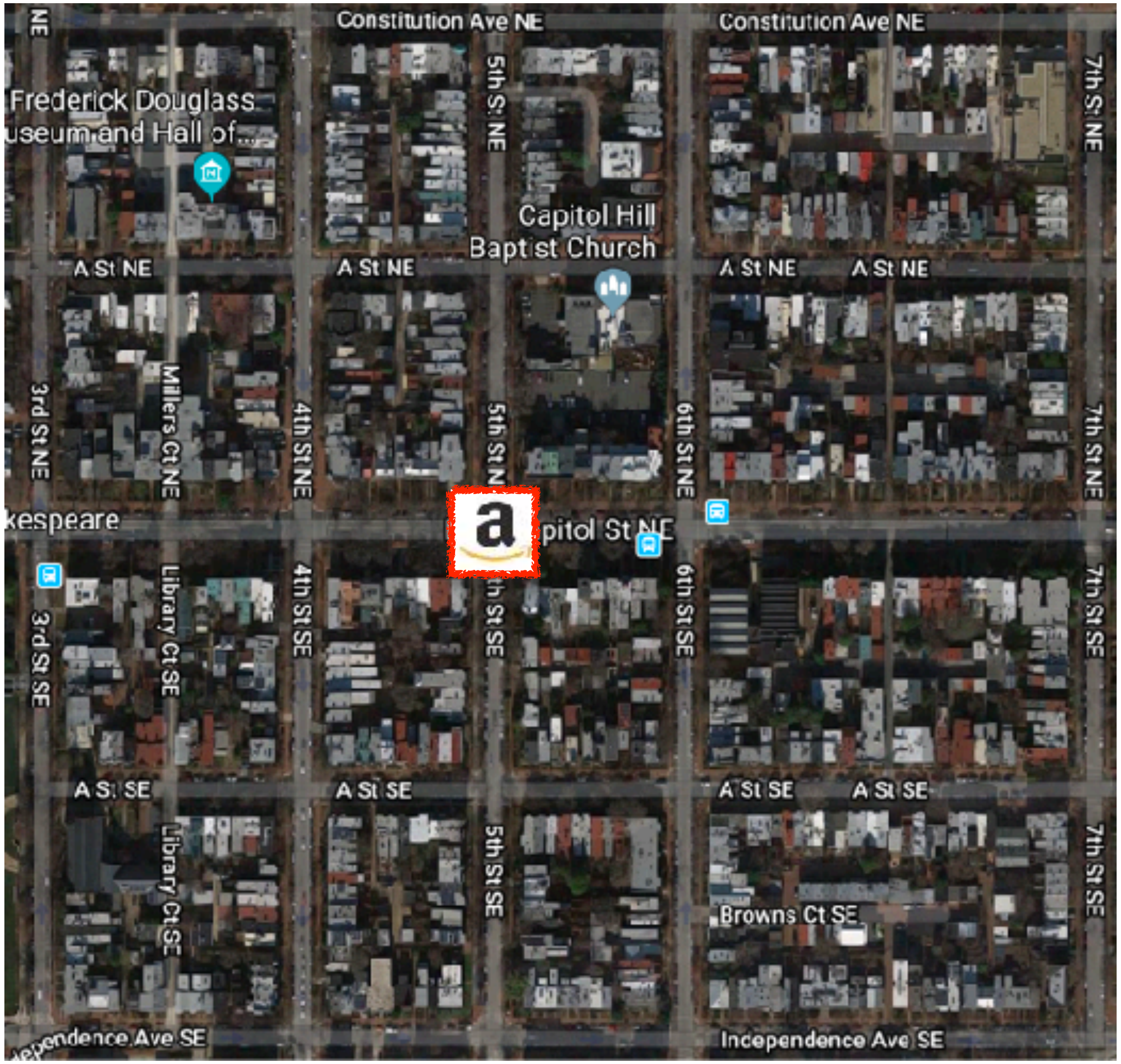
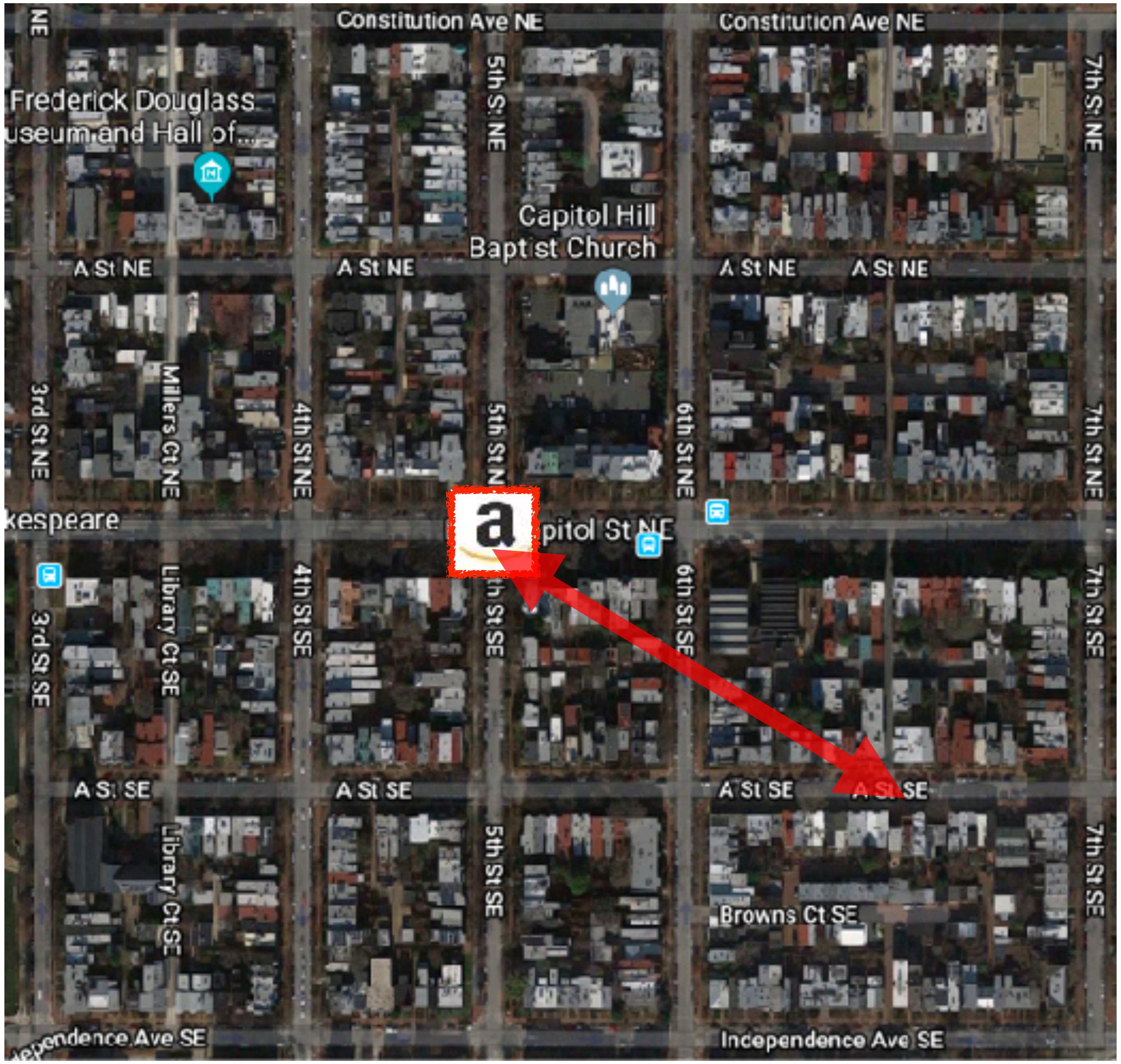


Optimal location?

Minimize average distance!



Optimal location?

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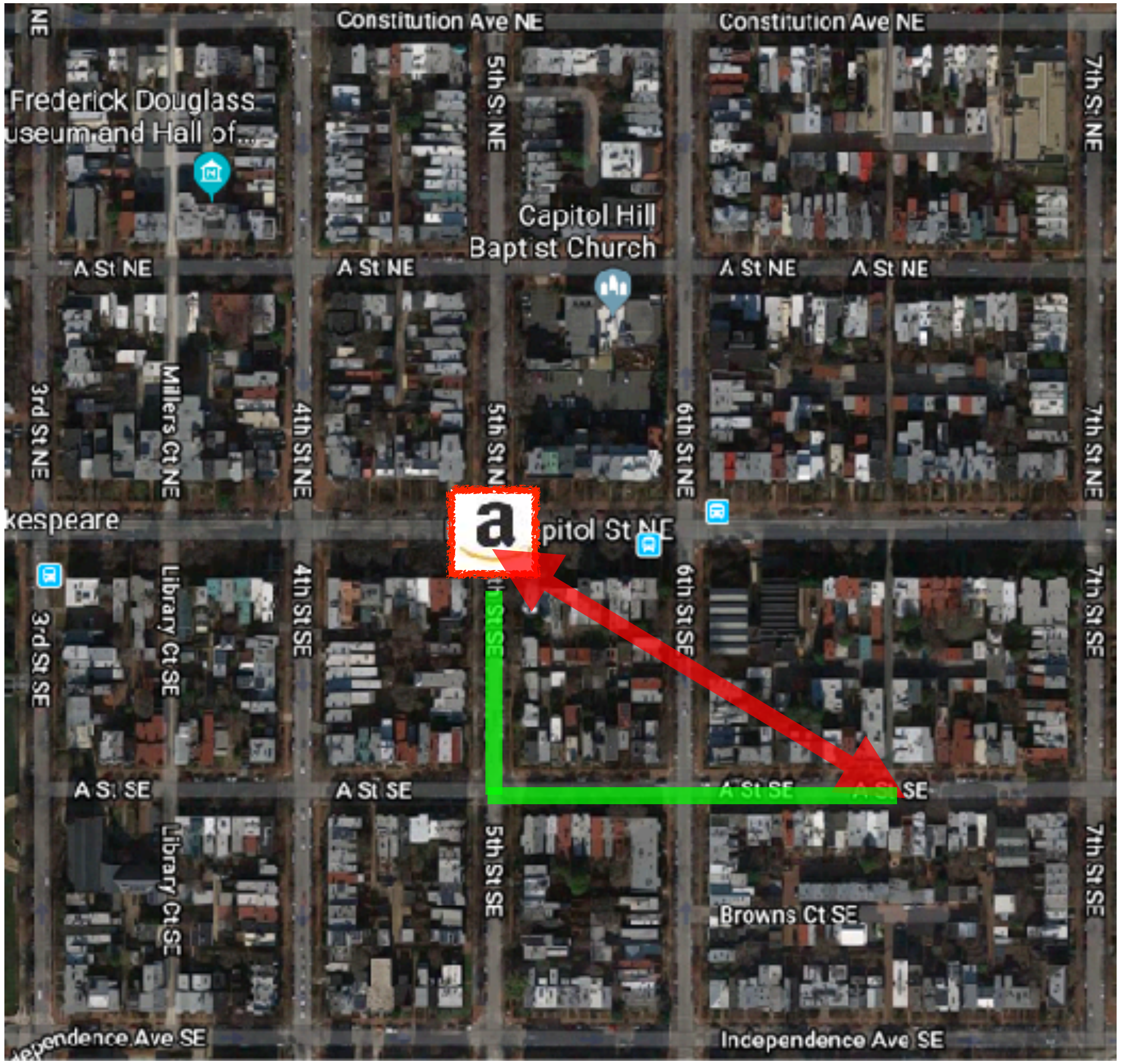
Motivation



Optimal location?

Minimize average distance!

History

History

Ueber den

Standort der Industrien

Von

Alfred Weber

Erster Teil

Reine Theorie des Standorts

Mit einem mathematischen Anhang
von
Georg Pick

64 Abbildungen

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History

Problem 7.1:

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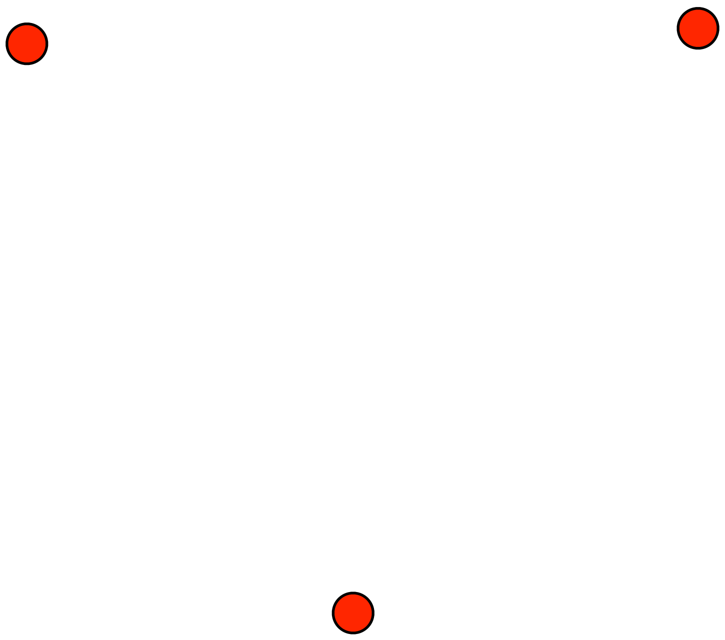

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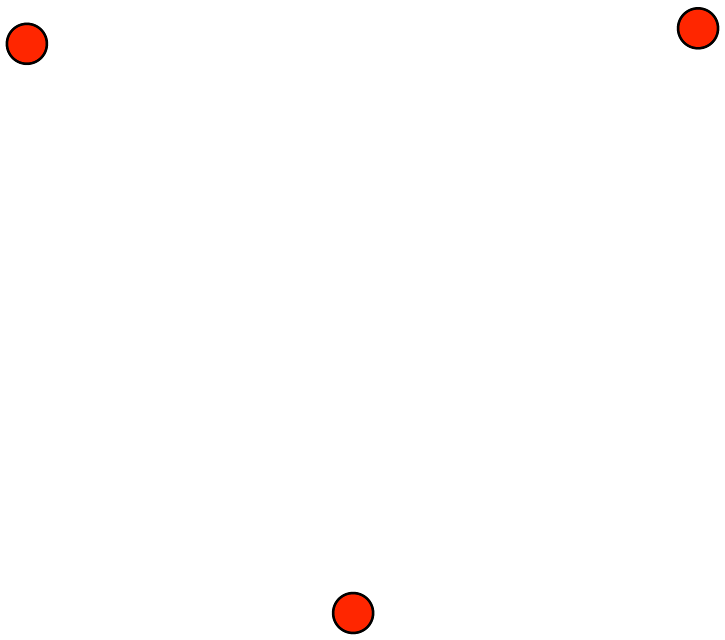

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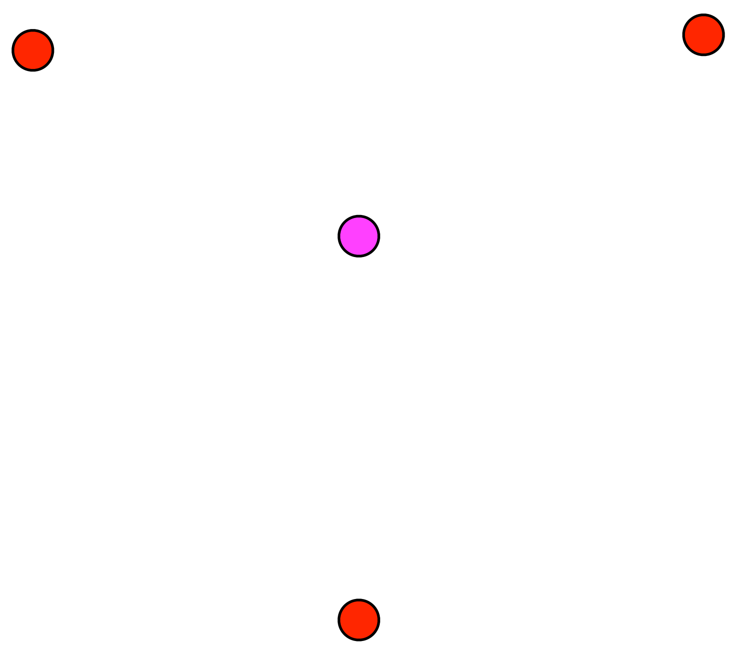

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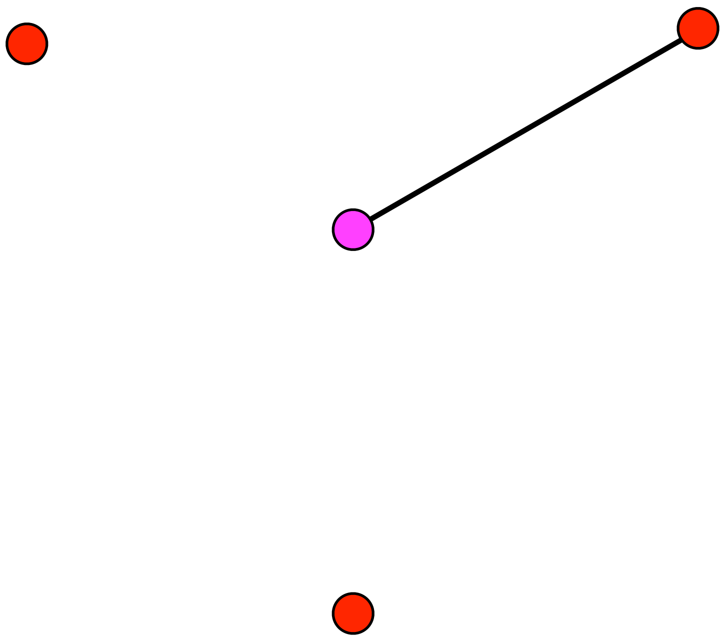

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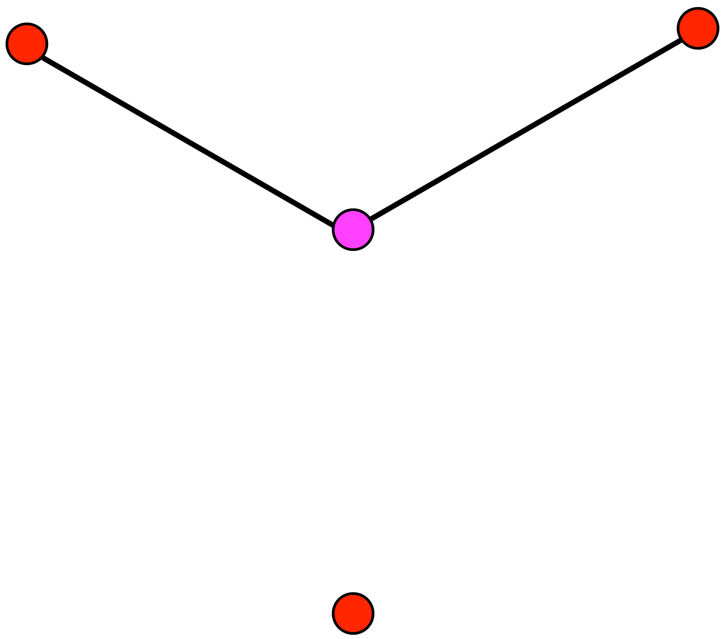

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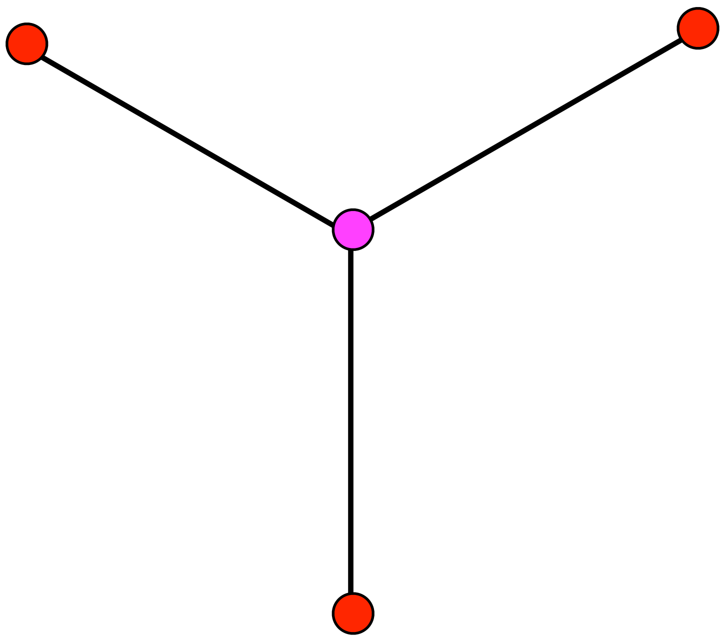

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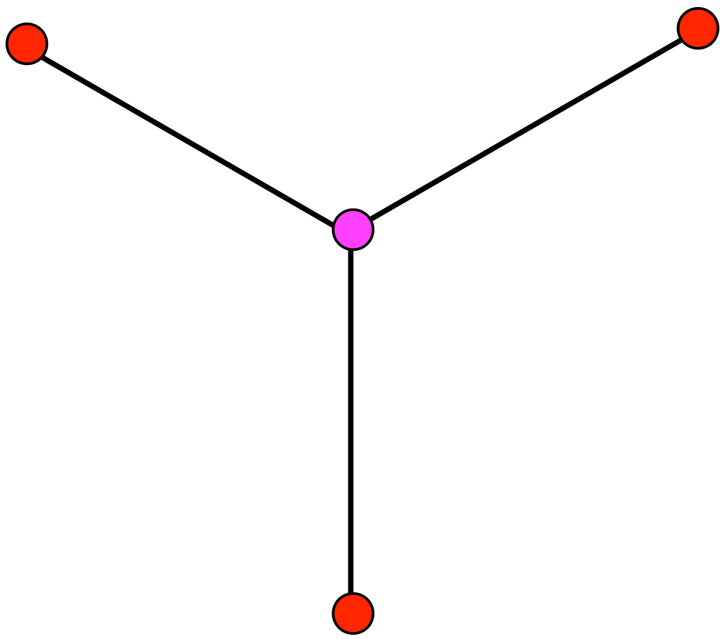

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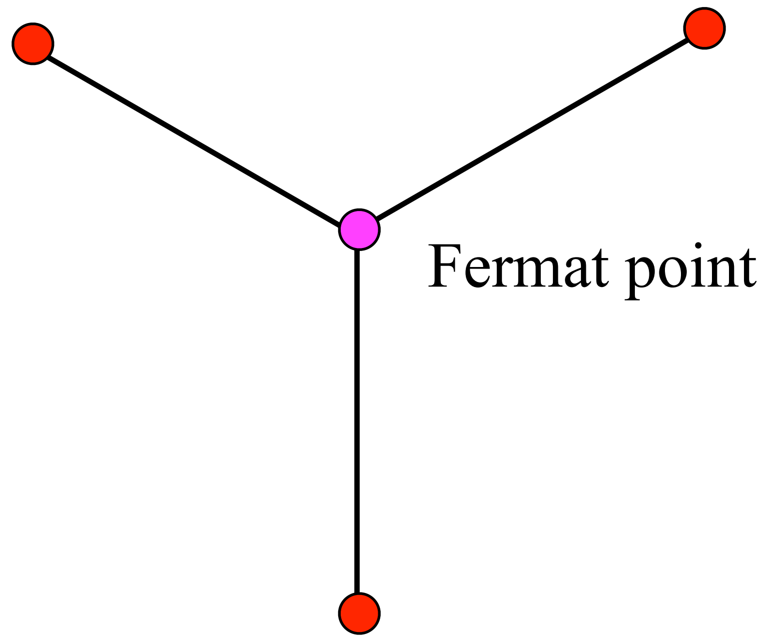

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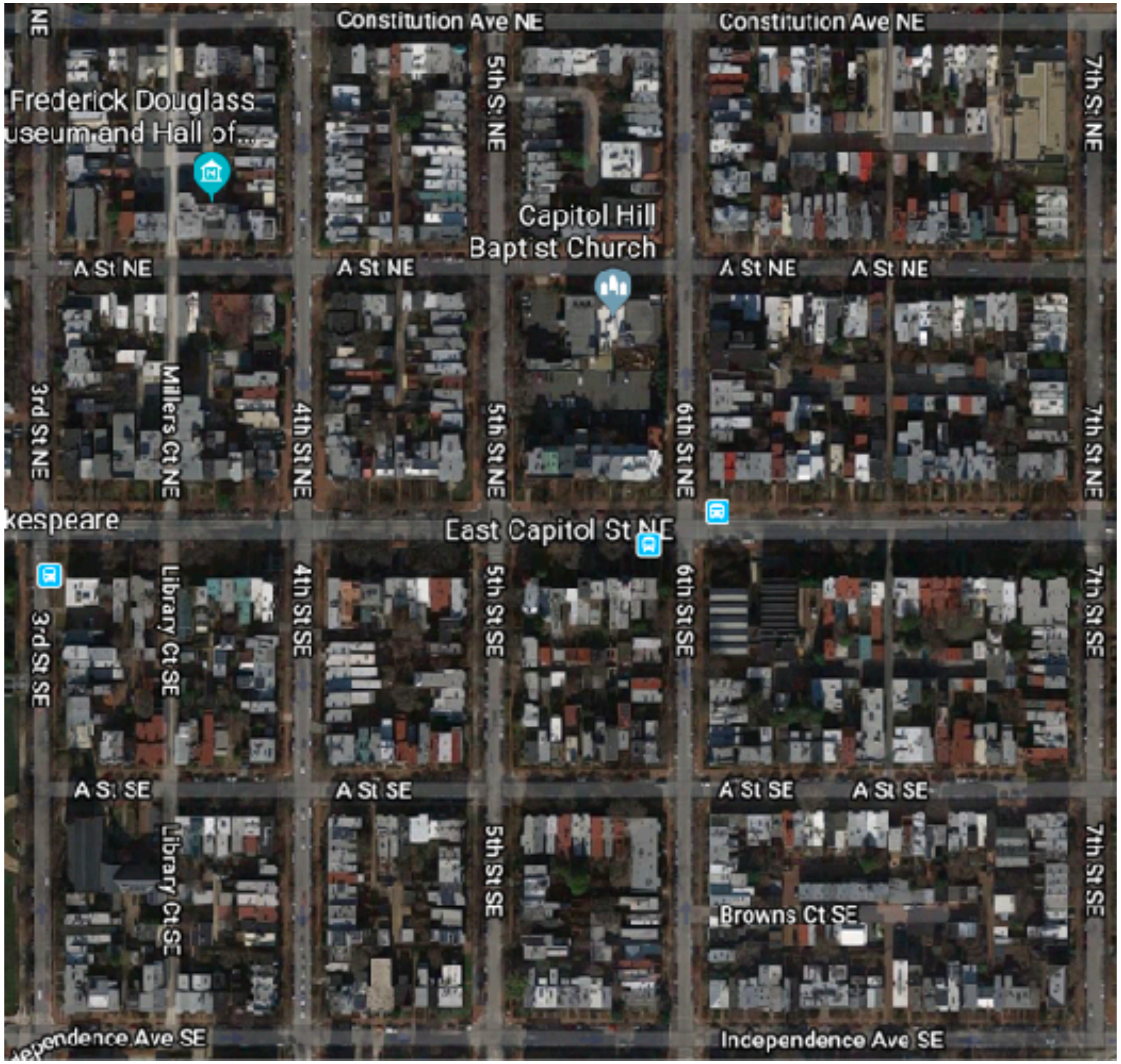
Manhattan distances



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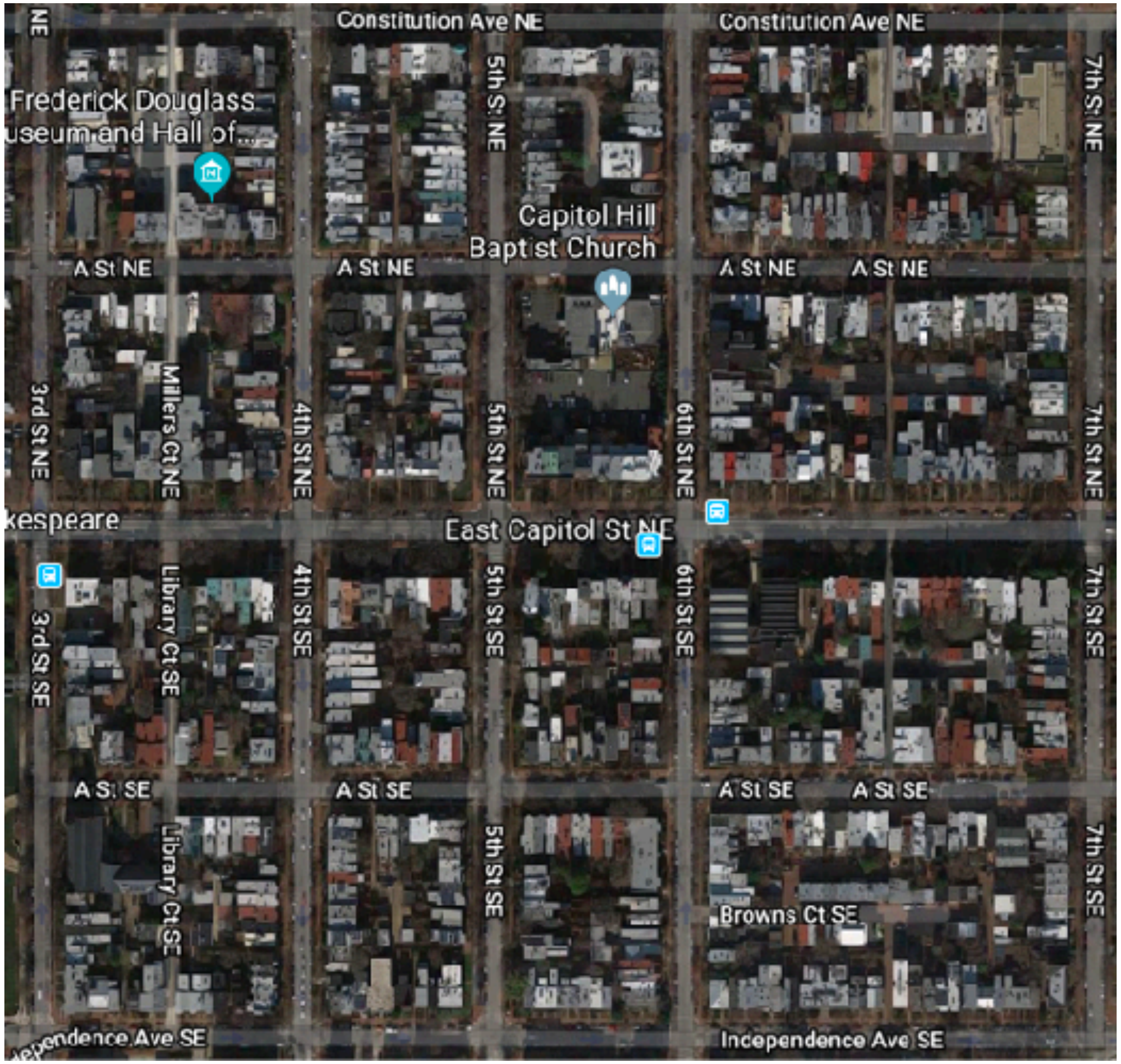
Manhattan distances



American cities: „Manhattan distance“

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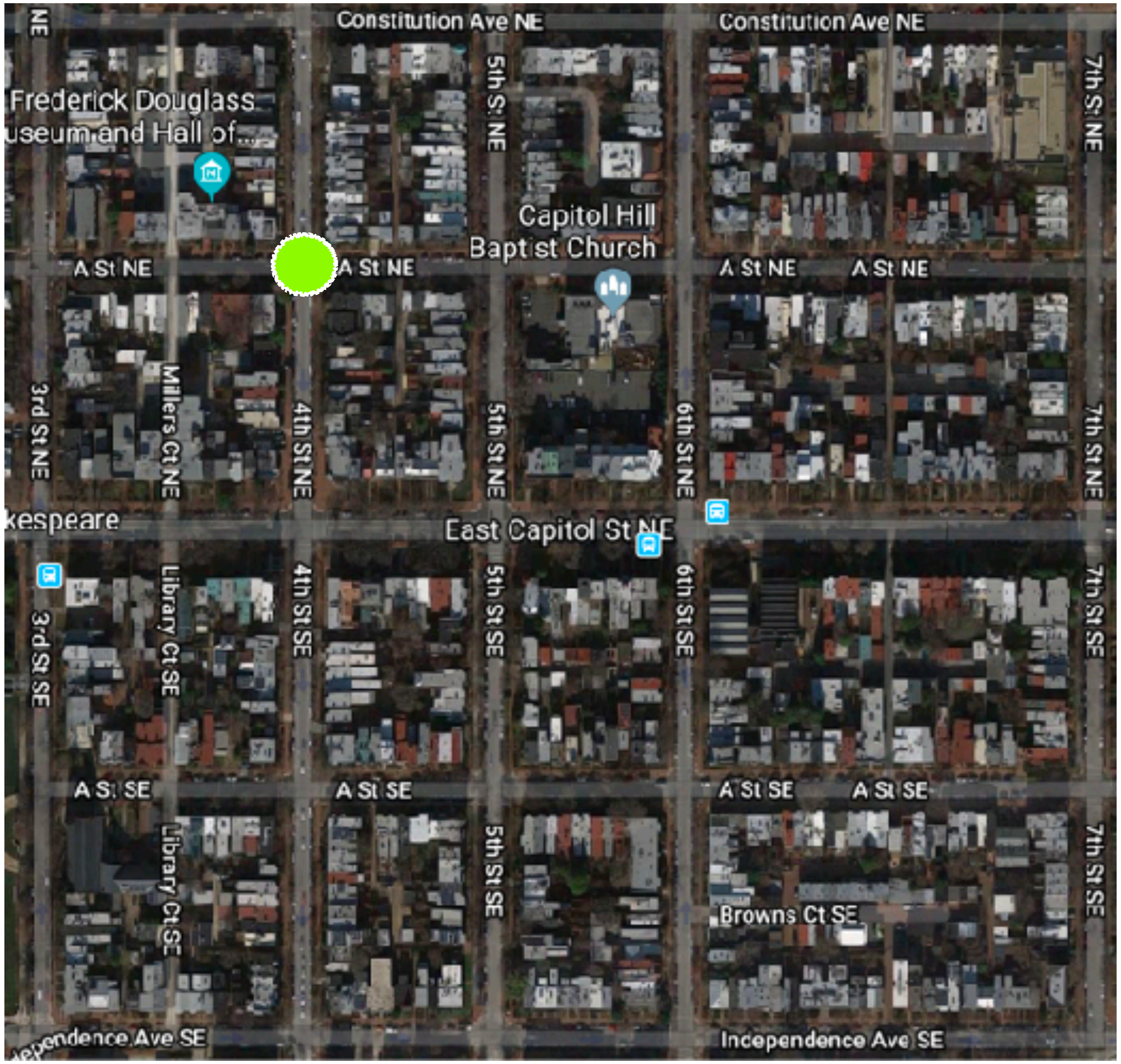

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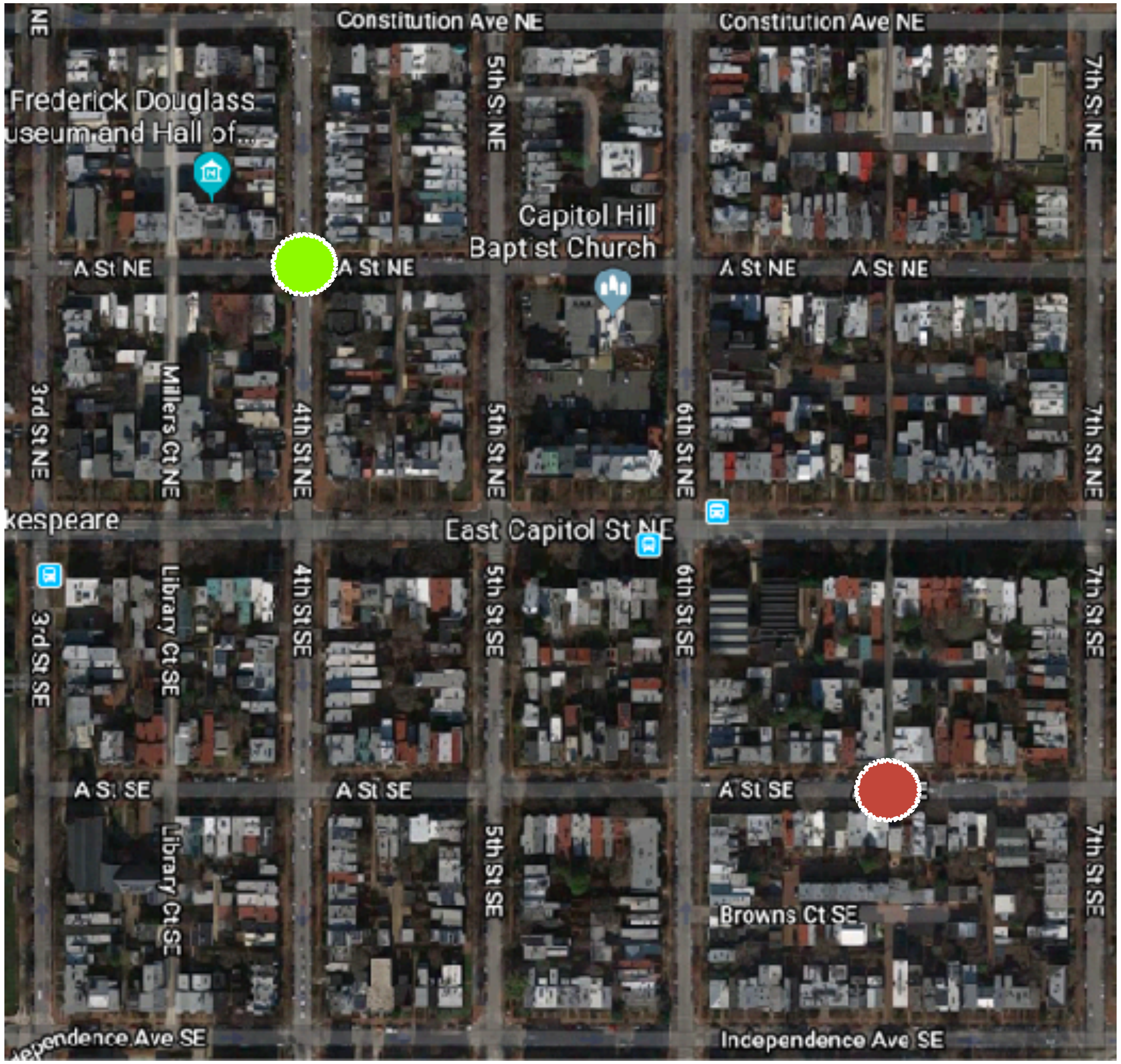

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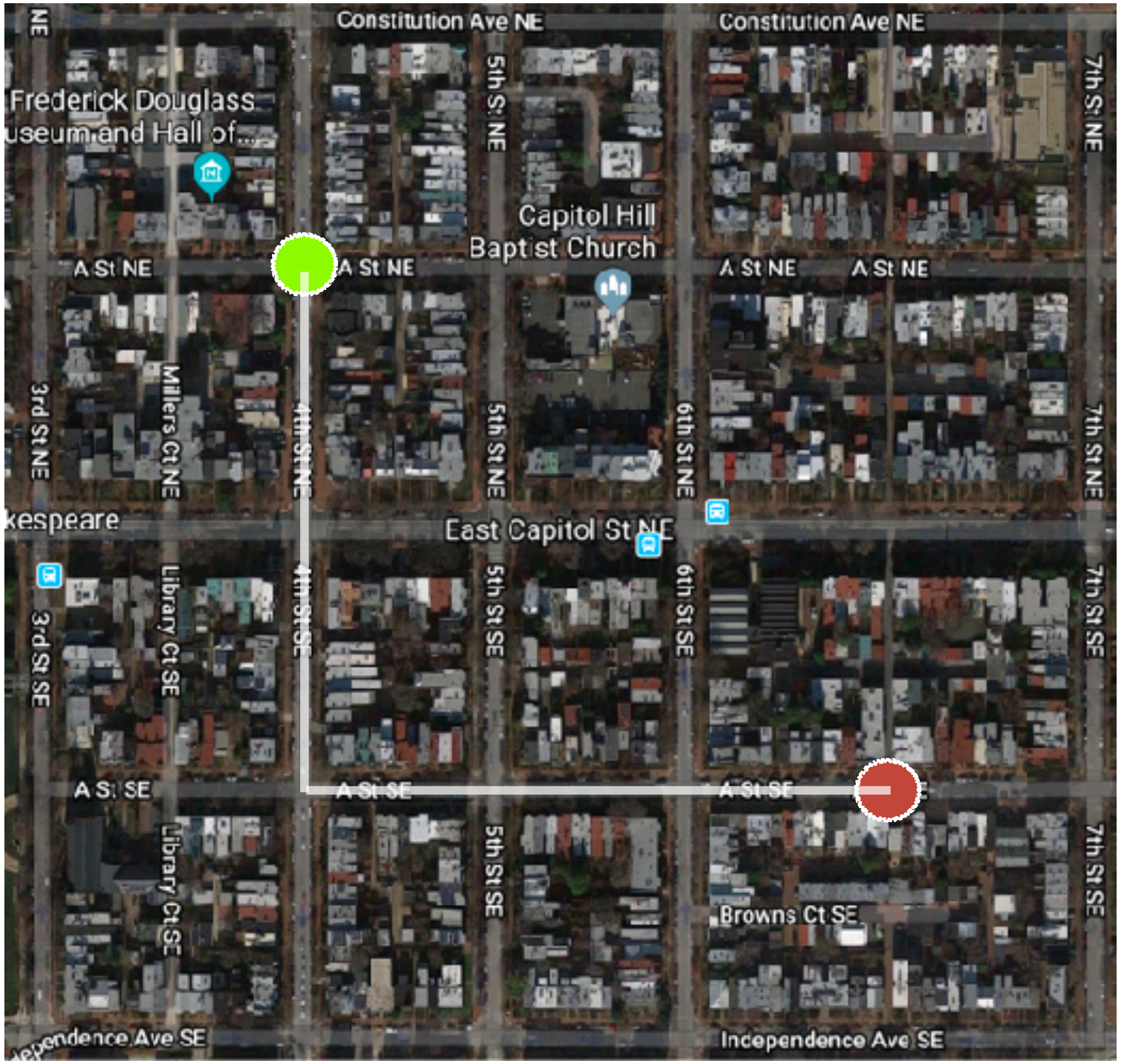

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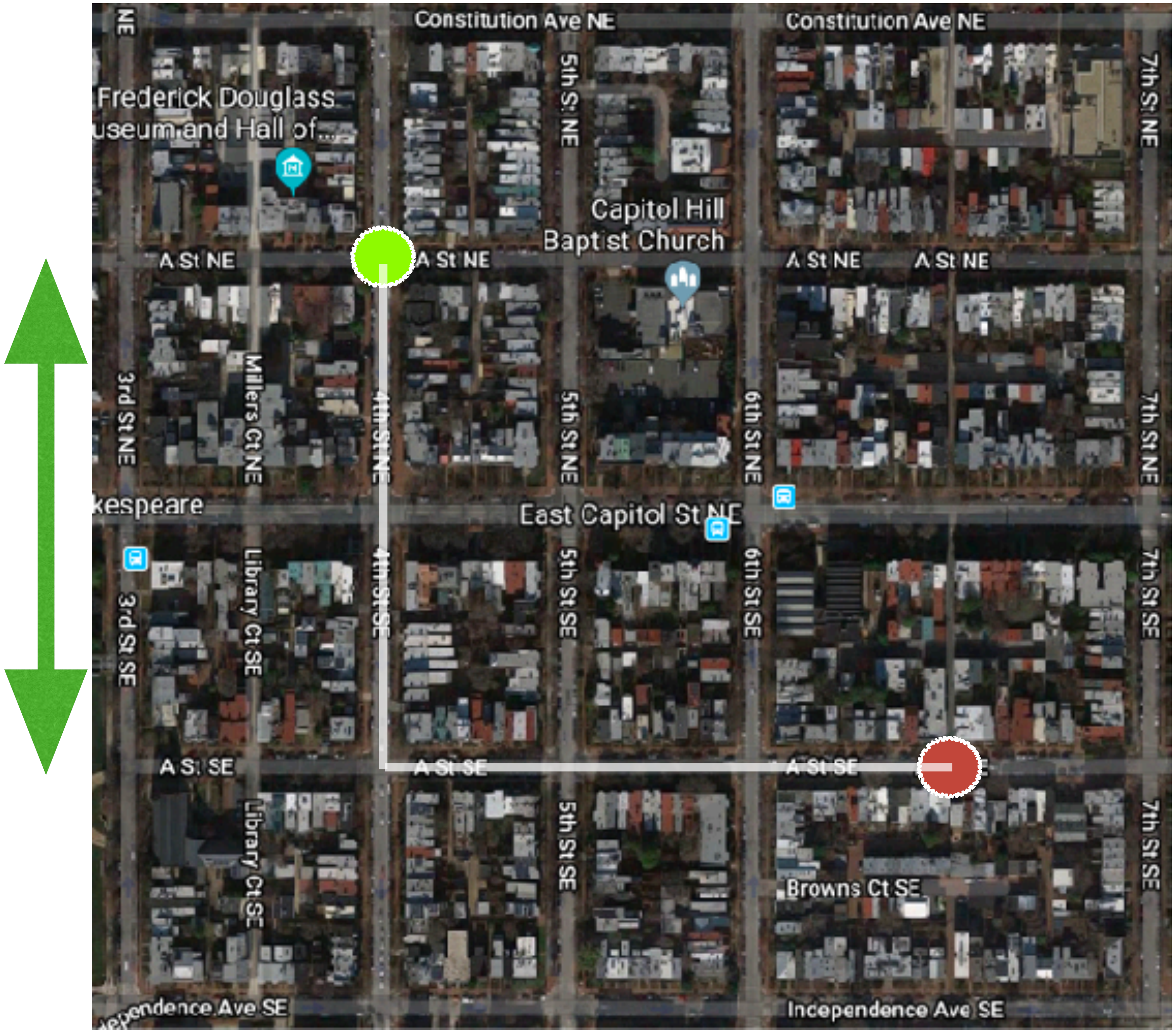

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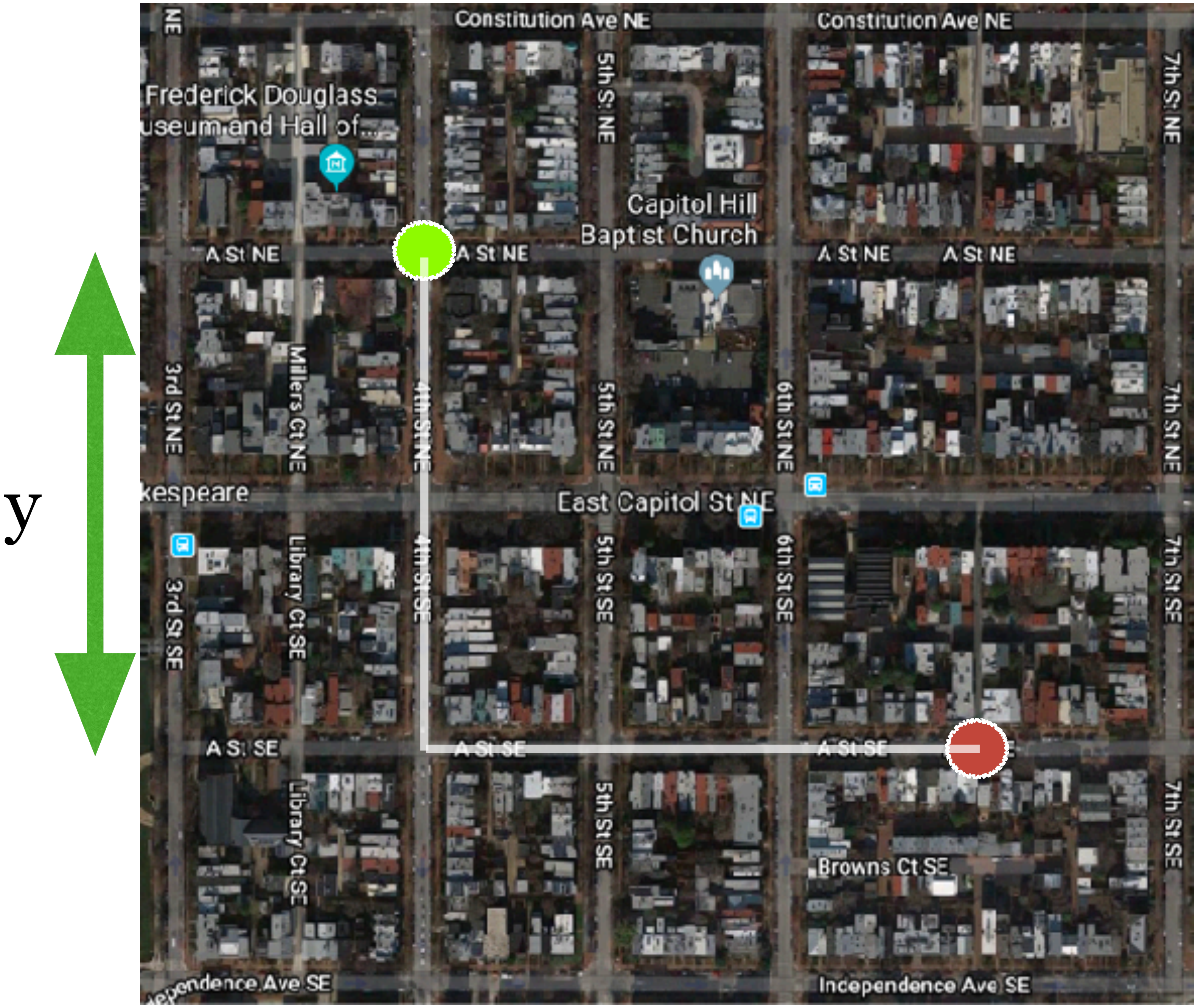

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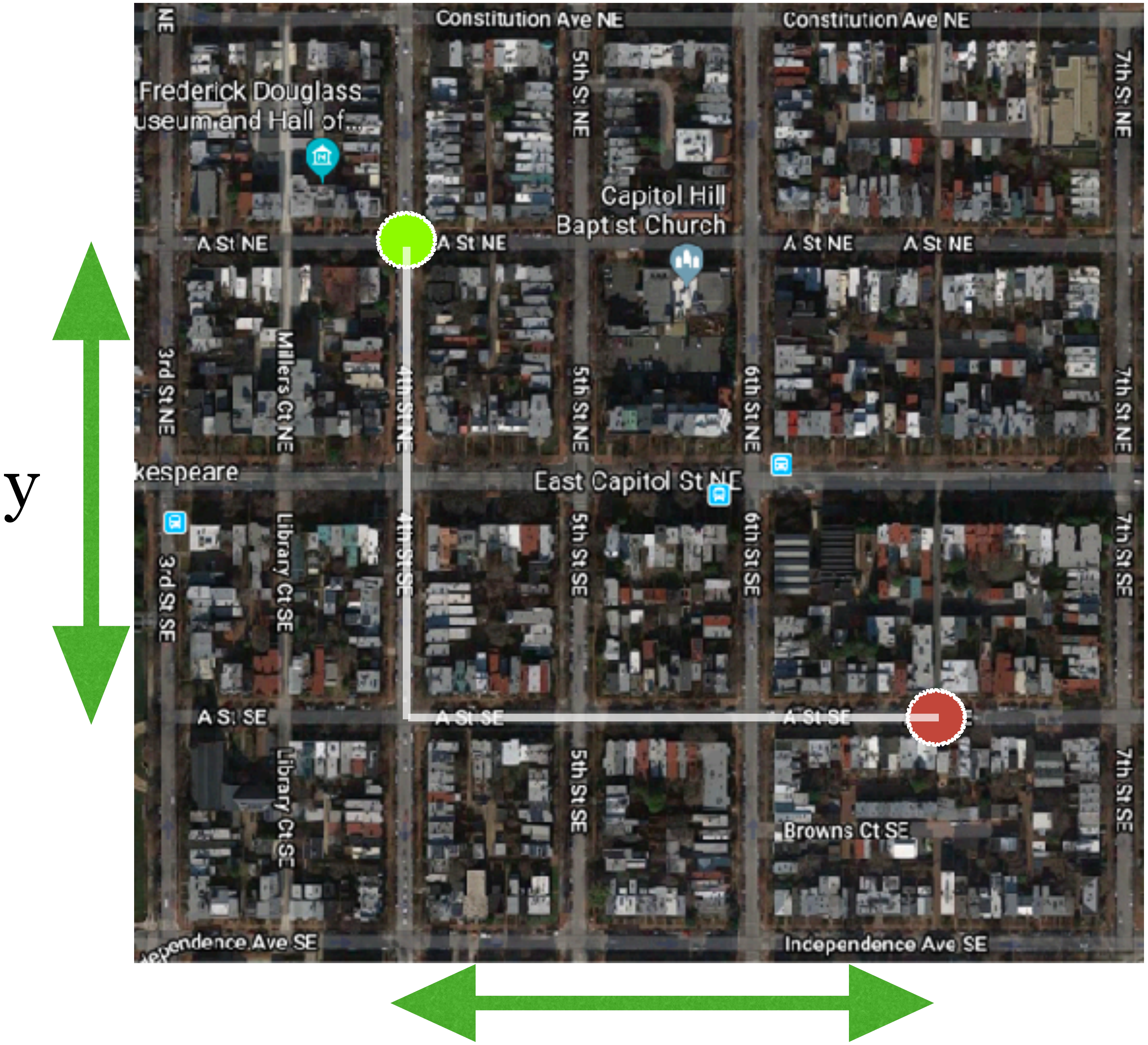

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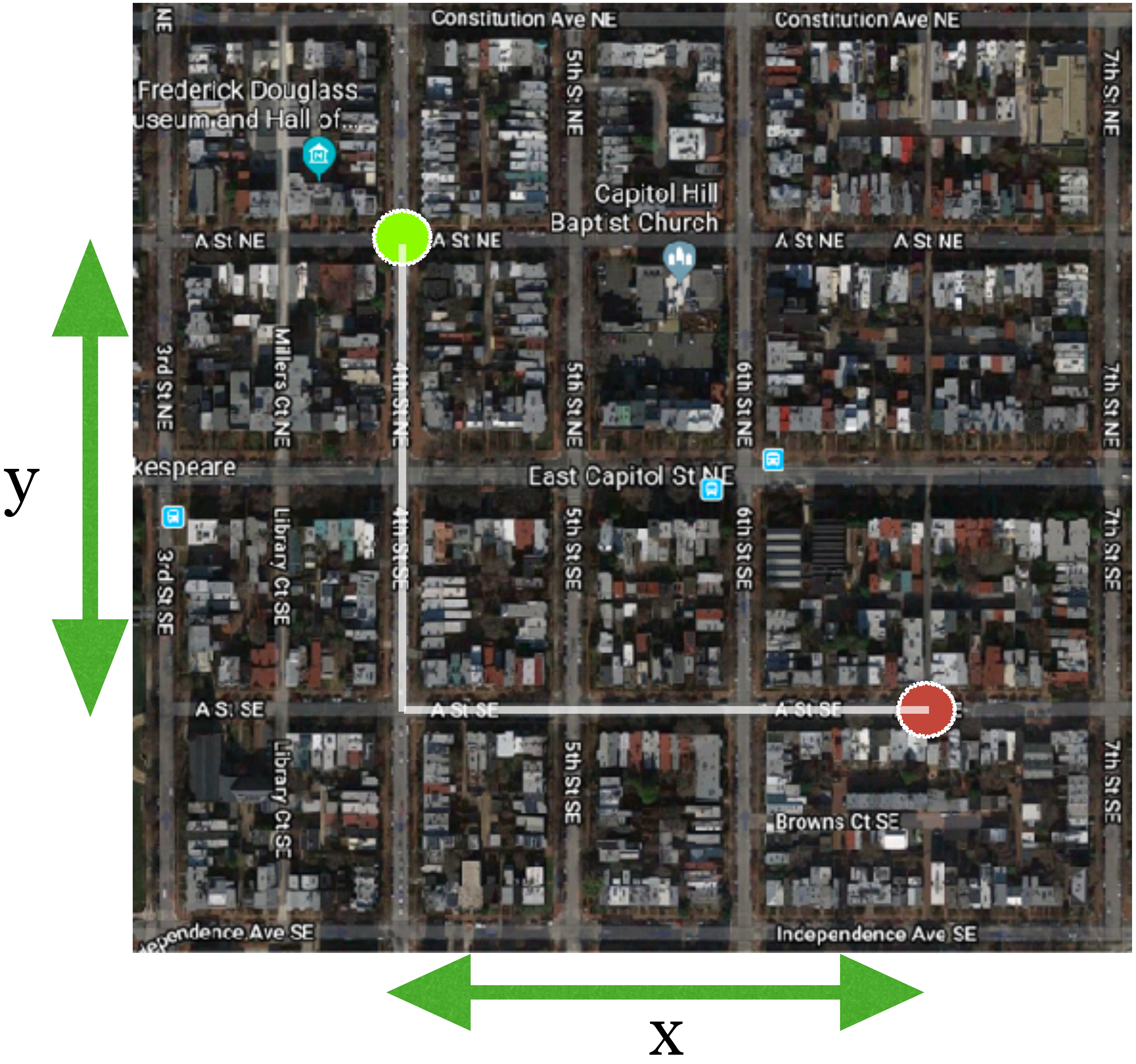

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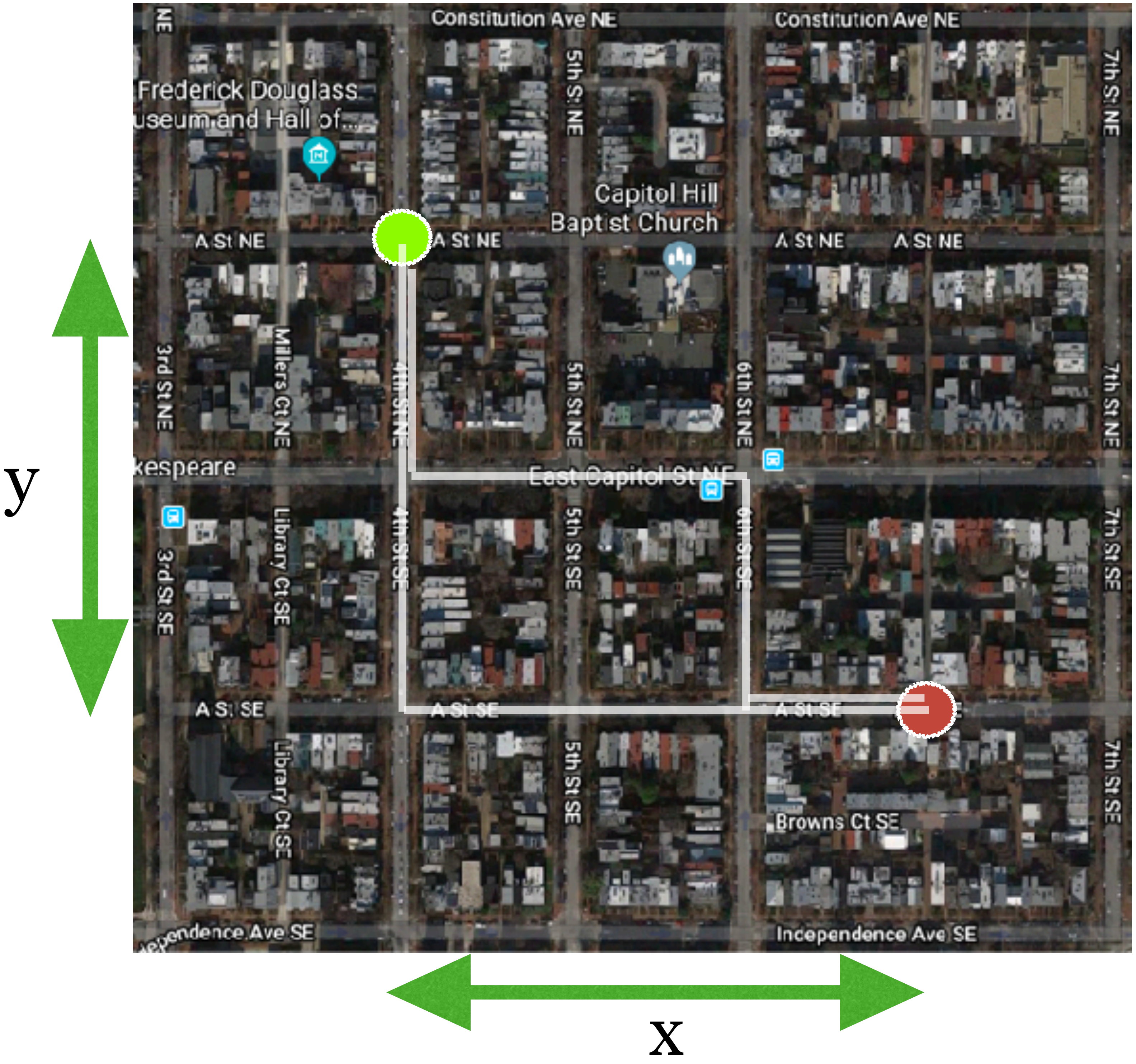

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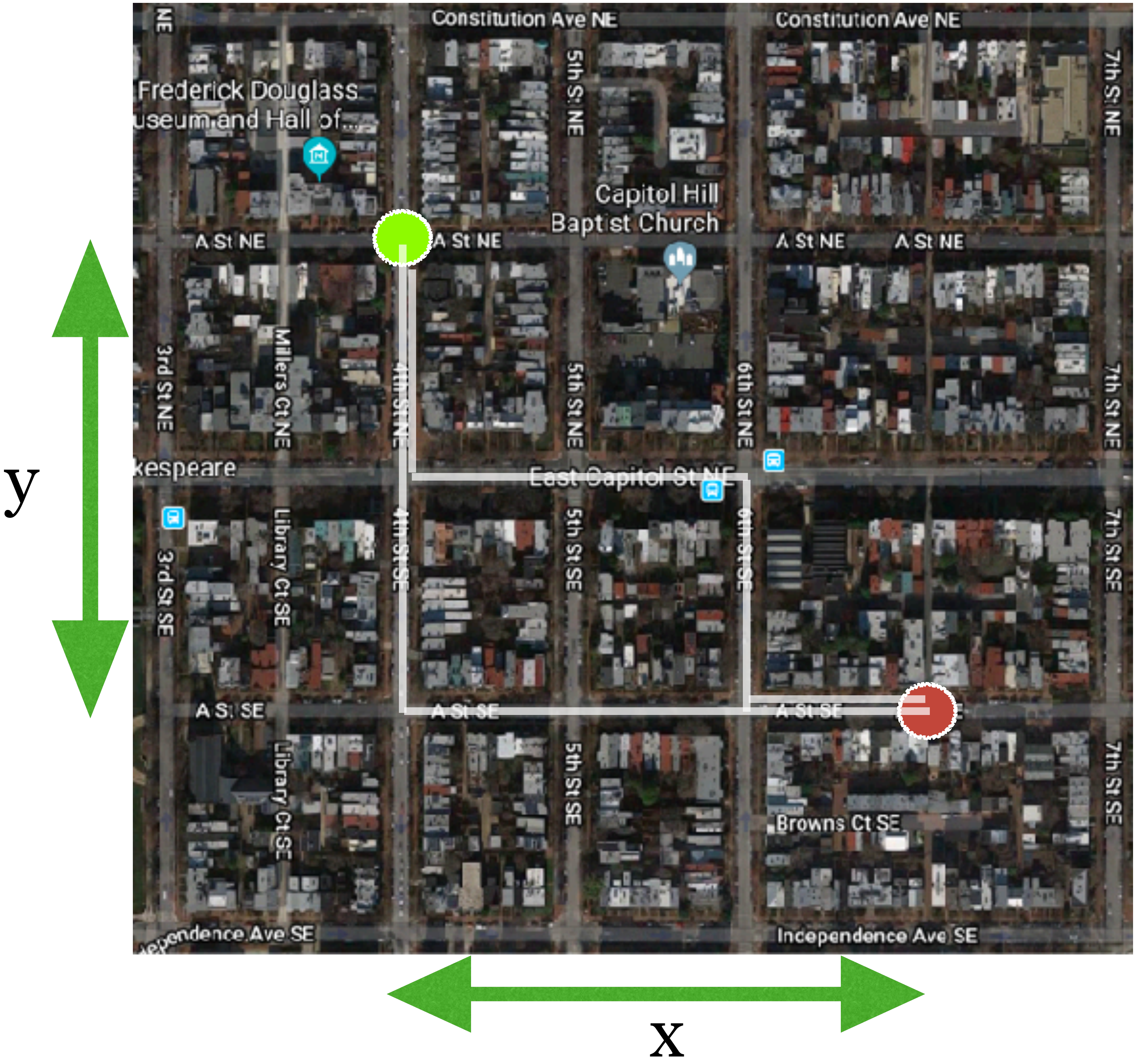
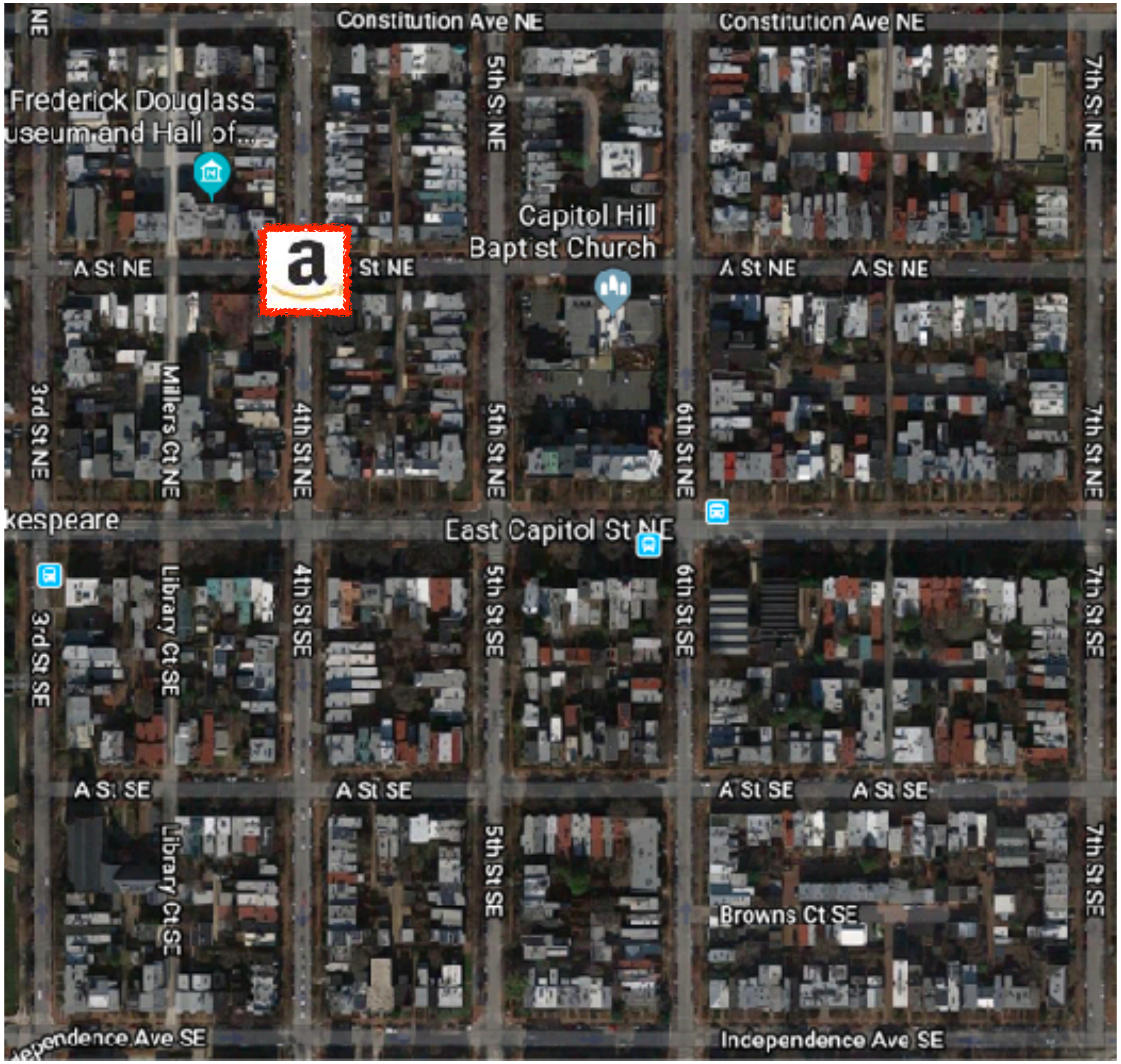


American cities: „Manhattan distance“

Consider x- und y-position separately!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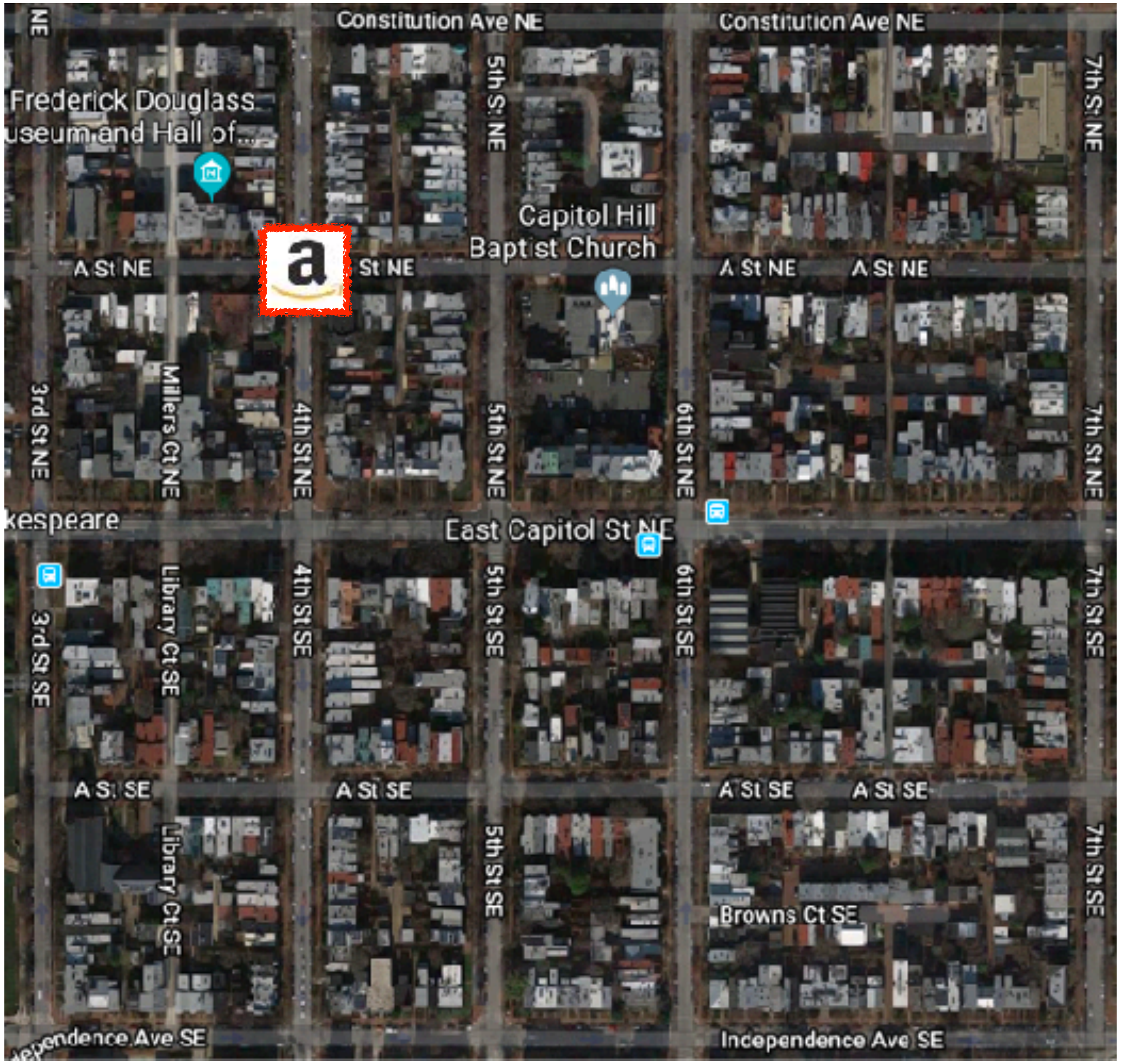

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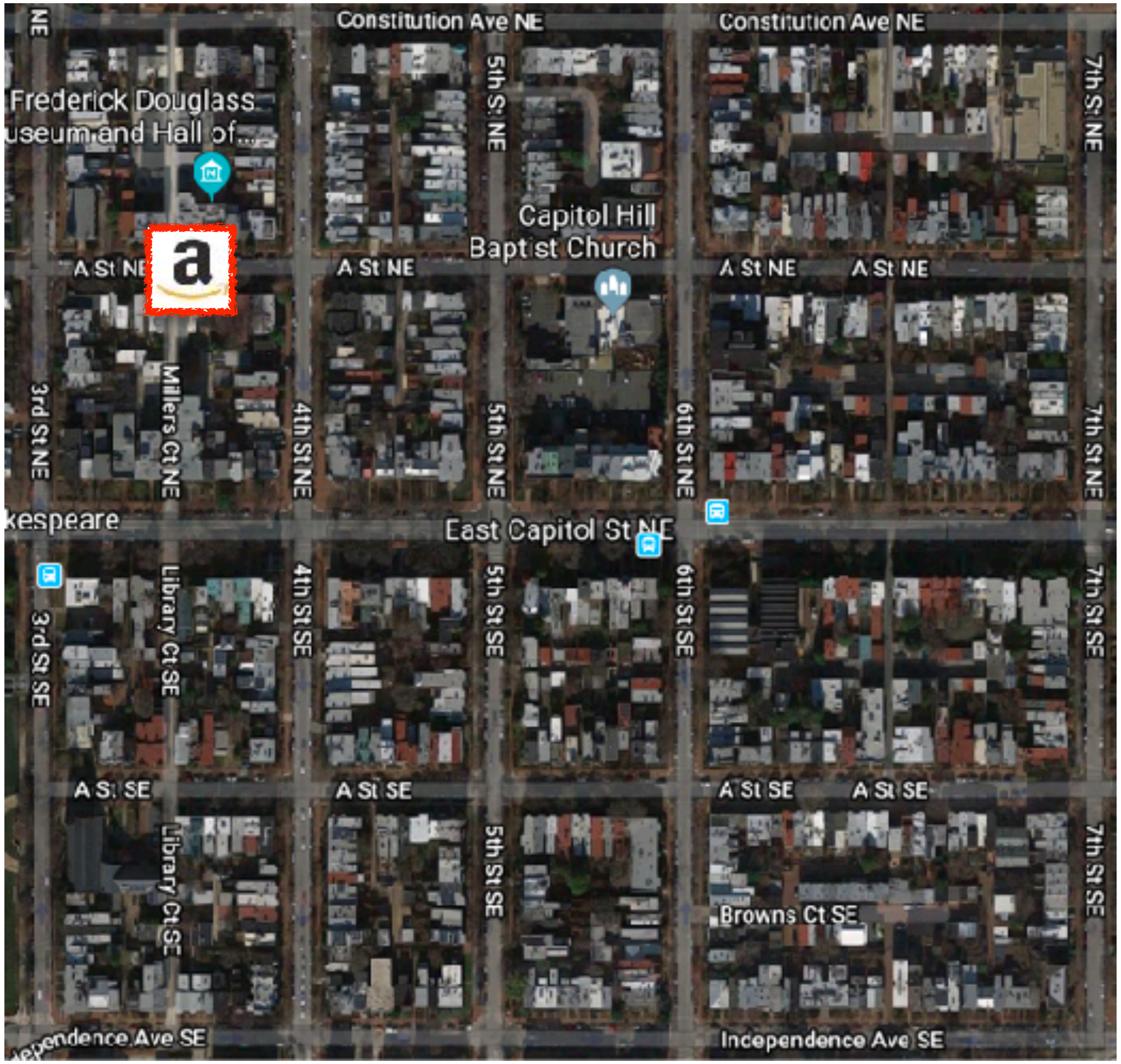

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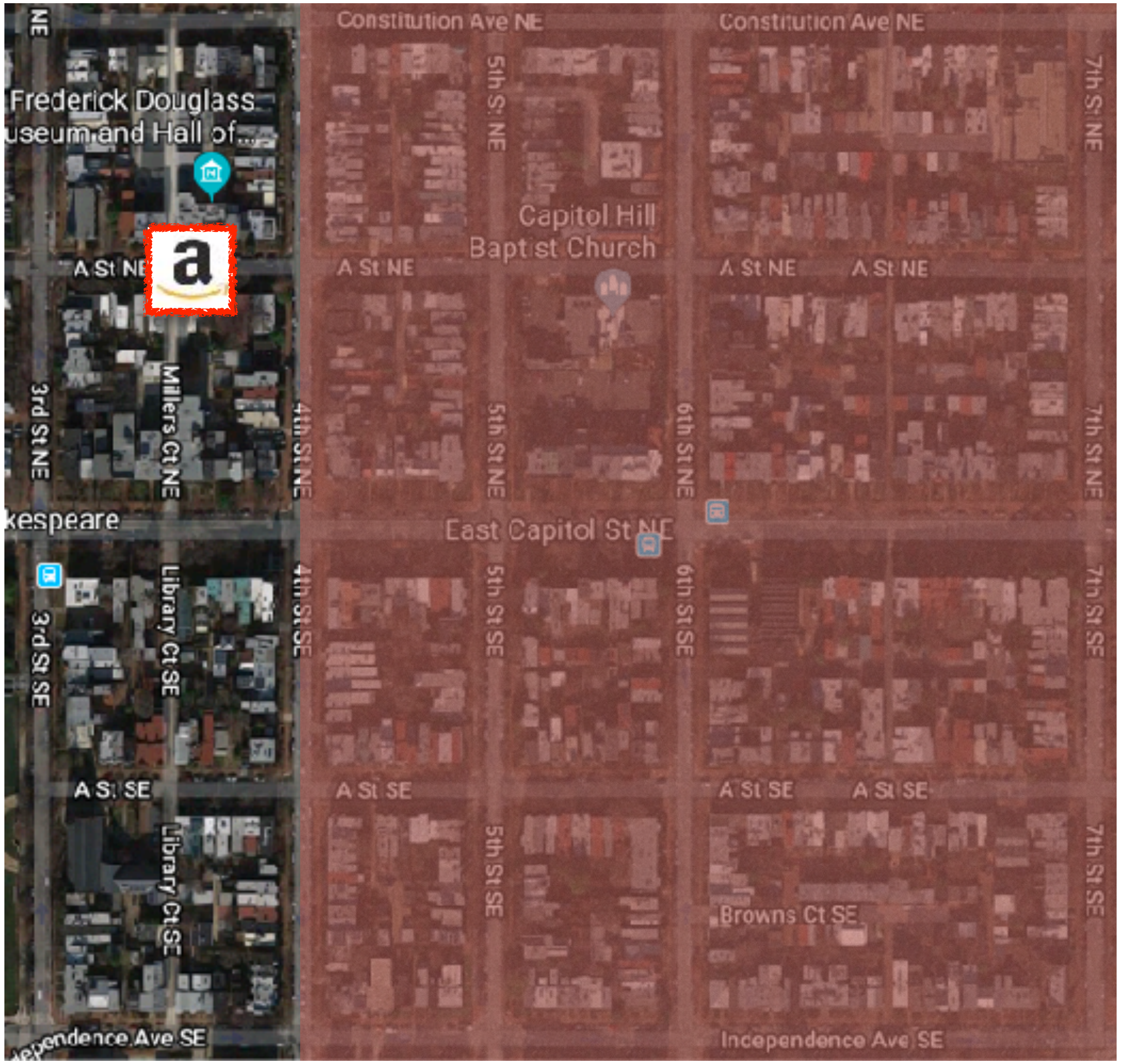

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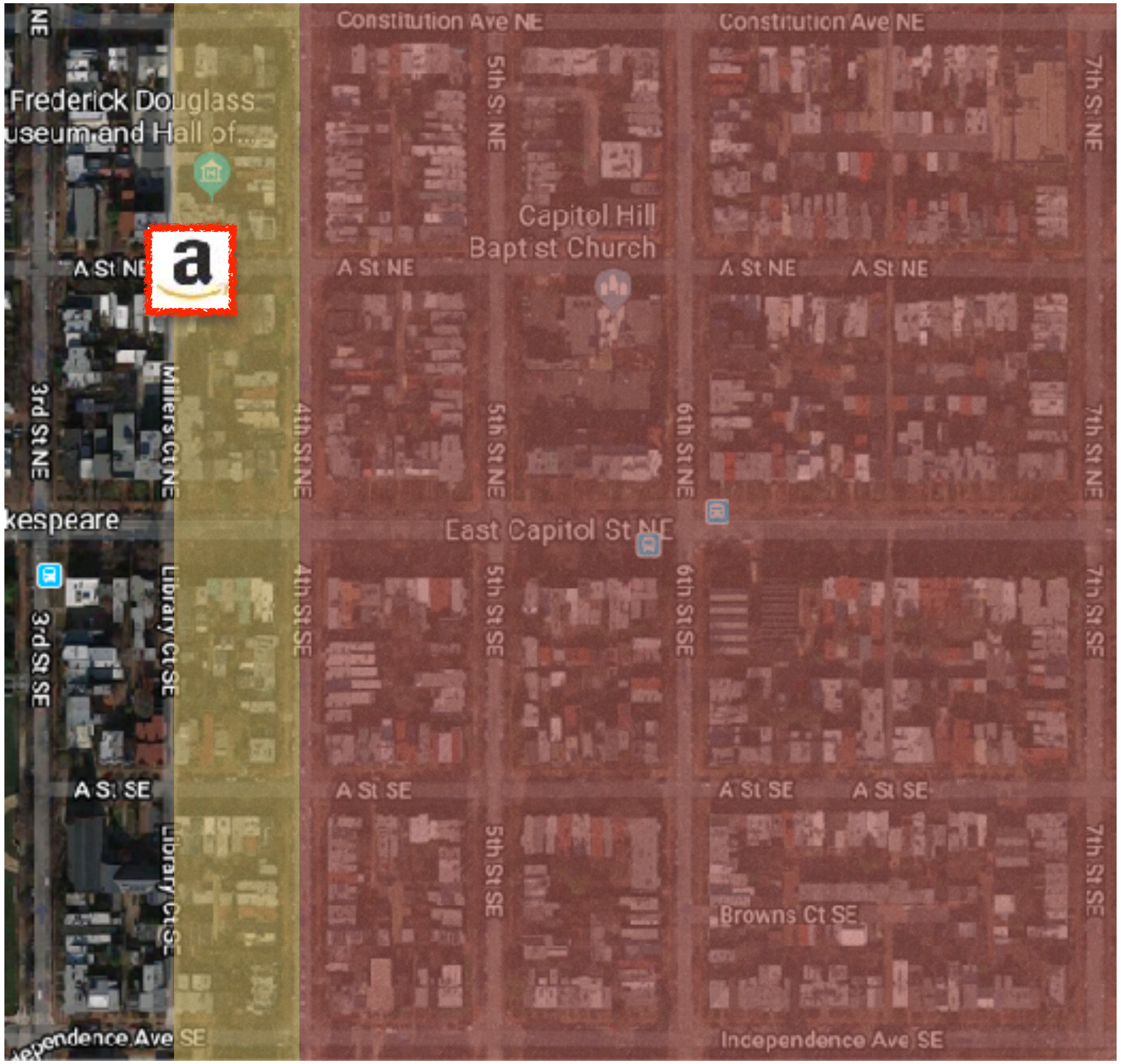

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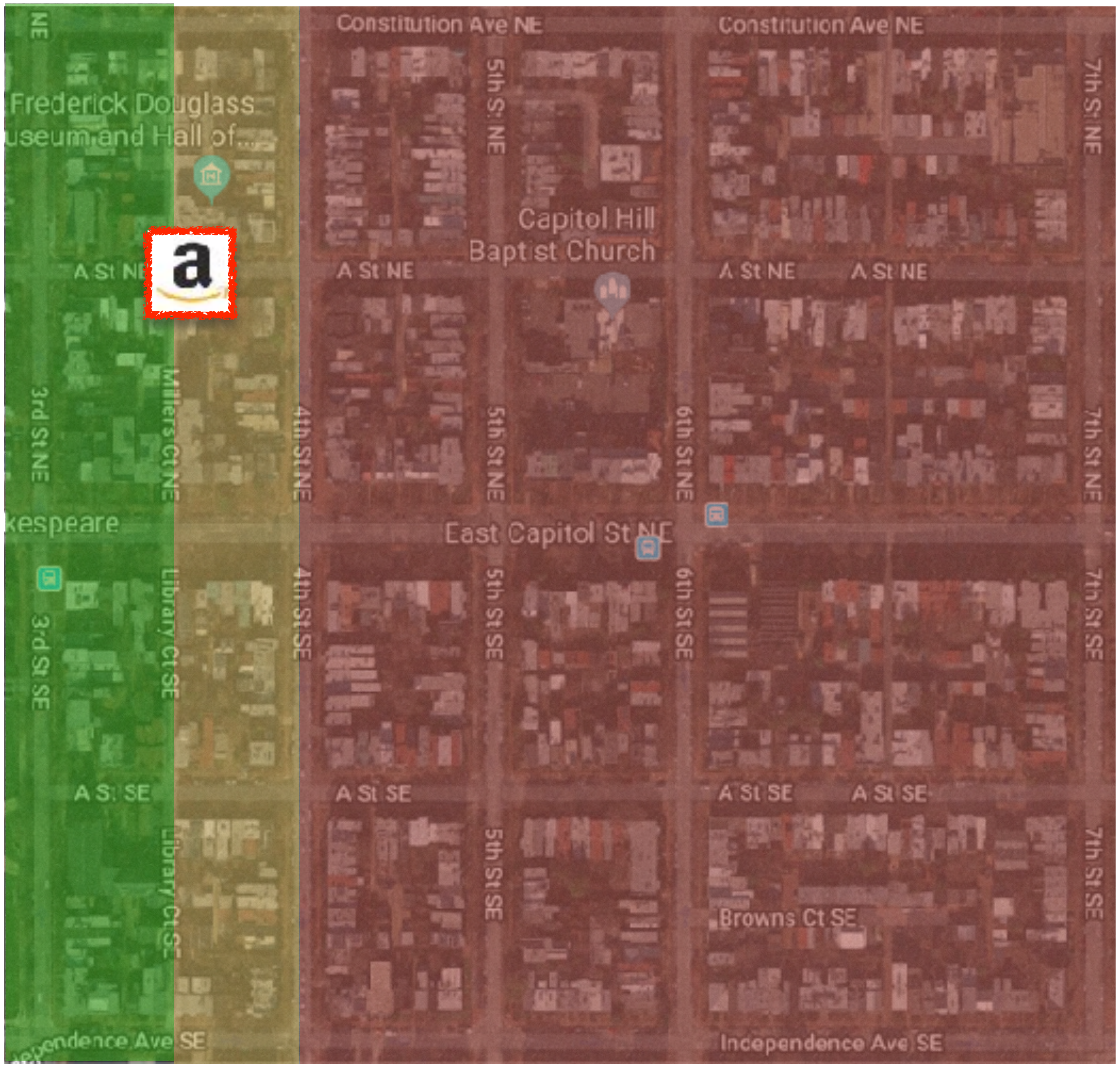

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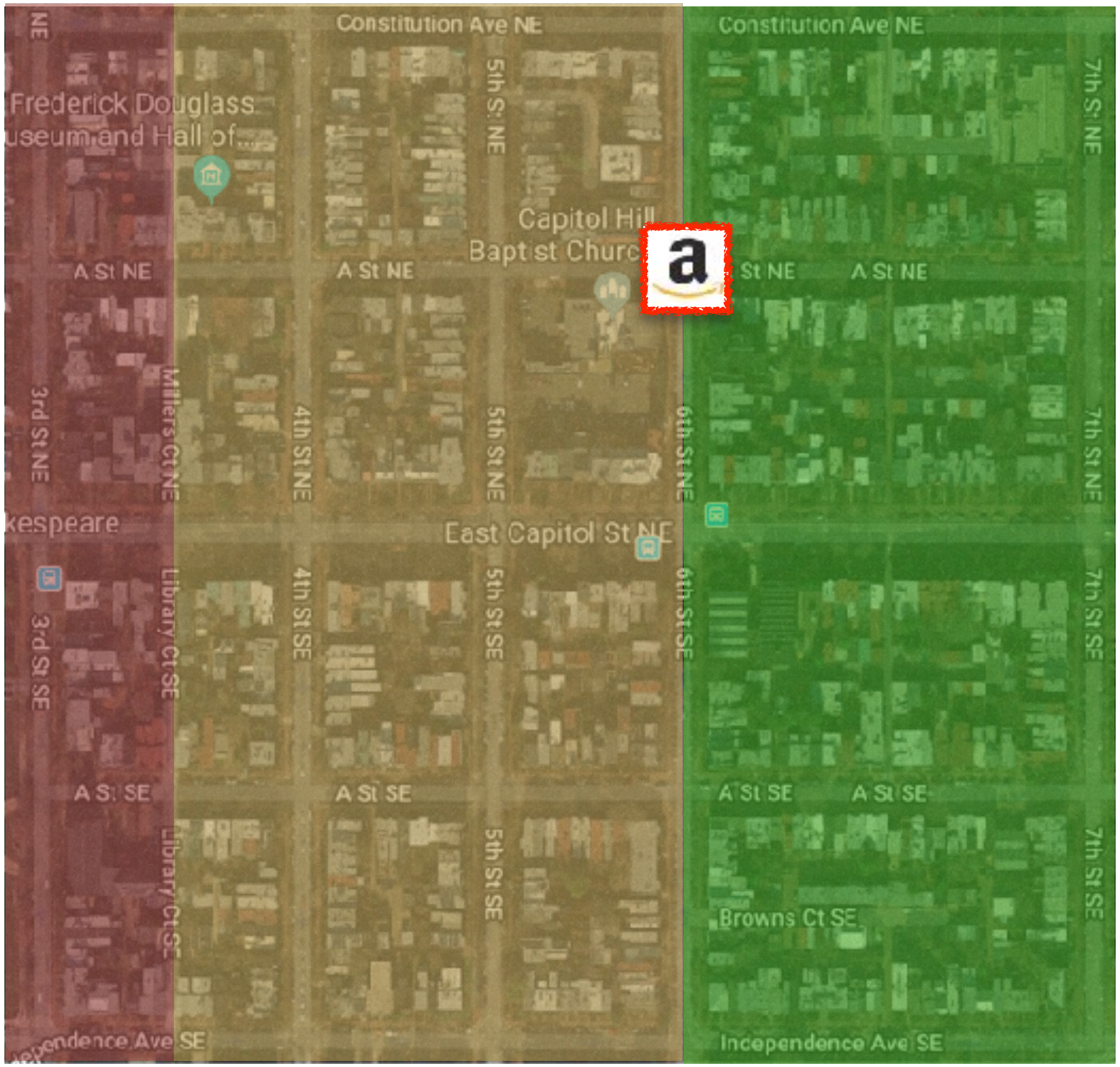

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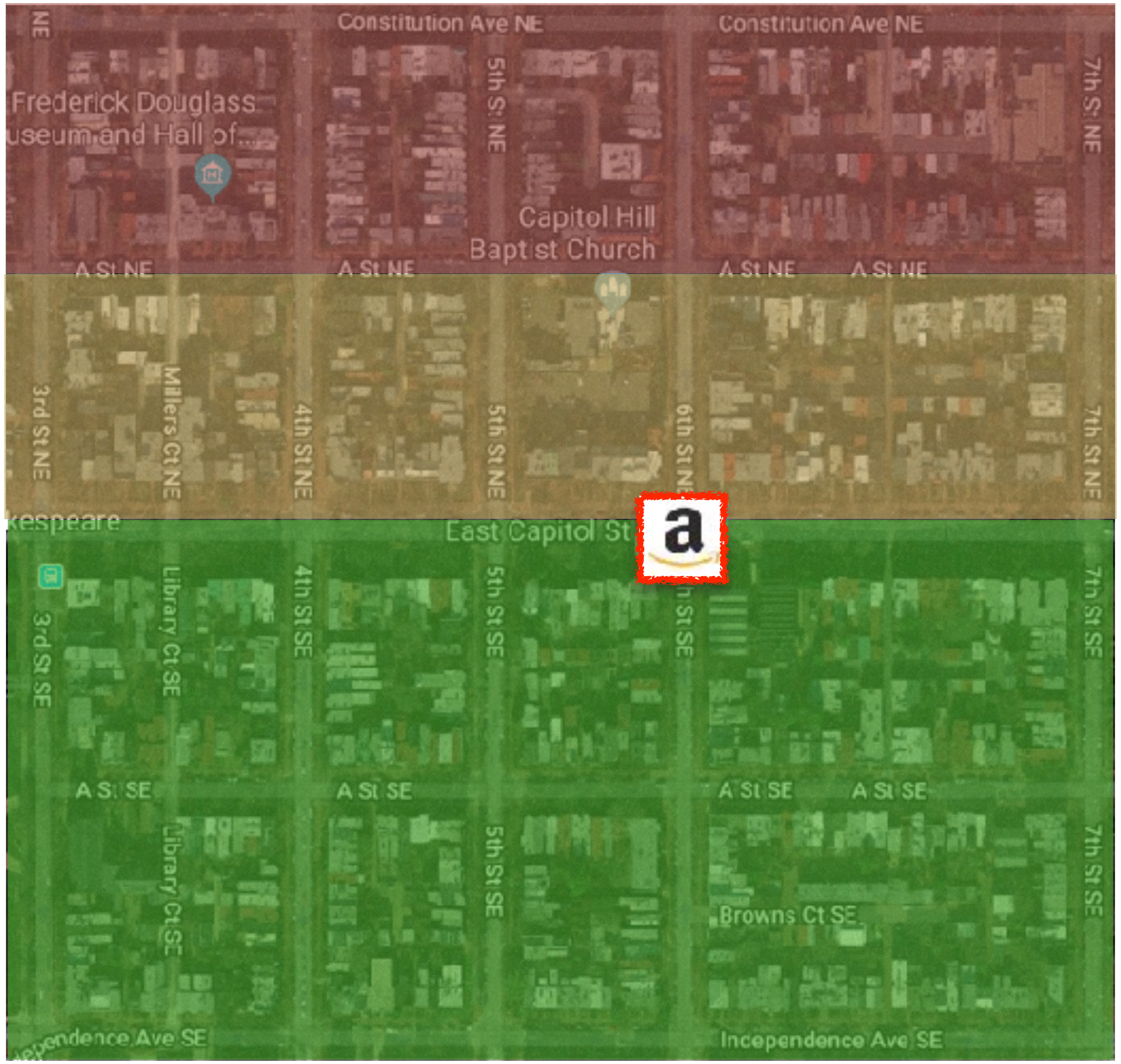

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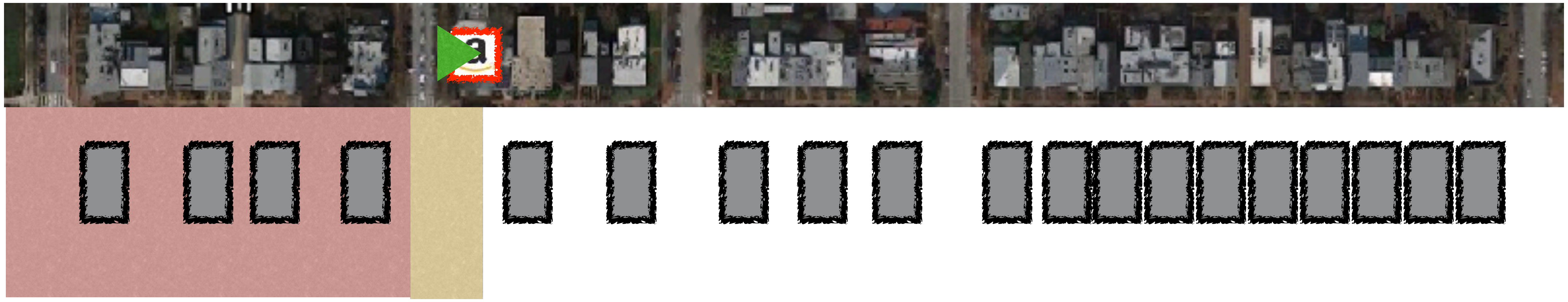

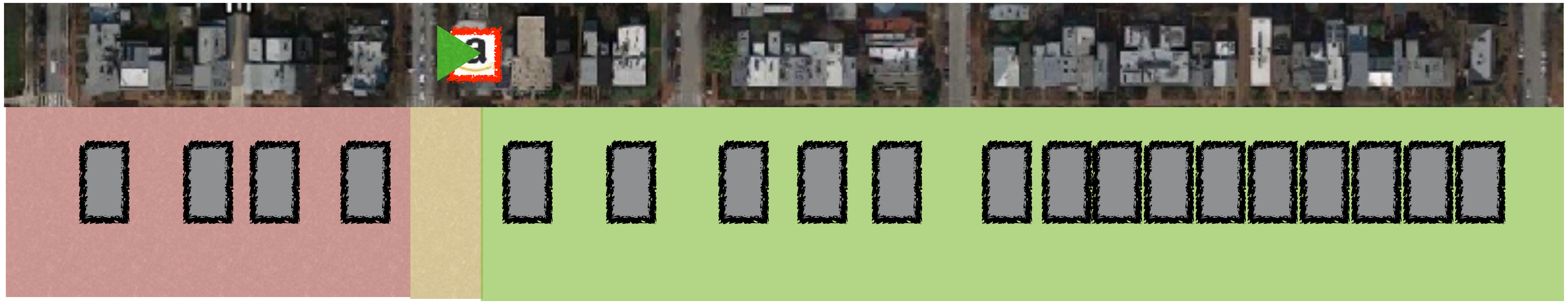

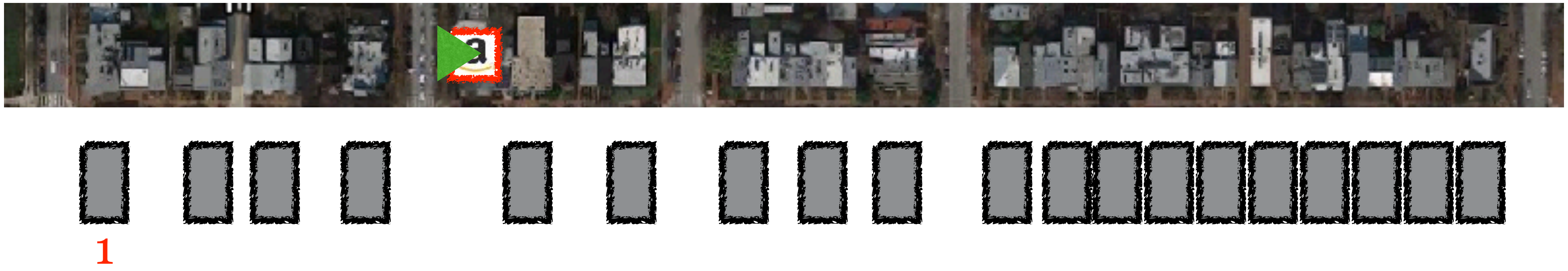
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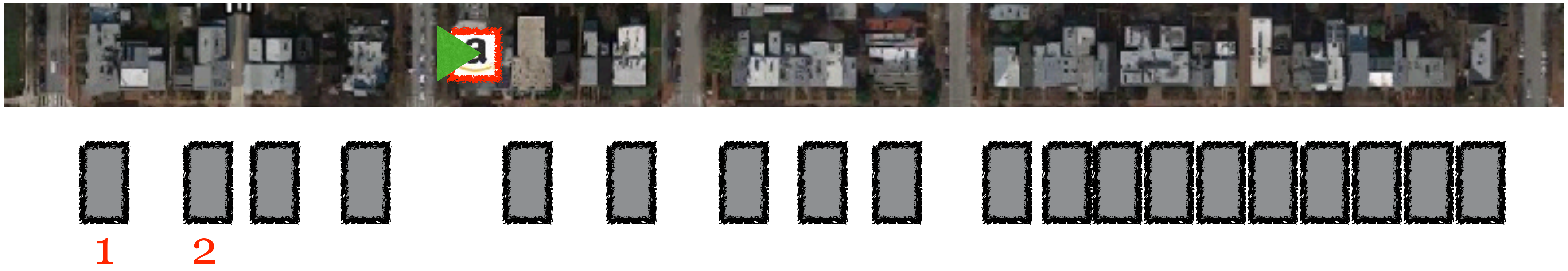
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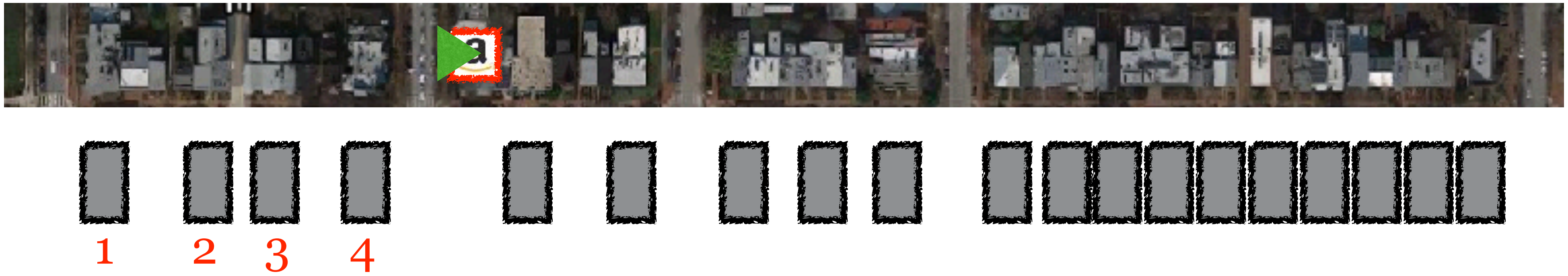
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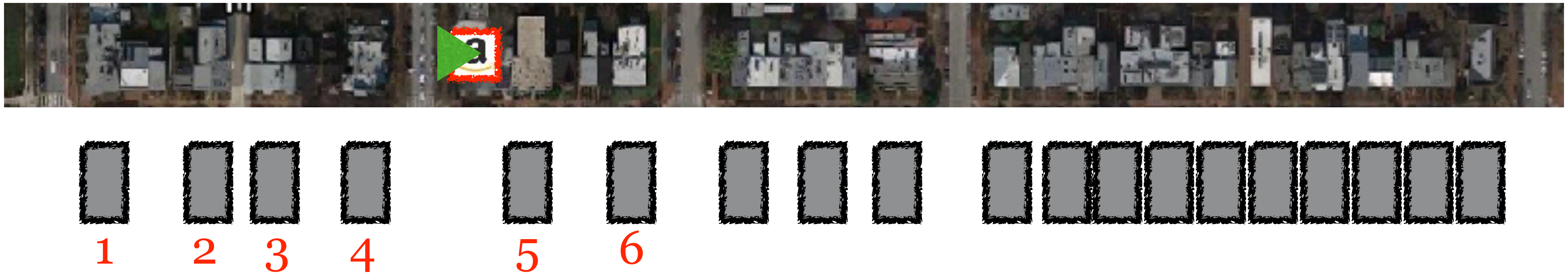
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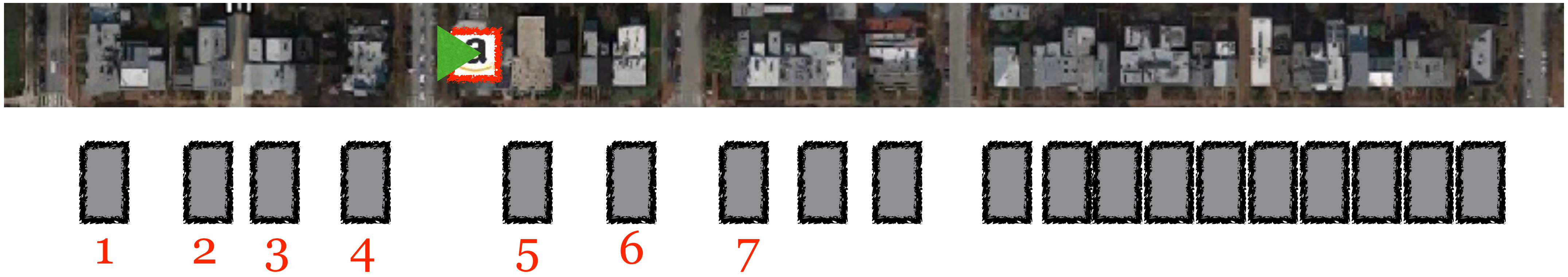
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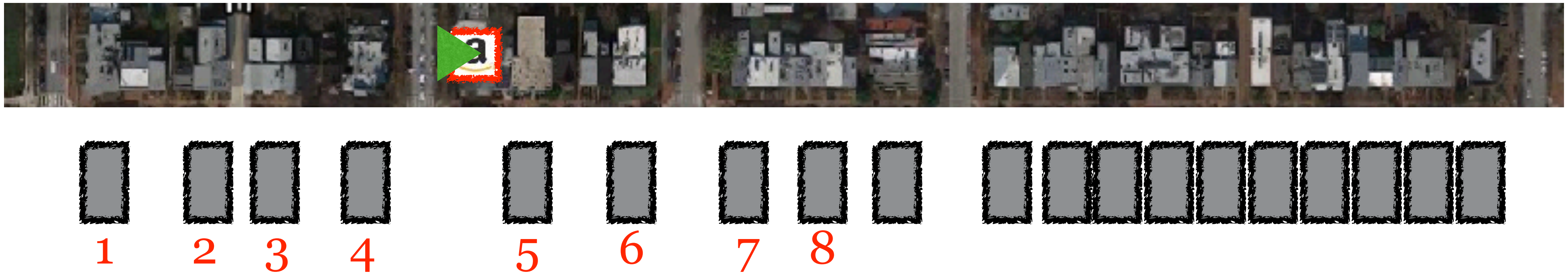
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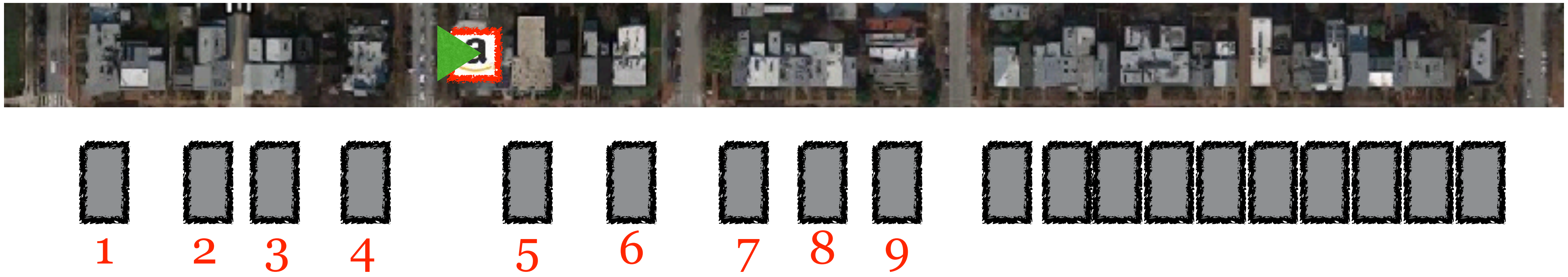
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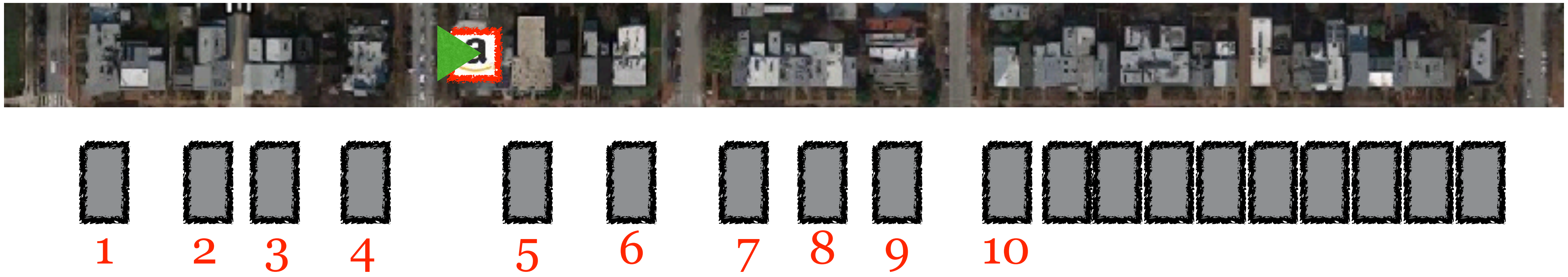
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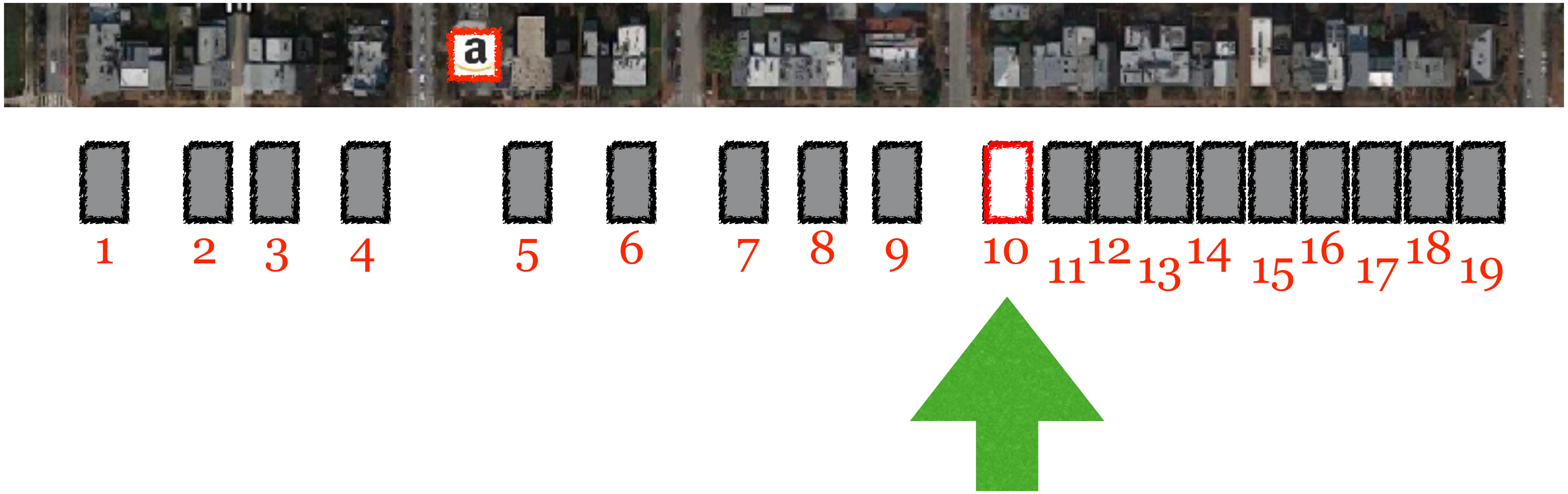
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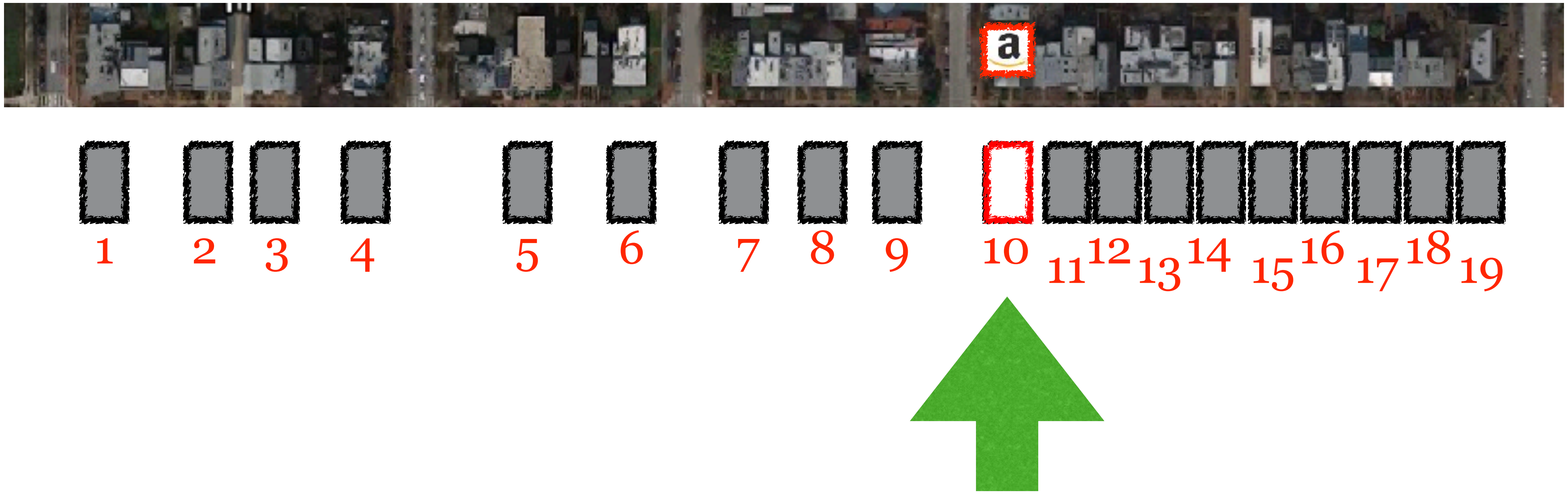

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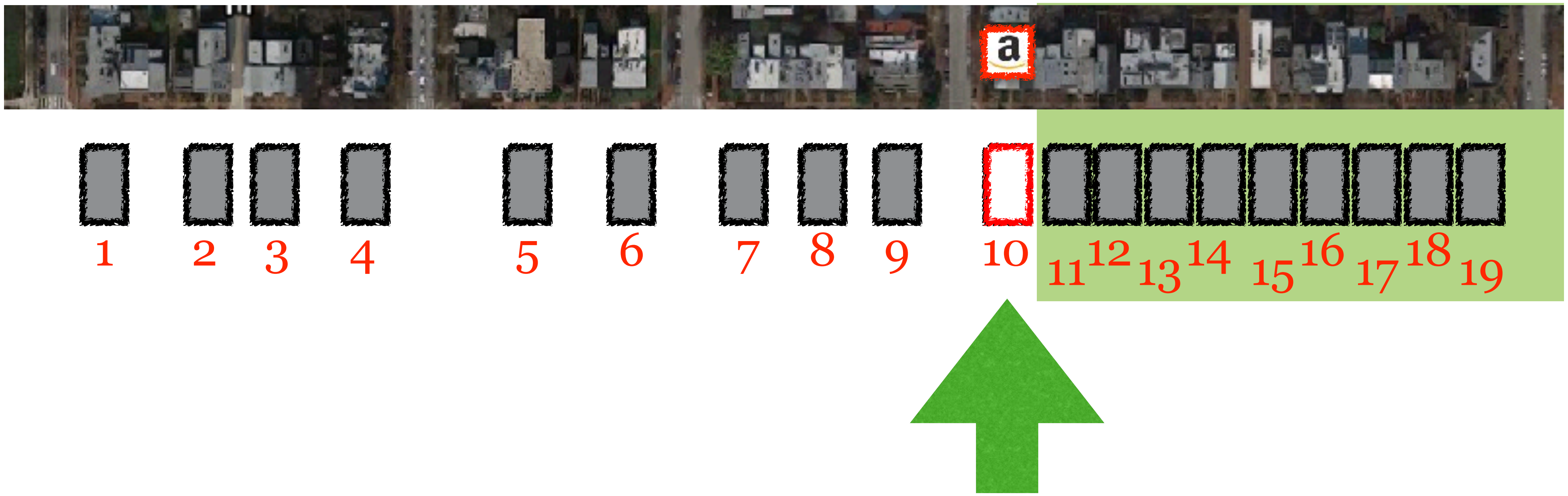
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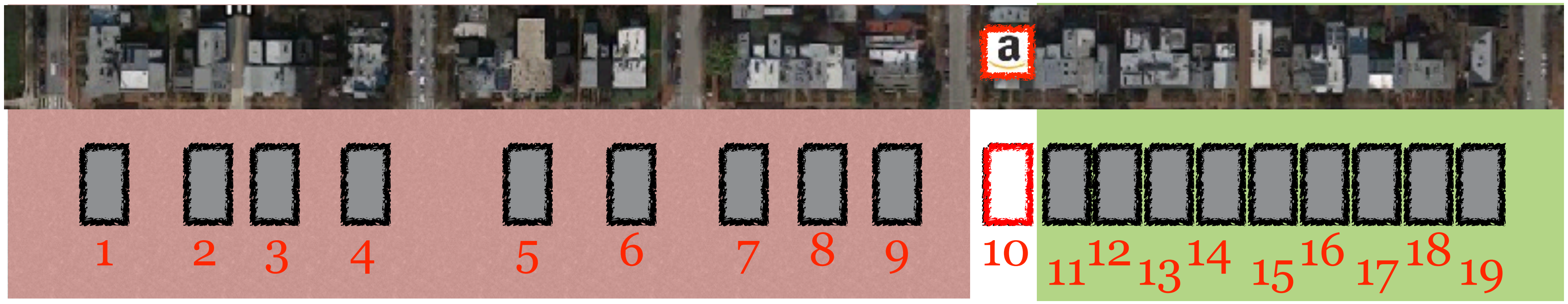
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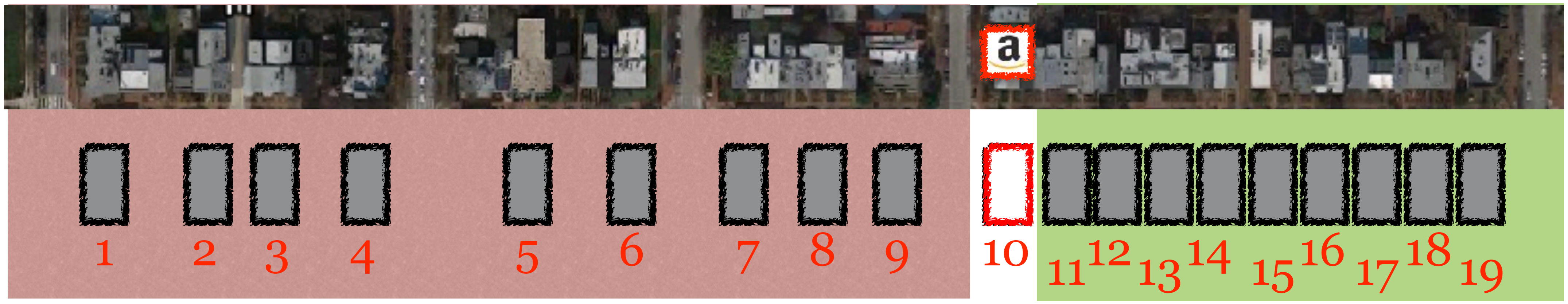
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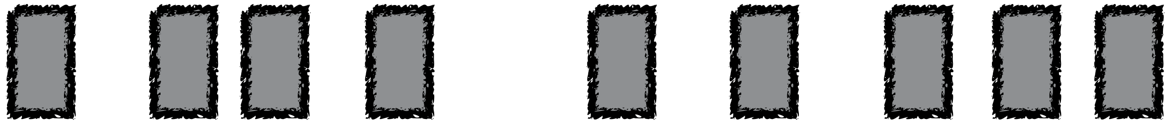


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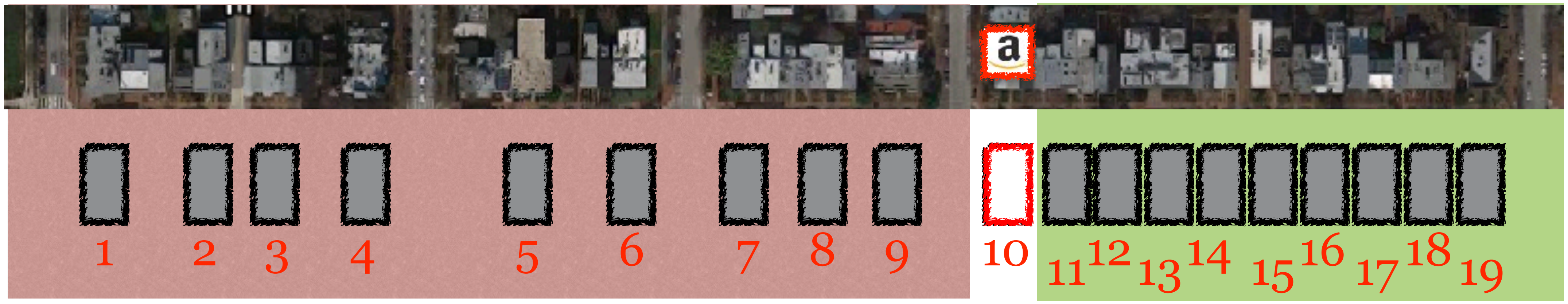
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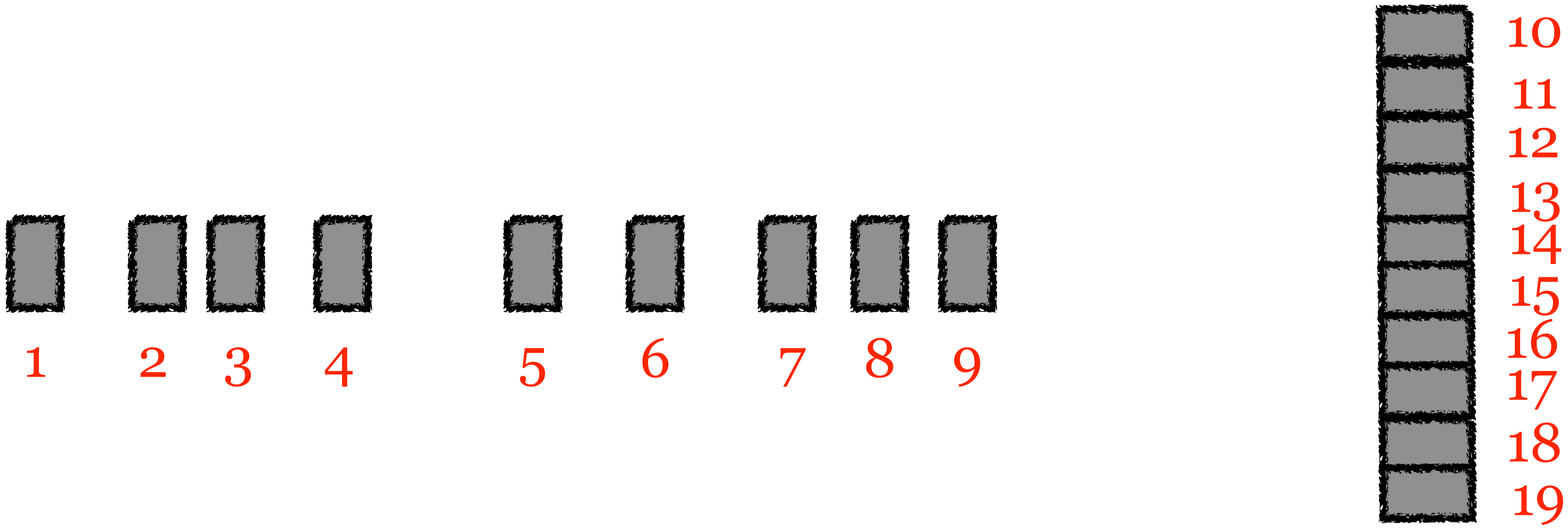
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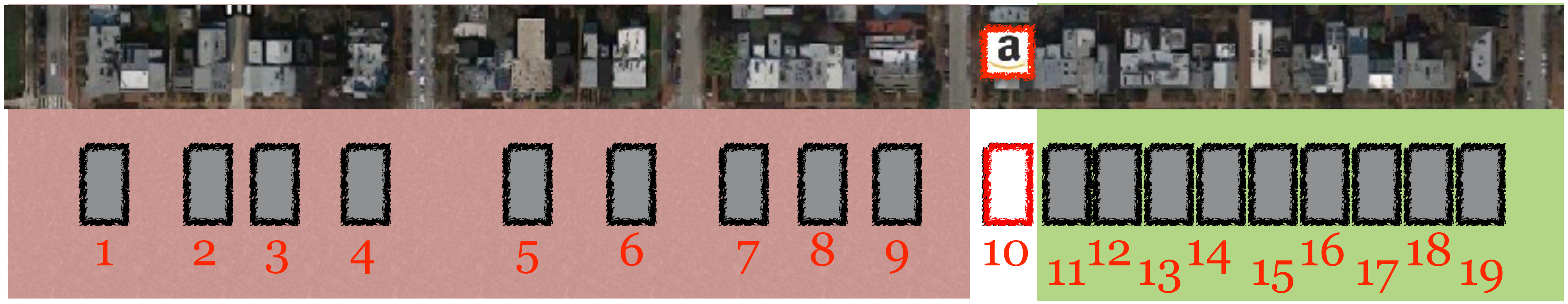

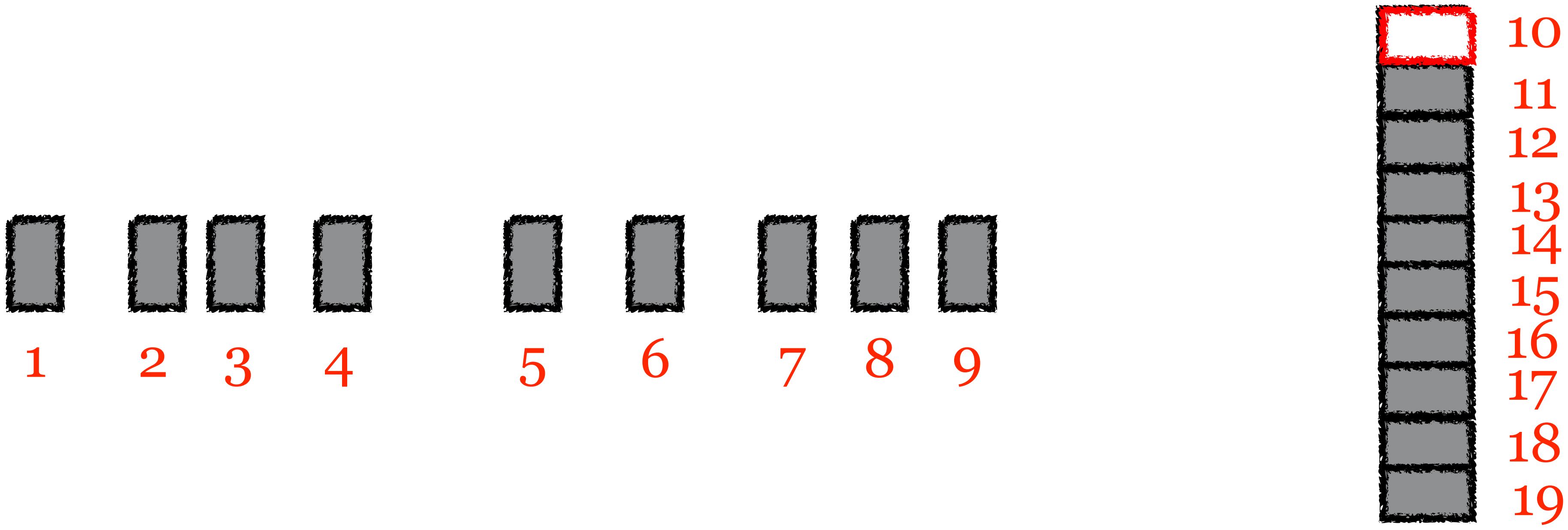

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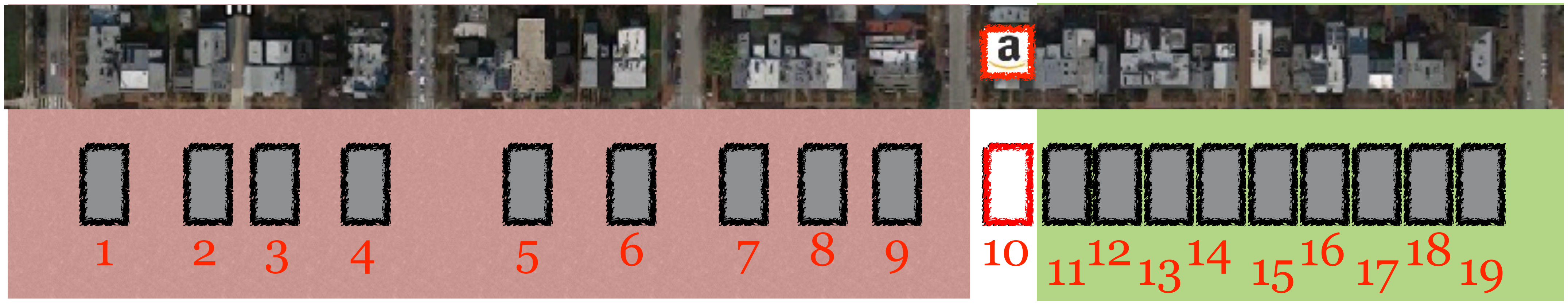
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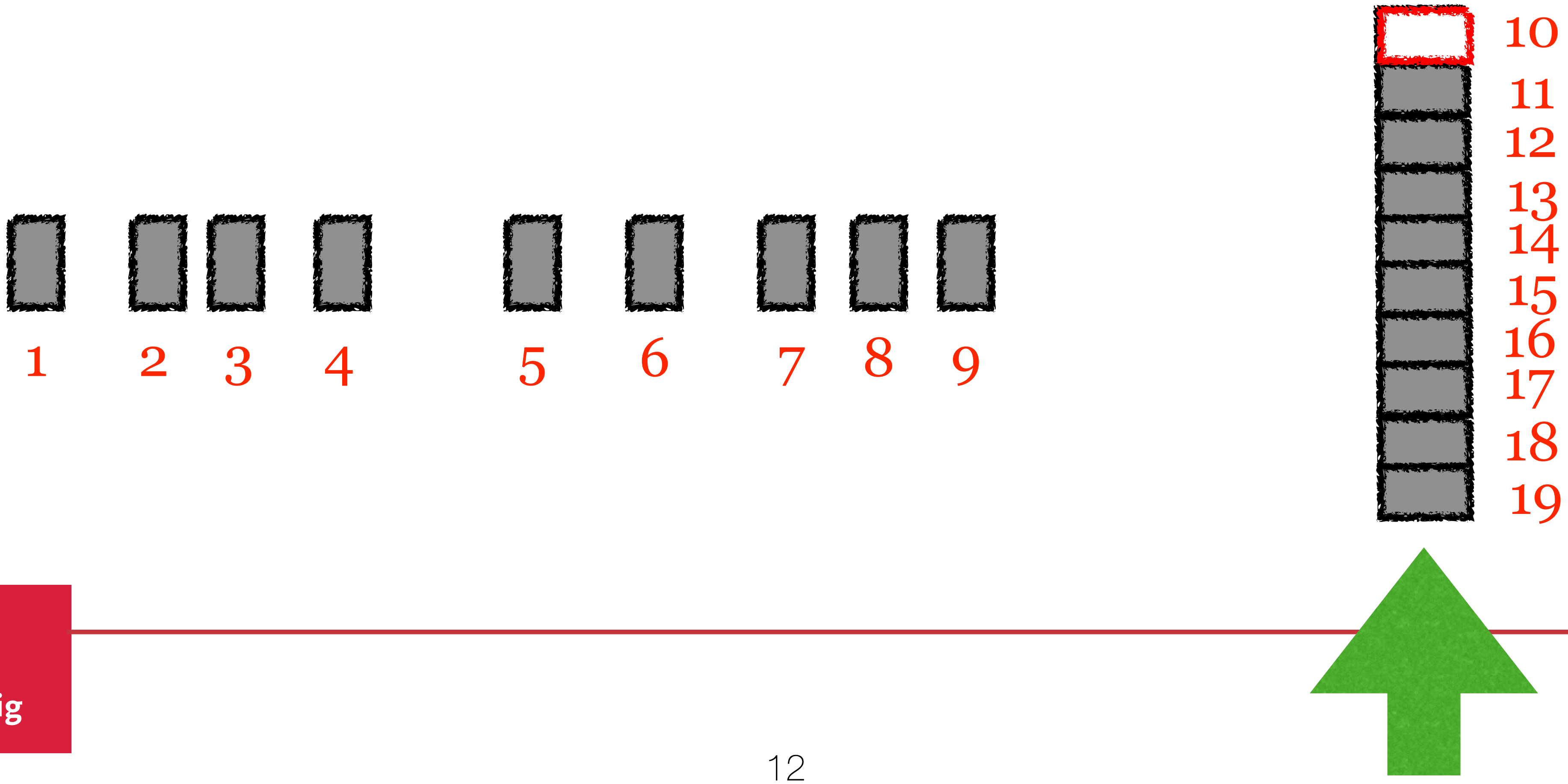
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1. Introduction
2. Manhattan distances
3. Euclidean distances
4. Galois and Bajaj
5. Continuous sets

Fermat again

Technische
Universität
Braunschweig

Fermat again

Problem 7.1:

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<u>Problem 7.1:</u>

Given: A set P of points in $\mathbb{R}^2$

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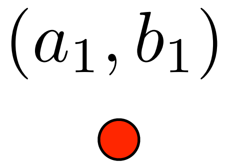

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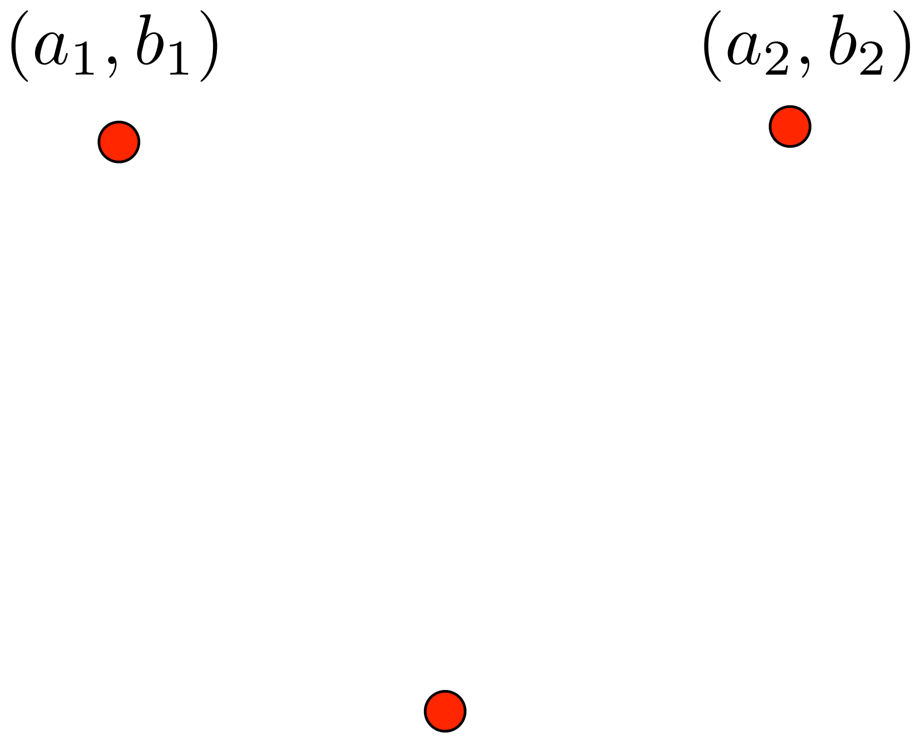

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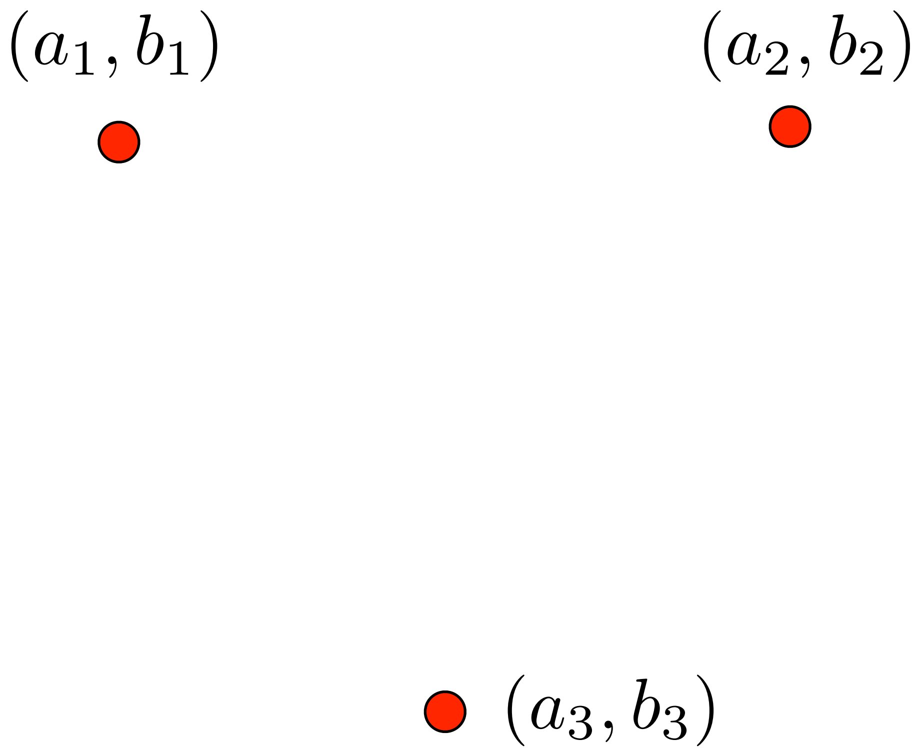

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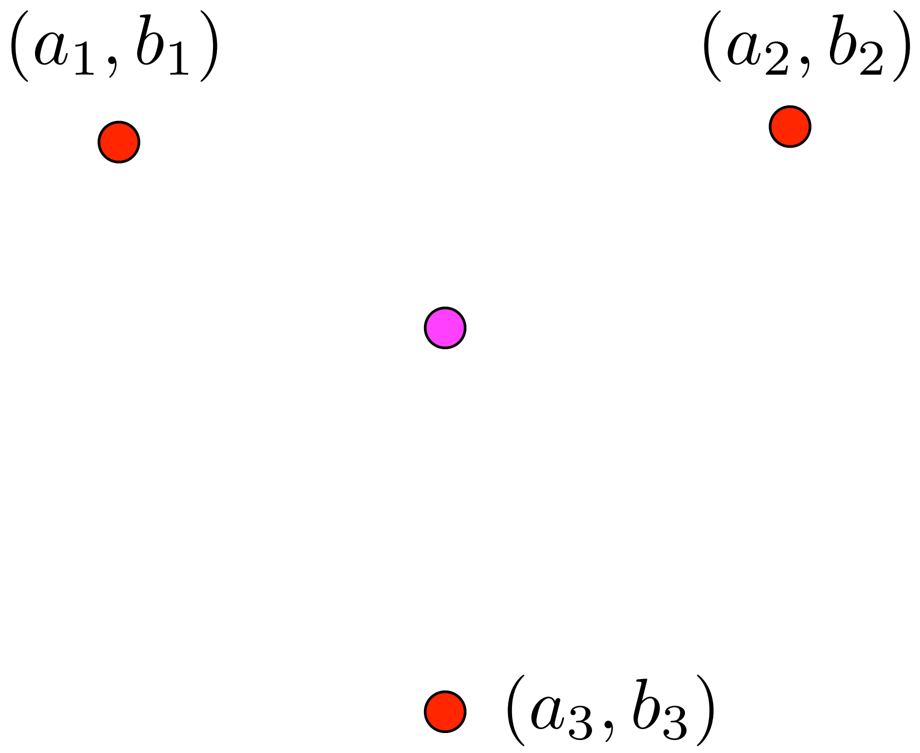

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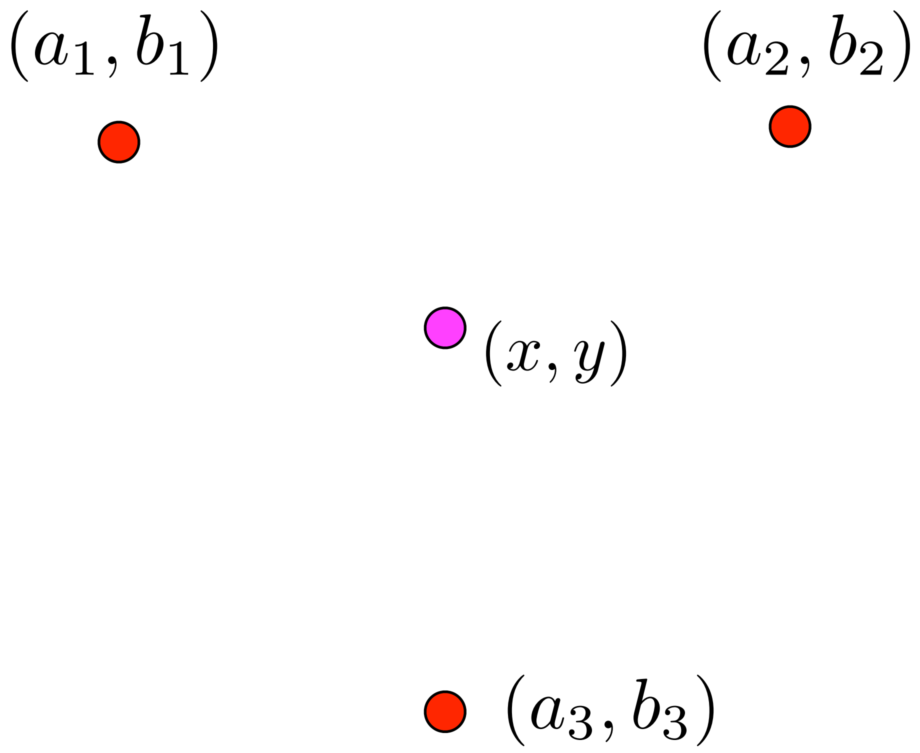

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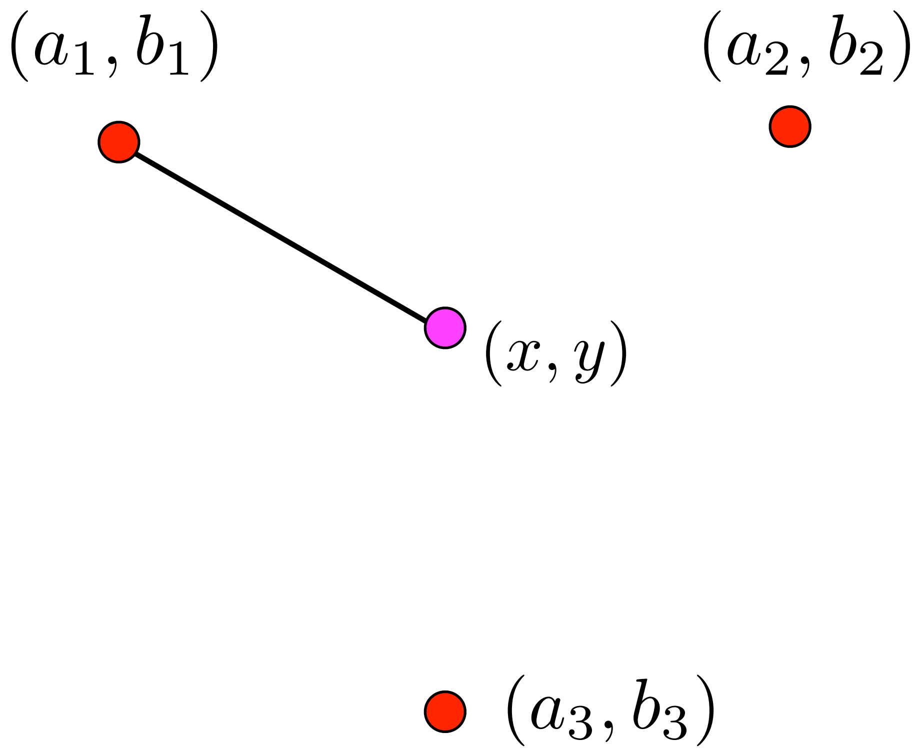

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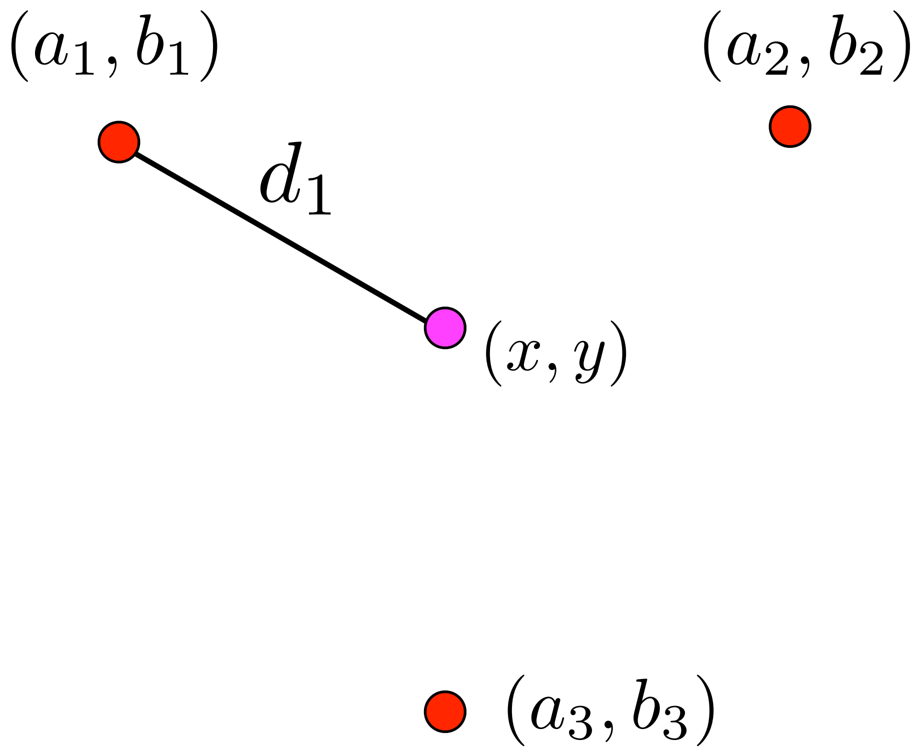

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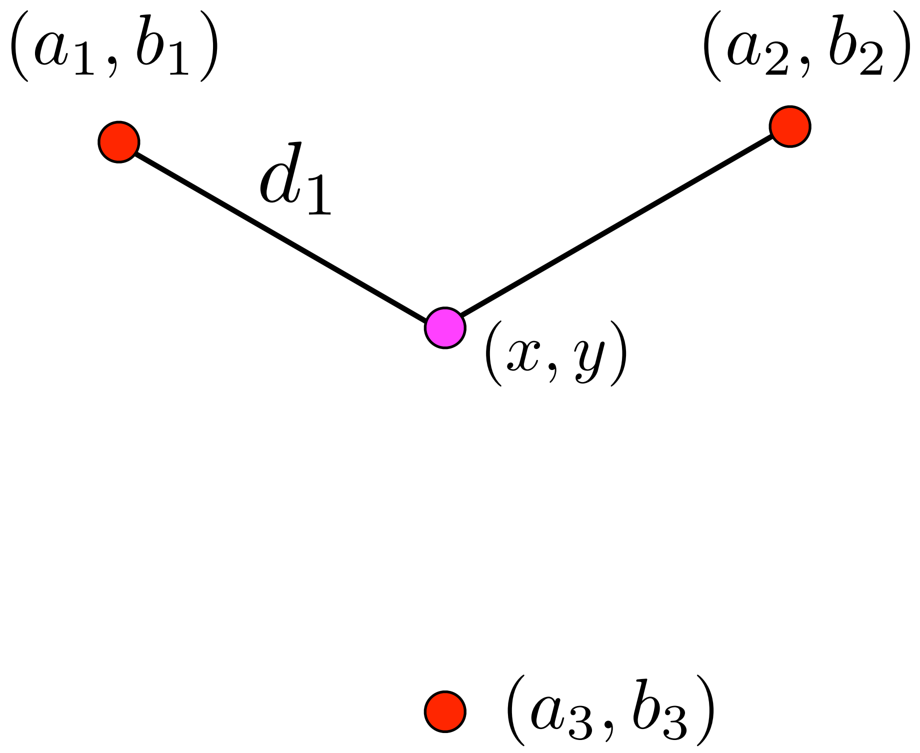

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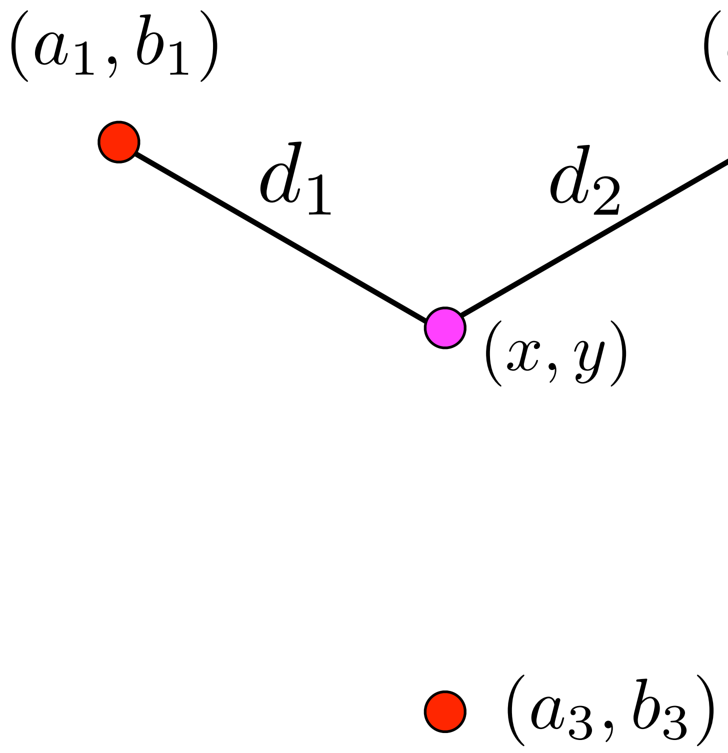

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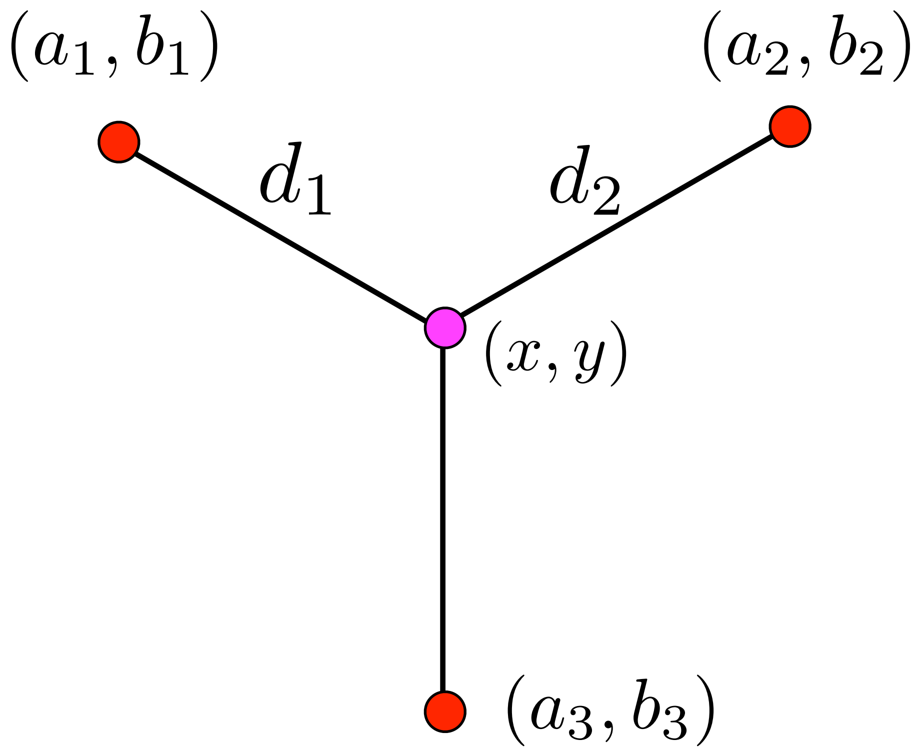

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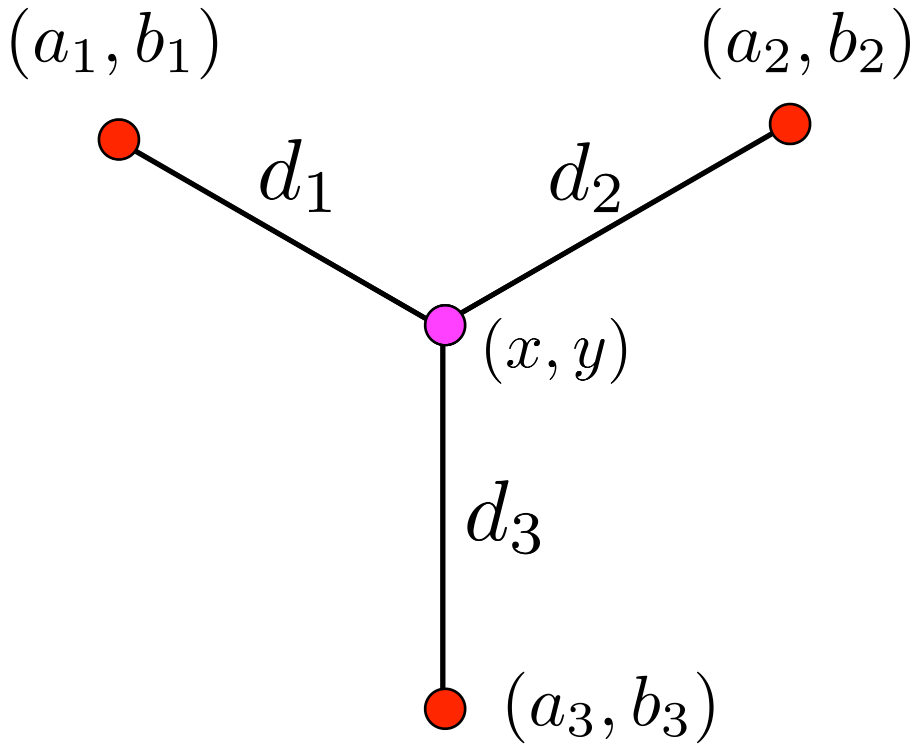

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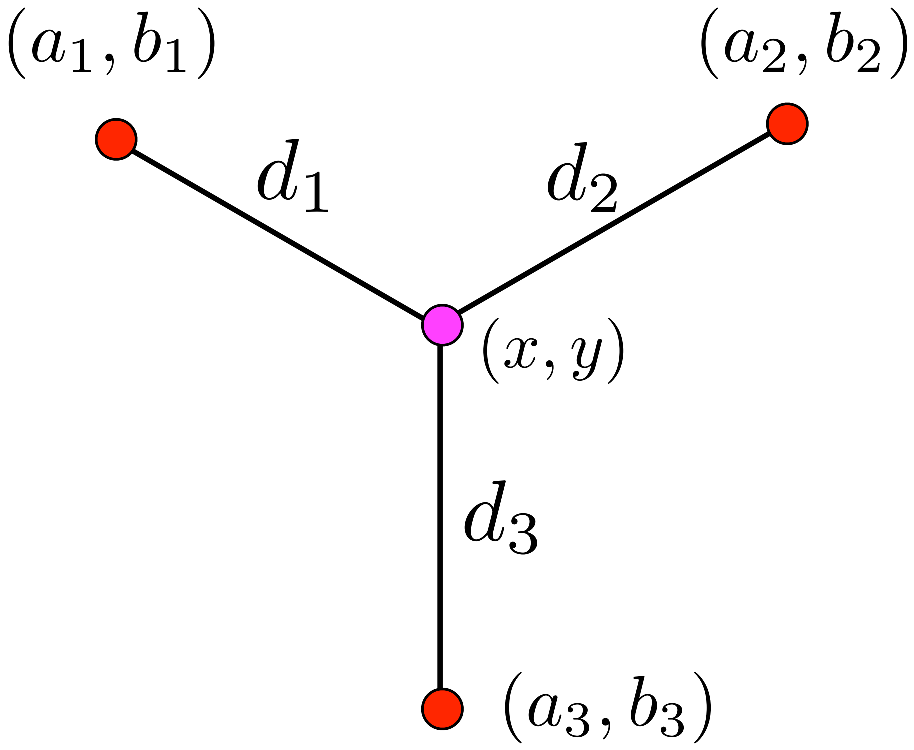

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Function:

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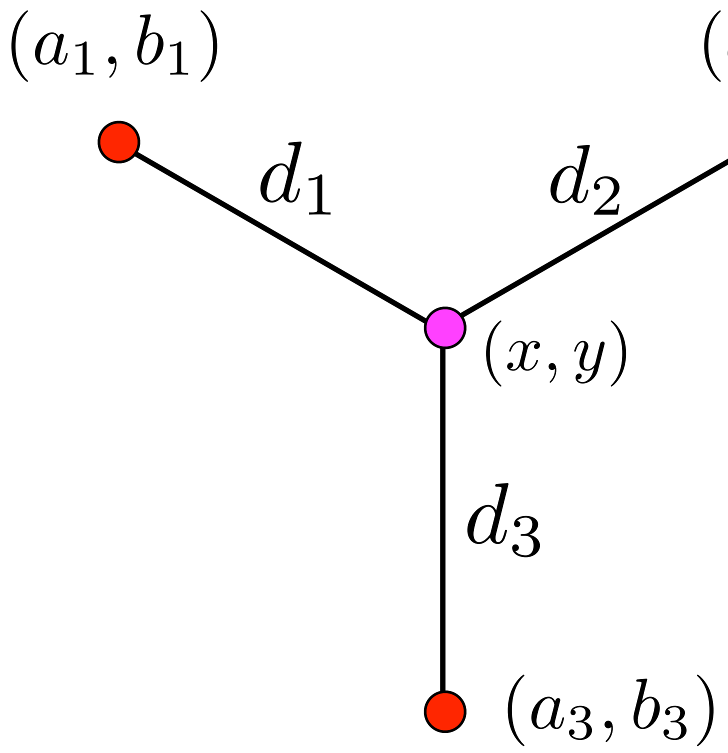

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Function:

$$f(x,y) = d_1 + d_2 + d_3 = \sqrt{(x-a_1)^2 + (y-b_1)^2} + \sqrt{(x-a_2)^2 + (y-b_2)^2} + \sqrt{(x-a_3)^2 + (y-b_3)^2}.$$

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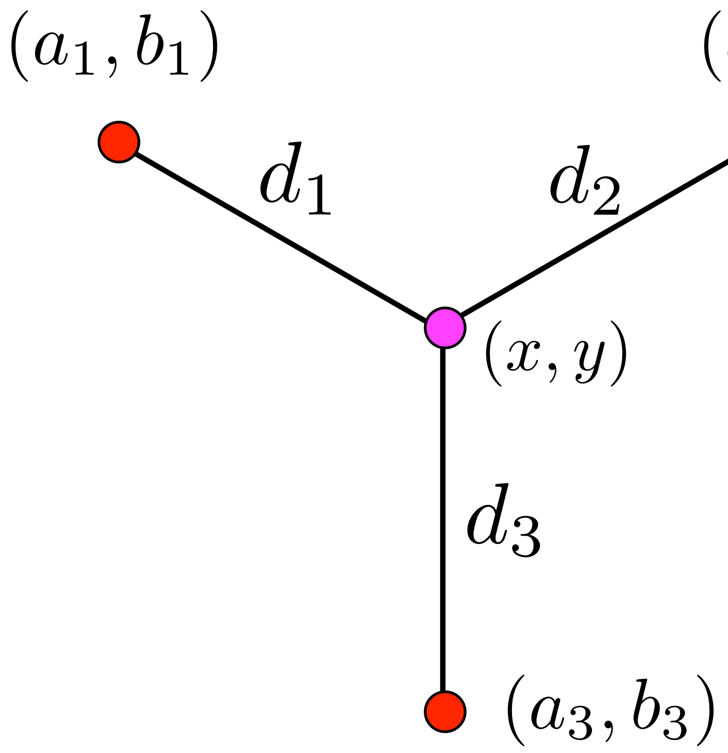

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Wanted: A location c that minimizes the total distance to the given points

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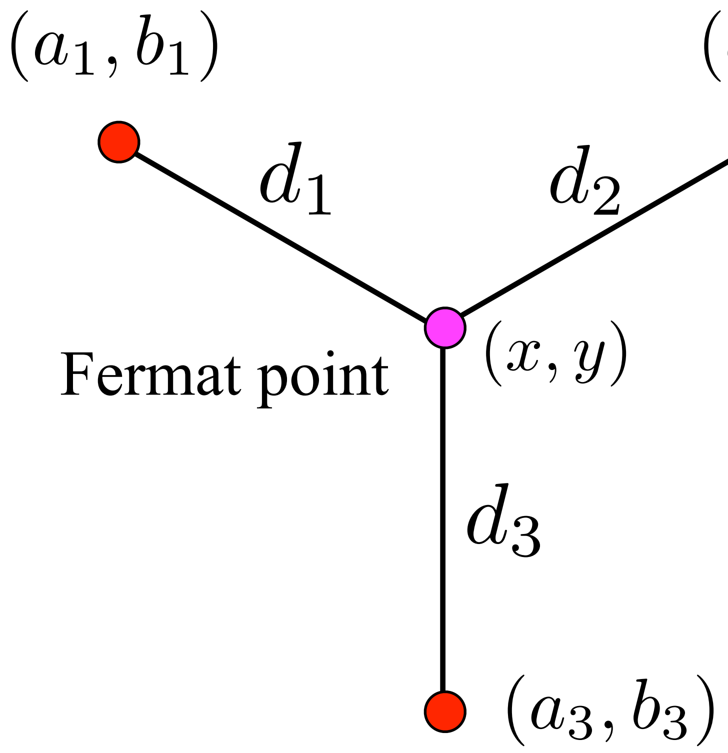

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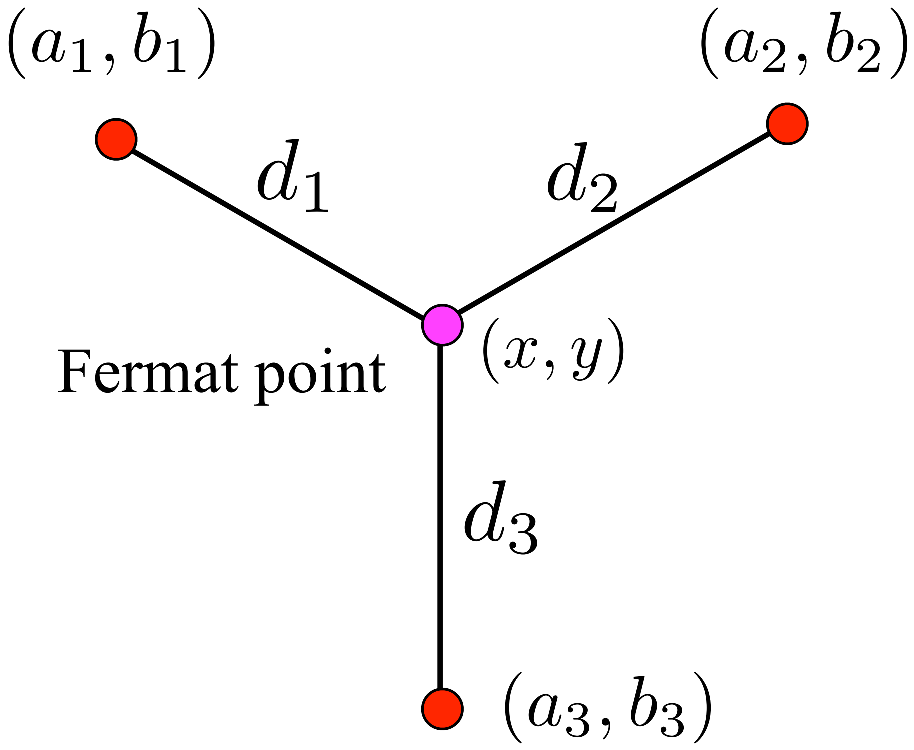

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Partial derivatives:

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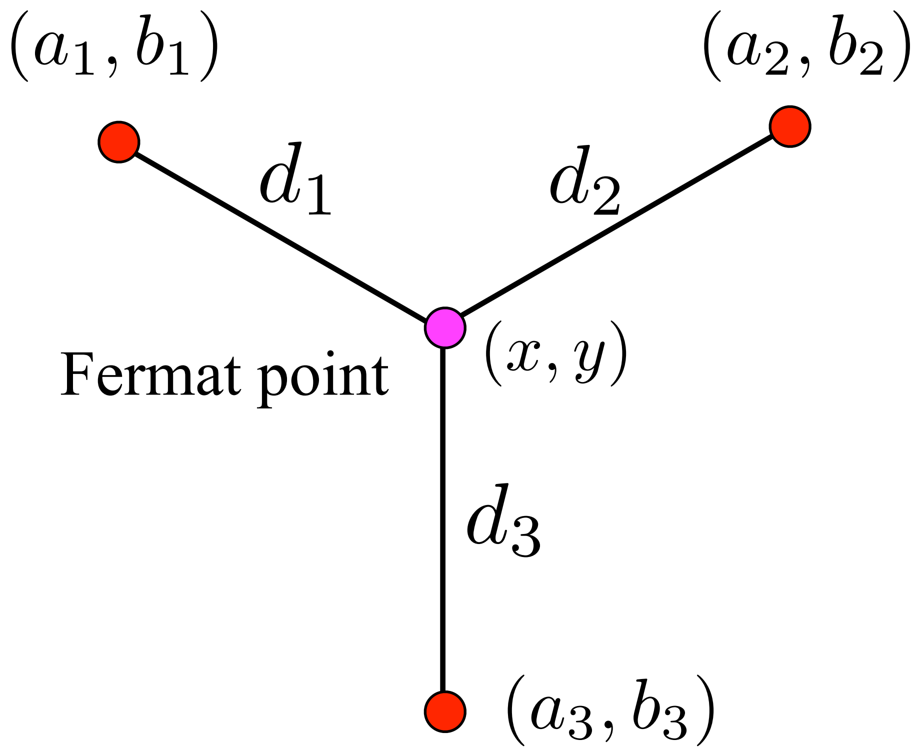

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$$\frac{\partial f(x,y)}{\partial x} = \frac{(x-a_1)}{\sqrt{(x-a_1)^2 + (y-b_1)^2}} + \frac{(x-a_2)}{\sqrt{(x-a_2)^2 + (y-b_2)^2}} + \frac{(x-a_3)}{\sqrt{(x-a_3)^2 + (y-b_3)^2}}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{(y-b_1)}{\sqrt{(x-a_1)^2 + (y-b_1)^2}} + \frac{(y-b_2)}{\sqrt{(x-a_2)^2 + (y-b_2)^2}} + \frac{(y-b_3)}{\sqrt{(x-a_3)^2 + (y-b_3)^2}}$$

Derivatives geometrically

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2D derivative: (negative) gradient!

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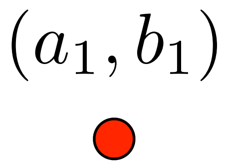

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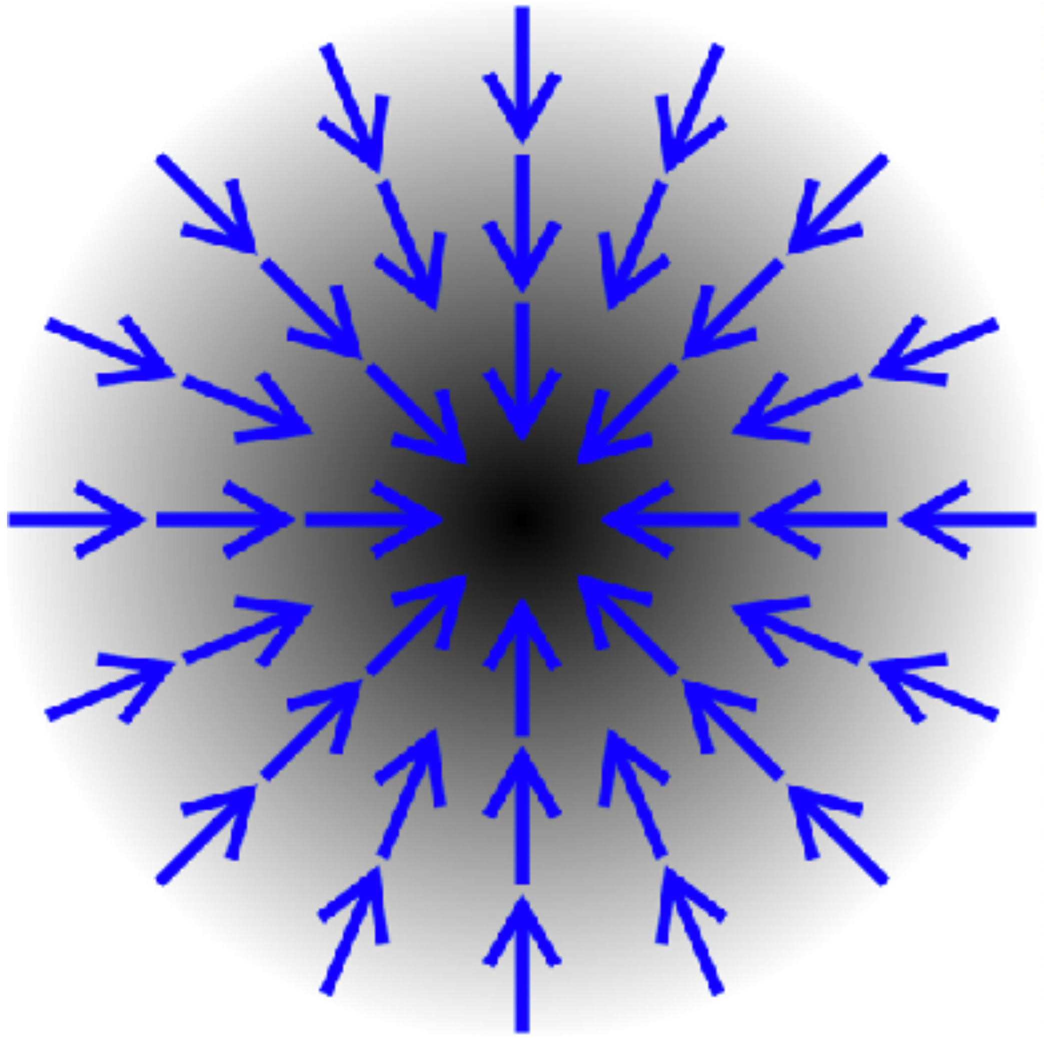

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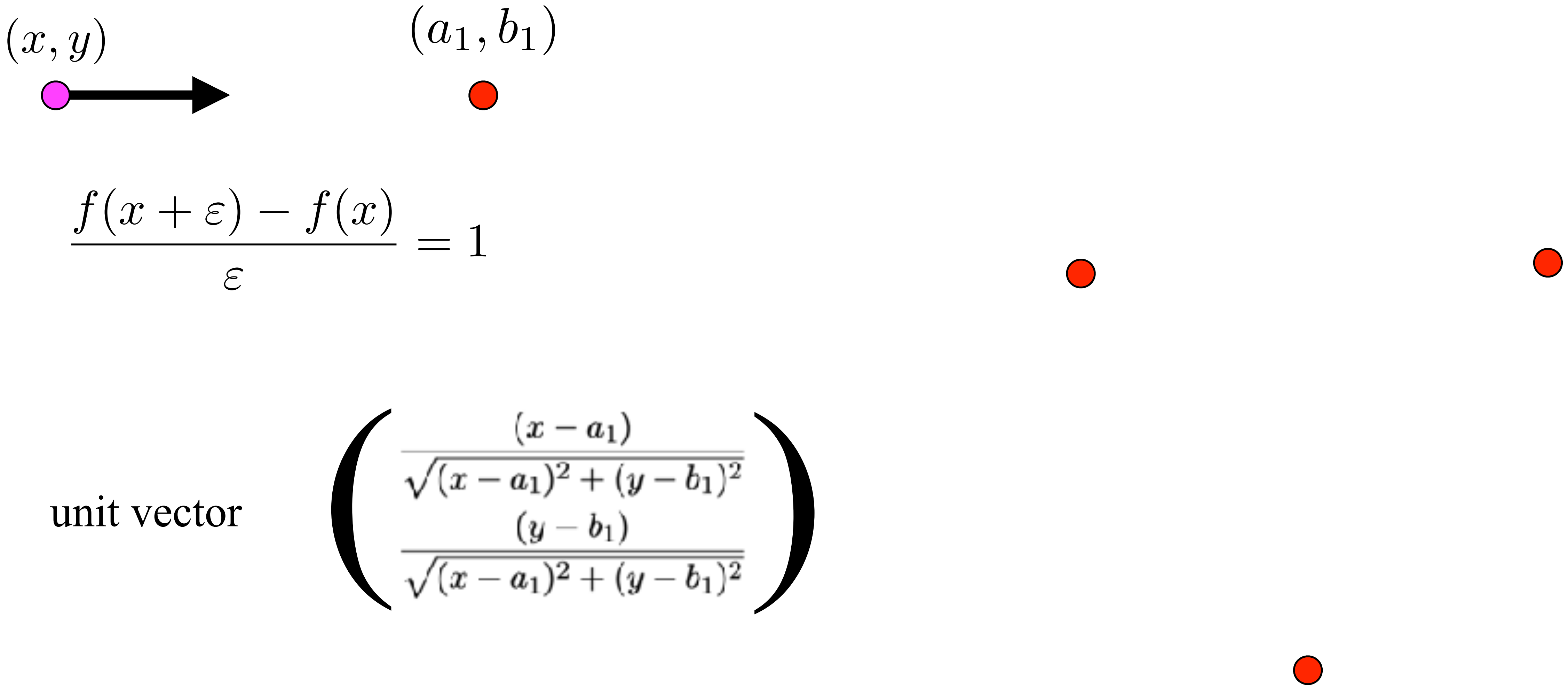

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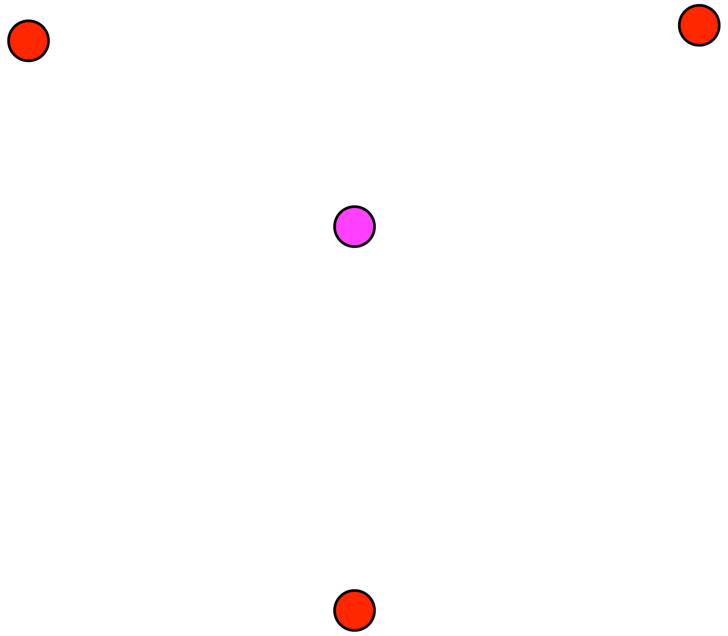

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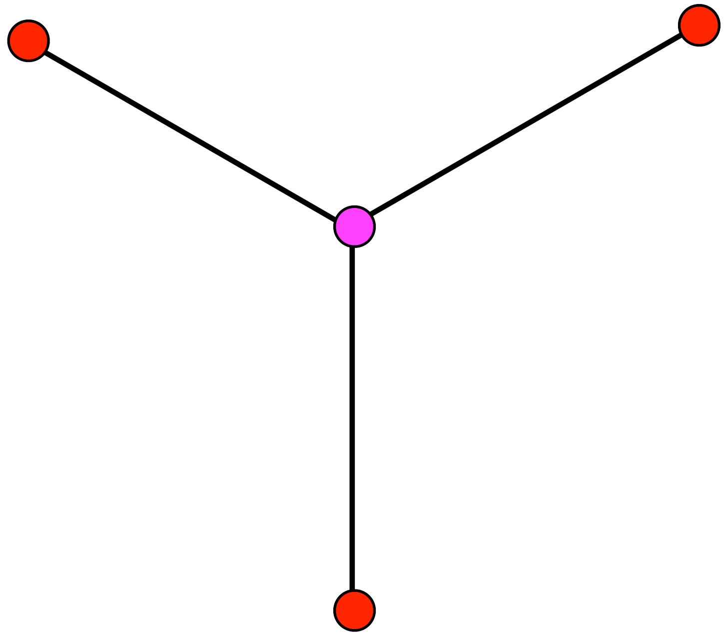

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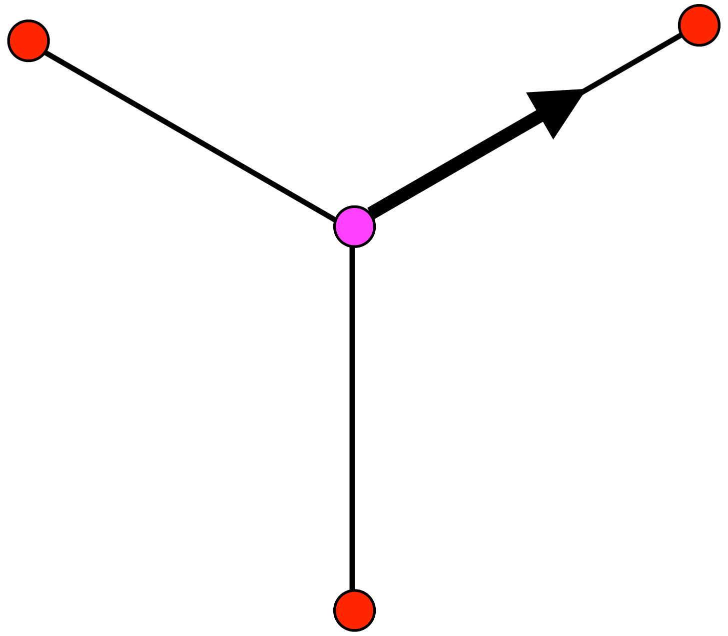

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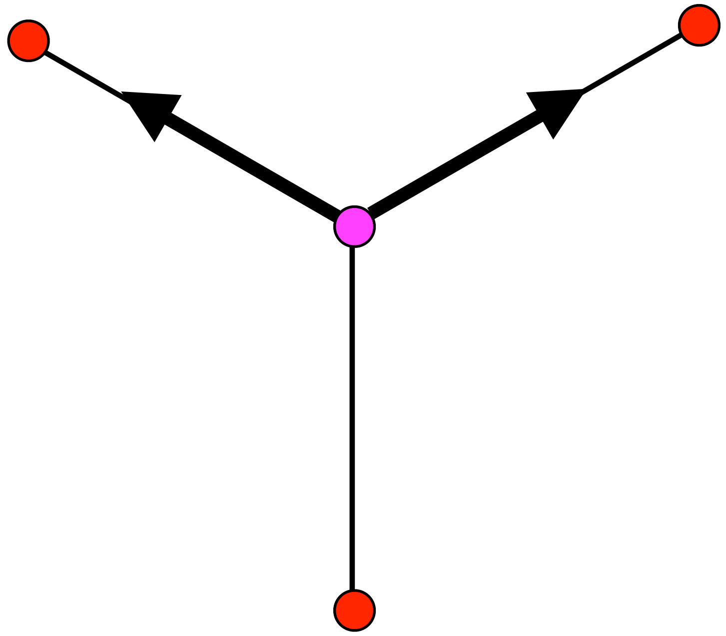

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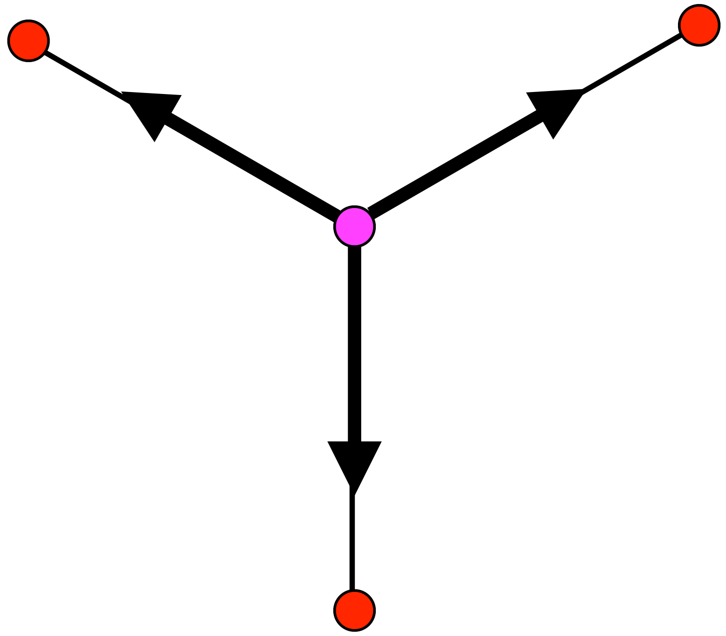

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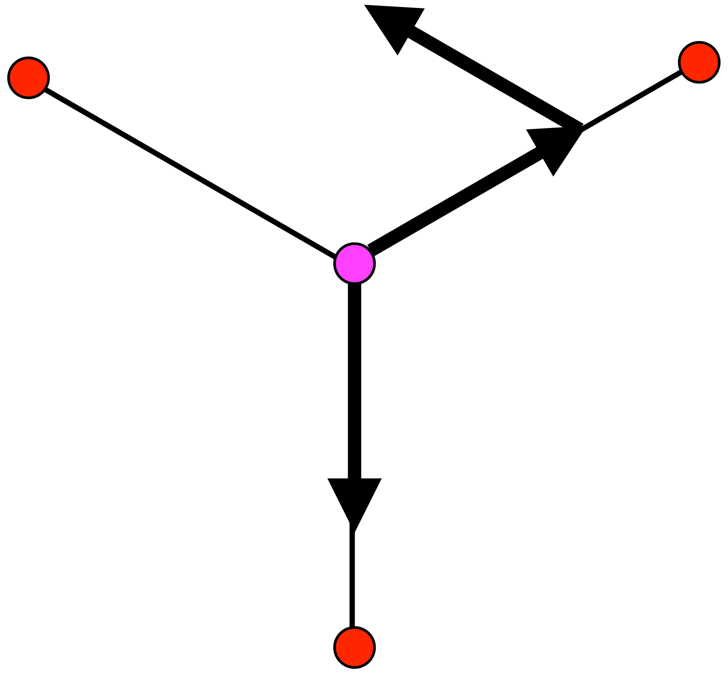

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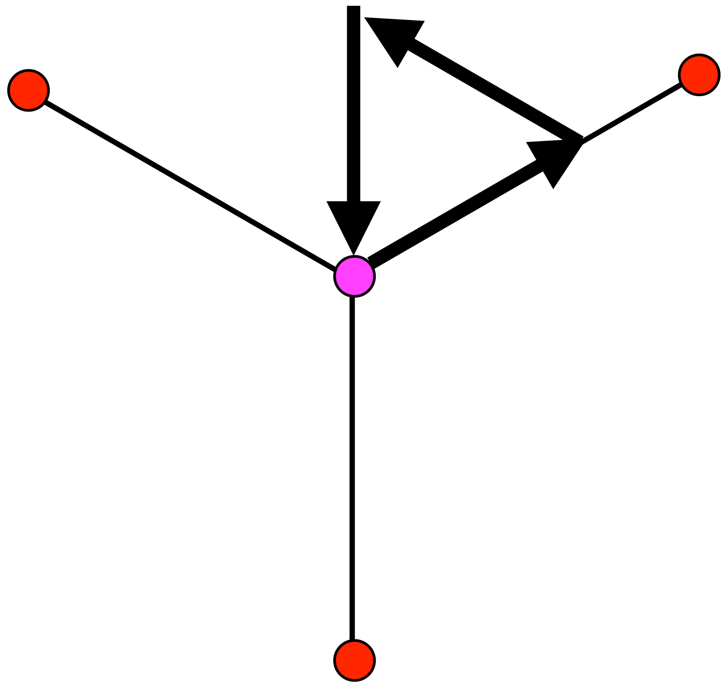

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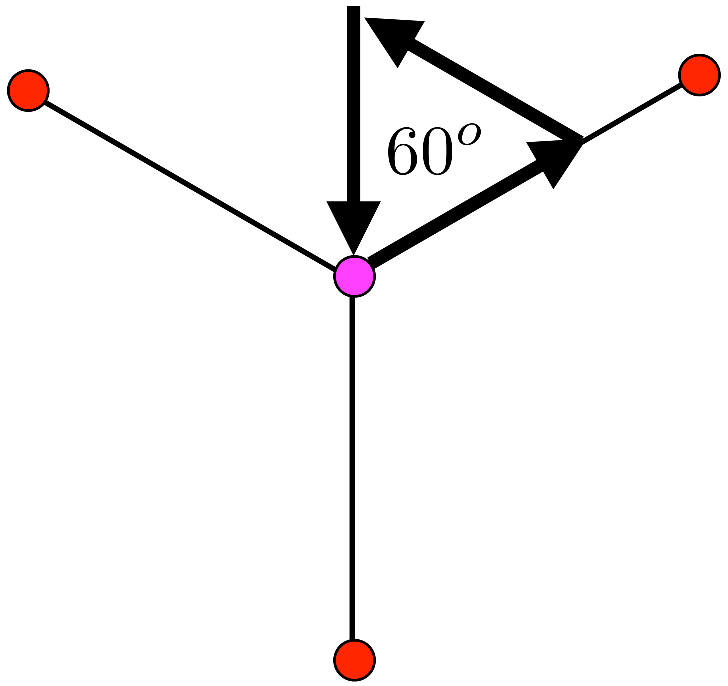

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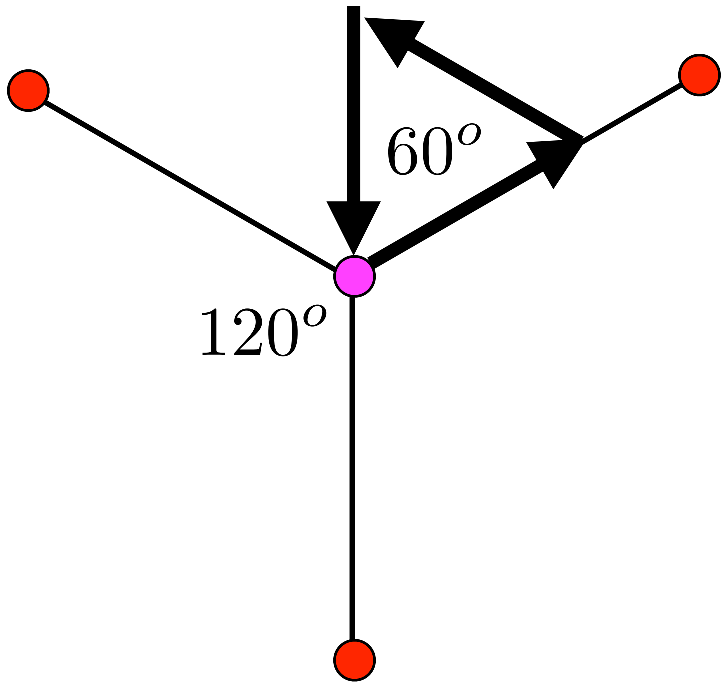
$$\frac{f(x+\varepsilon) - f(x)}{\varepsilon} = 1$$

unit vector $\left(\begin{array}{c} \frac{(x-a_1)}{\sqrt{(x-a_1)^2+(y-b_1)^2}} \\ \frac{(y-b_1)}{\sqrt{(x-a_1)^2+(y-b_1)^2}} \end{array} \right)$



Derivatives geometrically

2D derivative: (negative) gradient!

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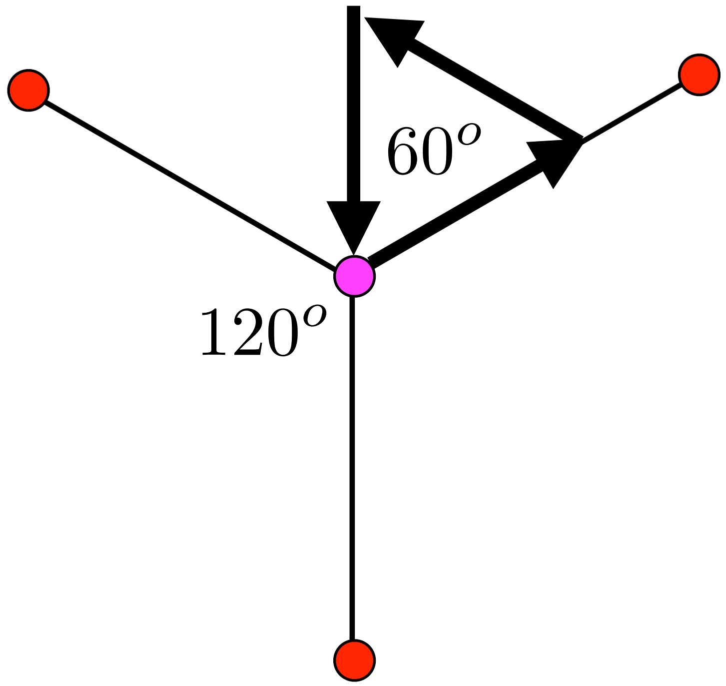

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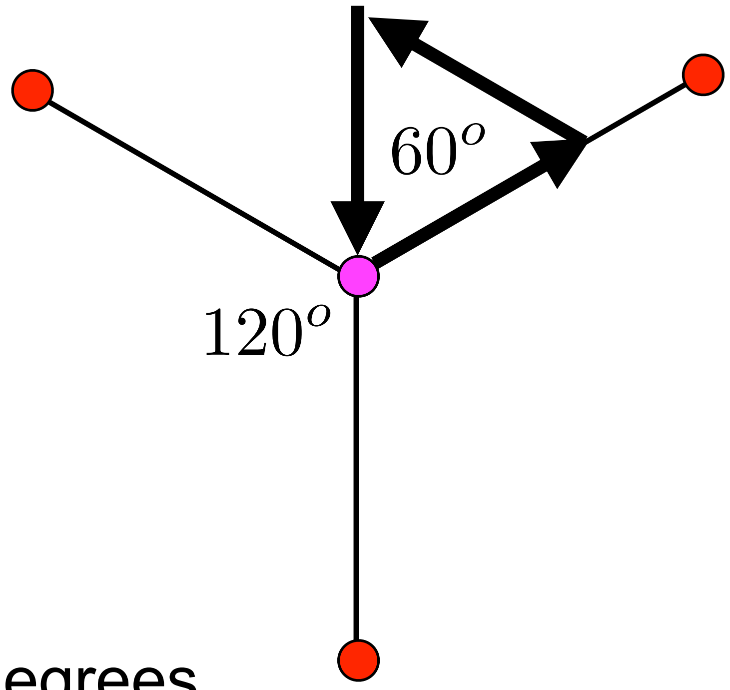

Theorem 7.2:

In an optimal Steiner tree, edges meet at angles of 120 degrees.

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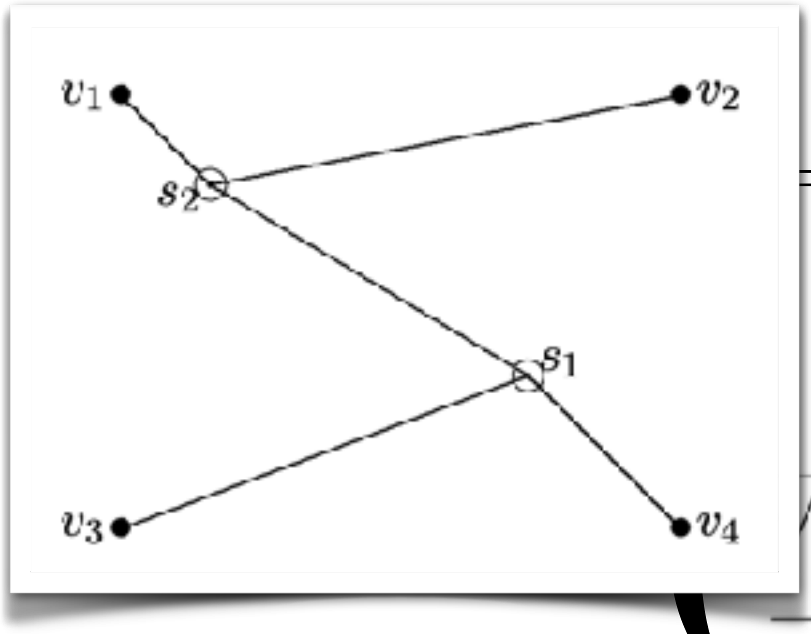

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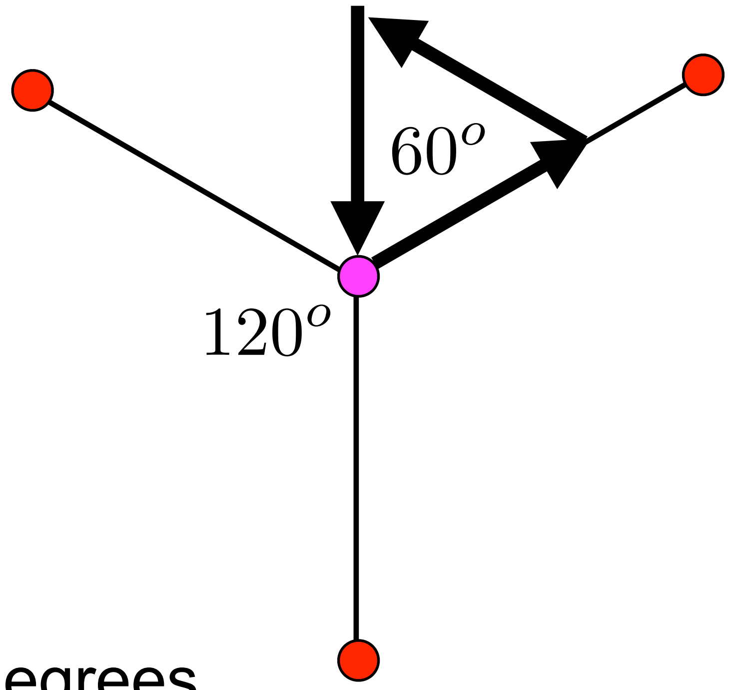

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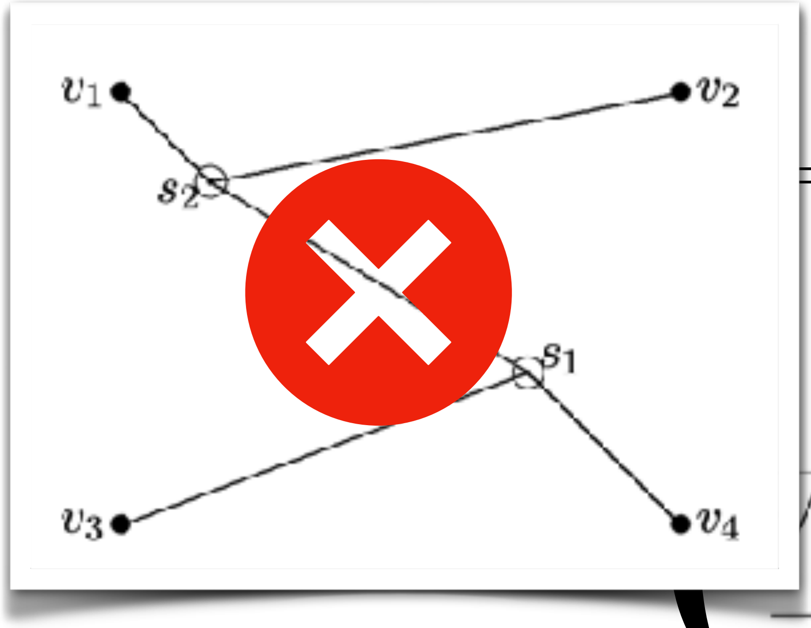

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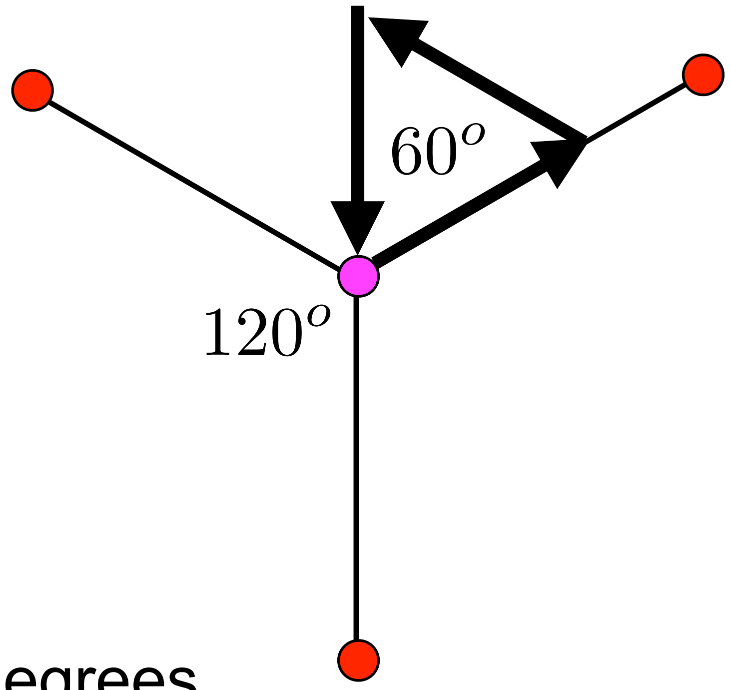

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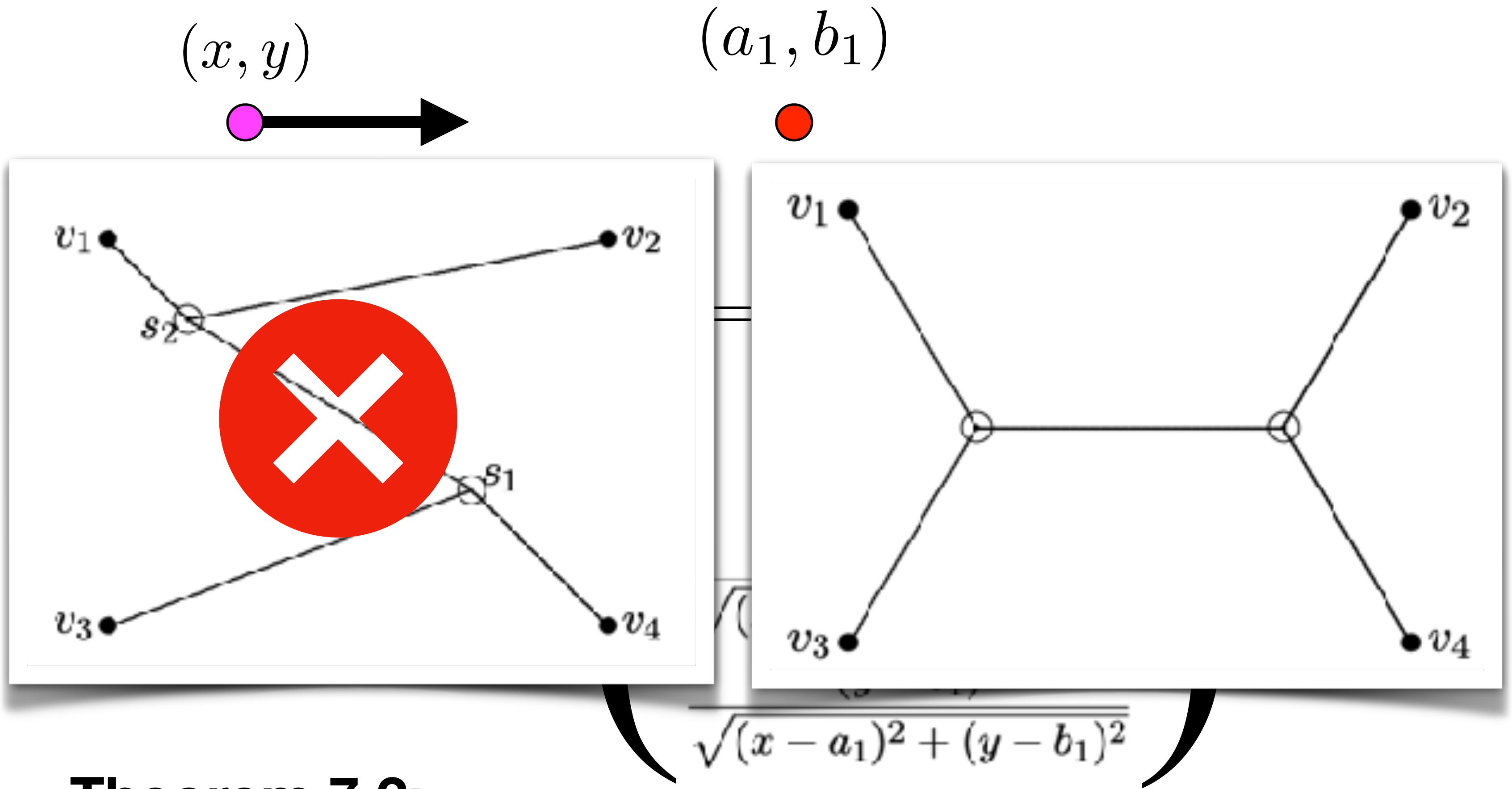

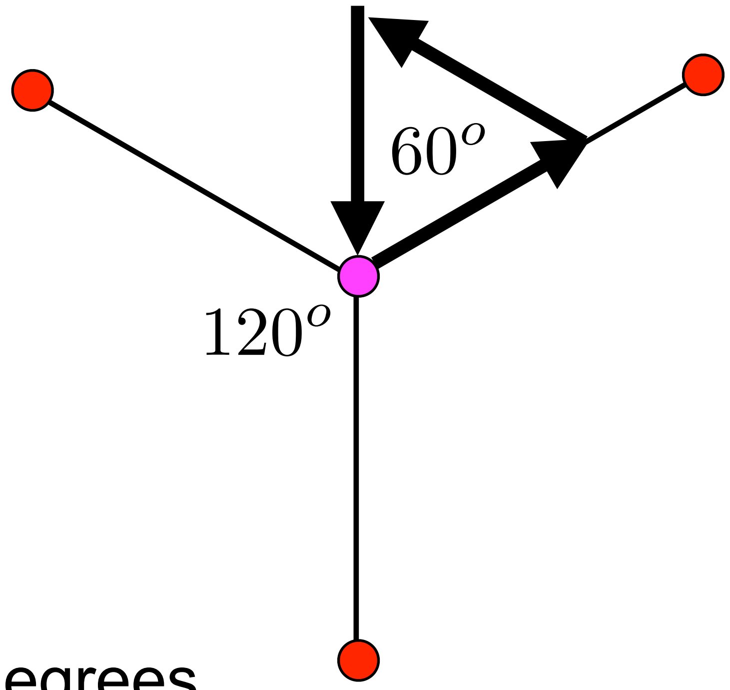

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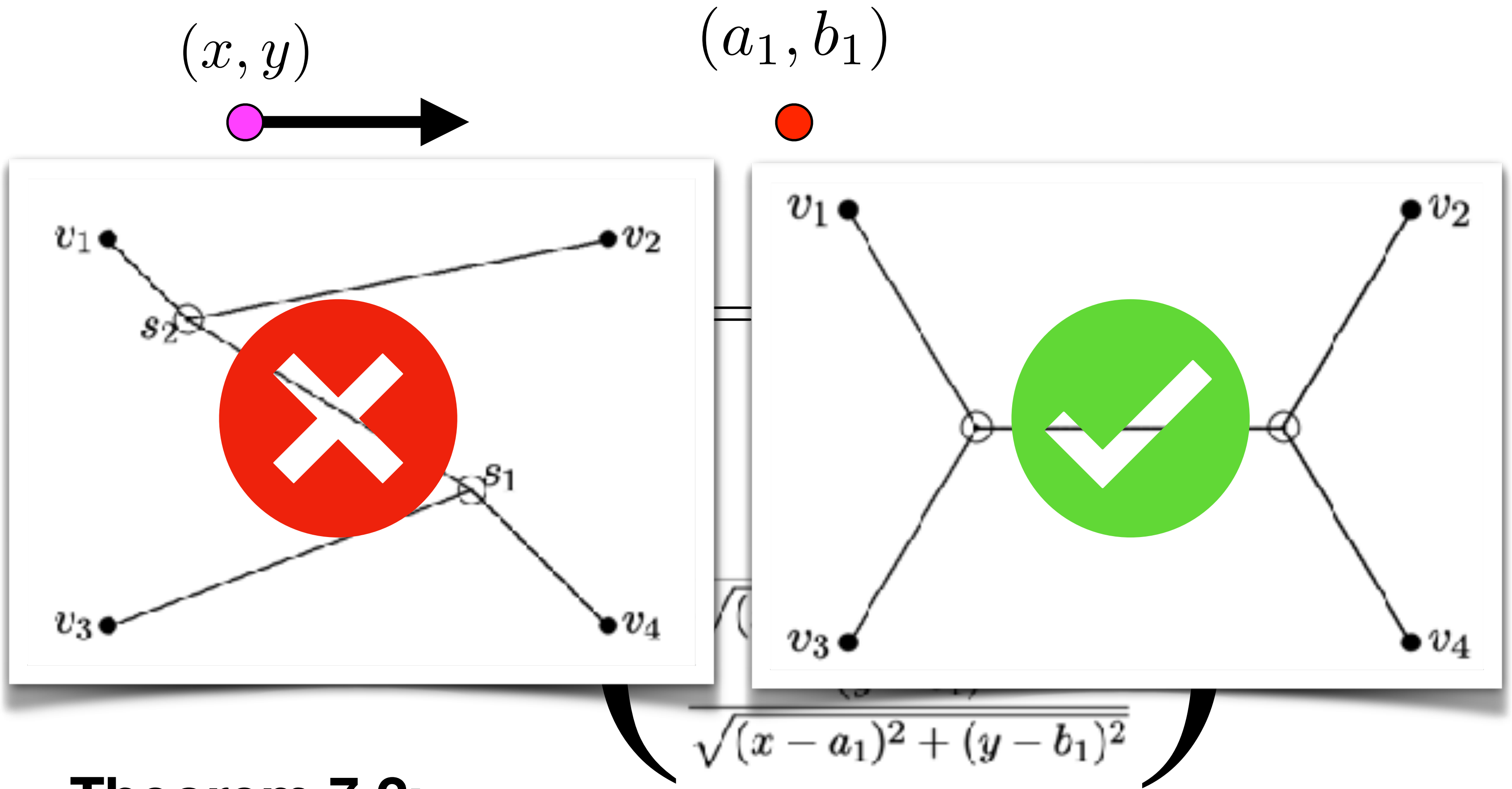

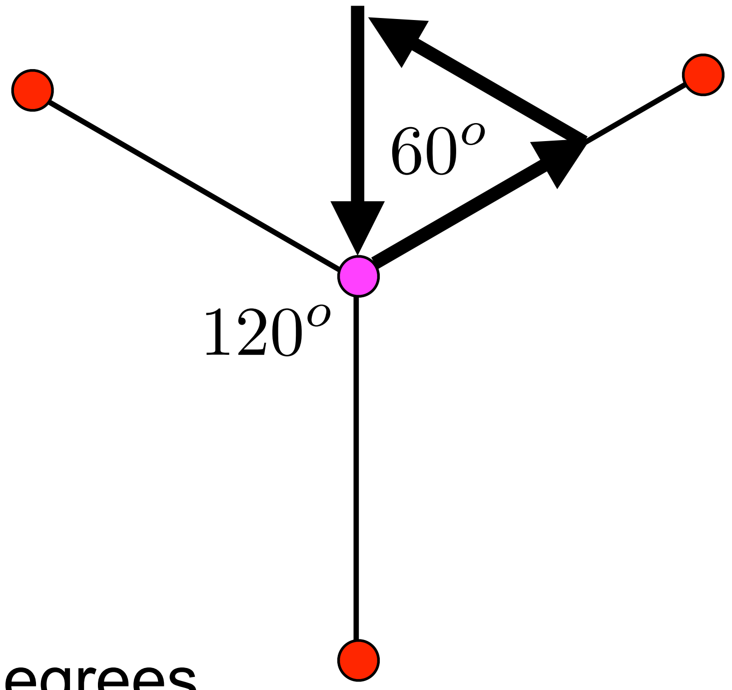

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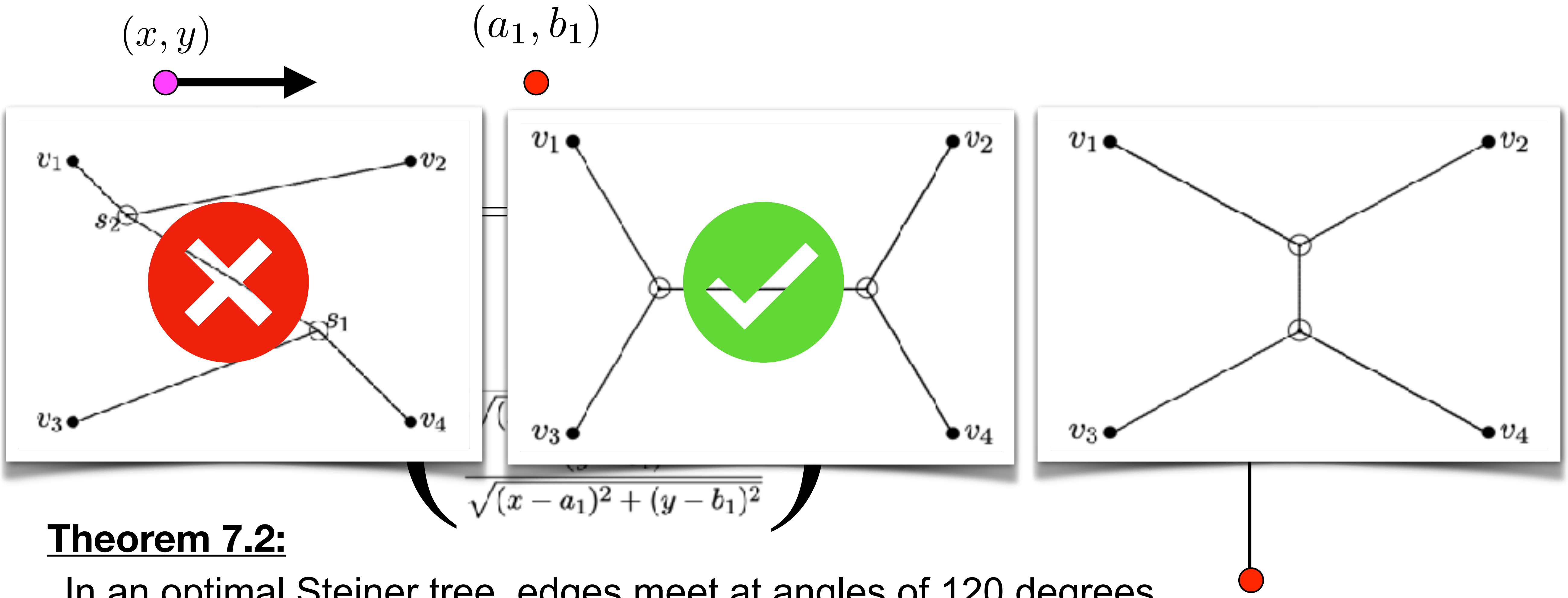

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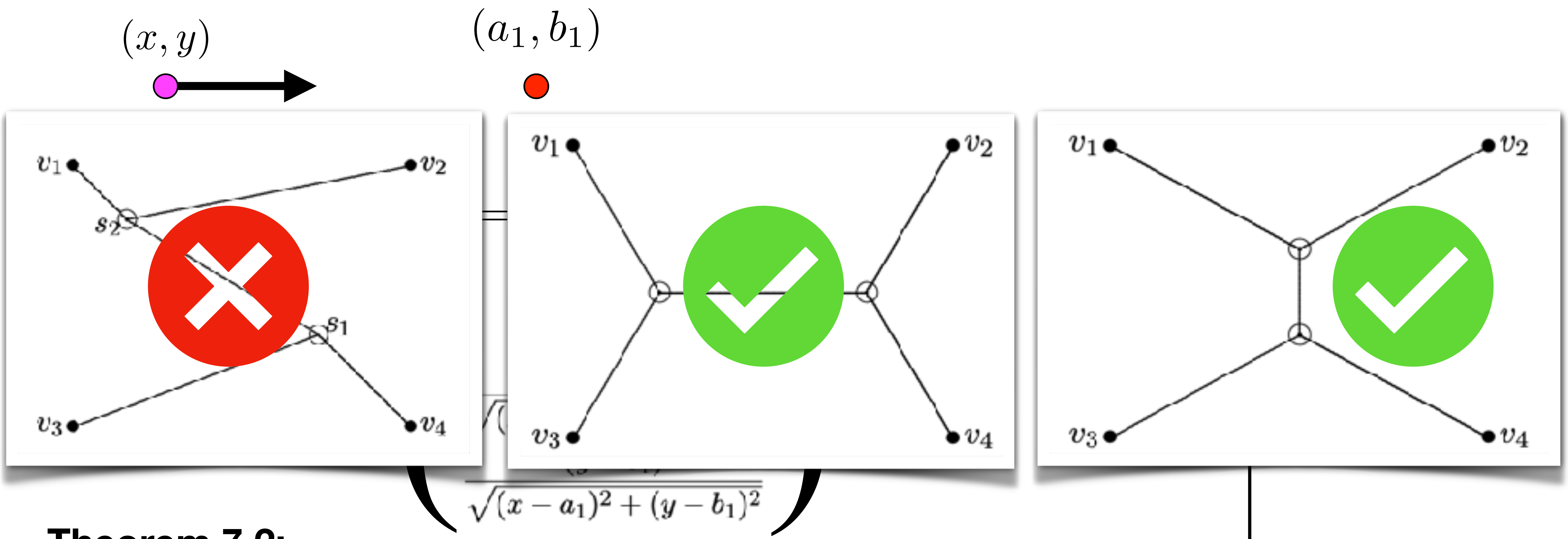

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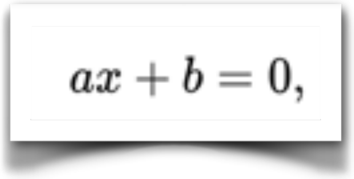
Overview

1. Introduction
2. Manhattan distances
3. Euclidean distances
4. Galois and Bajaj
5. Continuous sets

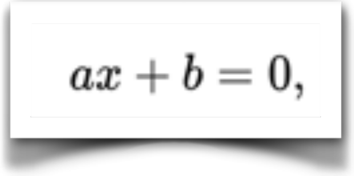
Solving polynomial equations

Solving polynomial equations



$$ax + b = 0,$$

Solving polynomial equations



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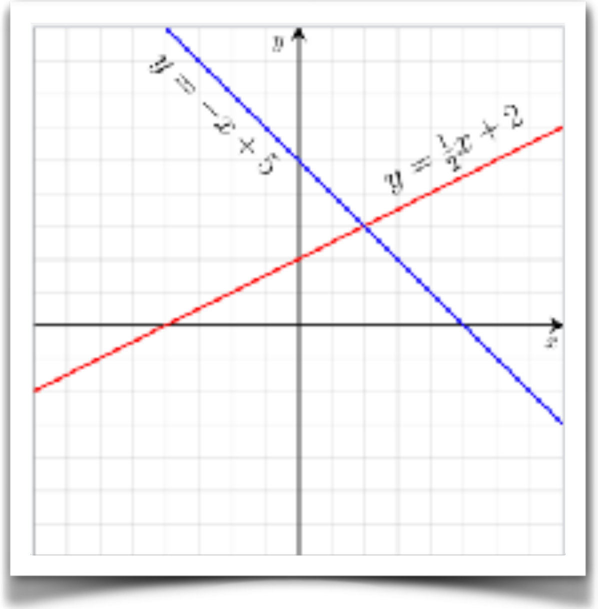

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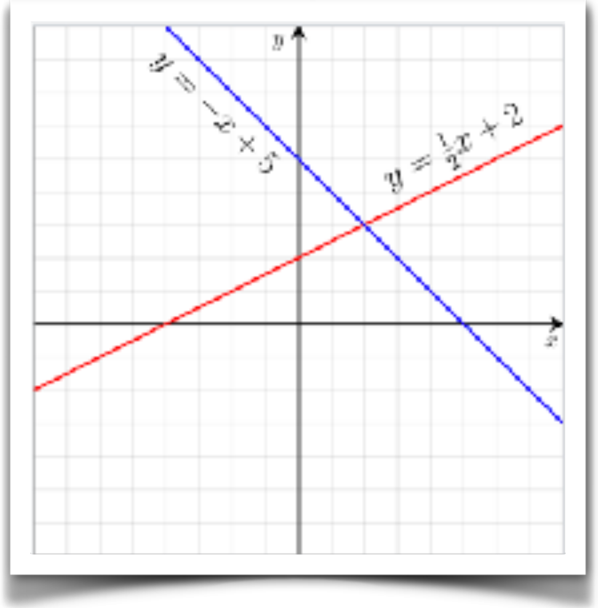

$$x = -\frac{b}{a}$$

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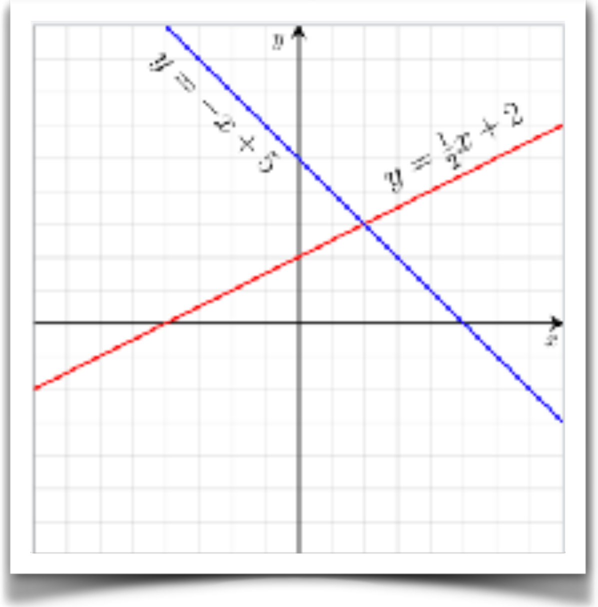

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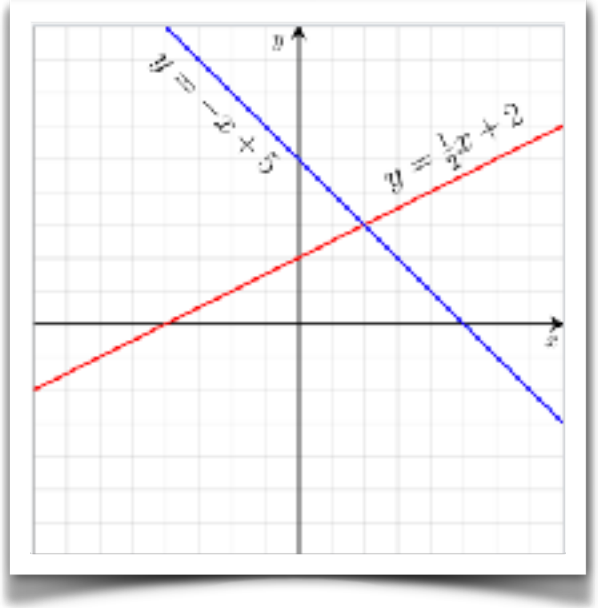

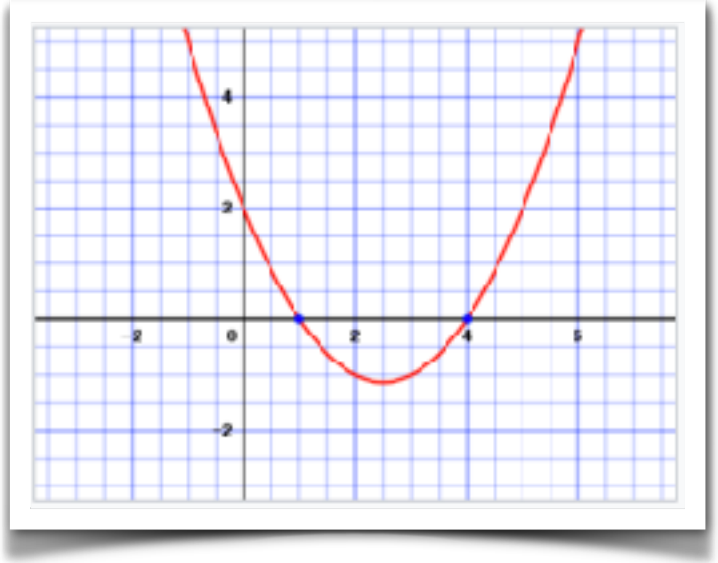

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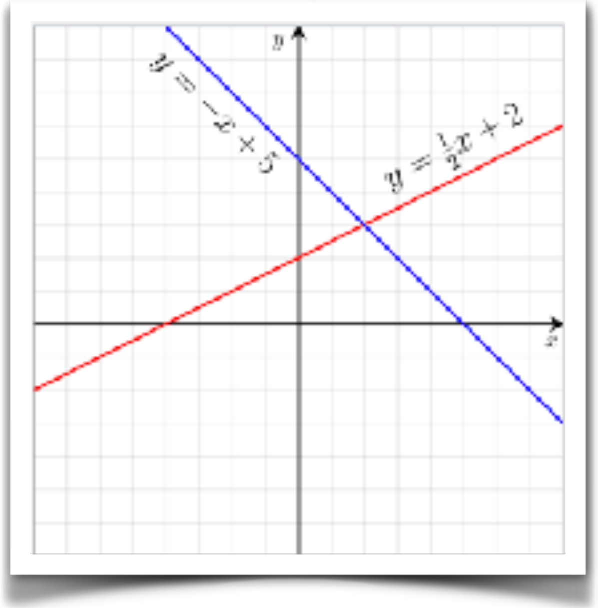

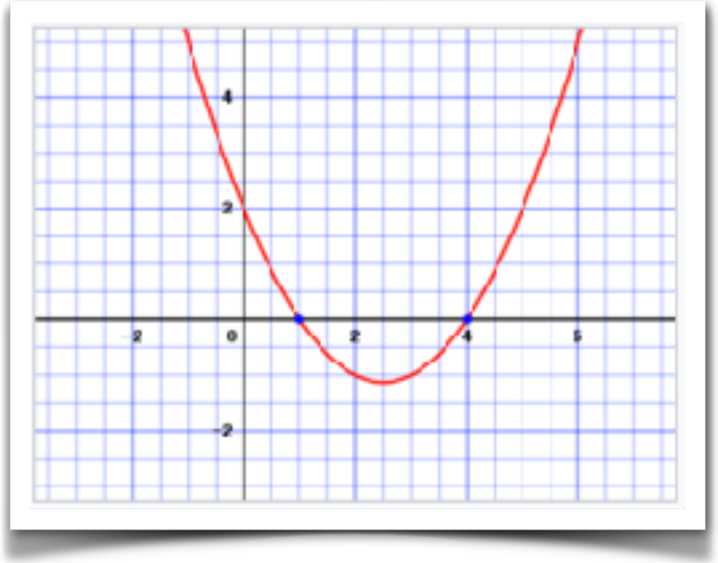

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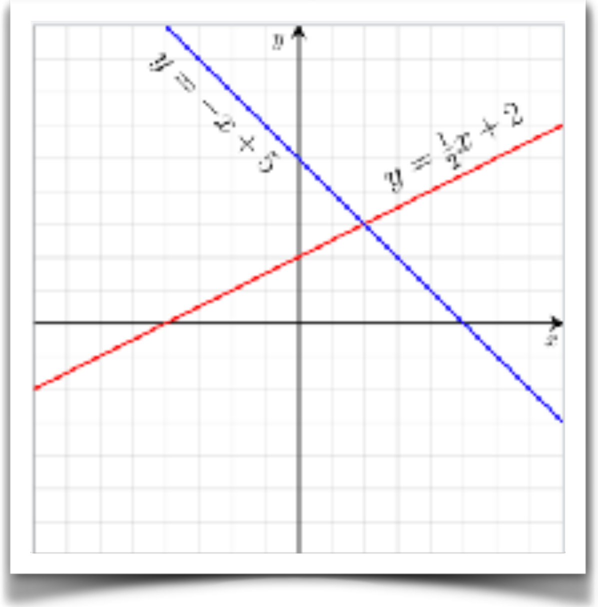

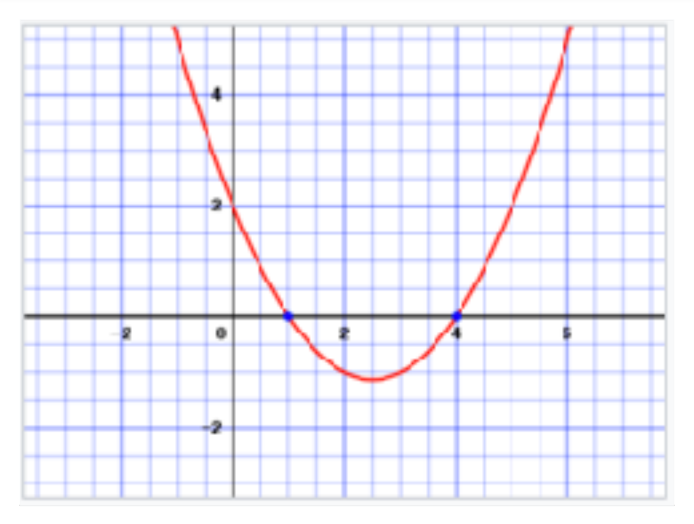

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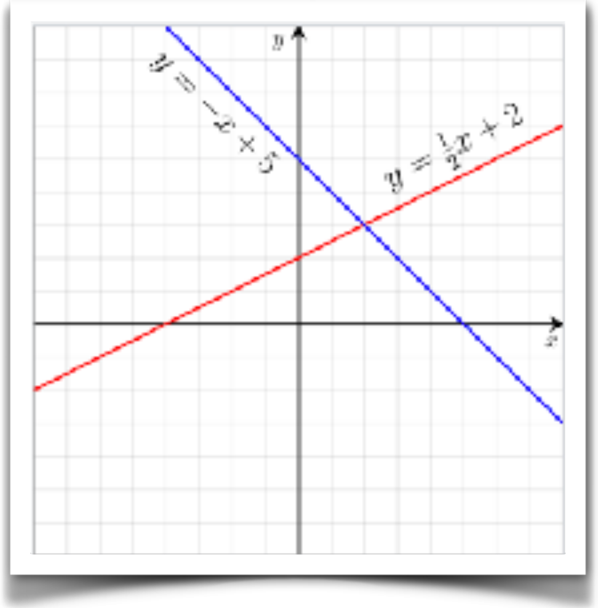

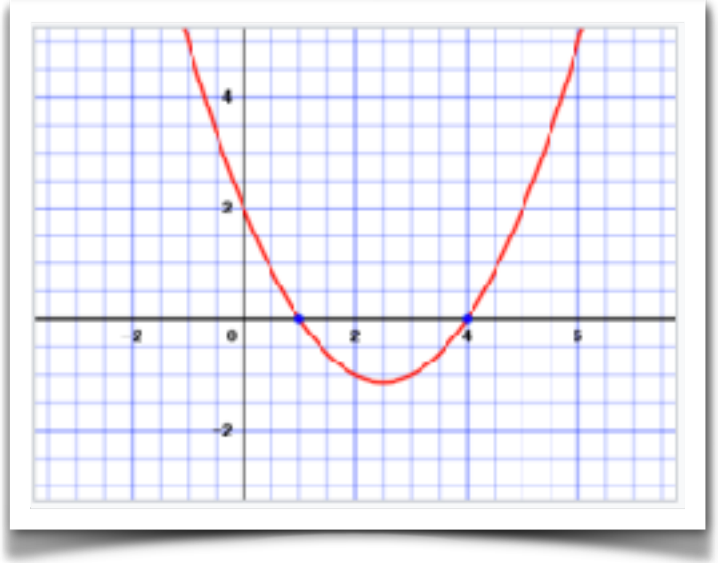

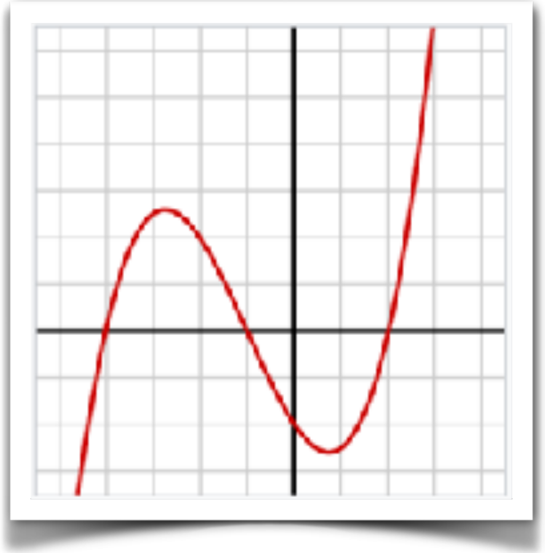

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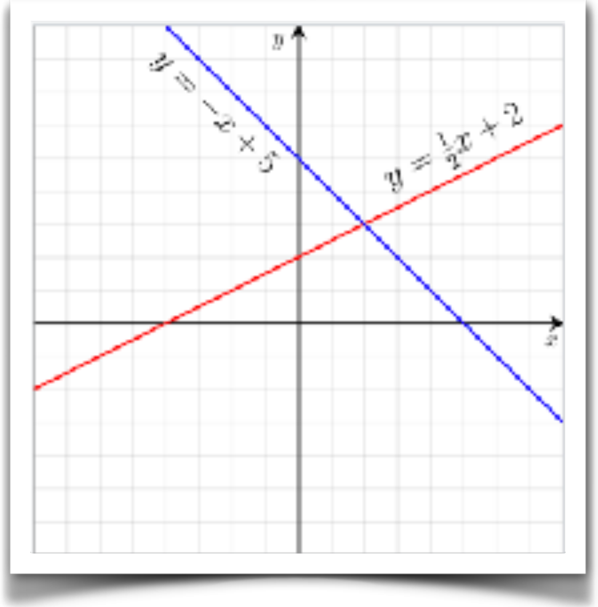

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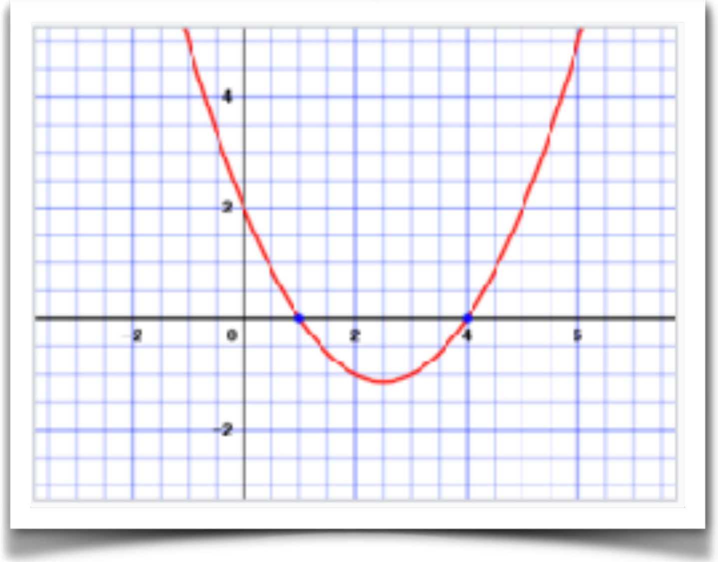

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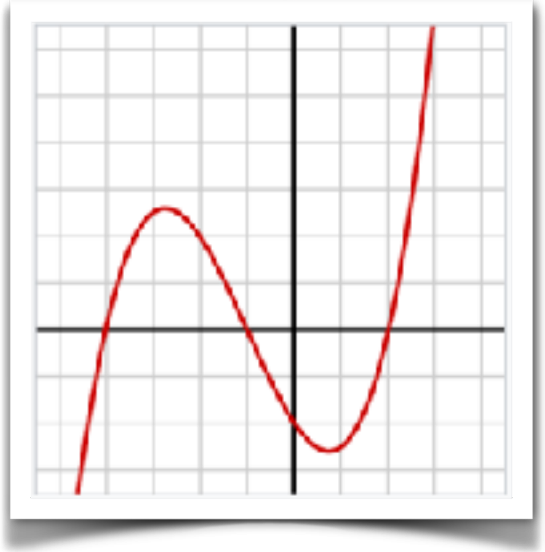

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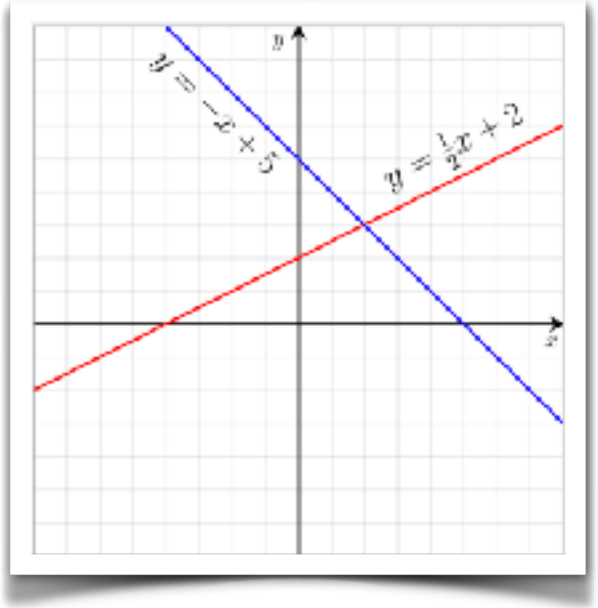

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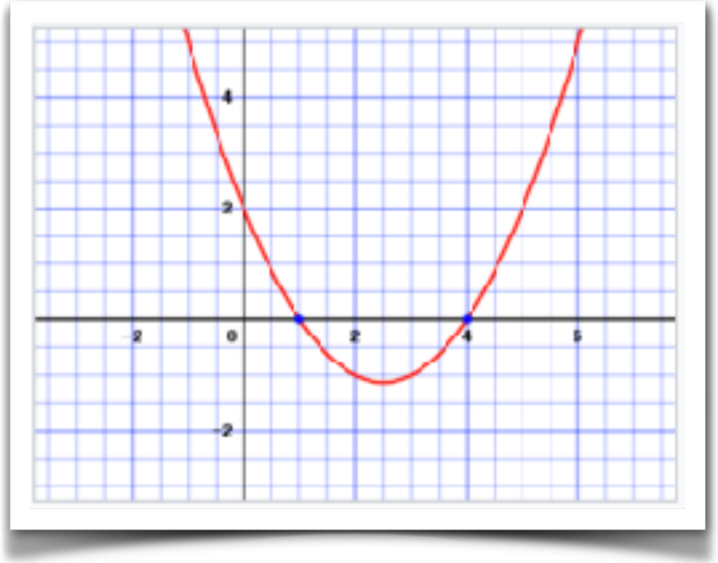

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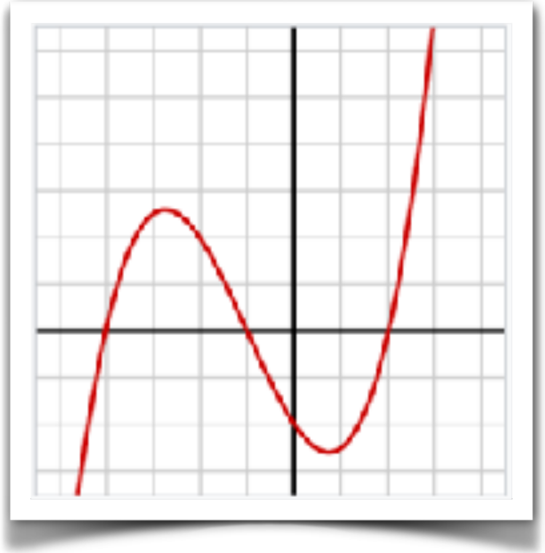

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$$\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Solving polynomial equations

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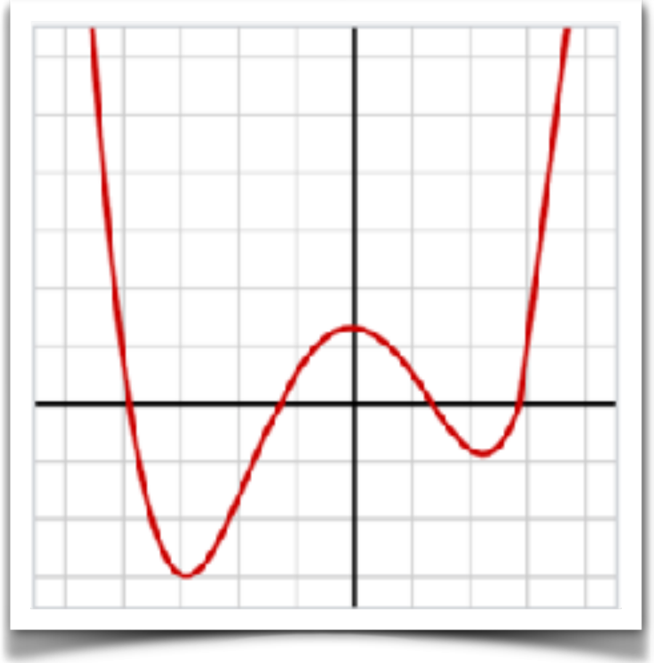

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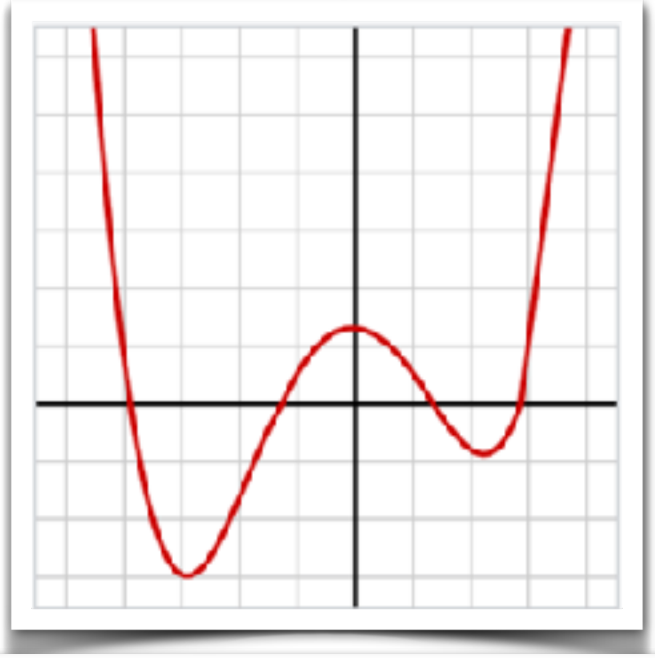

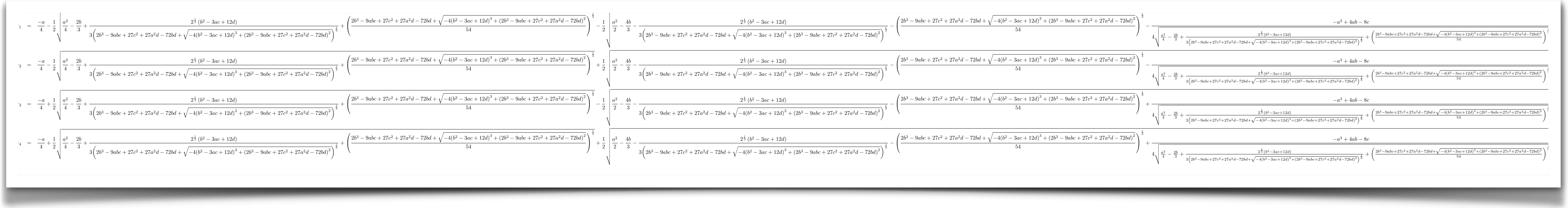

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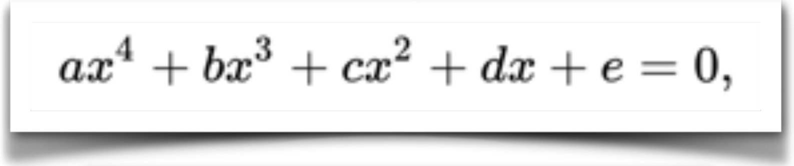

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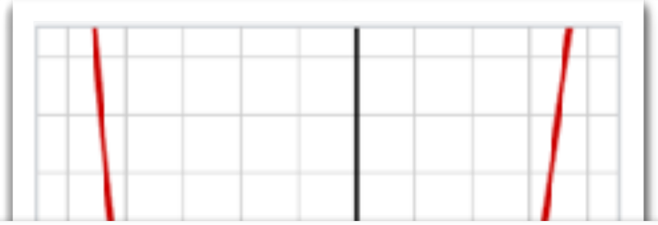

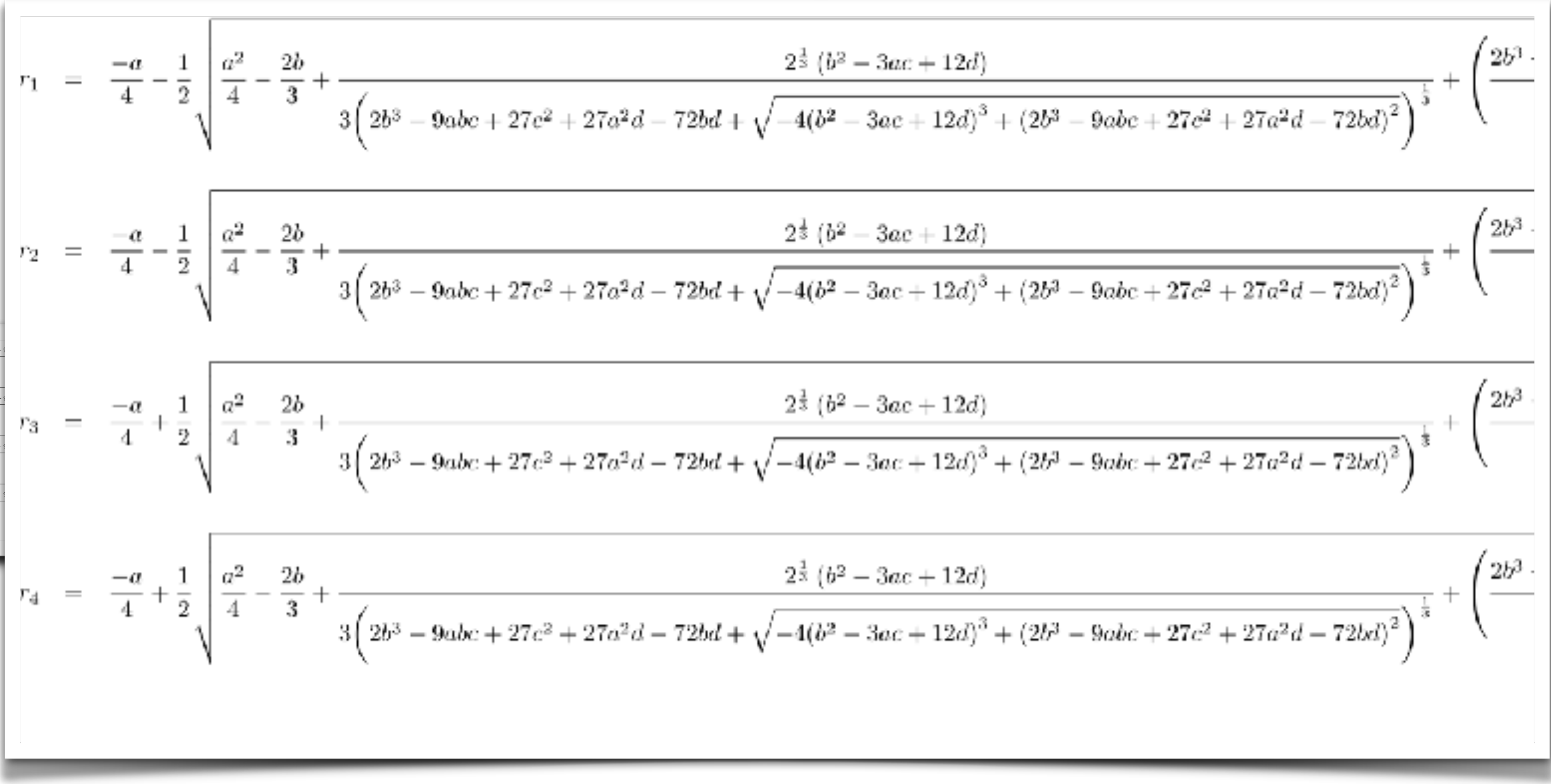

$$r_1 = \frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}\right)^{\frac{1}{3}}} + \Bigg(2b^3}$$

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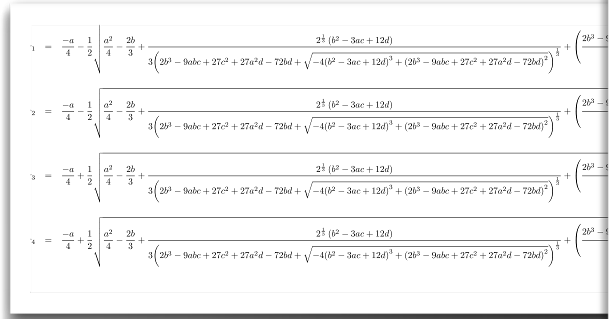
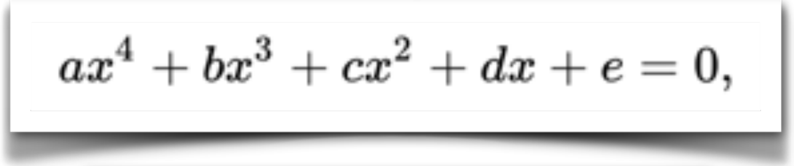


Solving polynomial equations



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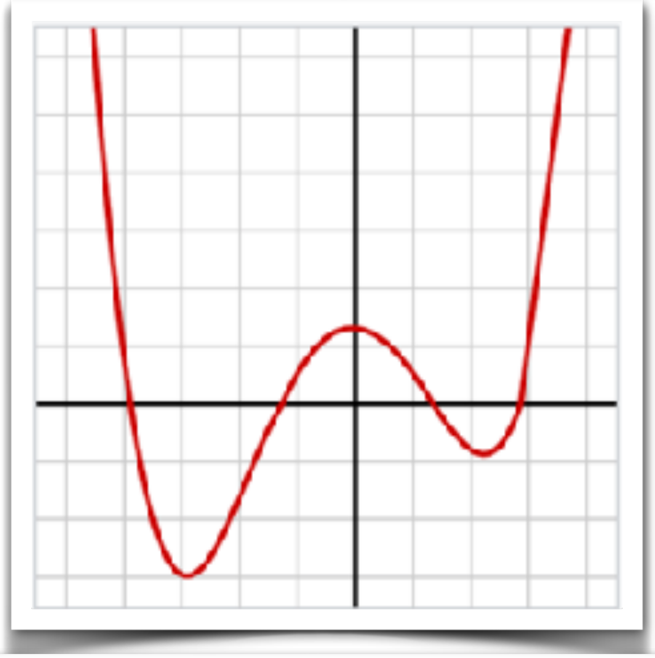

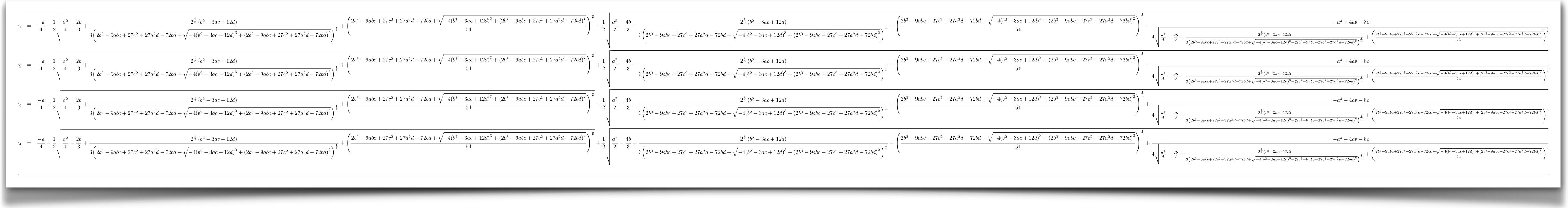

Solving quintic equations?

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Solving quintic equations?

$$x^5 - x - 1 = 0$$

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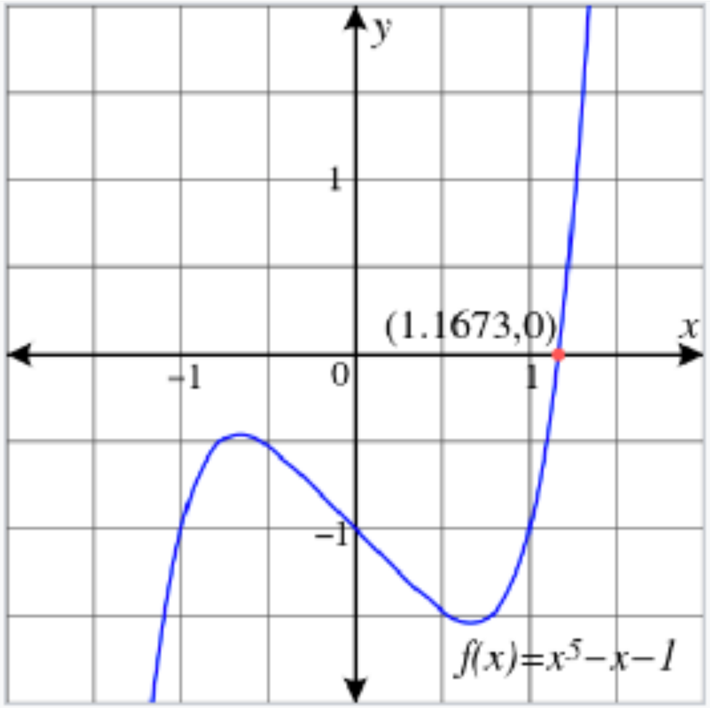


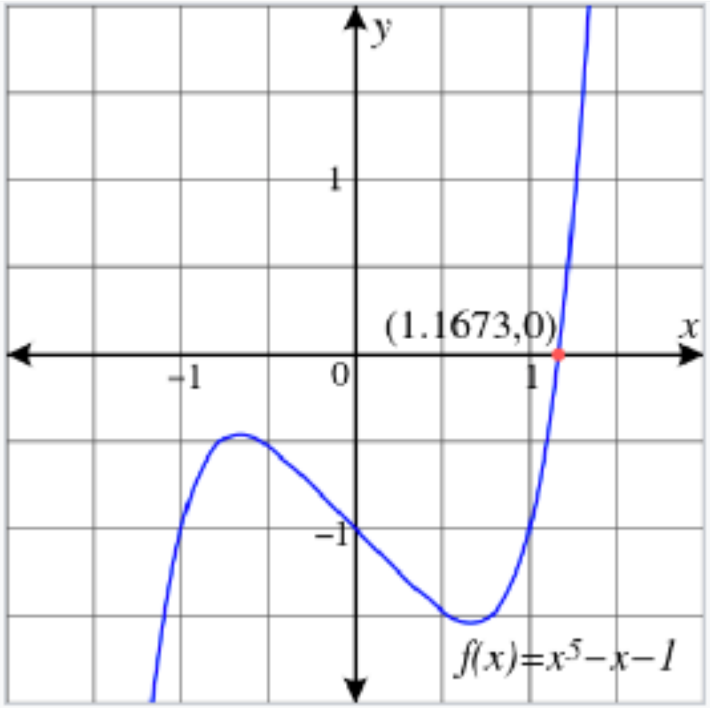
For the polynomial $f(x) = x^5 - x - 1$, the lone real root $x = 1.1673...$ is algebraic, but not expressible in terms of radicals. The other four roots are complex numbers.

Solving quintic equations?

Niels Henrik Abel

Born	5 August 1802 Nedstrand, Denmark-Norway
Died	6 April 1829 (aged 26) Froland, Norway

$$x^5 - x - 1 = 0$$



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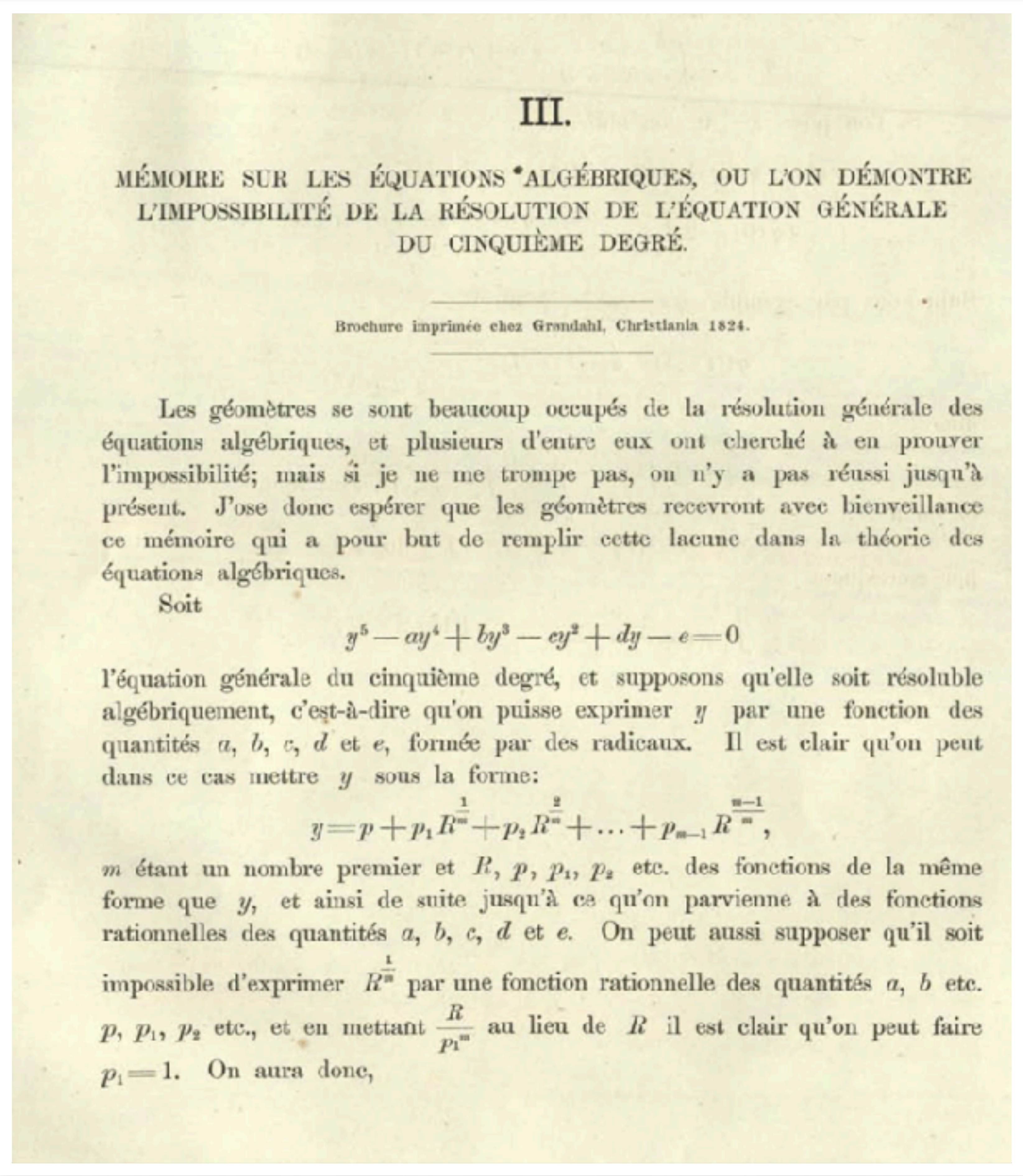


III.

MÉMOIRE SUR LES ÉQUATIONS ALGÉBRIQUES, OU L'ON DÉMONTRE L'IMPOSSIBILITÉ DE LA RÉSOLUTION DE L'ÉQUATION GÉNÉRALE DU CINQUIÈME DEGRÉ.

Brochure imprimée chez Grøndahl, Christiania 1824.

Les géomètres se sont beaucoup occupés de la résolution générale des équations algébriques, et plusieurs d'entre eux ont cherché à en prouver l'impossibilité; mais si je ne me trompe pas, on n'y a pas réussi jusqu'à présent. J'ose donc espérer que les géomètres recevront avec bienveillance ce mémoire qui a pour but de remplir cette lacune dans la théorie des équations algébriques.

Soit

$$y^5 - ay^4 + by^3 - cy^2 + dy - e = 0$$

l'équation générale du cinquième degré, et supposons qu'elle soit résoluble algébriquement, c'est-à-dire qu'on puisse exprimer y par une fonction des quantités a, b, c, d et e, formée par des radicaux. Il est clair qu'on peut dans ce cas mettre y sous la forme:

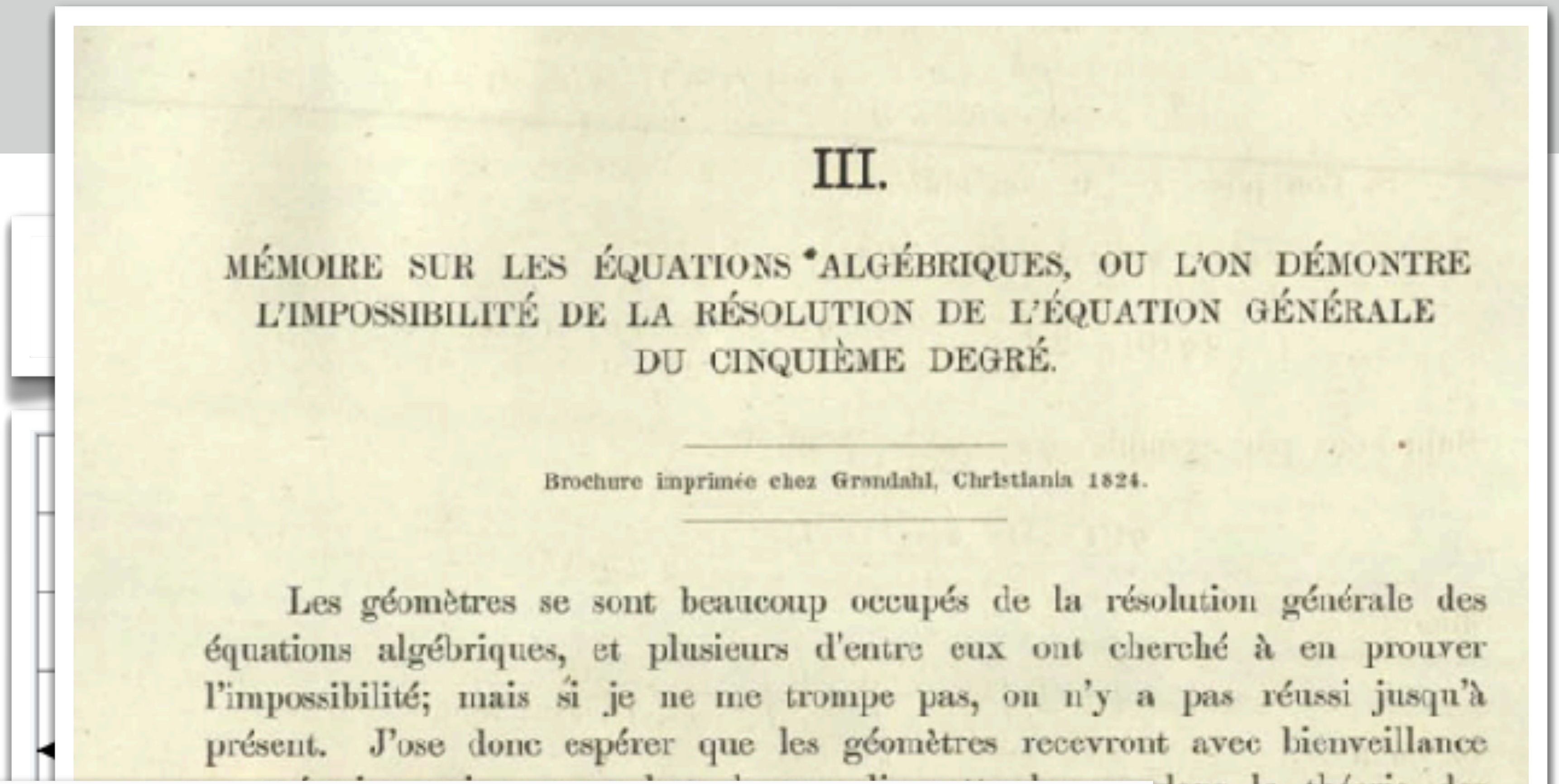
$$y = p + p_1 R^{\frac{1}{m}} + p_2 R^{\frac{2}{m}} + \ldots + p_{m-1} R^{\frac{m-1}{m}},$$

m étant un nombre premier et R, p, p_1, p_2 etc. des fonctions de la même forme que y, et ainsi de suite jusqu'à ce qu'on parvienne à des fonctions rationnelles des quantités a, b, c, d et e. On peut aussi supposer qu'il soit impossible d'exprimer $R^{\frac{1}{m}}$ par une fonction rationnelle des quantités a, b etc. p, p_1, p_2 etc., et en mettant $\frac{R}{p_1^m}$ au lieu de R il est clair qu'on peut faire $p_1 = 1$. On aura donc,

Niels Henrik Abel

Born 5 August 18
Nedstrand,

Died 6 April 1829
Froland, No



III.

MÉMOIRE SUR LES ÉQUATIONS ALGÉBRIQUES, OU L'ON DÉMONTRE L'IMPOSSIBILITÉ DE LA RÉSOLUTION DE L'ÉQUATION GÉNÉRALE DU CINQUIÈME DEGRÉ.

Brochure imprimée chez Grøndahl, Christiania 1824.

Les géomètres se sont beaucoup occupés de la résolution générale des équations algébriques, et plusieurs d'entre eux ont cherché à en prouver l'impossibilité; mais si je ne me trompe pas, on n'y a pas réussi jusqu'à présent. J'ose donc espérer que les géomètres recevront avec bienveillance

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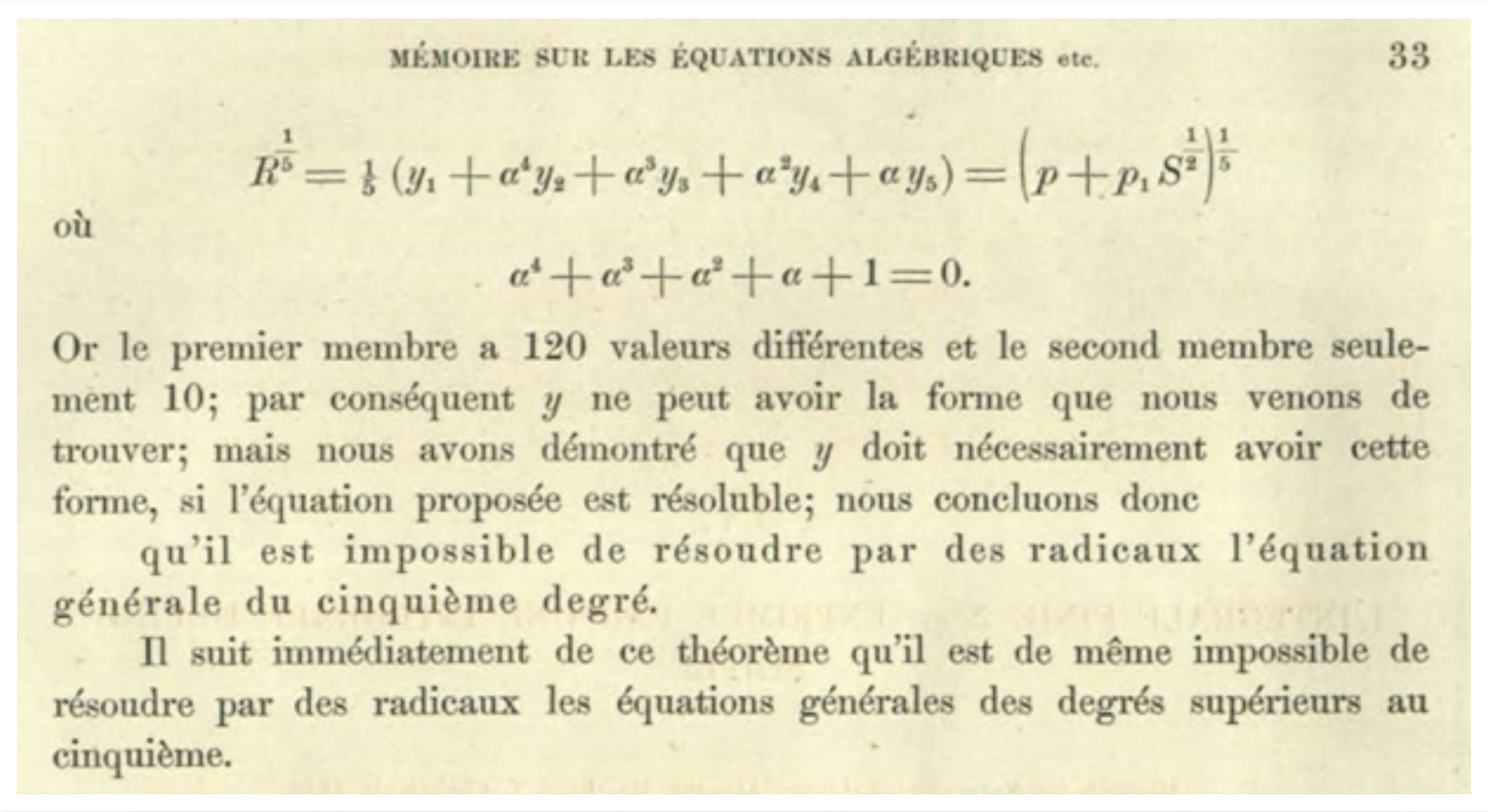


$$R^{\frac{1}{5}} = \tfrac{1}{5}(y_1 + \alpha^4 y_2 + \alpha^3 y_3 + \alpha^2 y_4 + \alpha y_5) = \left(p + p_1 S^{\frac{1}{2}}\right)^{\frac{1}{5}}$$

où

$$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0.$$

Or le premier membre a 120 valeurs différentes et le second membre seulement 10; par conséquent y ne peut avoir la forme que nous venons de trouver; mais nous avons démontré que y doit nécessairement avoir cette forme, si l'équation proposée est résoluble; nous concluons donc

qu'il est impossible de résoudre par des radicaux l'équation générale du cinquième degré.

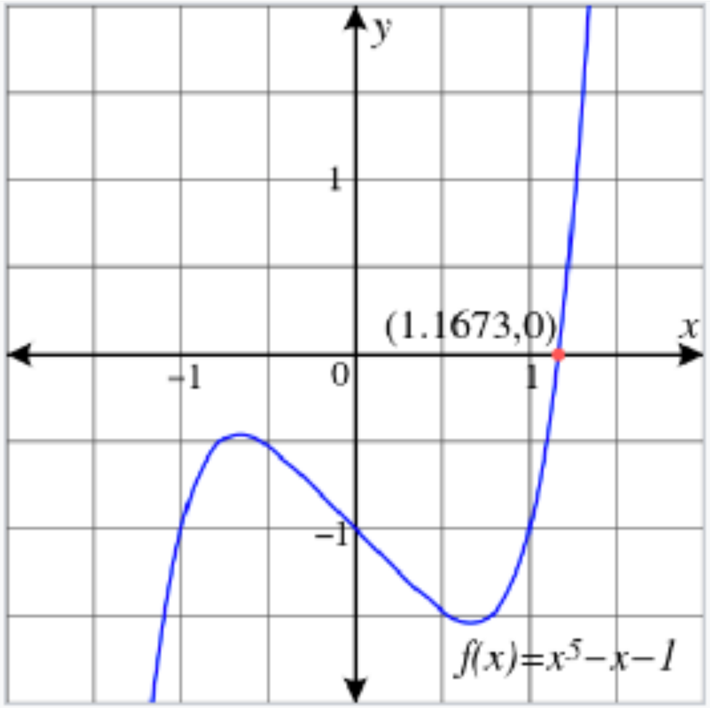
Il suit immédiatement de ce théorème qu'il est de même impossible de résoudre par des radicaux les équations générales des degrés supérieurs au cinquième.

Solving quintic equations?

Niels Henrik Abel

Born	5 August 1802 Nedstrand, Denmark-Norway
Died	6 April 1829 (aged 26) Froland, Norway

$$x^5 - x - 1 = 0$$



For the polynomial $f(x) = x^5 - x - 1$, the lone real root $x = 1.1673...$ is algebraic, but not expressible in terms of radicals. The other four roots are complex numbers.

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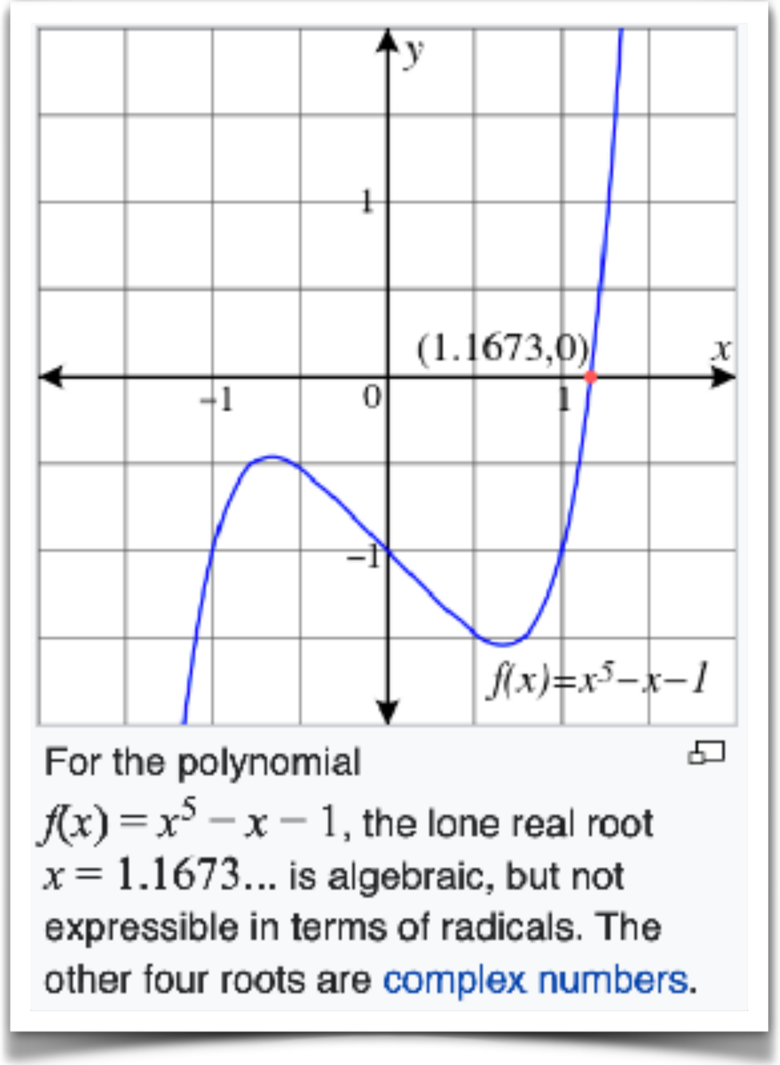

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Évariste Galois

A portrait of Évariste Galois aged about 15

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Died	31 May 1832 (aged 20) Paris, Kingdom of France

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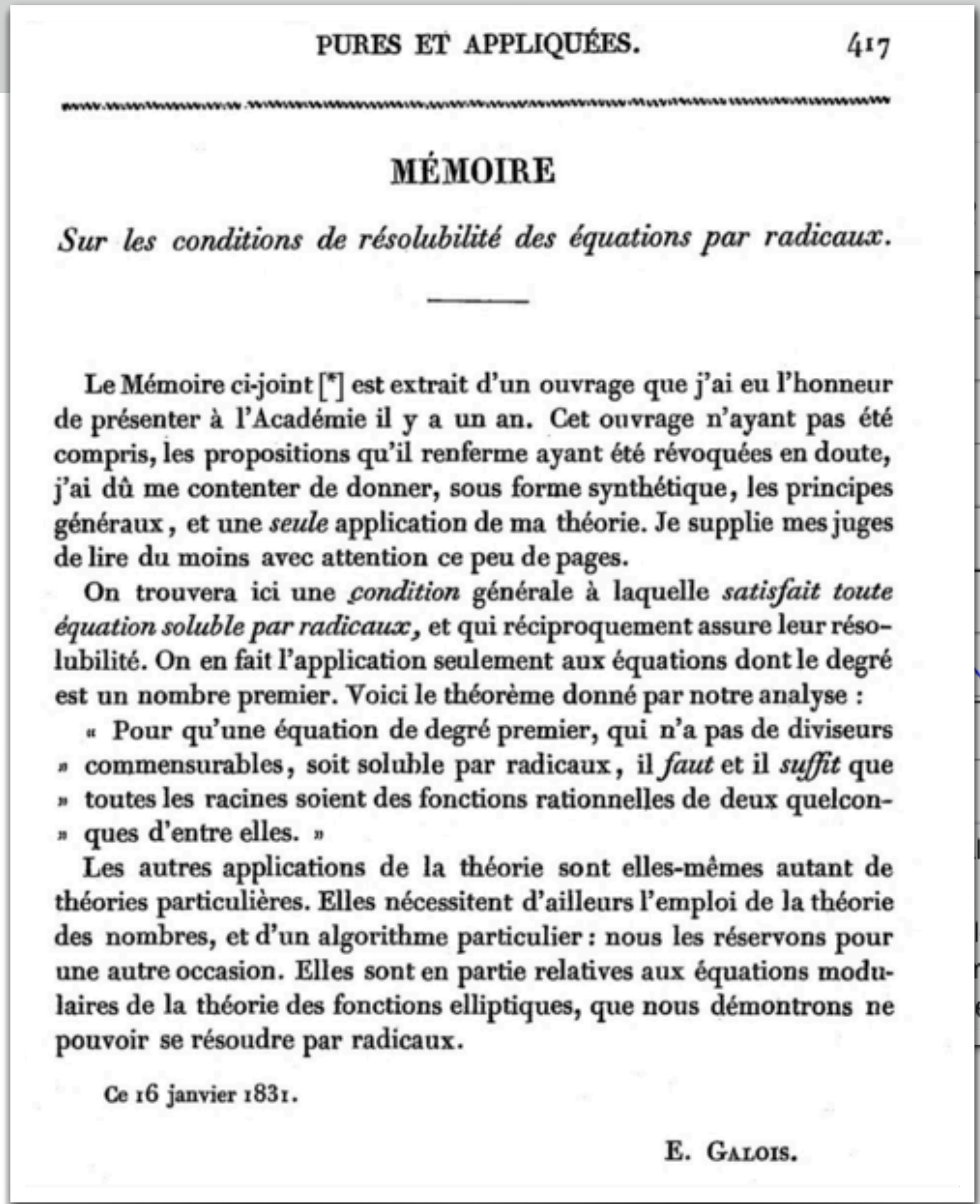


MÉMOIRE

Sur les conditions de résolubilité des équations par radicaux.

Le Mémoire ci-joint [*] est extrait d'un ouvrage que j'ai eu l'honneur de présenter à l'Académie il y a un an. Cet ouvrage n'ayant pas été compris, les propositions qu'il renferme ayant été révoquées en doute, j'ai dû me contenter de donner, sous forme synthétique, les principes généraux, et une *seule* application de ma théorie. Je supplie mes juges de lire du moins avec attention ce peu de pages.

On trouvera ici une *condition* générale à laquelle *satisfait toute équation soluble par radicaux*, et qui réciproquement assure leur résolubilité. On en fait l'application seulement aux équations dont le degré est un nombre premier. Voici le théorème donné par notre analyse :

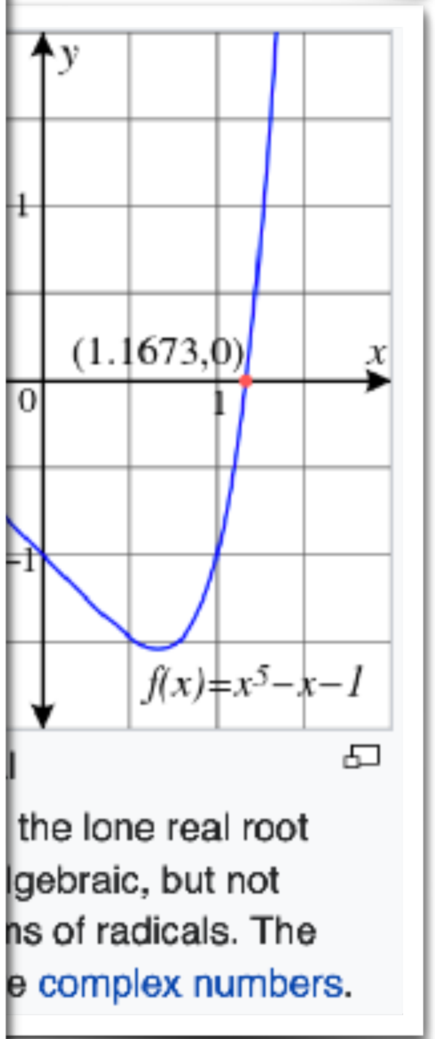
« Pour qu'une équation de degré premier, qui n'a pas de diviseurs » commensurables, soit soluble par radicaux, il *faut* et il *suffit* que » toutes les racines soient des fonctions rationnelles de deux quelcon- » ques d'entre elles. »

Les autres applications de la théorie sont elles-mêmes autant de théories particulières. Elles nécessitent d'ailleurs l'emploi de la théorie des nombres, et d'un algorithme particulier : nous les réservons pour une autre occasion. Elles sont en partie relatives aux équations modulaires de la théorie des fonctions elliptiques, que nous démontrons ne pouvoir se résoudre par radicaux.

Ce 16 janvier 1831.

E. Galois.

$$- 1 = 0$$



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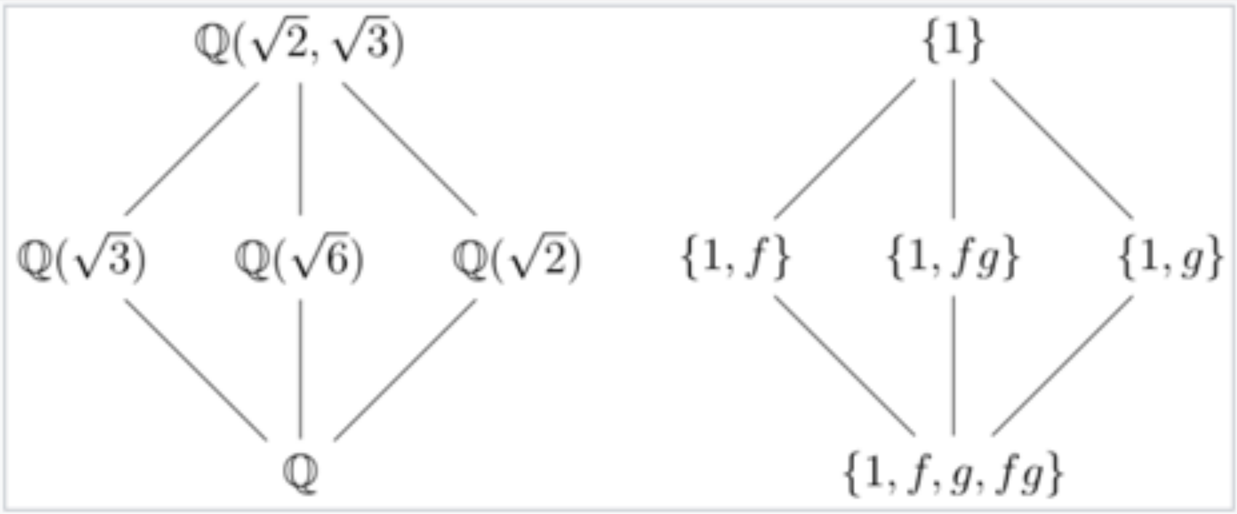

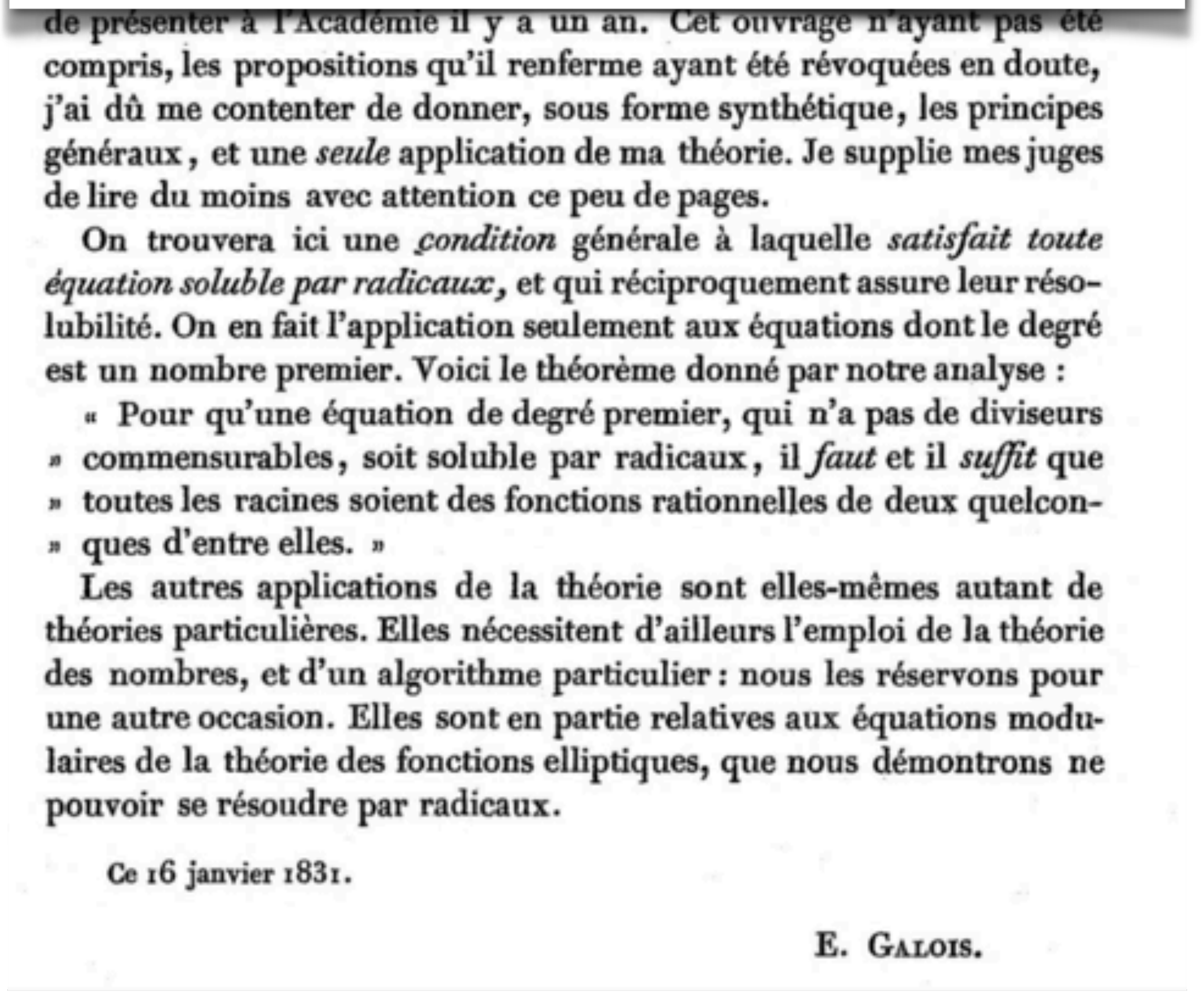

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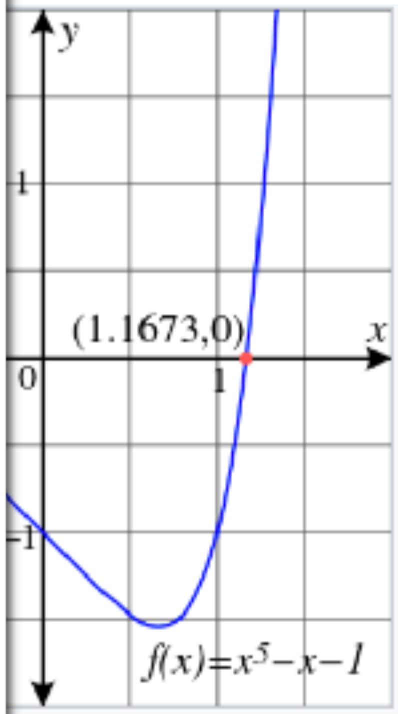

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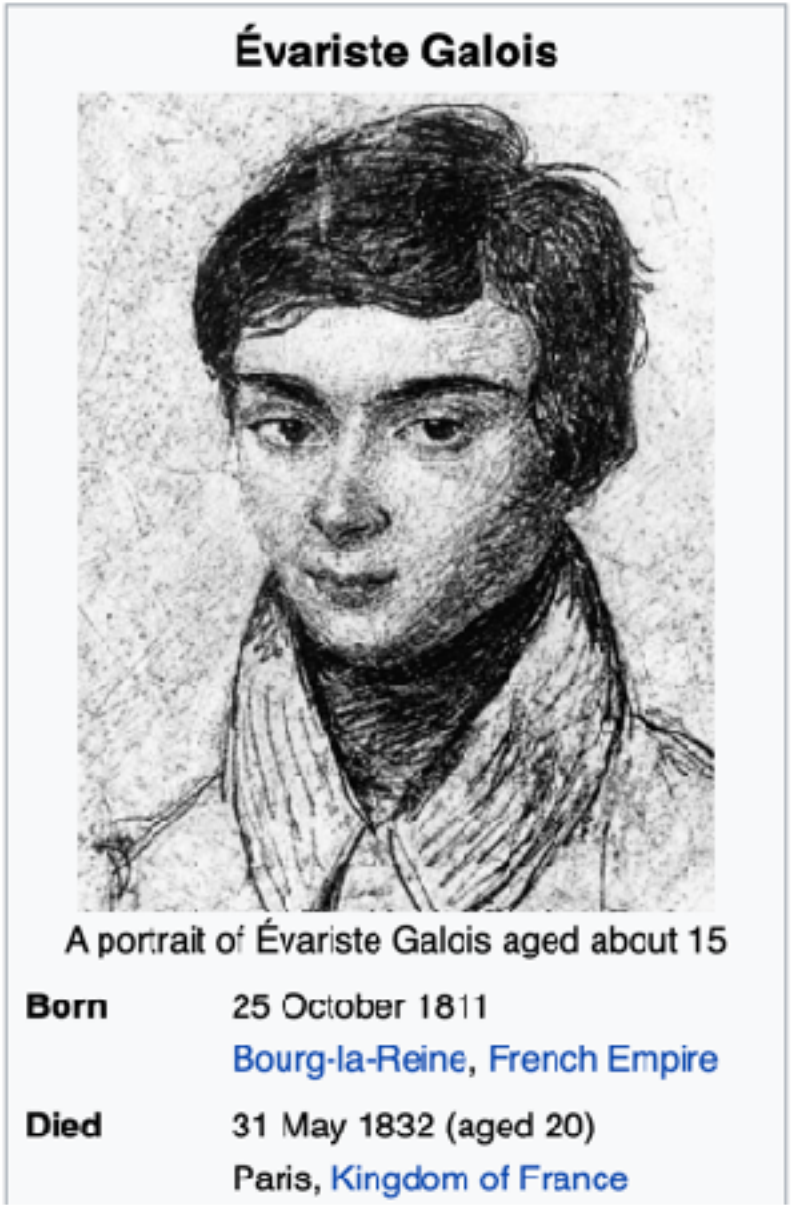

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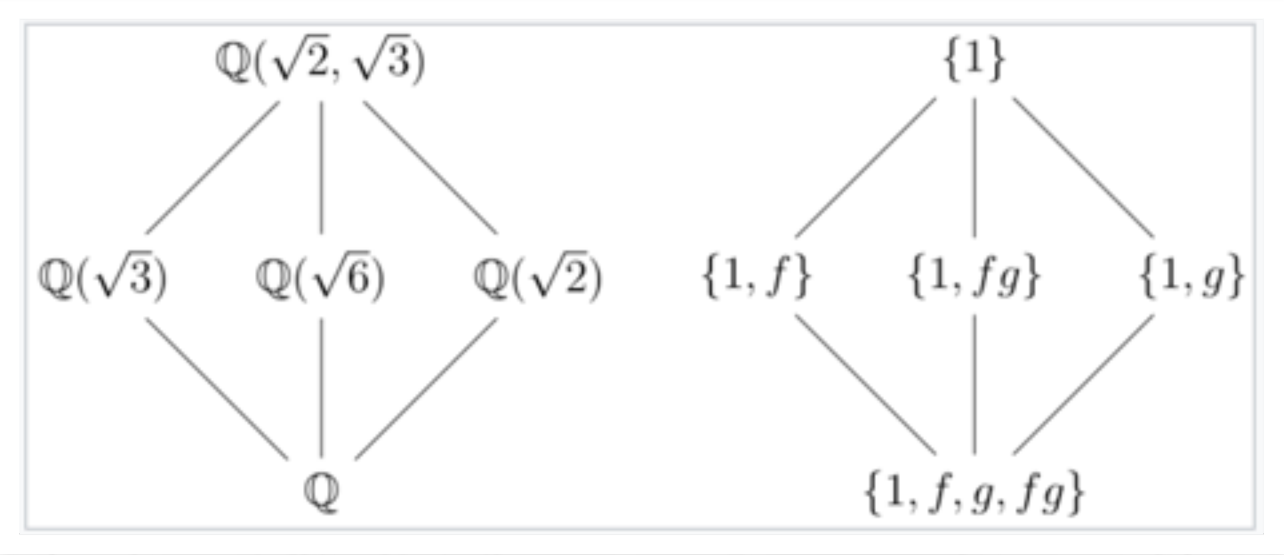

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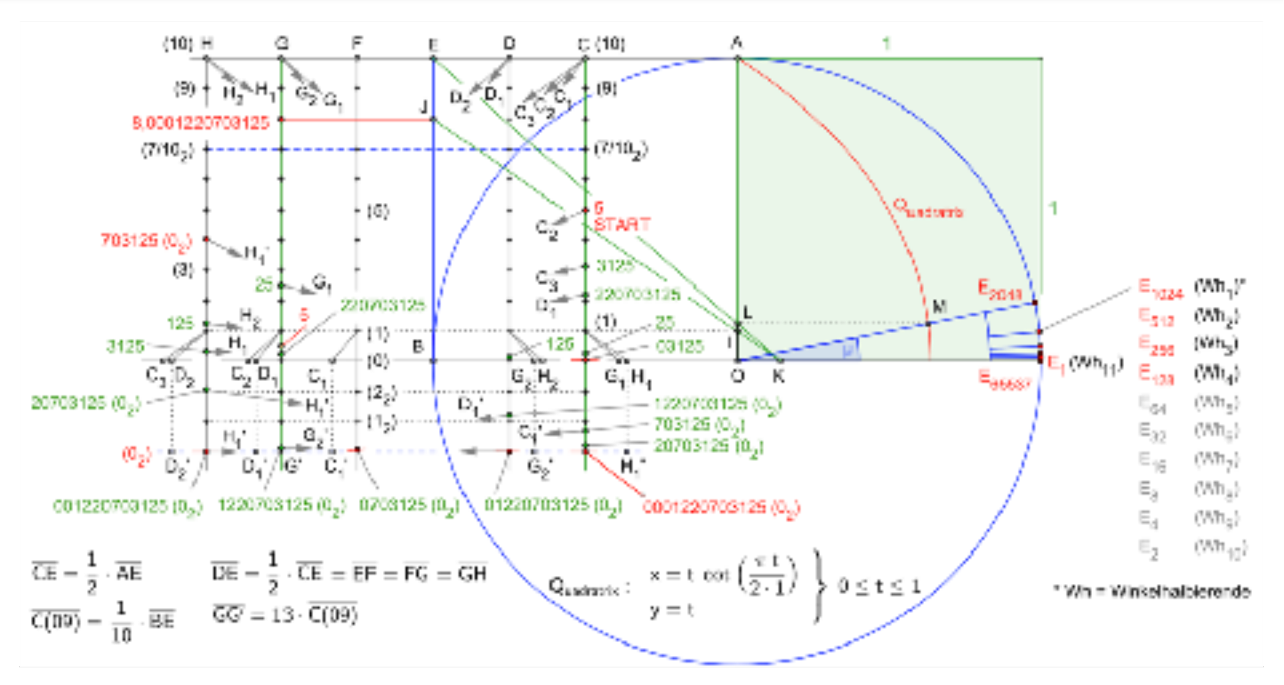

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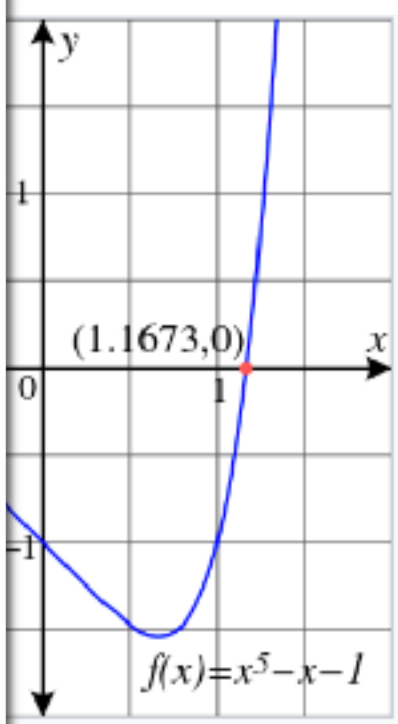

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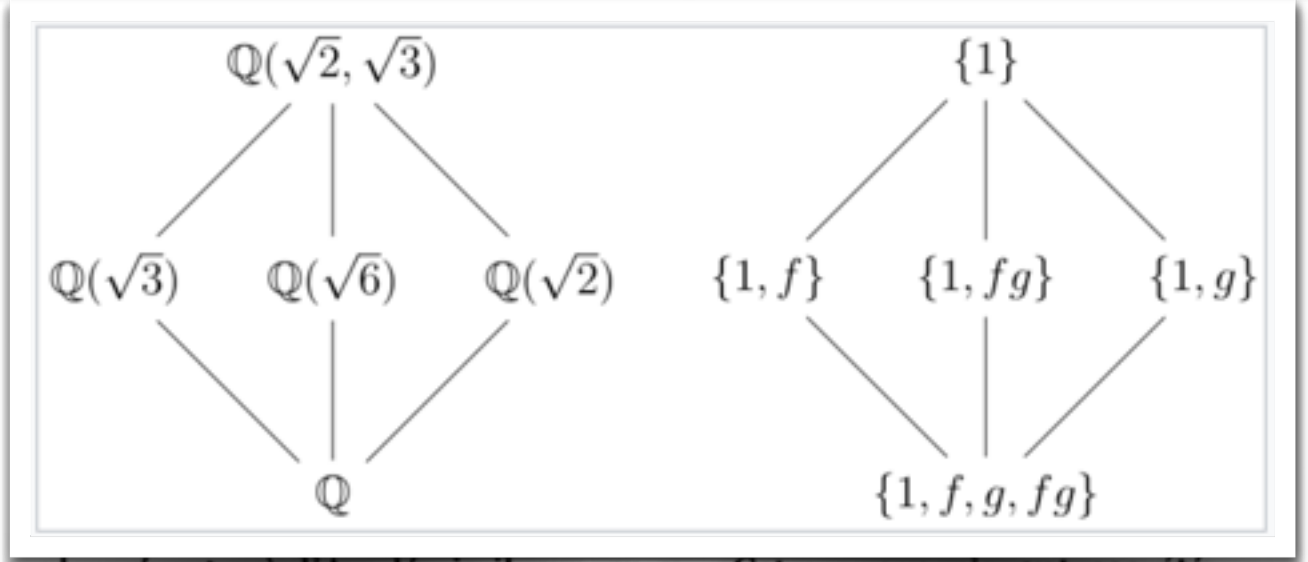

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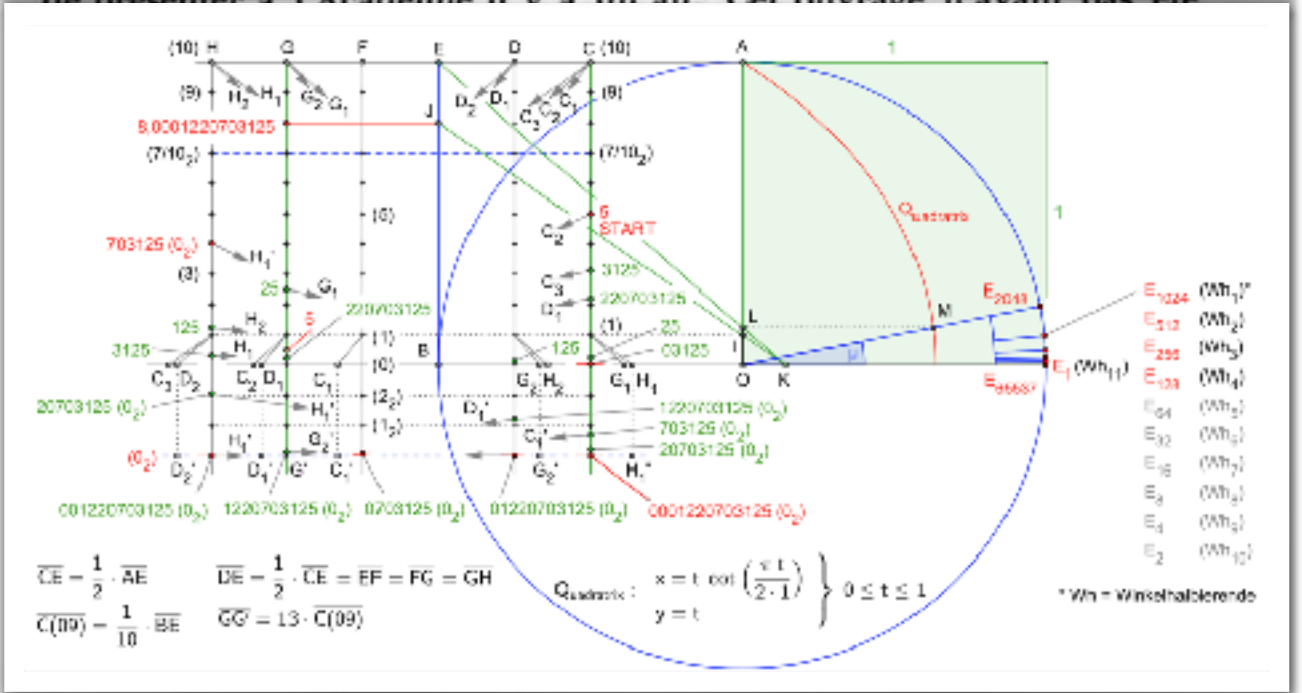
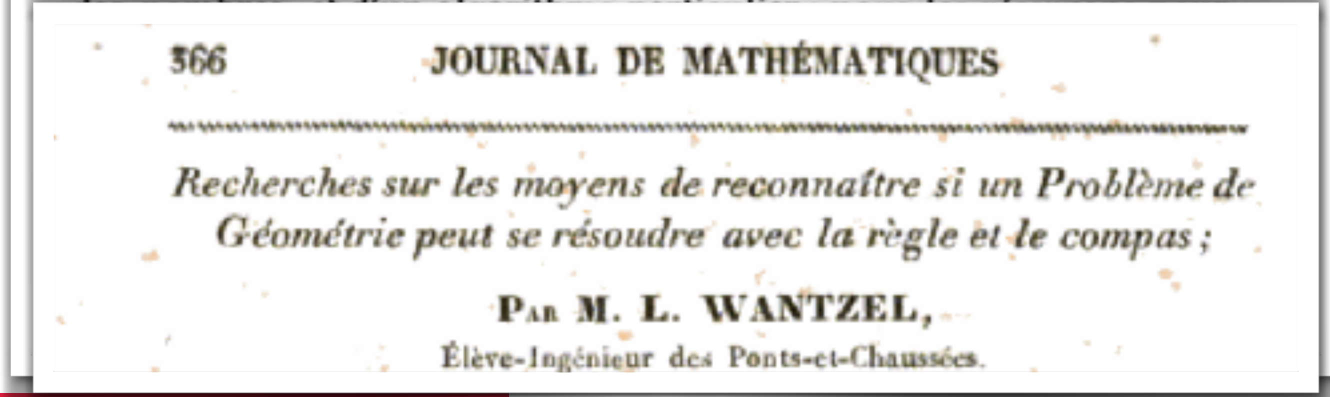


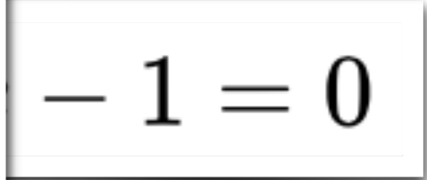


366 JOURNAL DE MATHÉMATIQUES

Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas;

PAR M. L. WANTZEL,

Élève-Ingénieur des Ponts-et-Chaussées.



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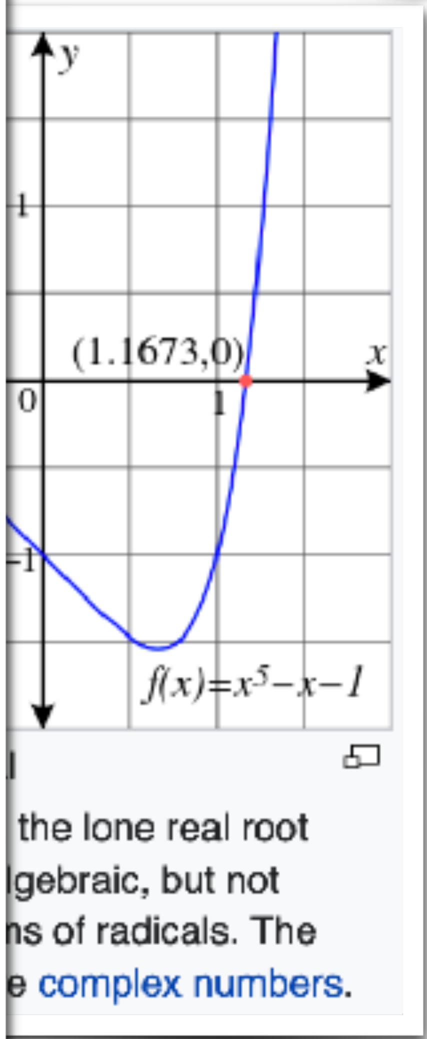

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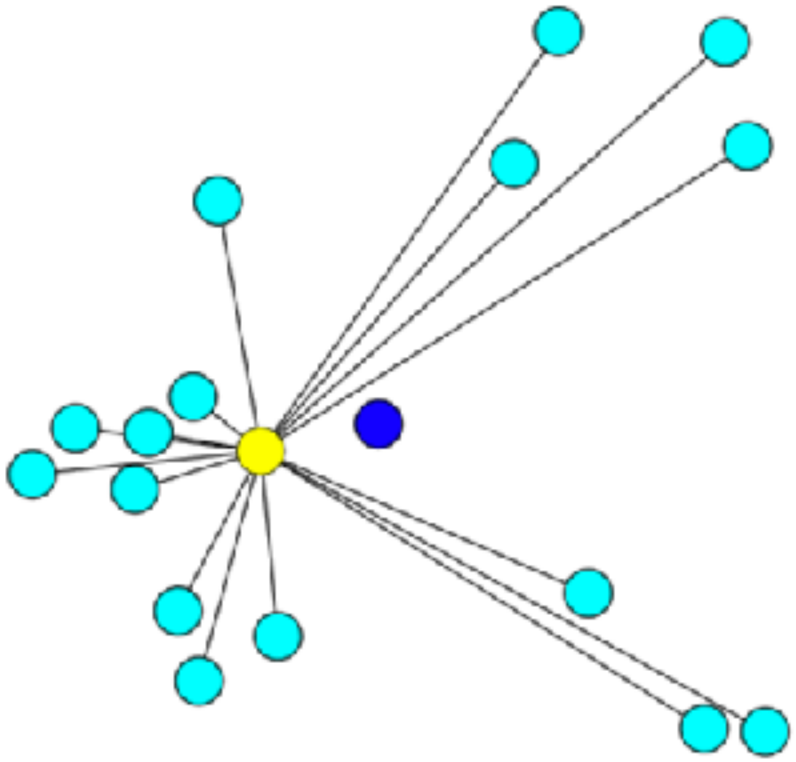

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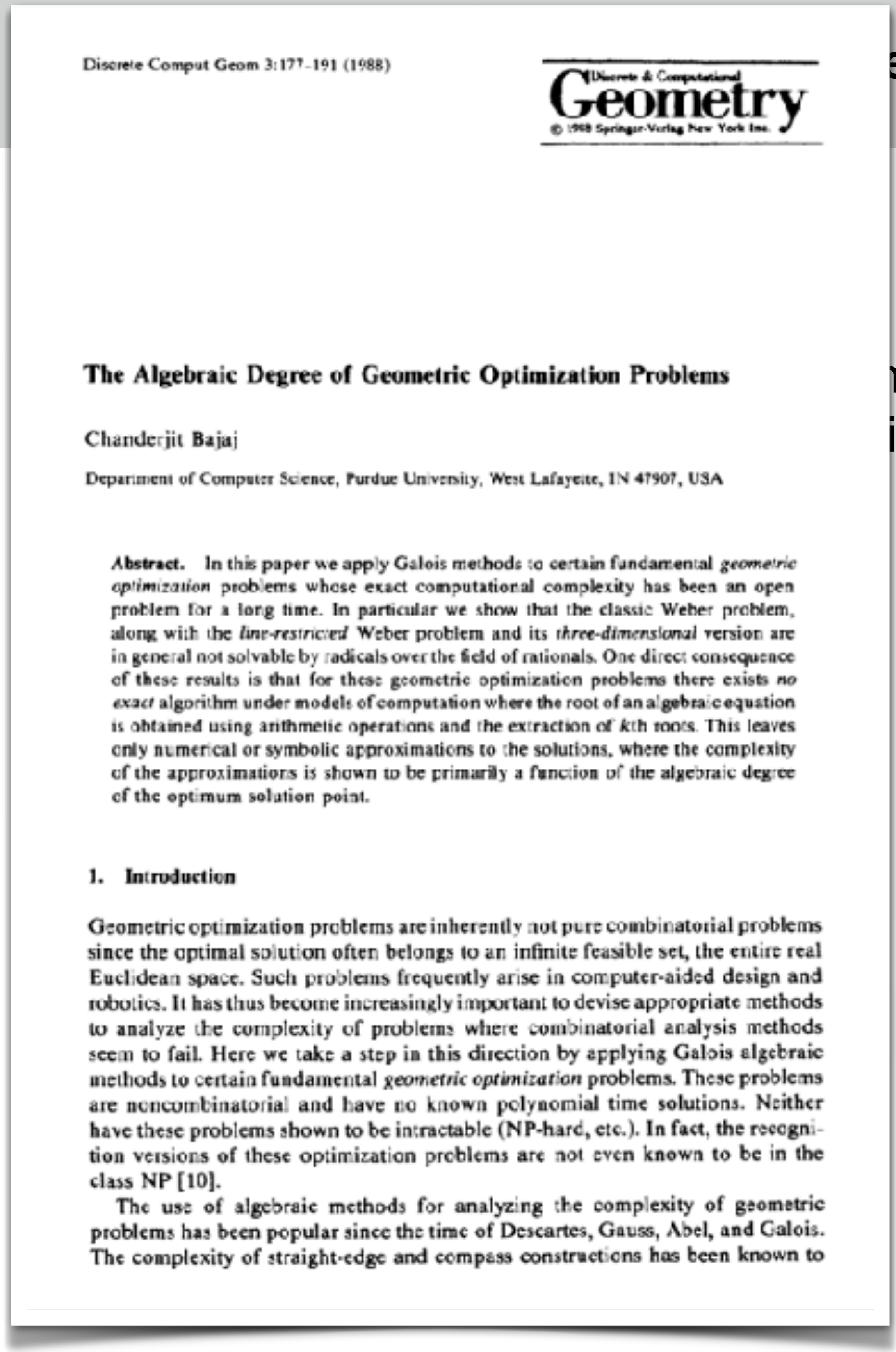

Discrete Comput Geom 3:177-191 (1988)

Discrete & Computational Geometry
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The Algebraic Degree of Geometric Optimization Problems

Chanderjit Bajaj

Department of Computer Science, Purdue University, West Lafayette, IN 47907, USA

Abstract. In this paper we apply Galois methods to certain fundamental *geometric optimization* problems whose exact computational complexity has been an open problem for a long time. In particular we show that the classic Weber problem, along with the *line-restricted* Weber problem and its *three-dimensional* version are in general not solvable by radicals over the field of rationals. One direct consequence of these results is that for these geometric optimization problems there exists *no exact* algorithm under models of computation where the root of an algebraic equation is obtained using arithmetic operations and the extraction of *k*th roots. This leaves only numerical or symbolic approximations to the solutions, where the complexity of the approximations is shown to be primarily a function of the algebraic degree of the optimum solution point.

1. Introduction

Geometric optimization problems are inherently not pure combinatorial problems since the optimal solution often belongs to an infinite feasible set, the entire real Euclidean space. Such problems frequently arise in computer-aided design and robotics. It has thus become increasingly important to devise appropriate methods to analyze the complexity of problems where combinatorial analysis methods seem to fail. Here we take a step in this direction by applying Galois algebraic methods to certain fundamental *geometric optimization* problems. These problems are noncombinatorial and have no known polynomial time solutions. Neither have these problems shown to be intractable (NP-hard, etc.). In fact, the recognition versions of these optimization problems are not even known to be in the class NP [10].

The use of algebraic methods for analyzing the complexity of geometric problems has been popular since the time of Descartes, Gauss, Abel, and Galois. The complexity of straight-edge and compass constructions has been known to

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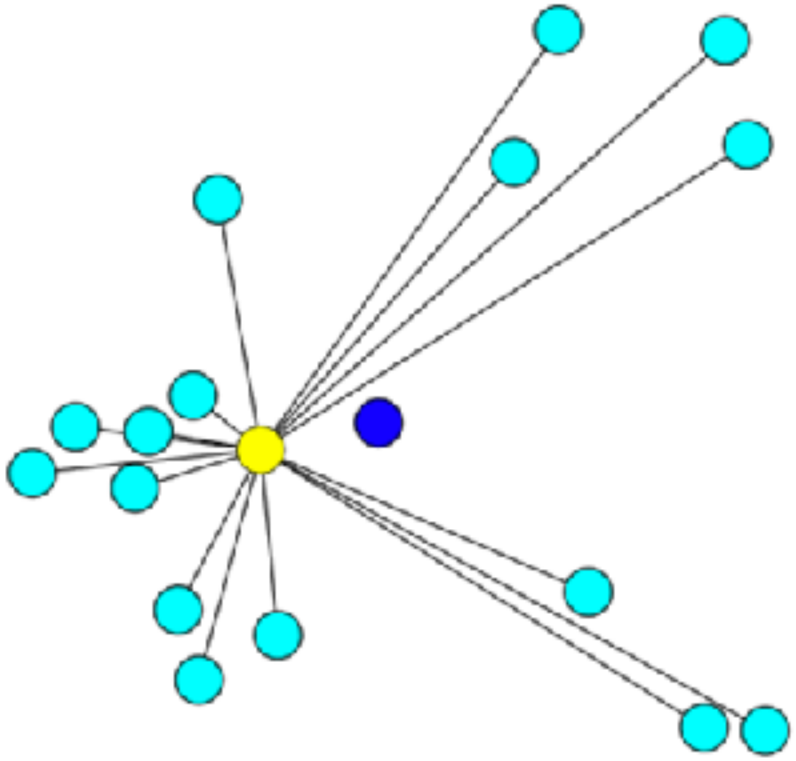

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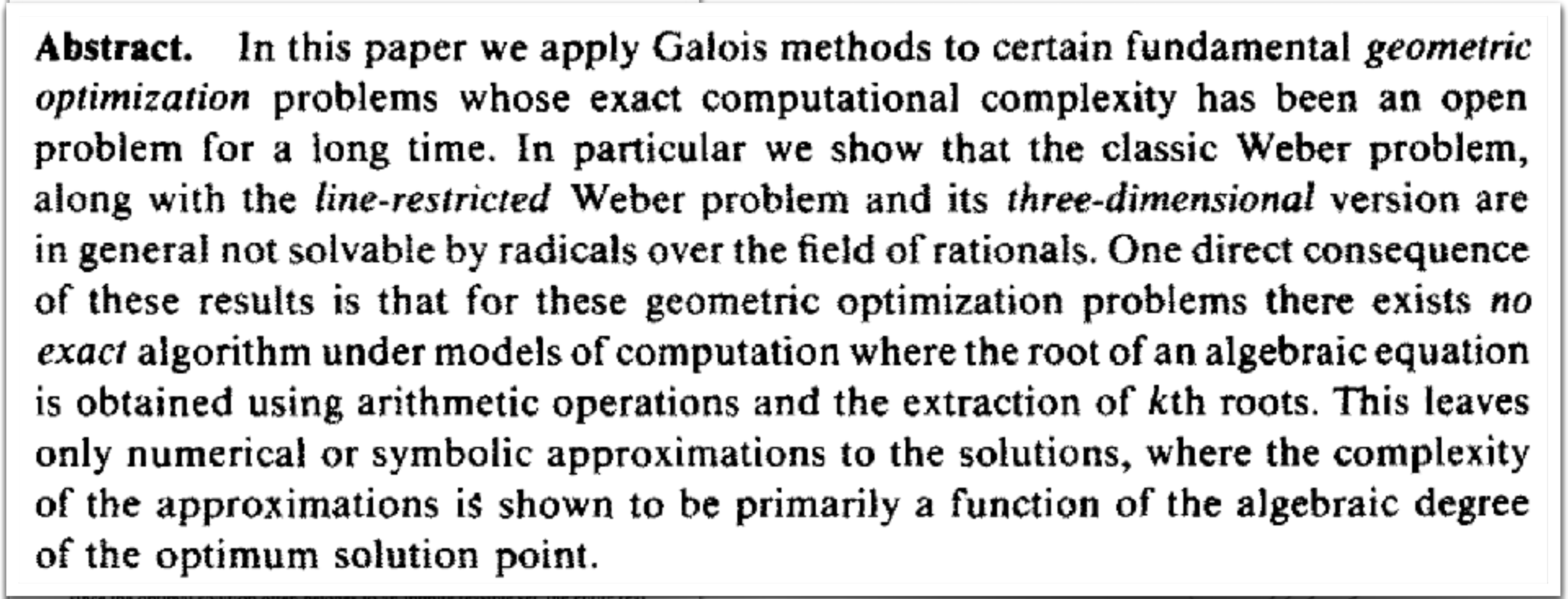

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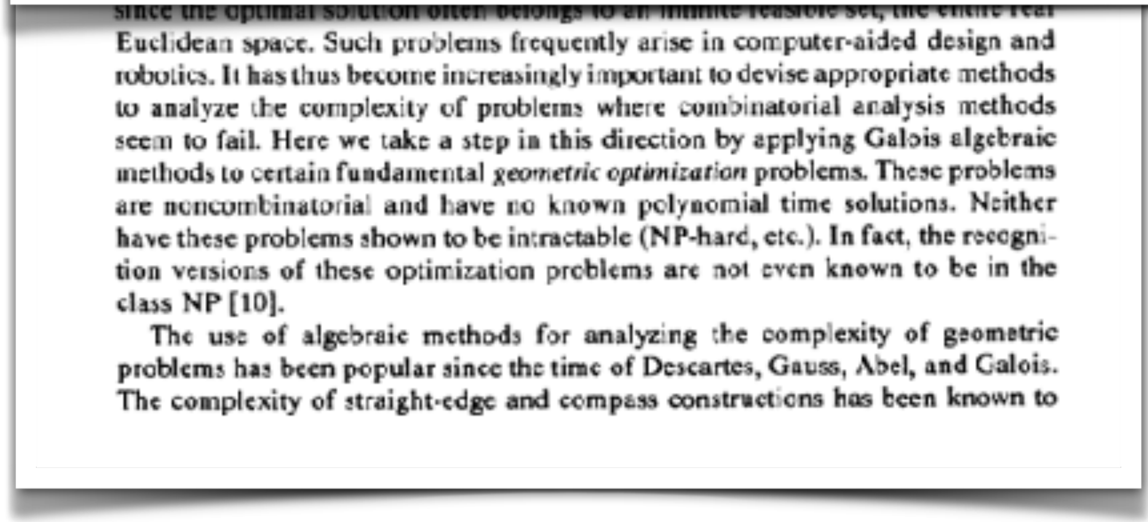

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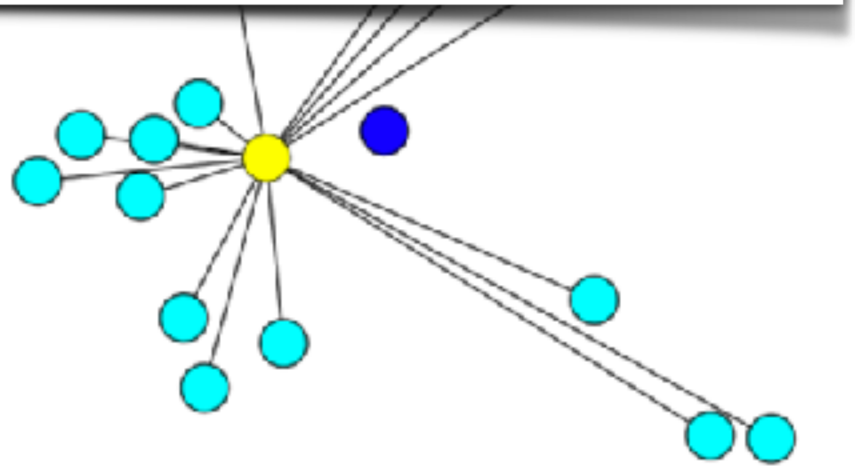

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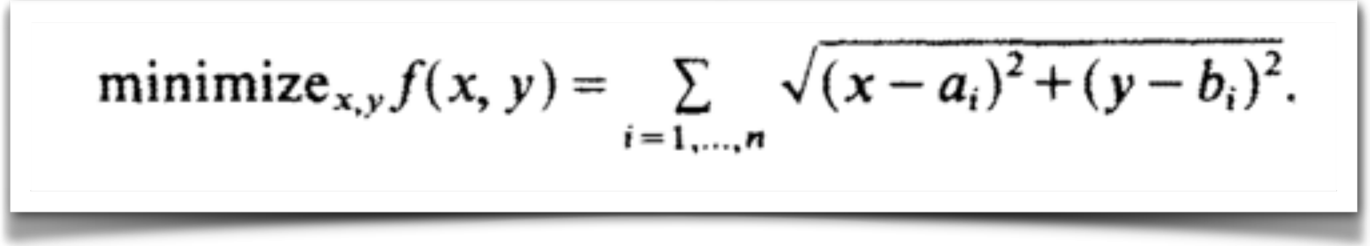
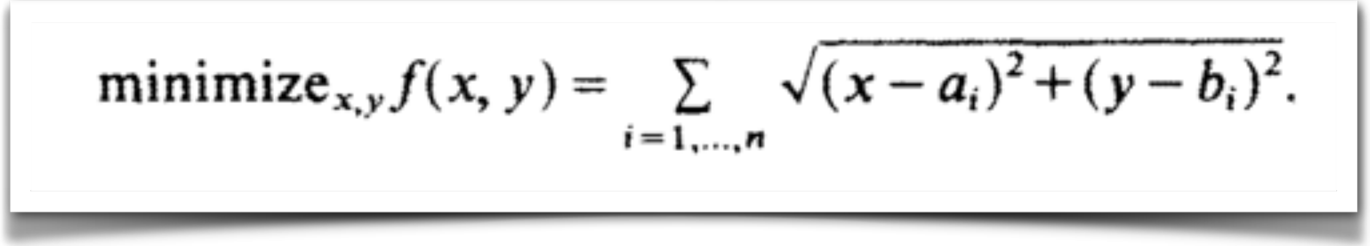


$$\text{minimize}_{x,y} f(x, y) = \sum_{i=1,\ldots,n} \sqrt{(x-a_i)^2+(y-b_i)^2}.$$

[Bajaj 1988]



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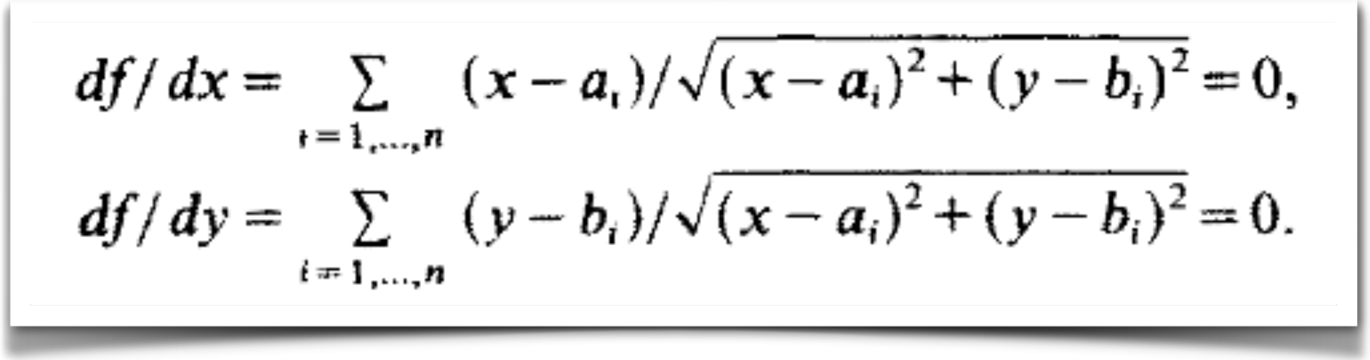

$$df/dx = \sum_{i=1,\dots,n} (x-a_i)/\sqrt{(x-a_i)^2+(y-b_i)^2}=0,$$
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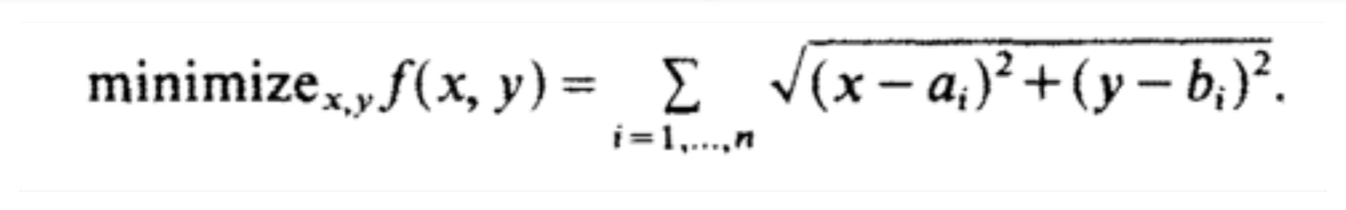

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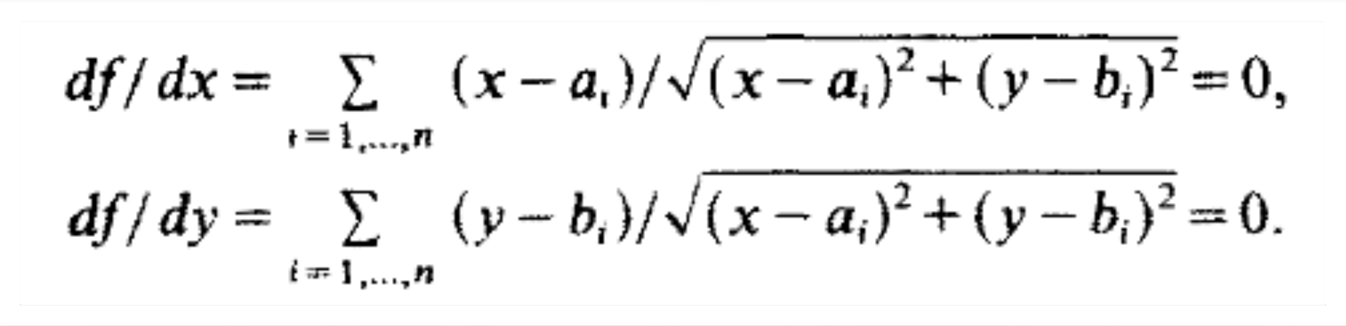

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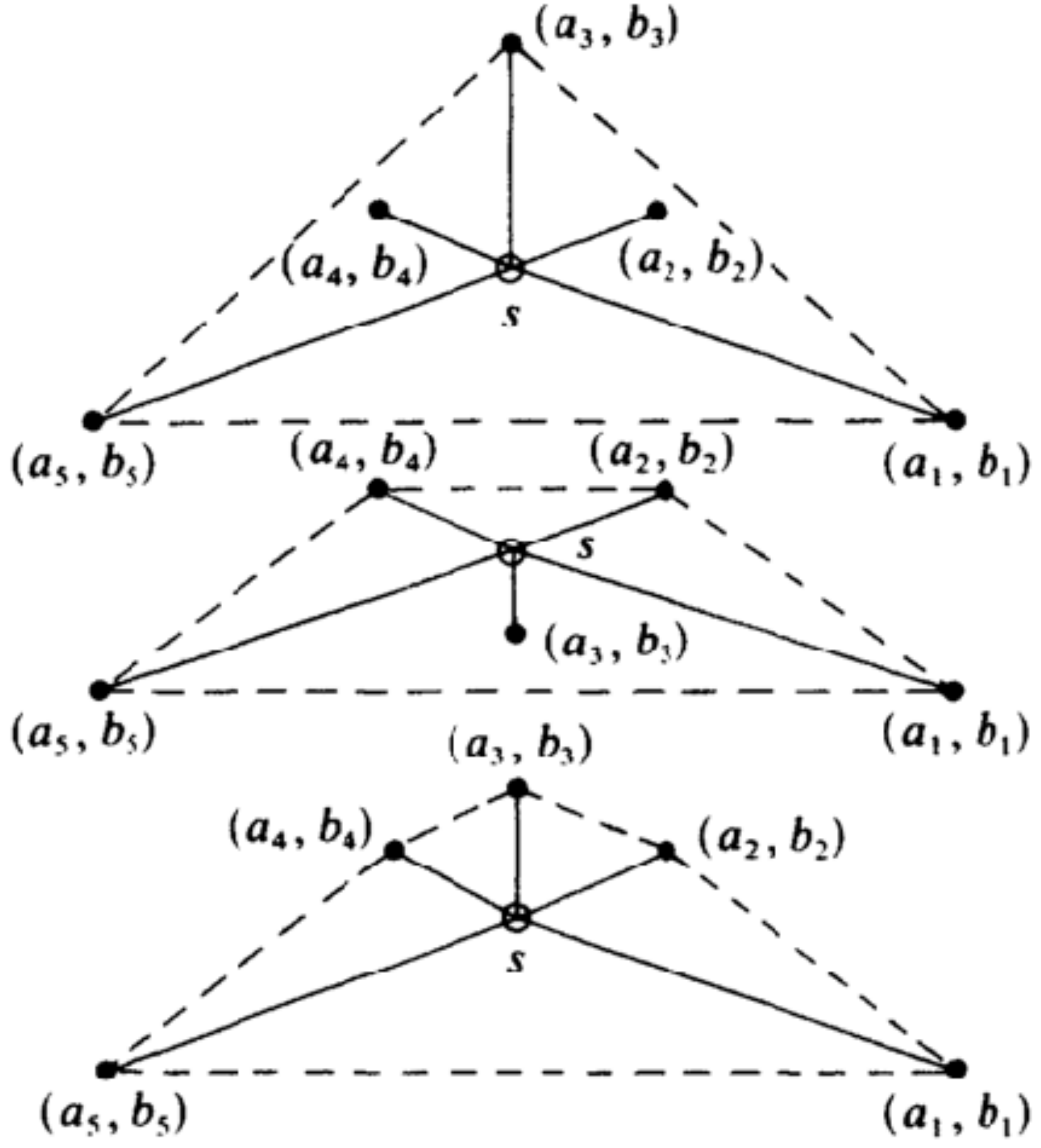
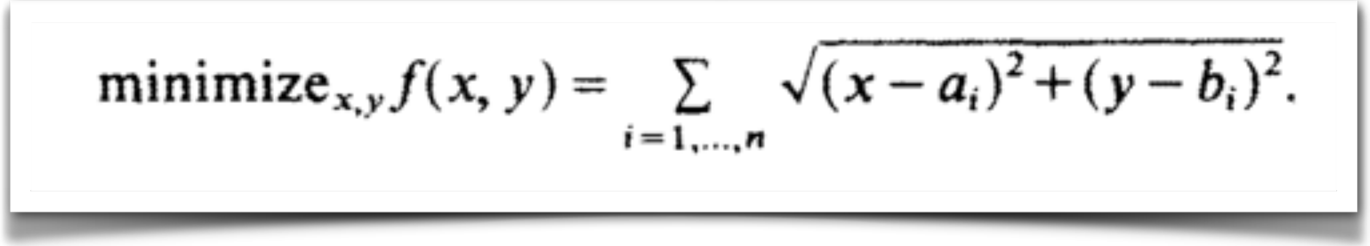


Fig. 4. Symmetric configurations of five points.

[Bajaj 1988]



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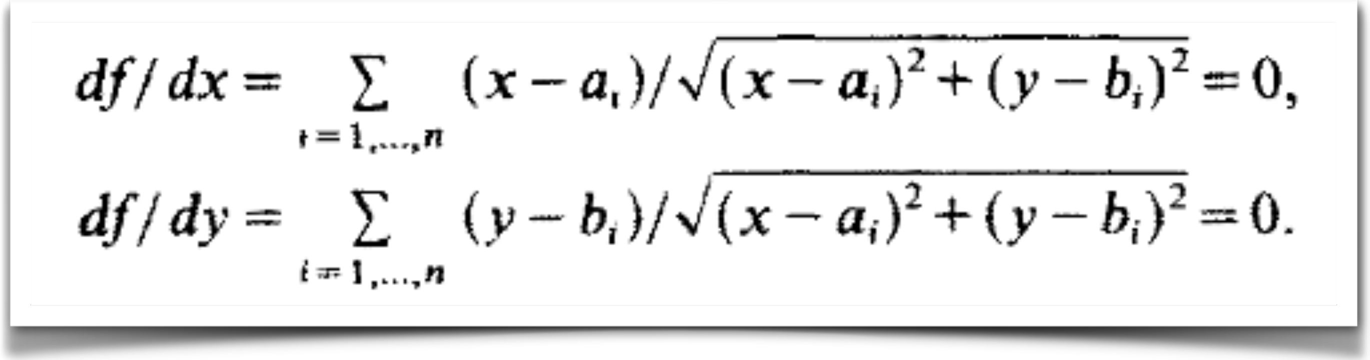

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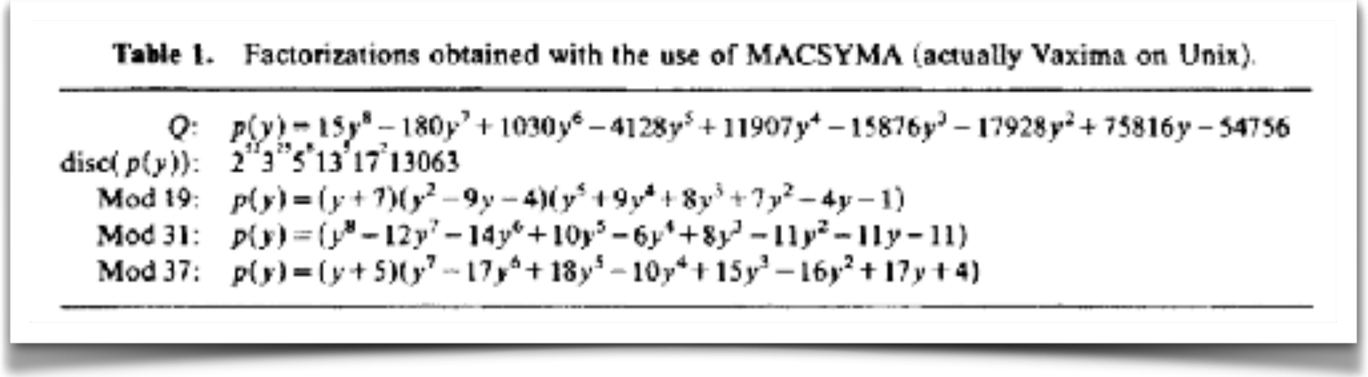
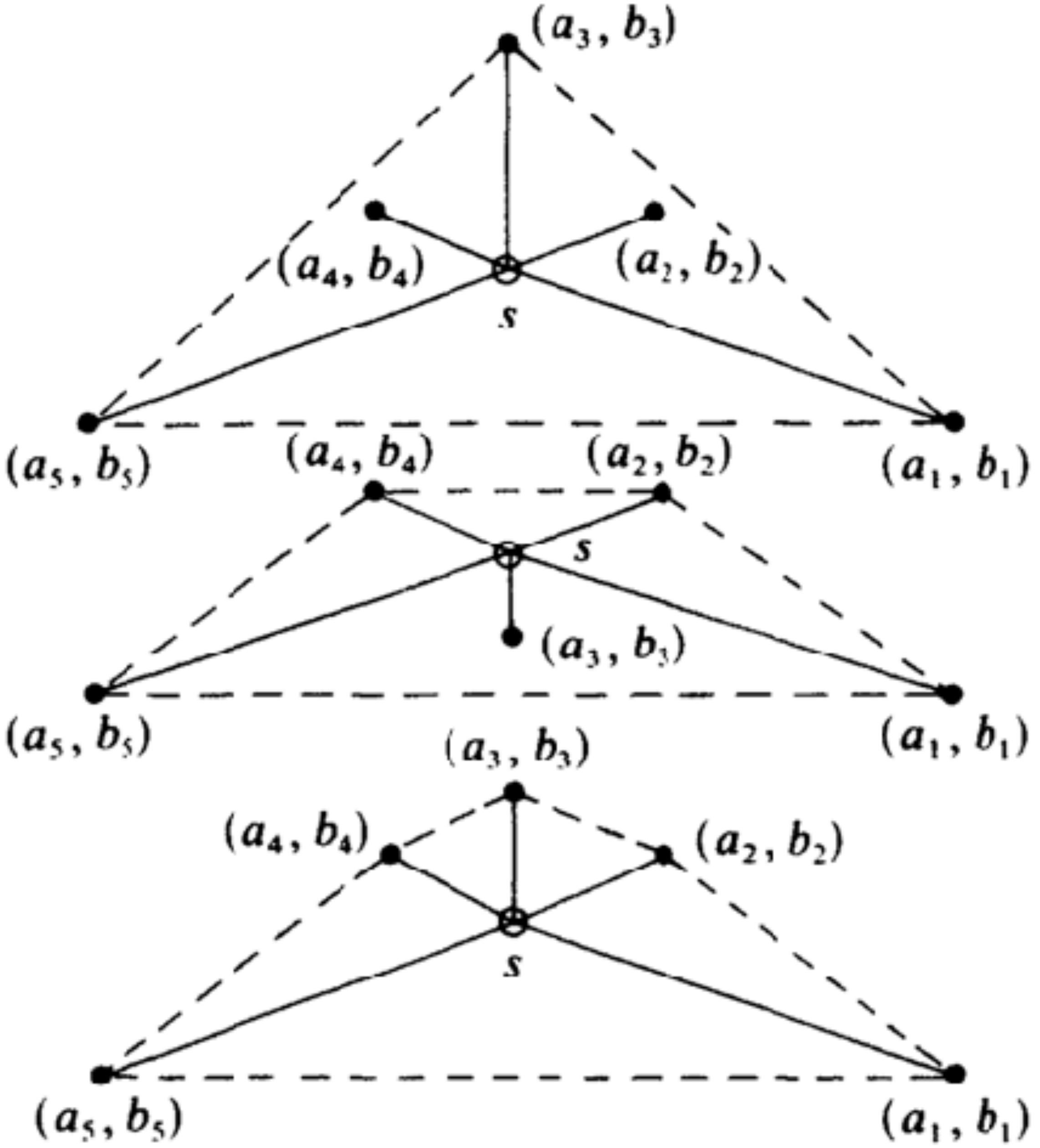


Table 1. Factorizations obtained with the use of MACSYMA (actually Vaxima on Unix).

Q:	$p(y) = 15y^8 - 180y^7 + 1030y^6 - 4128y^5 + 11907y^4 - 15876y^3 - 17928y^2 + 75816y - 54756$
disc($p(y)$):	$2^{32}3^{25}5^{8}13^{5}17^{2}13063$
Mod 19:	$p(y) = (y+7)(y^2-9y-4)(y^5+9y^4+8y^3+7y^2-4y-1)$
Mod 31:	$p(y) = (y^8-12y^7-14y^6+10y^5-6y^4+8y^3-11y^2-11y-11)$
Mod 37:	$p(y) = (y+5)(y^7-17y^6+18y^5-10y^4+15y^3-16y^2+17y+4)$



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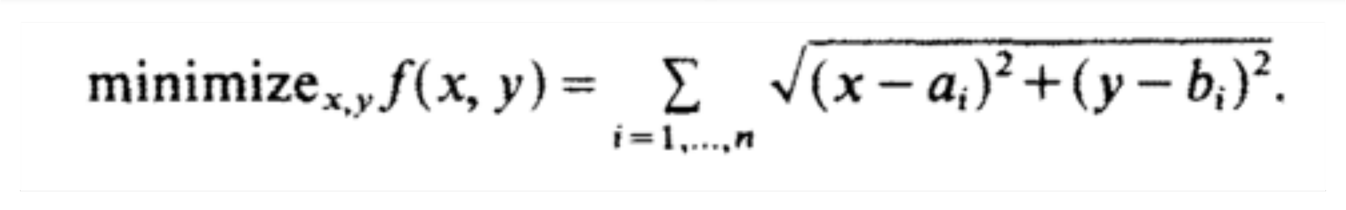

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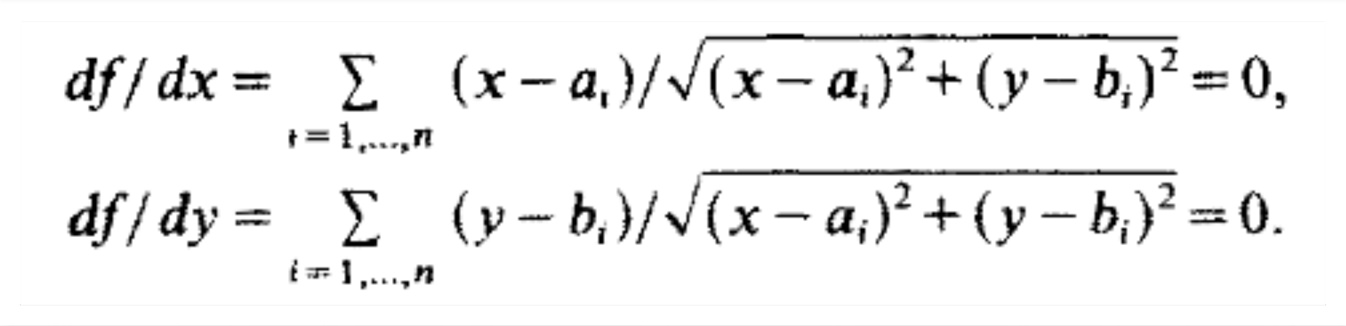


$$df/dx = \sum_{i=1,\dots,n} (x-a_i)/\sqrt{(x-a_i)^2+(y-b_i)^2} = 0,$$
$$df/dy = \sum_{i=1,\dots,n} (y-b_i)/\sqrt{(x-a_i)^2+(y-b_i)^2} = 0.$$

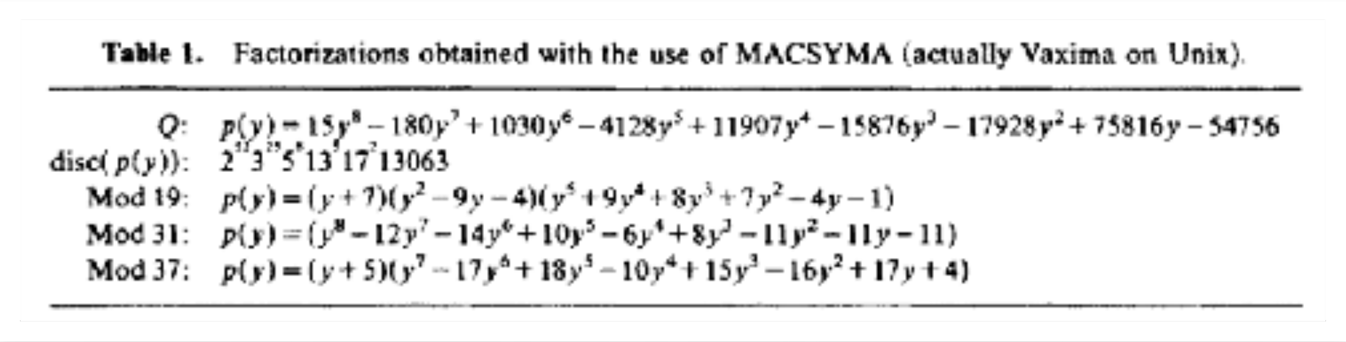


Table 1. Factorizations obtained with the use of MACSYMA (actually Vaxima on Unix).

Q:	$p(y) = 15y^8 - 180y^7 + 1030y^6 - 4128y^5 + 11907y^4 - 15876y^3 - 17928y^2 + 75816y - 54756$
disc($p(y)$):	$2^{41}3^{25}5^{8}13^{5}17^{2}13063$
Mod 19:	$p(y) = (y+7)(y^2-9y-4)(y^5+9y^4+8y^3+7y^2-4y-1)$
Mod 31:	$p(y) = (y^8-12y^7-14y^6+10y^5-6y^4+8y^3-11y^2-11y-11)$
Mod 37:	$p(y) = (y+5)(y^7-17y^6+18y^5-10y^4+15y^3-16y^2+17y+4)$

Lemma 1. *The polynomial $p(y)$ (Table 1) is irreducible over Q.*

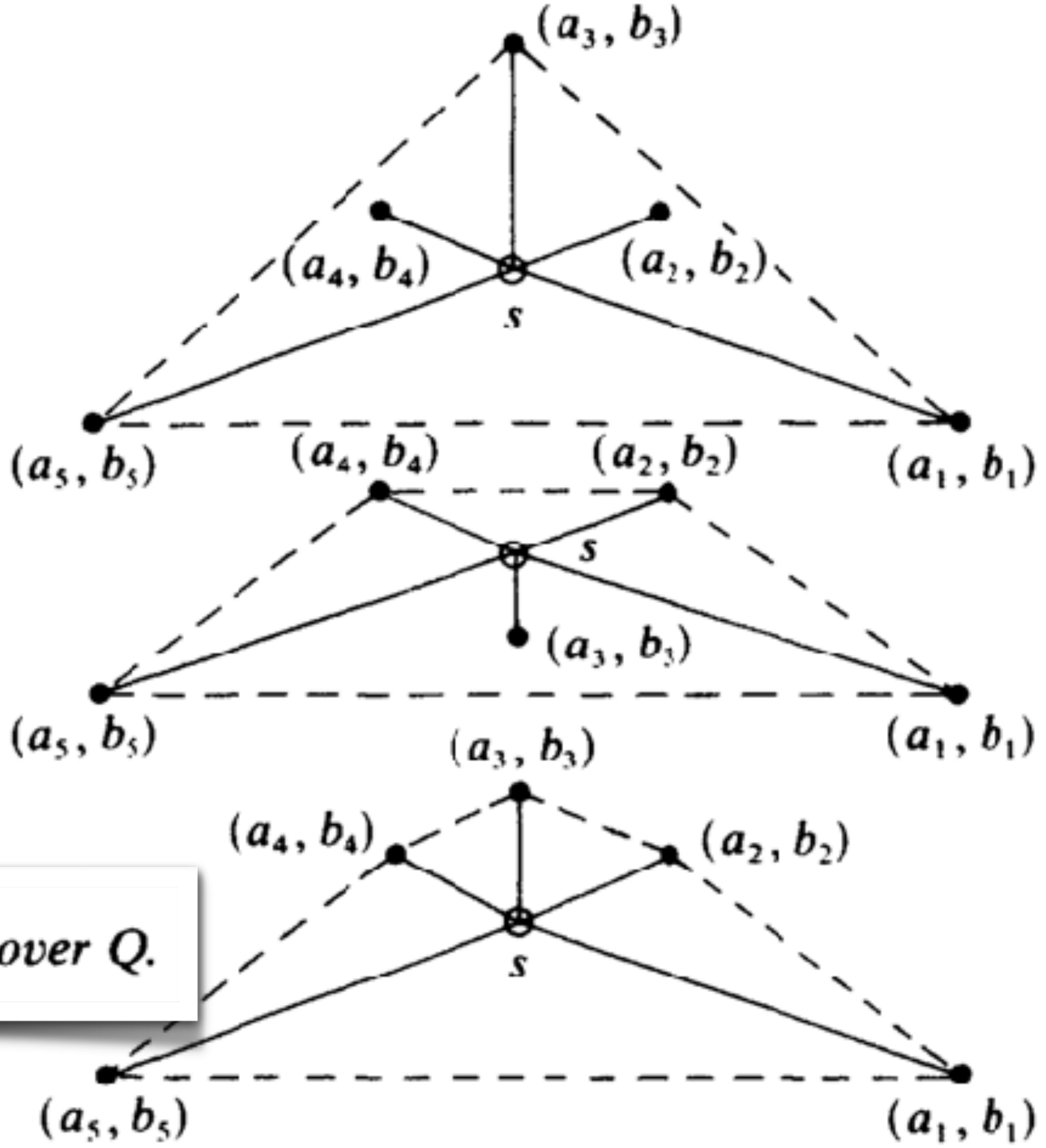
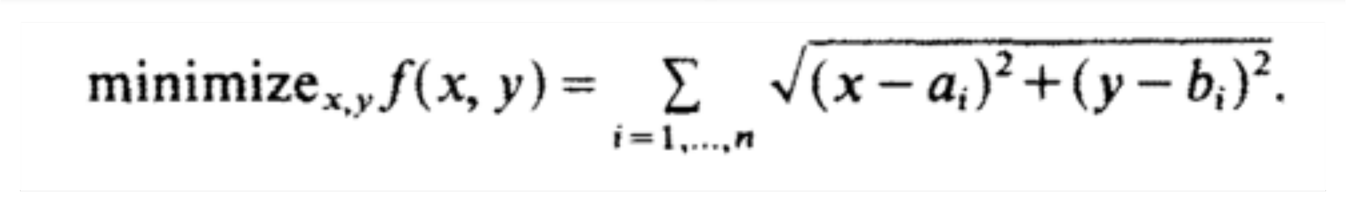


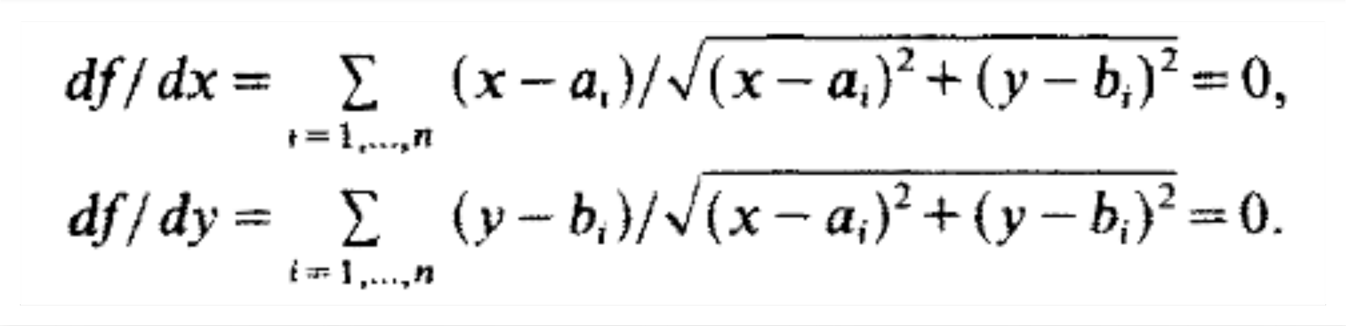
Fig. 4. Symmetric configurations of five points.

Technische Universität Braunschweig

[Bajaj 1988]



$$\text{minimize}_{x,y} f(x, y) = \sum_{i=1,\dots,n} \sqrt{(x-a_i)^2+(y-b_i)^2}.$$



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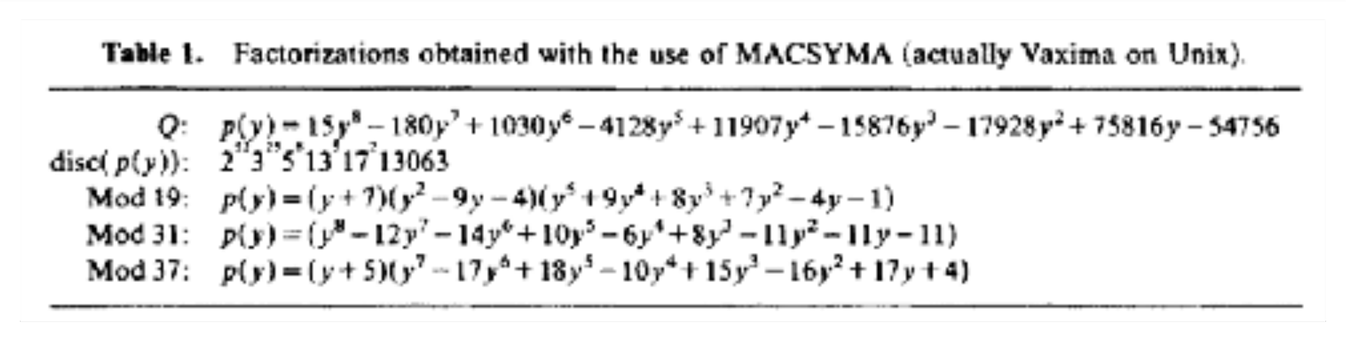

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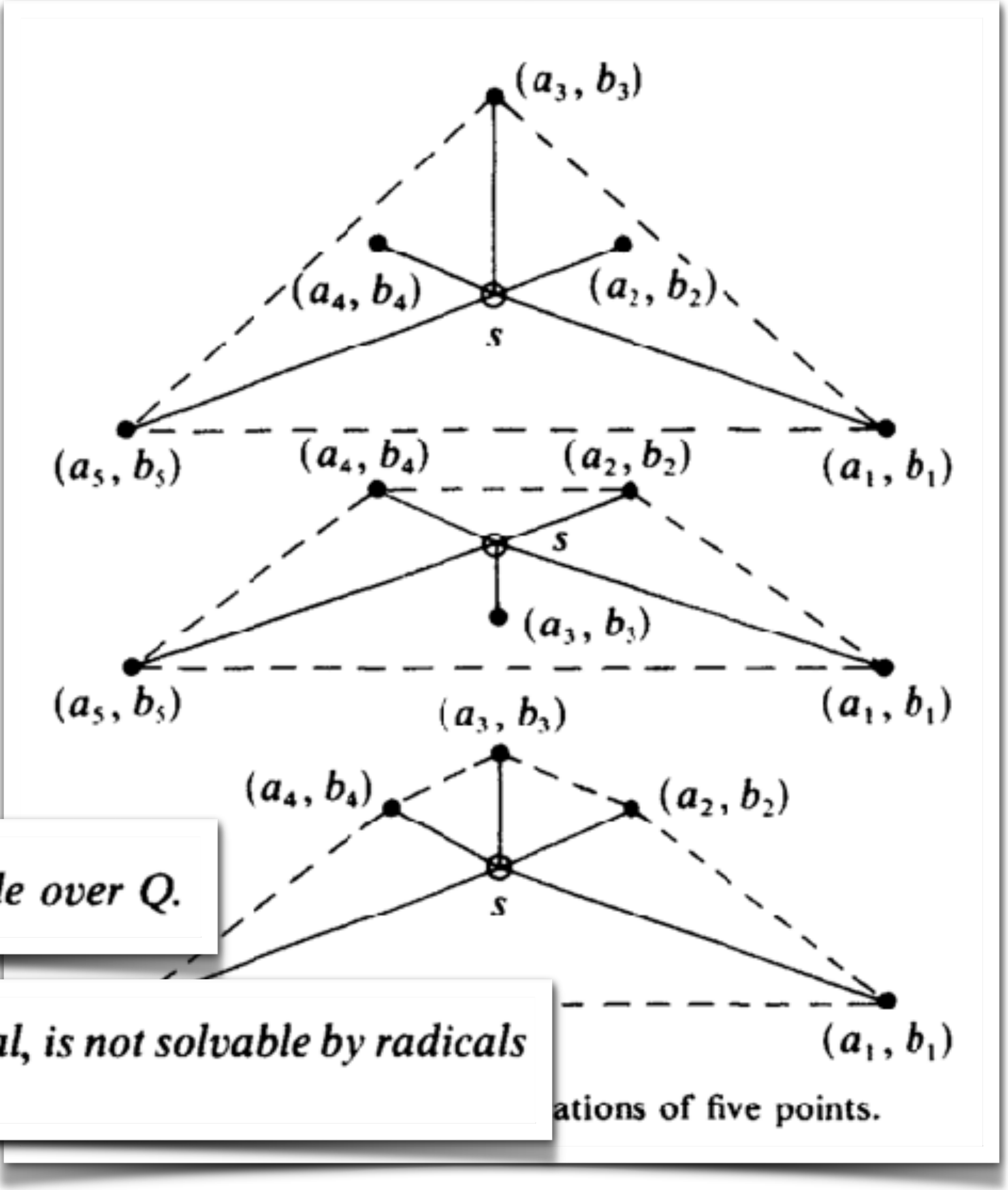
Theorem 11. *The generalized Weber problem, in general, is not solvable by radicals over Q for $n \geq 5$.*



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Solving a „hard“ problem to approximate an „easy“ one [Fekete, Meijer, Rohe, Tietze 2002]

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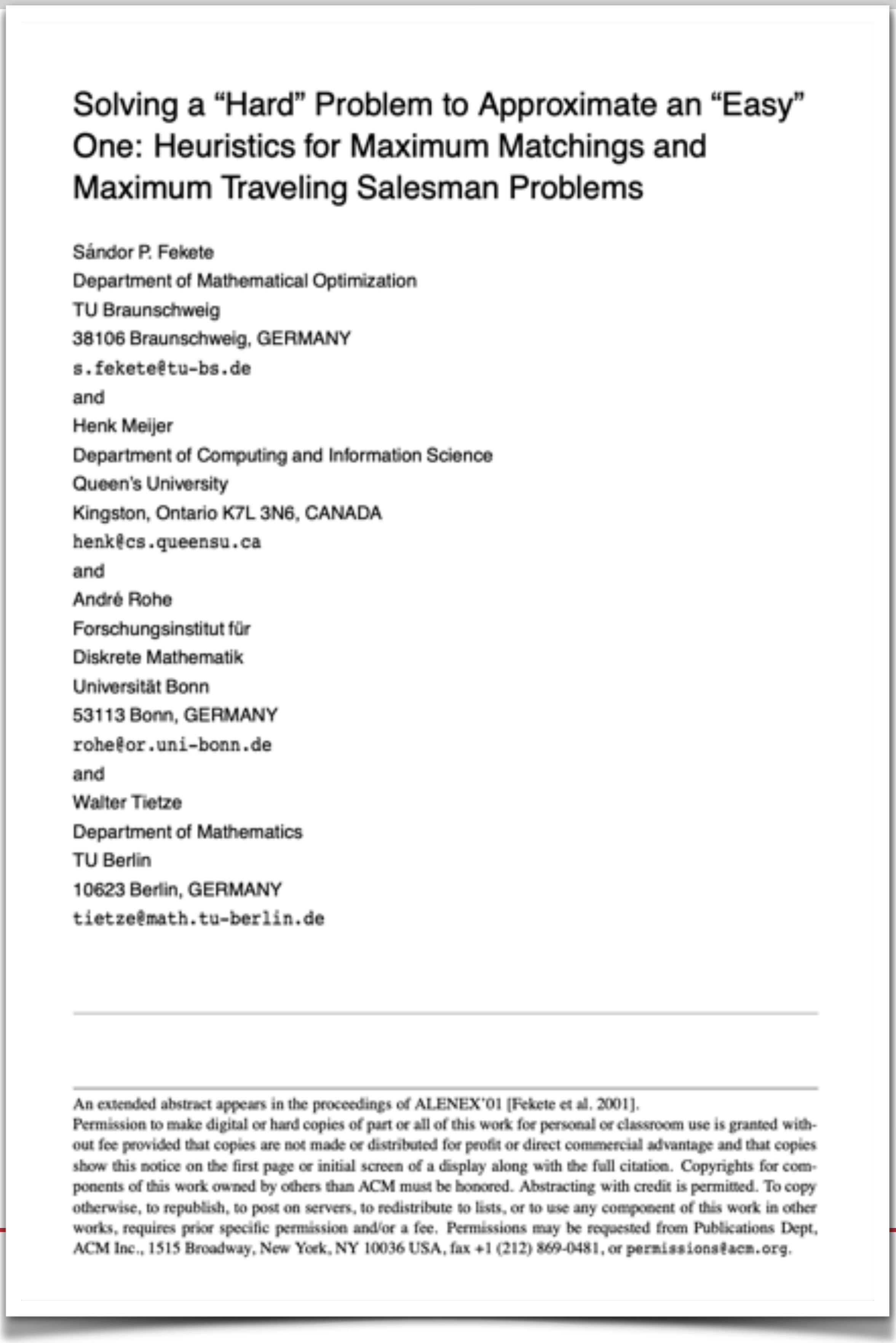

Solving a “Hard” Problem to Approximate an “Easy” One: Heuristics for Maximum Matchings and Maximum Traveling Salesman Problems

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An extended abstract appears in the proceedings of ALENEX’01 [Fekete et al. 2001].

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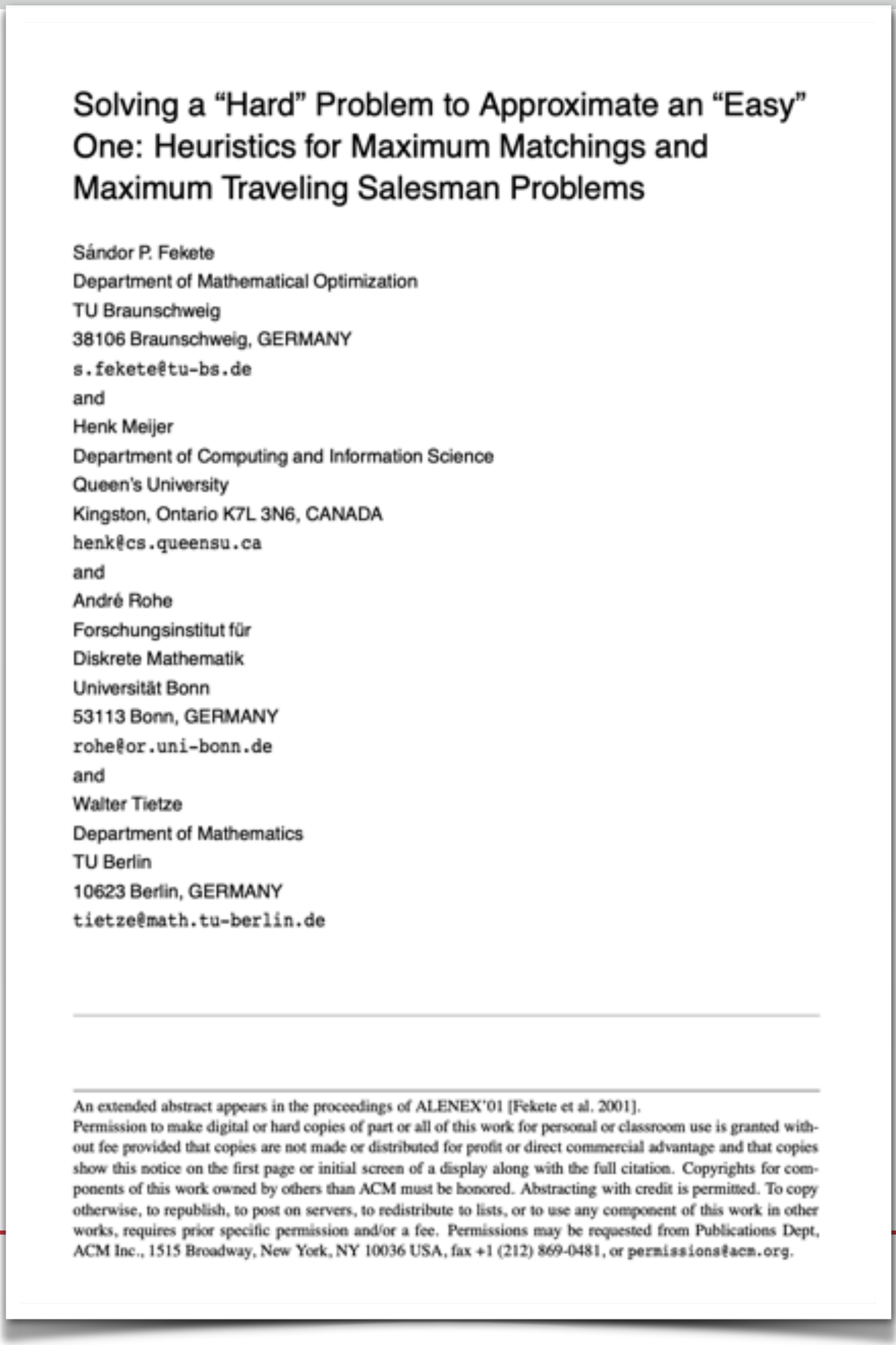

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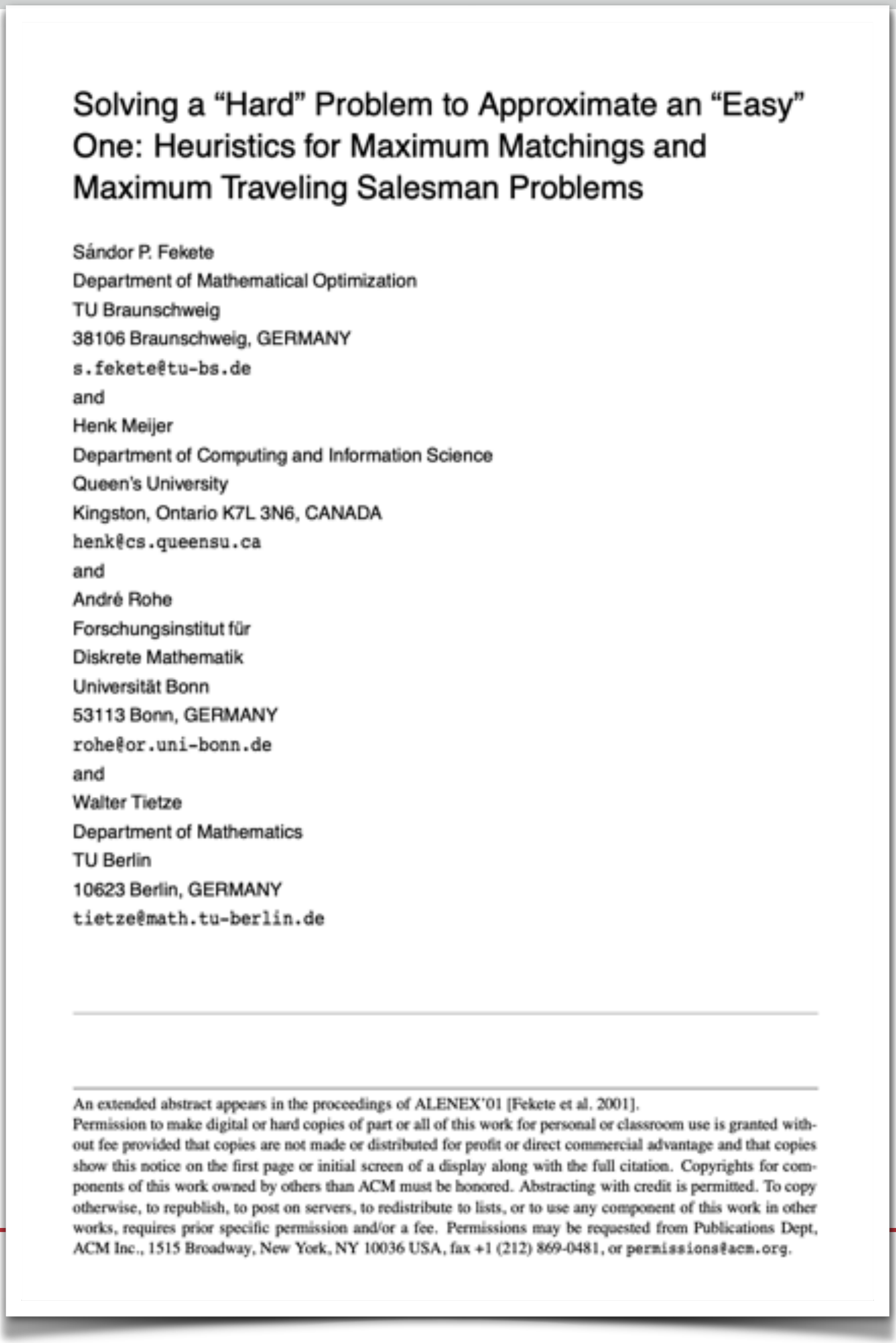

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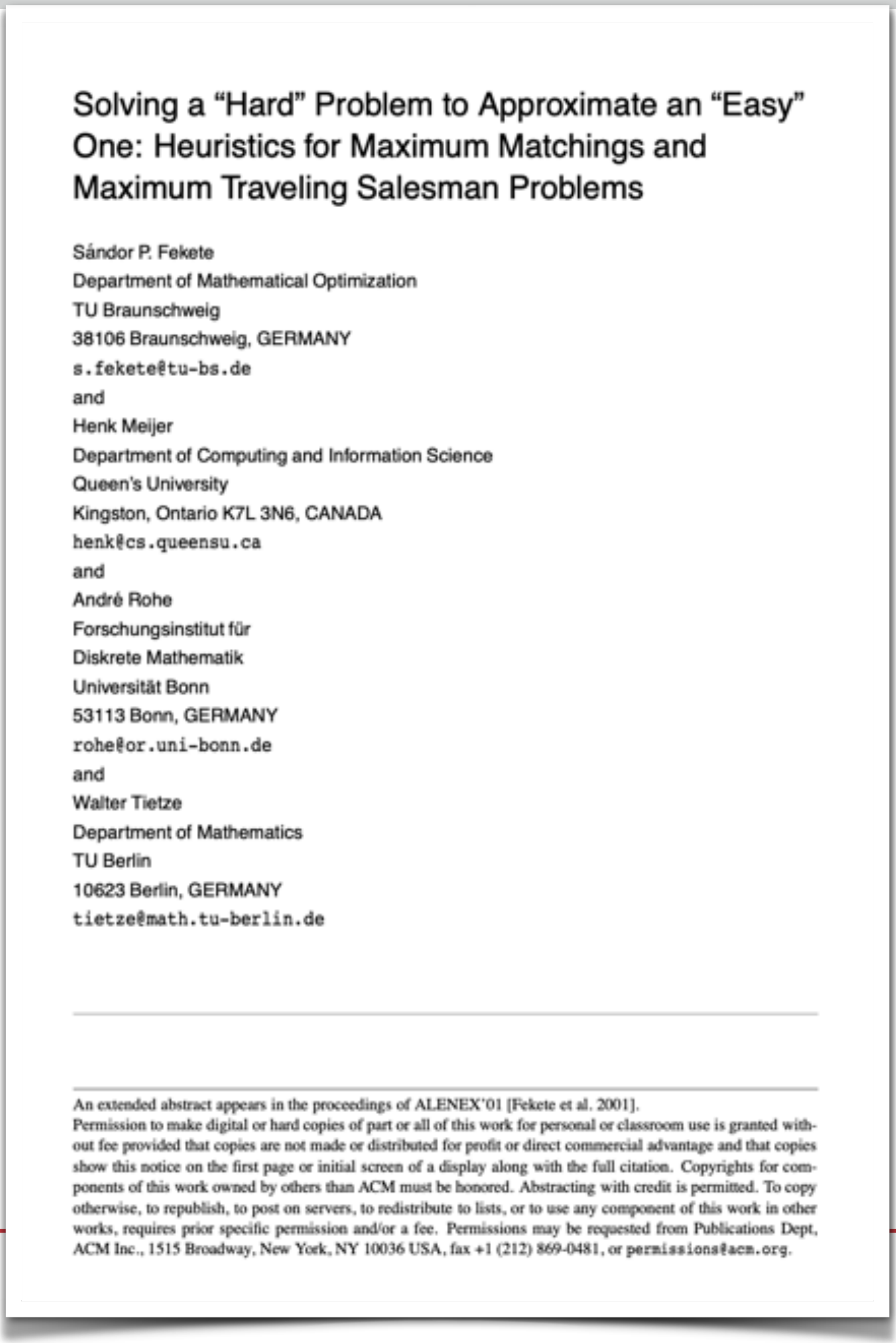
Given: A set P of points in $\mathbb{R}^2$

Wanted: A maximum-weight perfect matching of the given points



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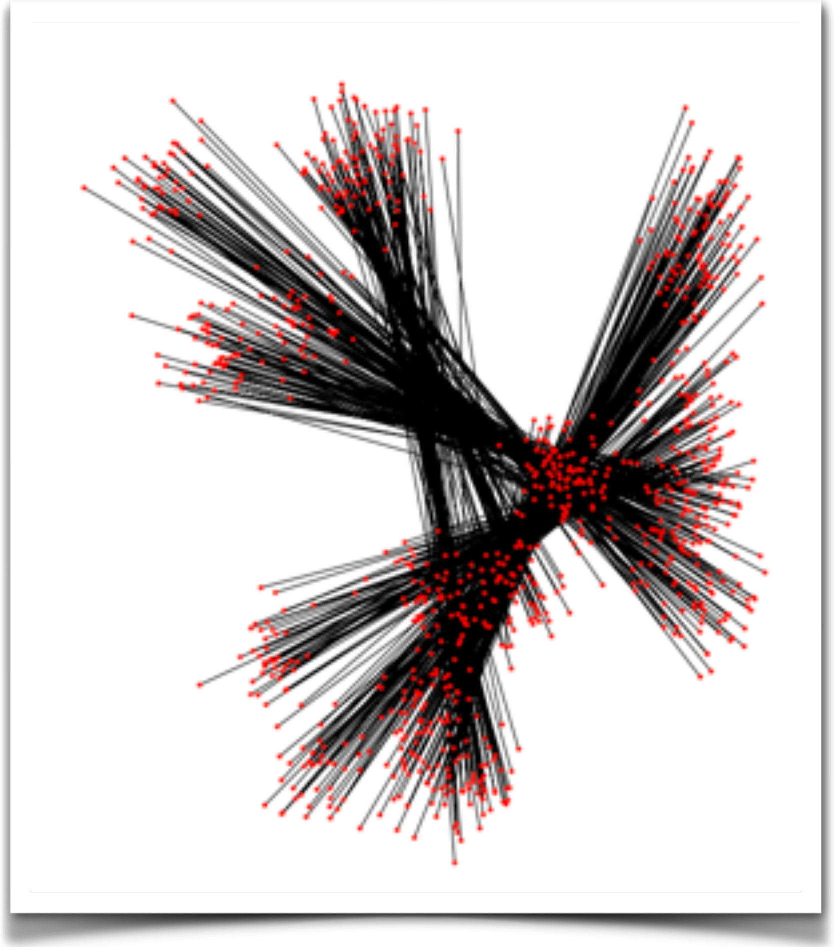

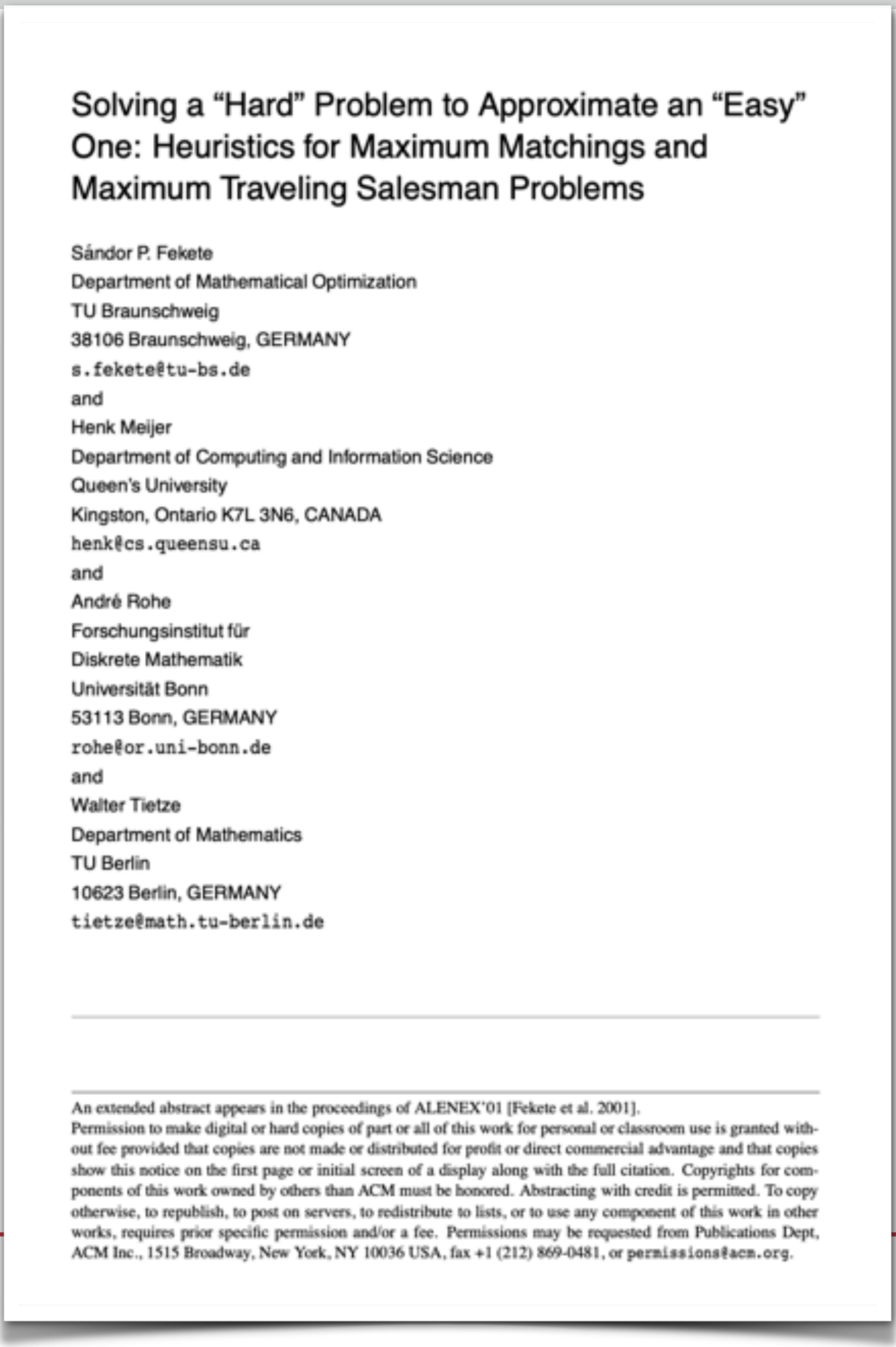

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1. **Introduction**
2. **Manhattan distances**
3. **Euclidean distances**
4. **Galois and Bajaj**
5. **Continuous sets**

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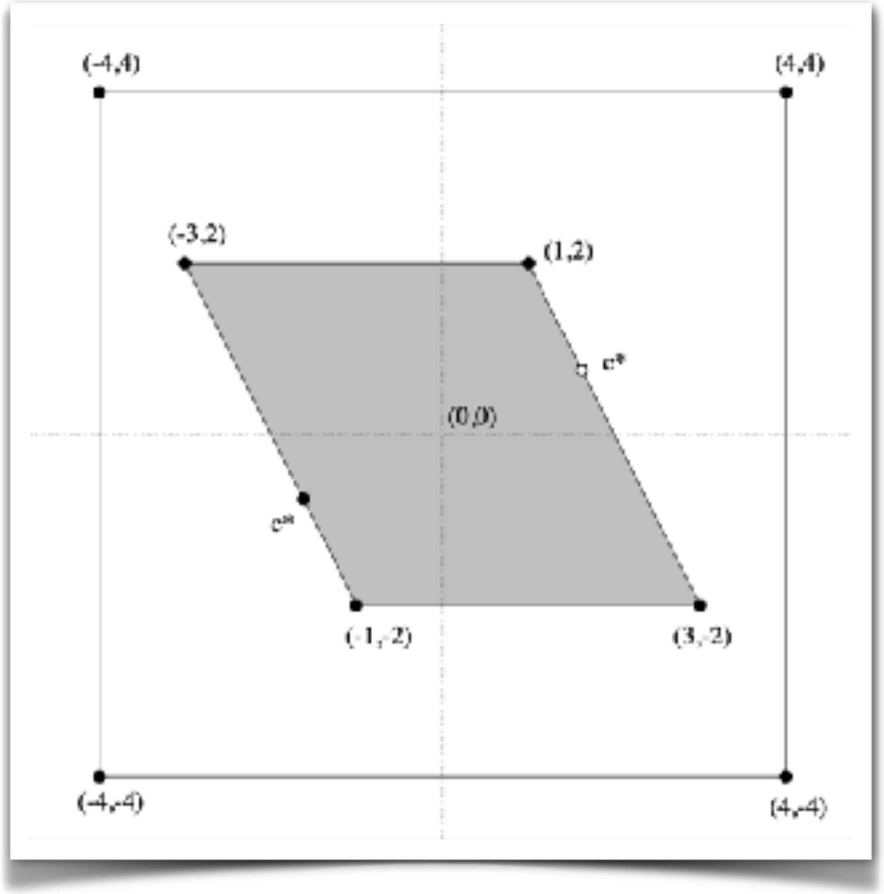

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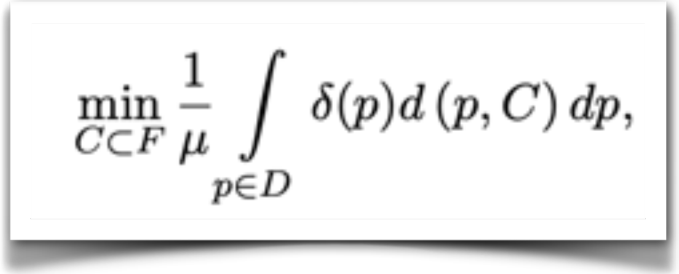

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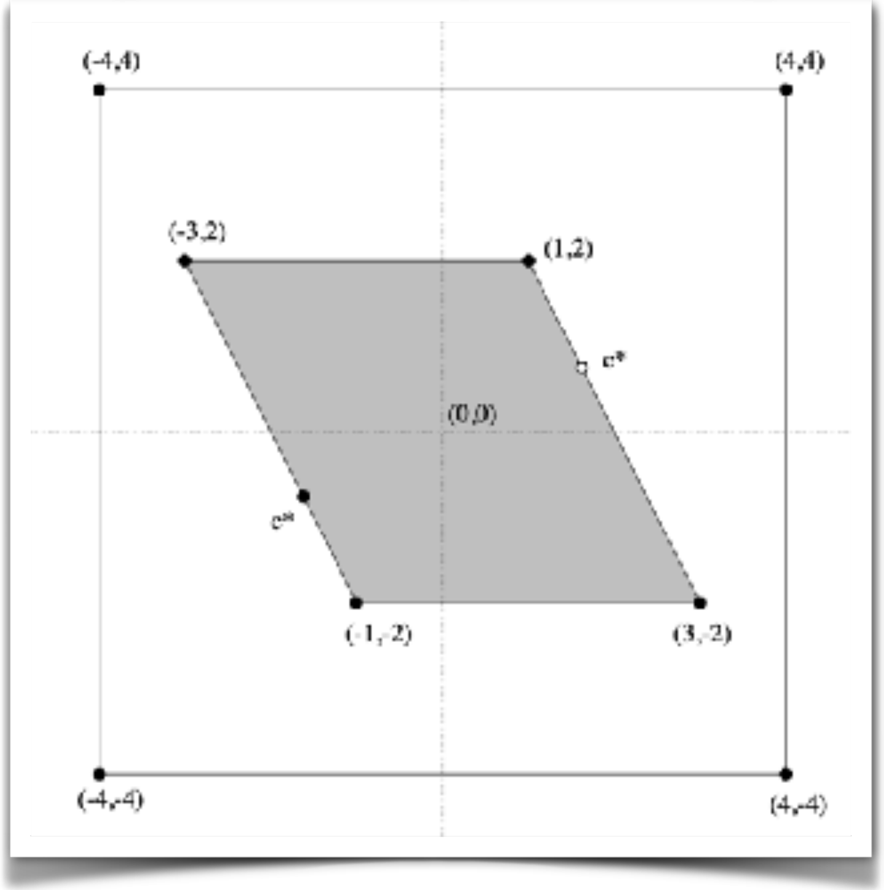

Continous location [Fekete, Mitchell, Weinbrecht/Beurer 2000/2005]

On the Continuous Weber and k-Median Problems

(Extended Abstract)

Sándor P. Fekete* Joseph S. B. Mitchell† Karin Weinbrecht‡

Abstract

We give the first *exact* algorithmic study of facility location problems that deal with finding a median for a continuum of demand points. In particular, we consider versions of the "continuous k-median (Weber) problem" where the goal is to select one or more center points that minimize the average distance to a set of points in a demand region. In such problems, the average is computed as an integral over the relevant region, versus the usual discrete sum of distances. The resulting facility location problems are inherently geometric, requiring analysis techniques of computational geometry. We provide polynomial-time algorithms for various versions of the L_1 1-median (Weber) problem. We also consider the multiple-center version of the L_1 k-median problem, which we prove is NP-hard for large k.

1 Introduction

> *"There are three important factors that determine the value of real estate – location, location, and location."*

There has been considerable study of facility location problems in the field of combinatorial optimization. In general, the input to these problems includes a weighted set D of demand locations (with weight distribution δ and total weight Λ), a set F of feasible facility locations, and a distance function d that measures cost between a pair of locations. In one important class of questions, the problem is to determine one or more feasible median locations $c \in F$ in order to minimize the average cost from the demand locations, $p \in D$, to the corresponding central points c_p that are nearest to p:

$$\min_{C \subseteq F} \frac{1}{\Lambda} \int_{p \in D} \delta(p) d(p, C)\, dp.$$

If there is one median point to be placed, the problem is known as the classical *Weber problem*; it was first discussed in Weber's 1909 book on the pure theory of location for industries [52] (see [54] for a modern survey). More generally, for a given number $k \geq 1$ of facilities, the problem is known as the *k-median problem*. A problem of similar type with a different objective function is the so-called *k-center problem*, where the goal is to find a set of k center locations such that the maximum distance of the demand set from the nearest center location is minimized.

With many practical motivations, geometric instances of facility location problems have attracted a major portion of the research to date. In these instances, the sets D of demand locations and F of feasible placements are modeled as points in some geometric space, typically $\Re^2$, with distances measured according to the Euclidean (L_2) or Manhattan (L_1) metric. In these geometric scenarios, it is natural to consider not only finite (discrete) sets F of feasible locations, but also (continuous) sets having positive area. For the classical Weber problem, the set F is the entire plane $\Re^2$, while D is some finite set of demand points.

Location theory distinguishes between discrete and continuous location theory (see [22]). However, for median problems, this distinction has mostly been applied to the set of feasible placements, distinguishing between discrete and continuous sets F. It is remarkable that, so far, continuous location theory of median problems has almost entirely treated discrete demand sets D [22, 45]. We should note that there are several studies in the literature that deal with k-center problems with continuous demand, e.g. see [36, 51], where demand arises from the continuous point sets along the edges in a graph. See [50] for results on the placement of k capacitated facilities serving a continuous demand on a one-dimensional interval. Also, k-center problems have been studied extensively in a geometric setting, see e.g. [1, 15, 23, 25, 26, 27, 28, 29, 30, 35, 37, 48, 49]. However, designing discrete algorithms for k-center problems can generally be expected to be more immediate than for k-median problems, since the set of demand points that determine a critical center location will usually form just a finite set of $d+1$ points in d-dimensional space.

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†jsbm@ams.sunysb.edu; http://www.ams.sunysb.edu/~jsbm. Department of Applied Mathematics and Statistics, State University of New York, Stony Brook, NY 11794-3600. Partially supported by NSF grant CCR-9732220, and by grants from Bridgeport Machines, Hughes Research Labs, NASA, Sandia National Labs, Seagull Technology, and Sun Microsystems.

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Computational Geometry 2000 Hong Kong China

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Given: A set P of points in $\mathbb{R}^2$

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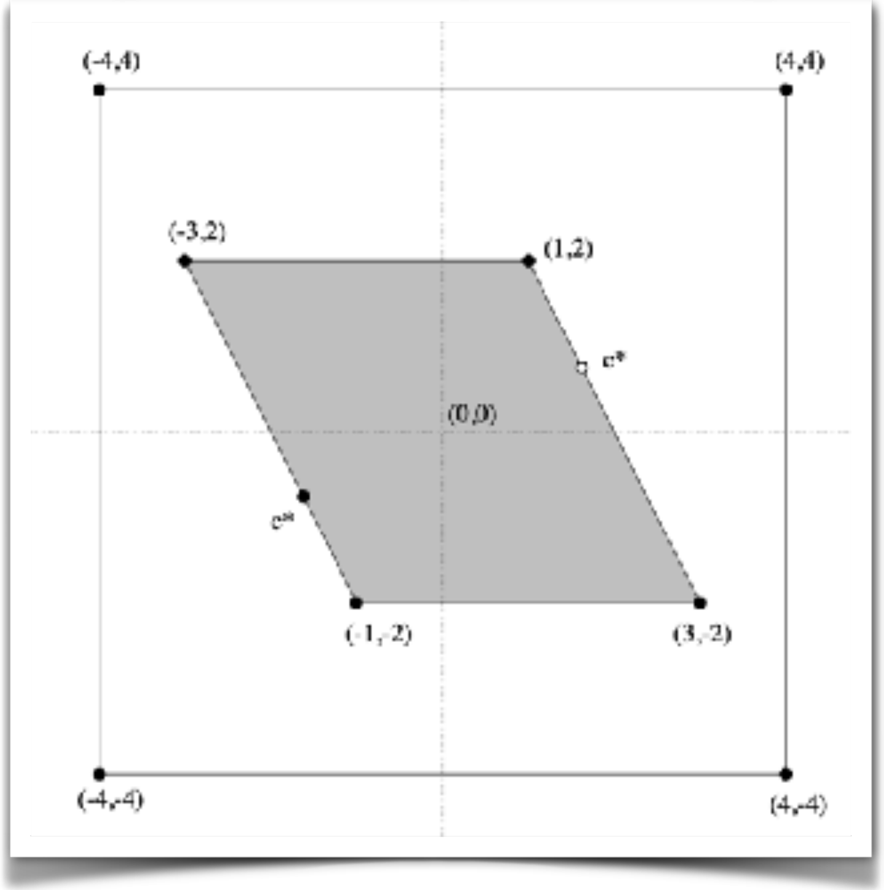

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On the Continuous Weber and k-Median Problems

(Extended Abstract)

Sándor P. Fekete* Joseph S. B. Mitchell† Karin Weinbrecht‡

On the Continuous Fermat-Weber Problem*

Sándor P. Fekete† Joseph S. B. Mitchell‡ Karin Beurer§

problems that deal with finding a median for a continuum of demand points. In particular, we consider versions of the "continuous k-median (Weber) problem" where the goal is to select one or more center points that minimize the average distance to a set of points in a demand region. In such problems, the average is computed as an integral over the relevant region, versus the usual discrete sum of distances. The resulting facility location problems are inherently geometric, requiring analysis techniques of computational geometry. We provide polynomial-time algorithms for various versions of the L_1 1-median (Weber) problem. We also consider the multiple-center version of the L_1 k-median problem, which we prove is NP-hard for large k.

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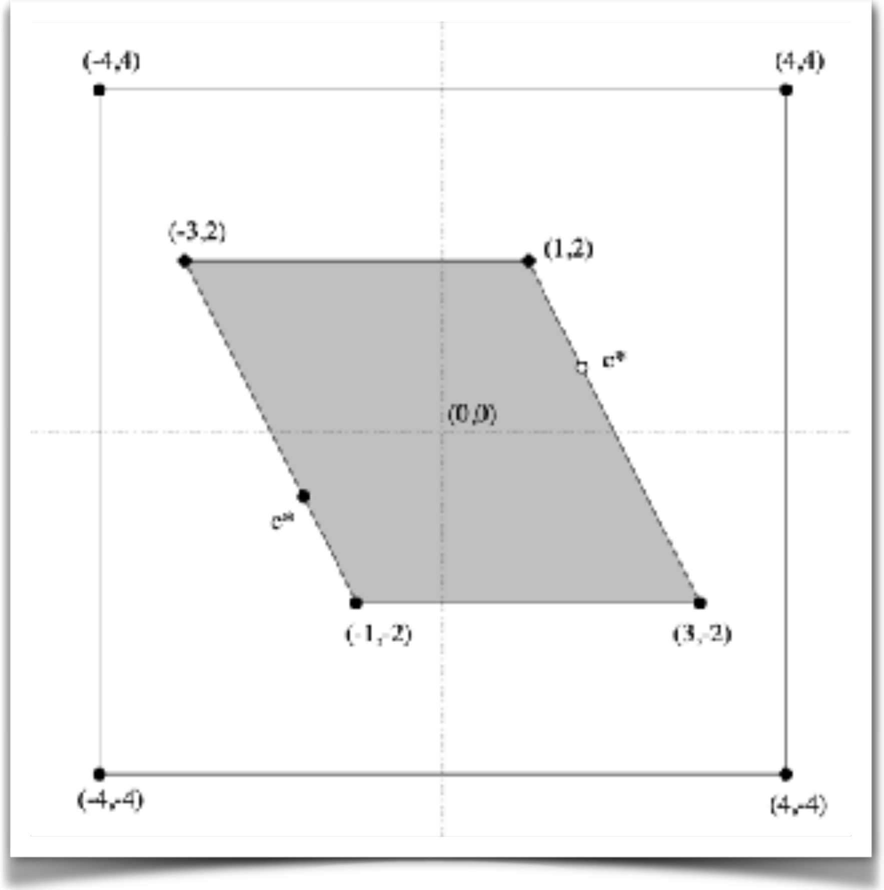

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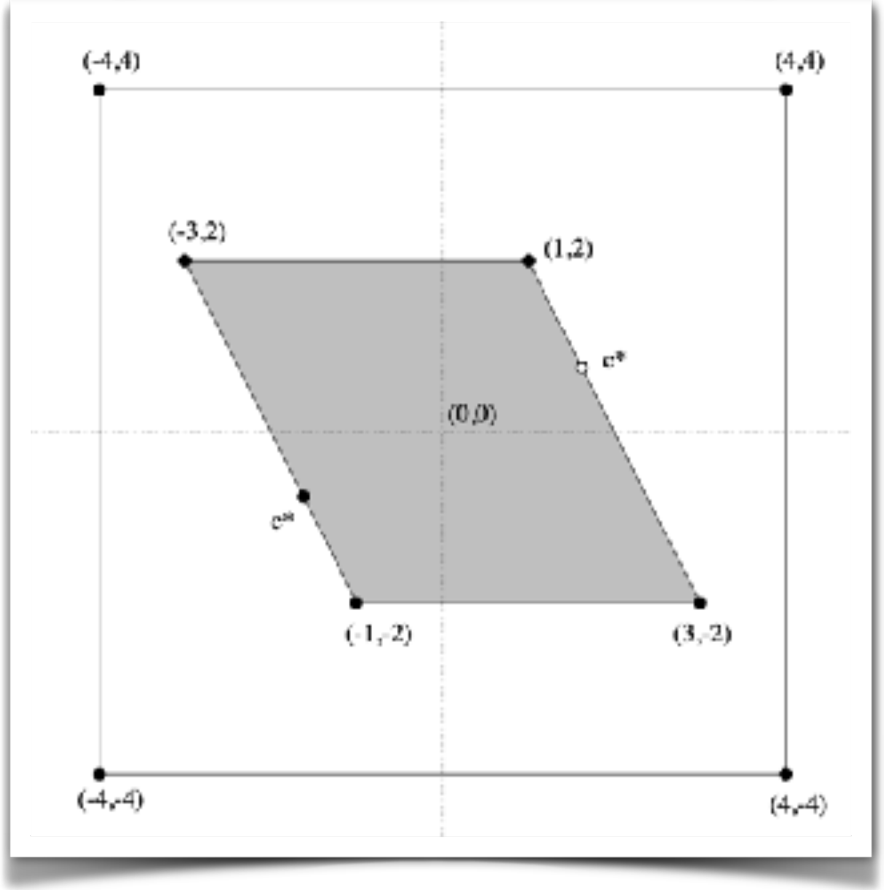
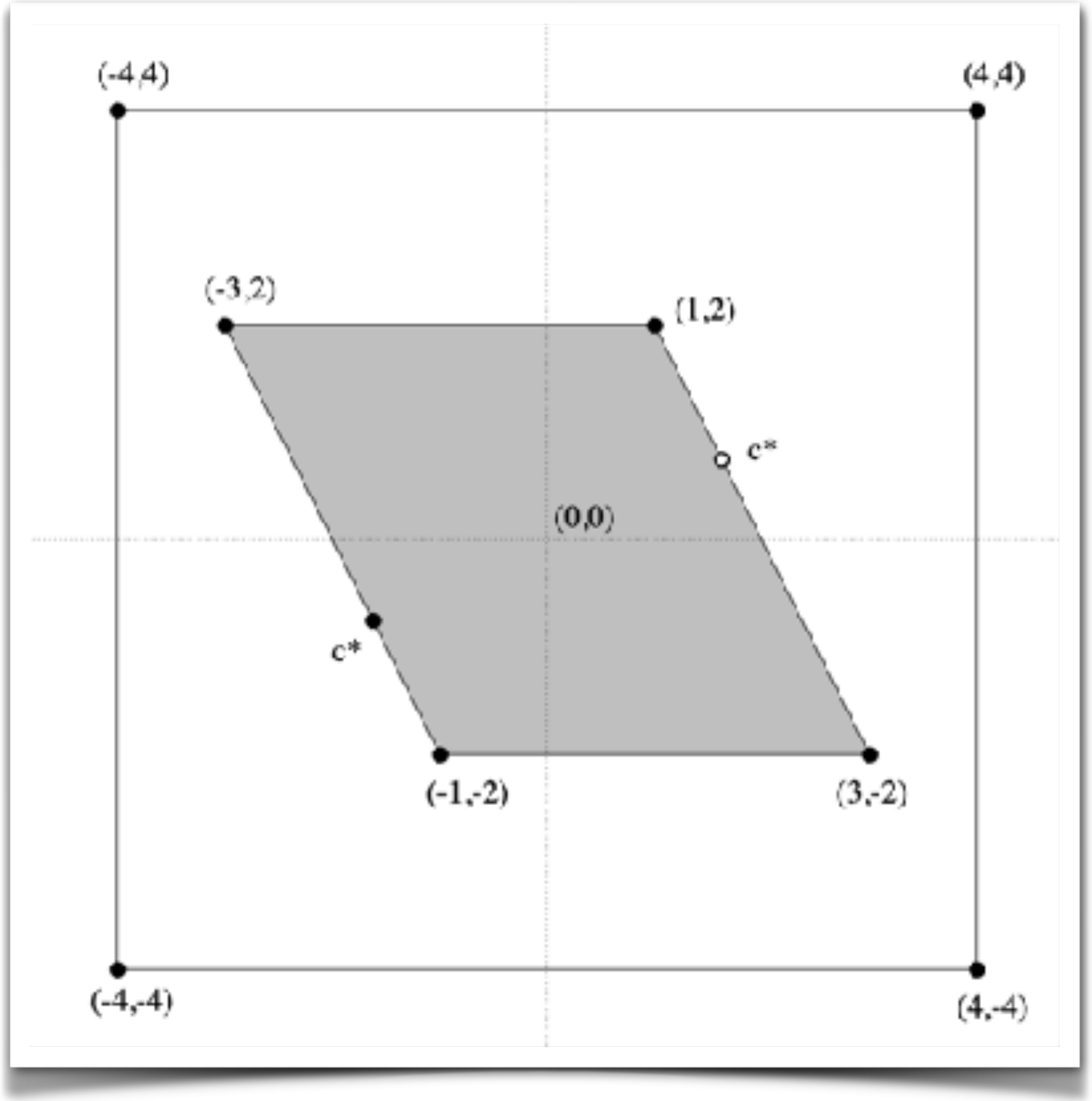


Issues



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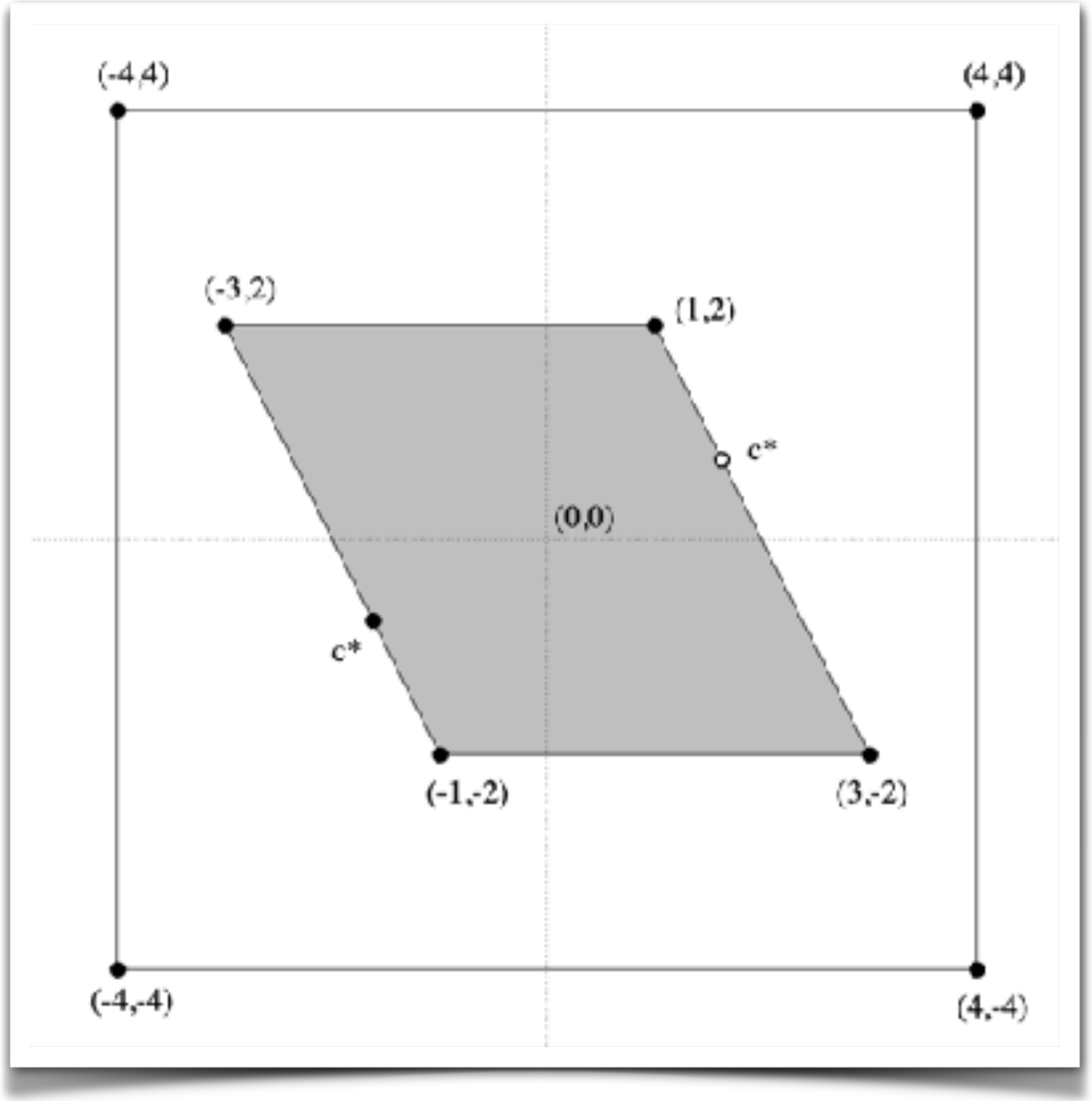
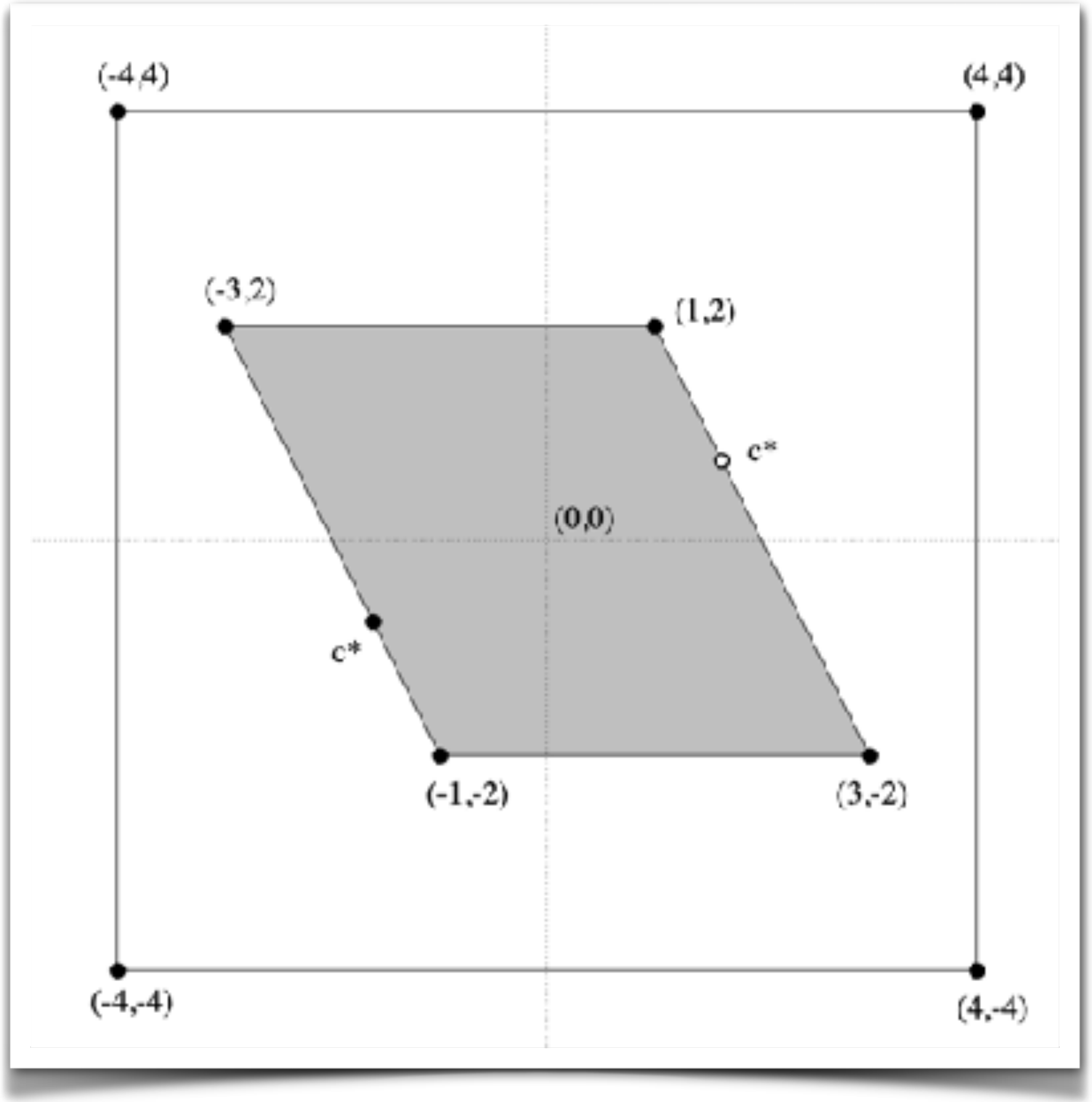


Issues:

- **L1-Median may not be feasible.**



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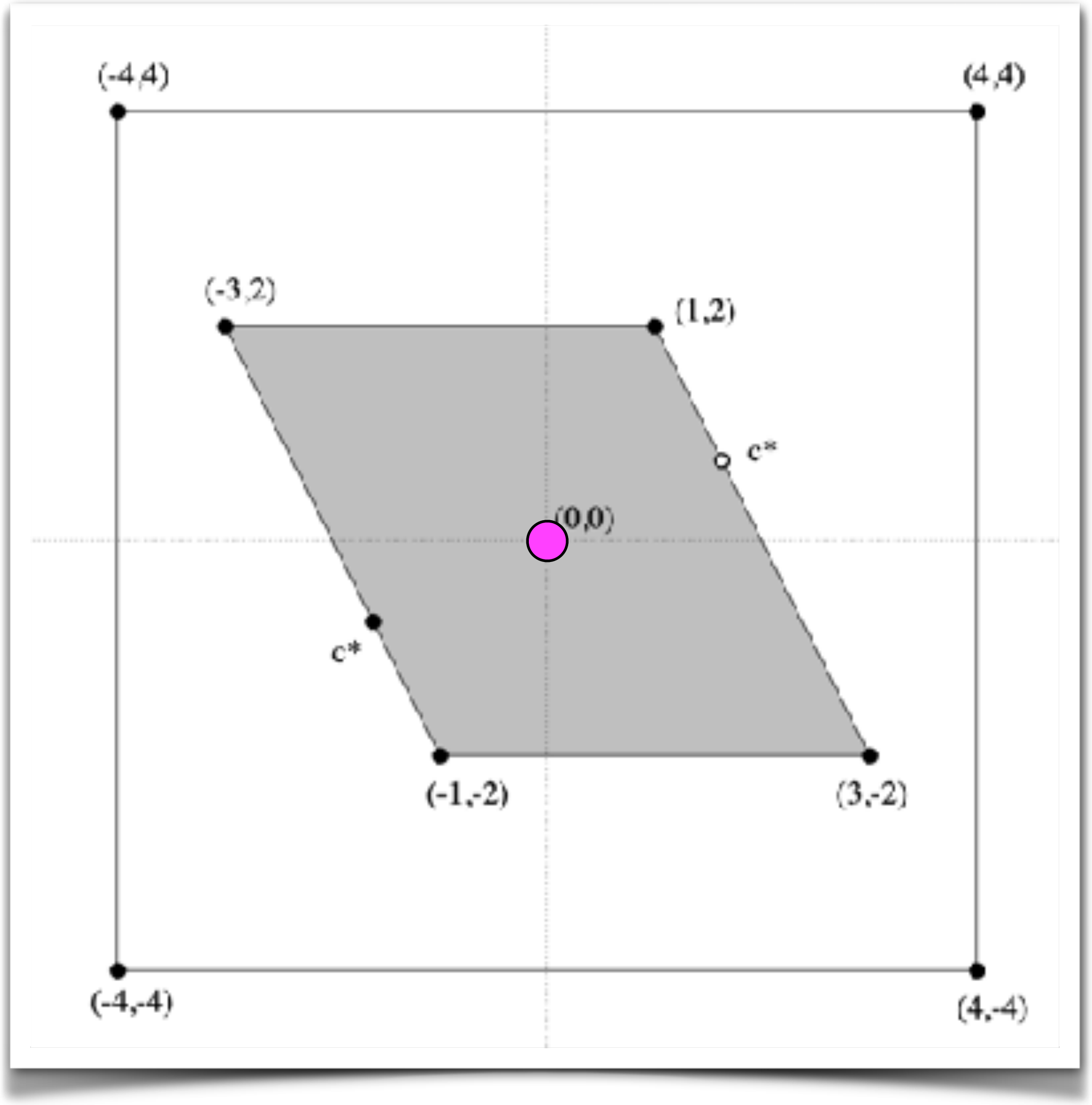

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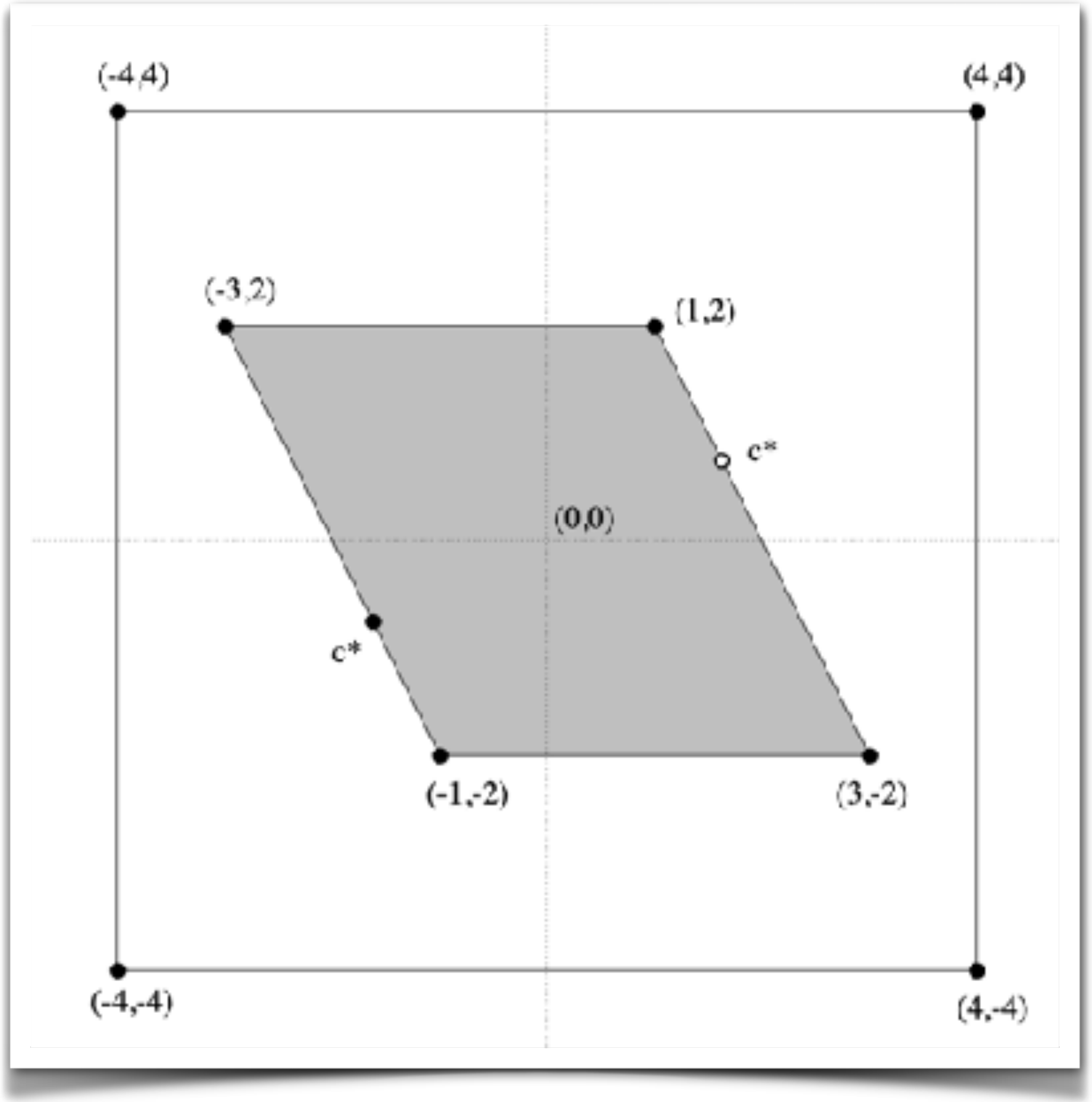

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Issues:

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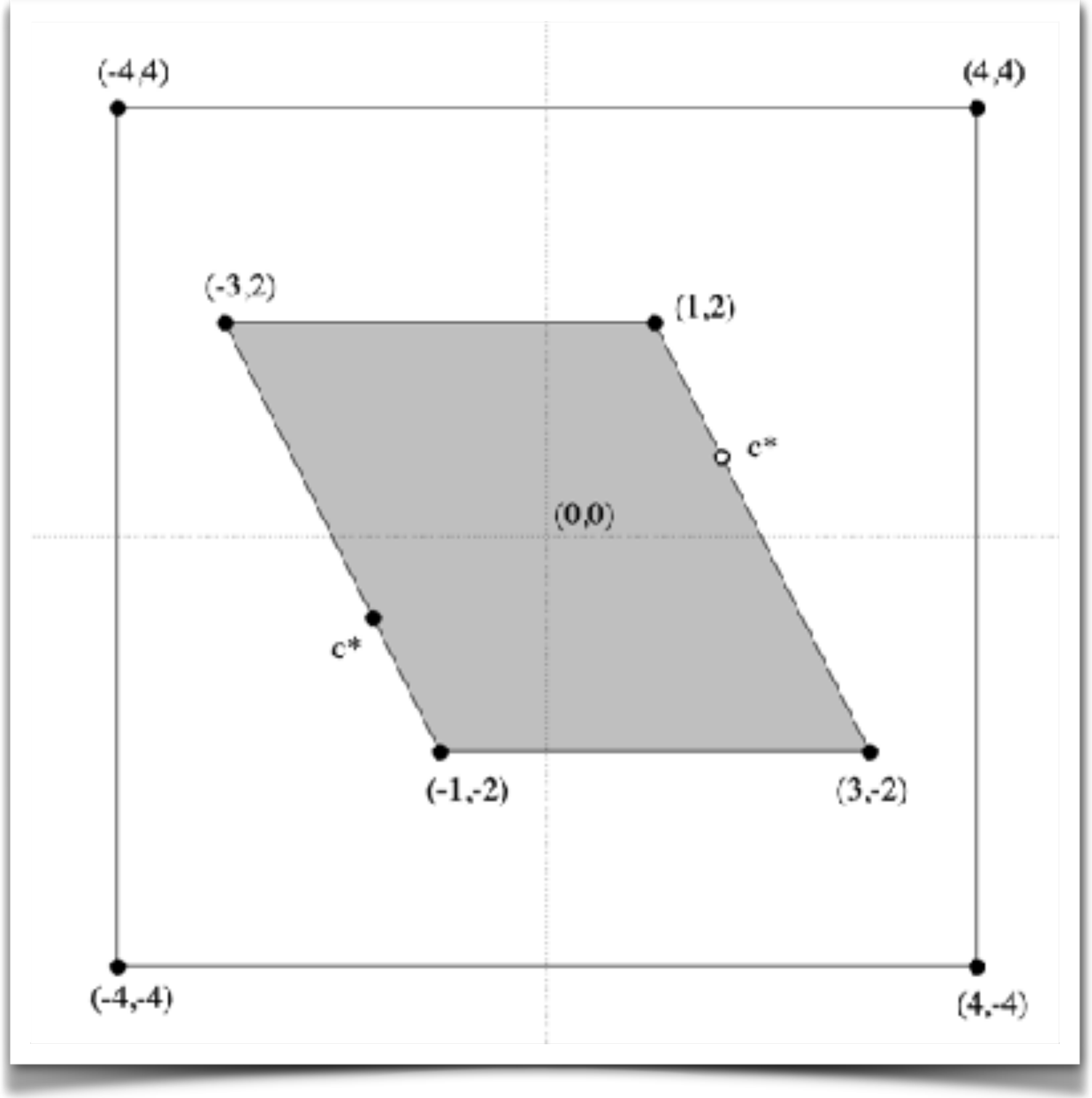

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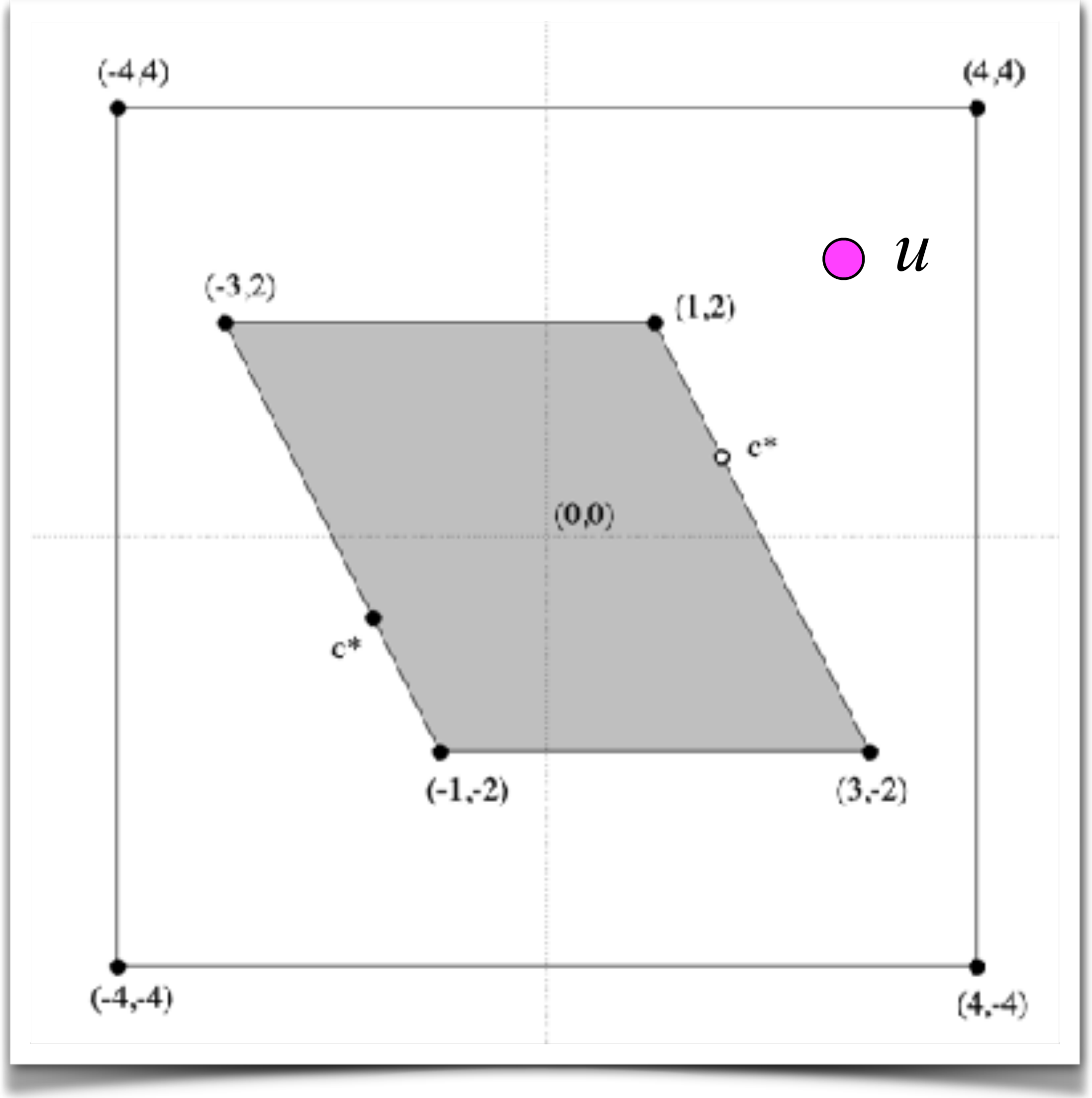

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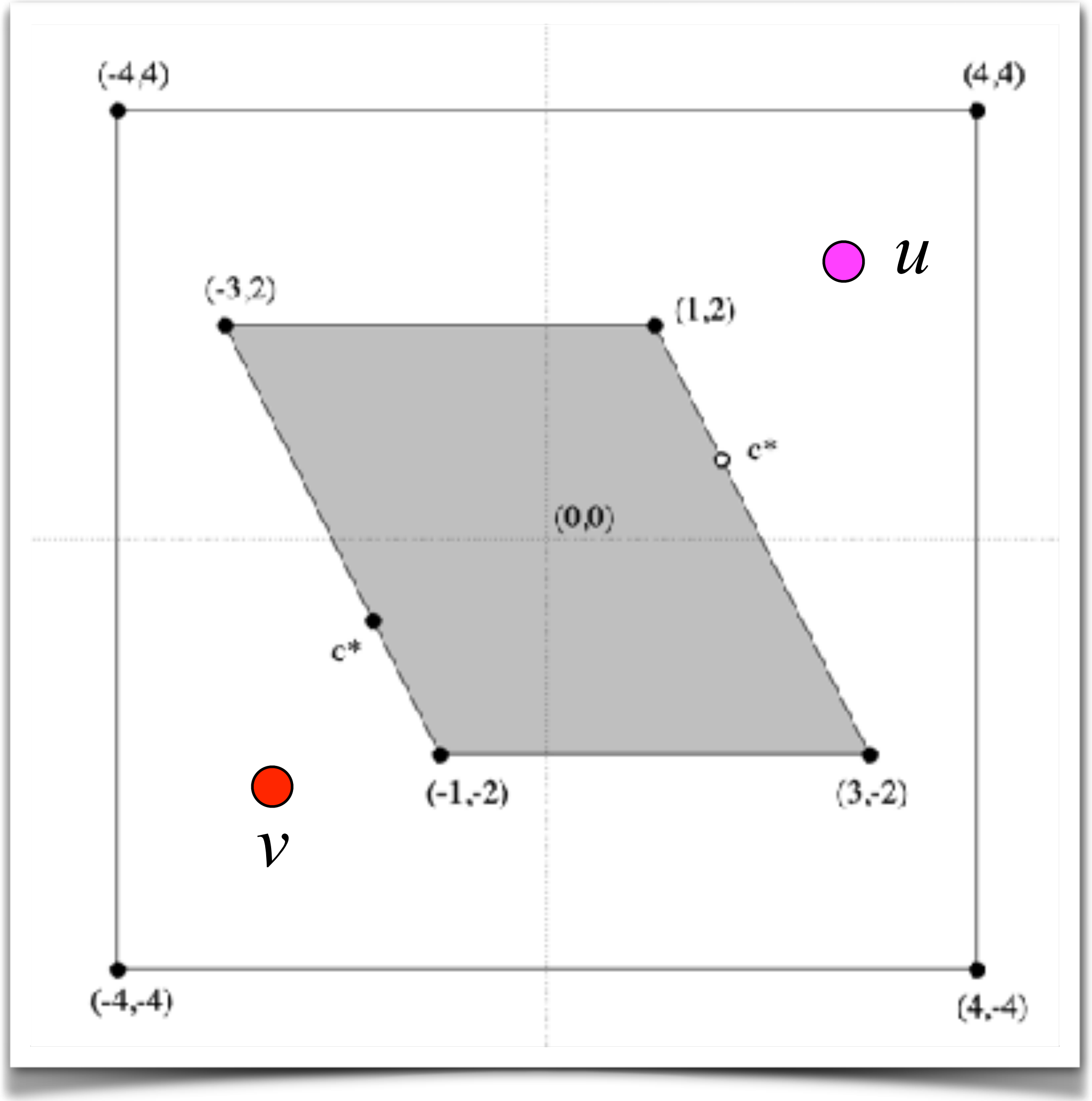

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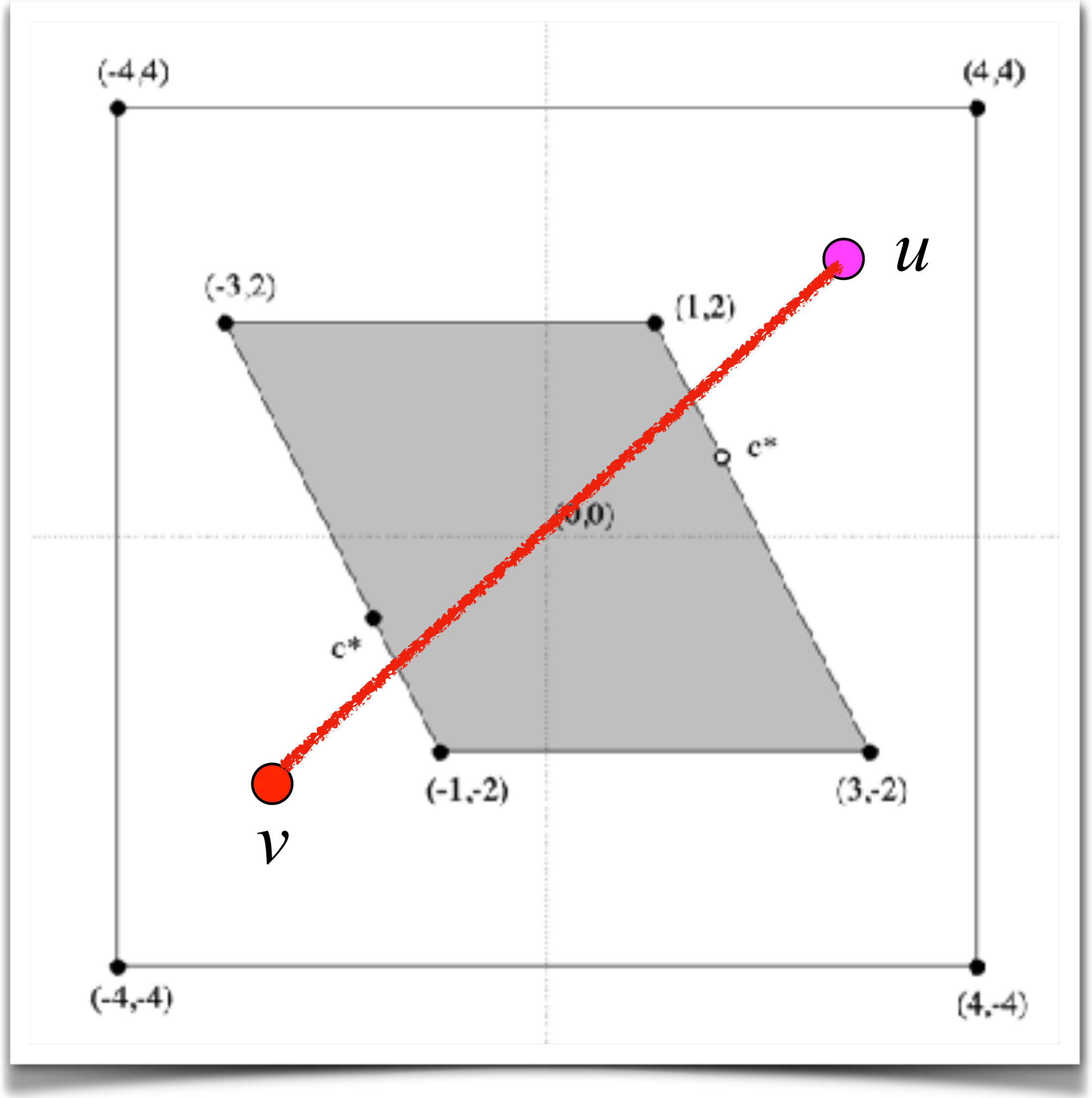
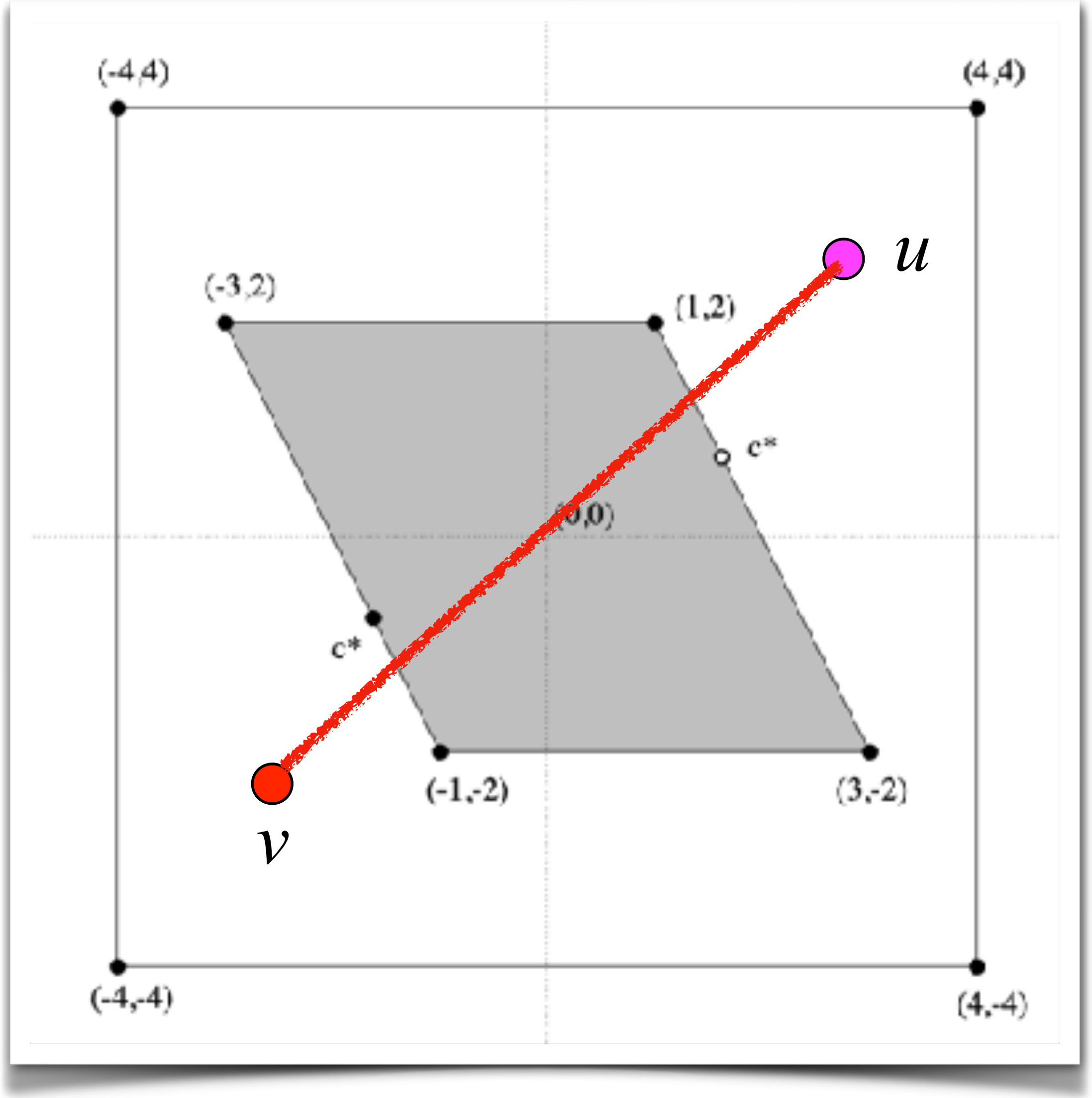


Issues:

- L1-Median may not be feasible.
- Distances may have to deal with obstacles:
  - straight-line distances



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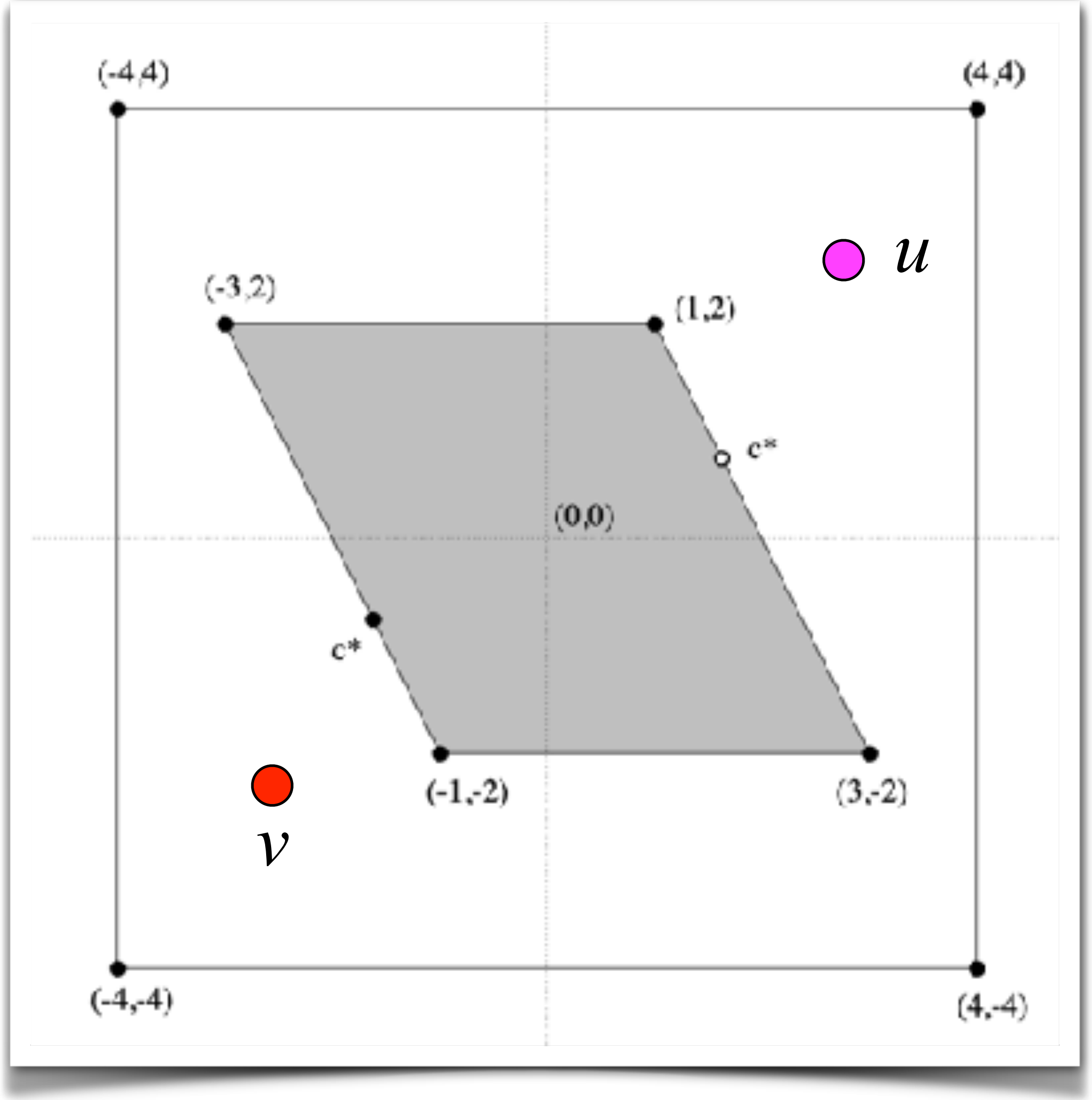

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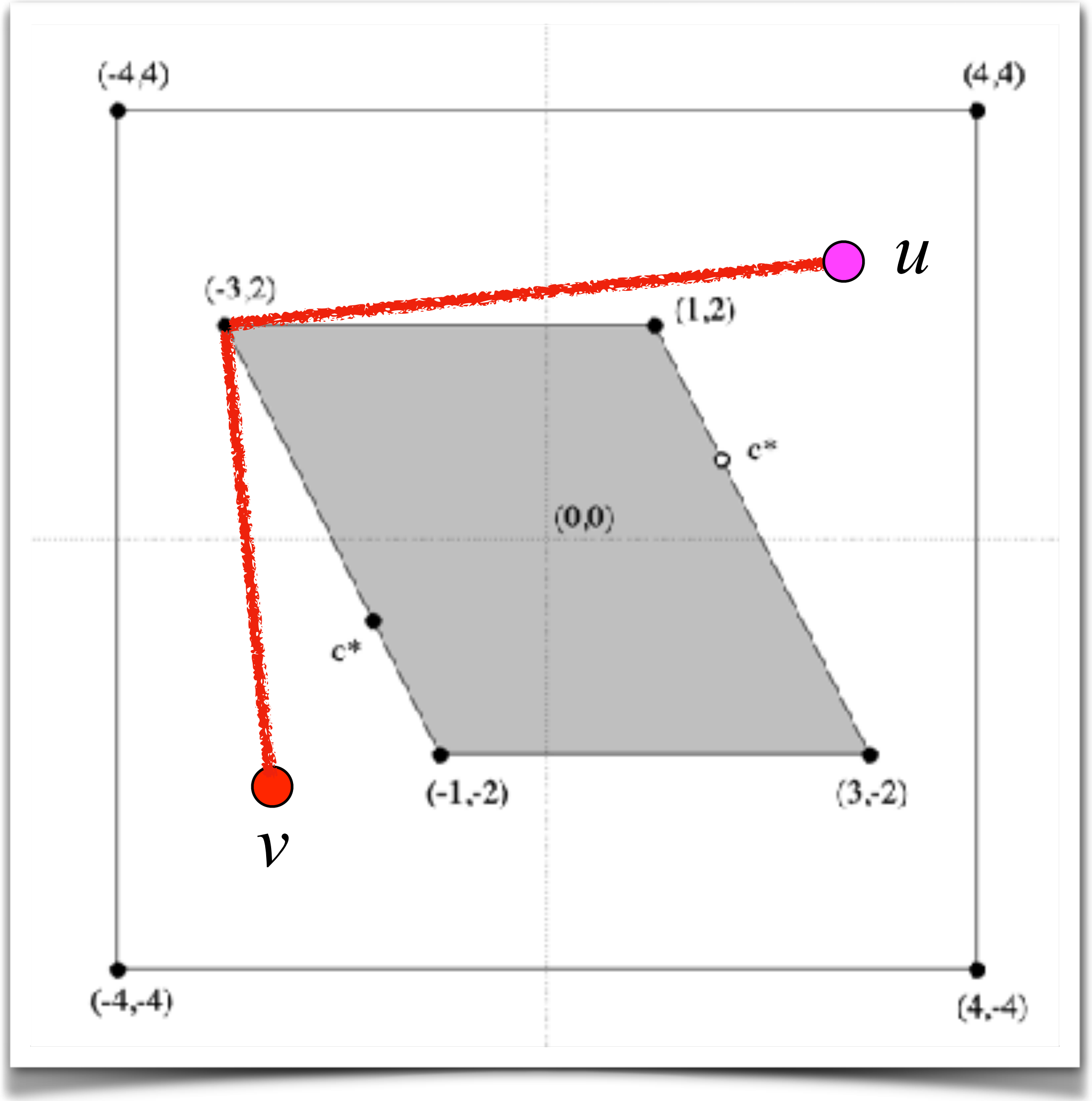
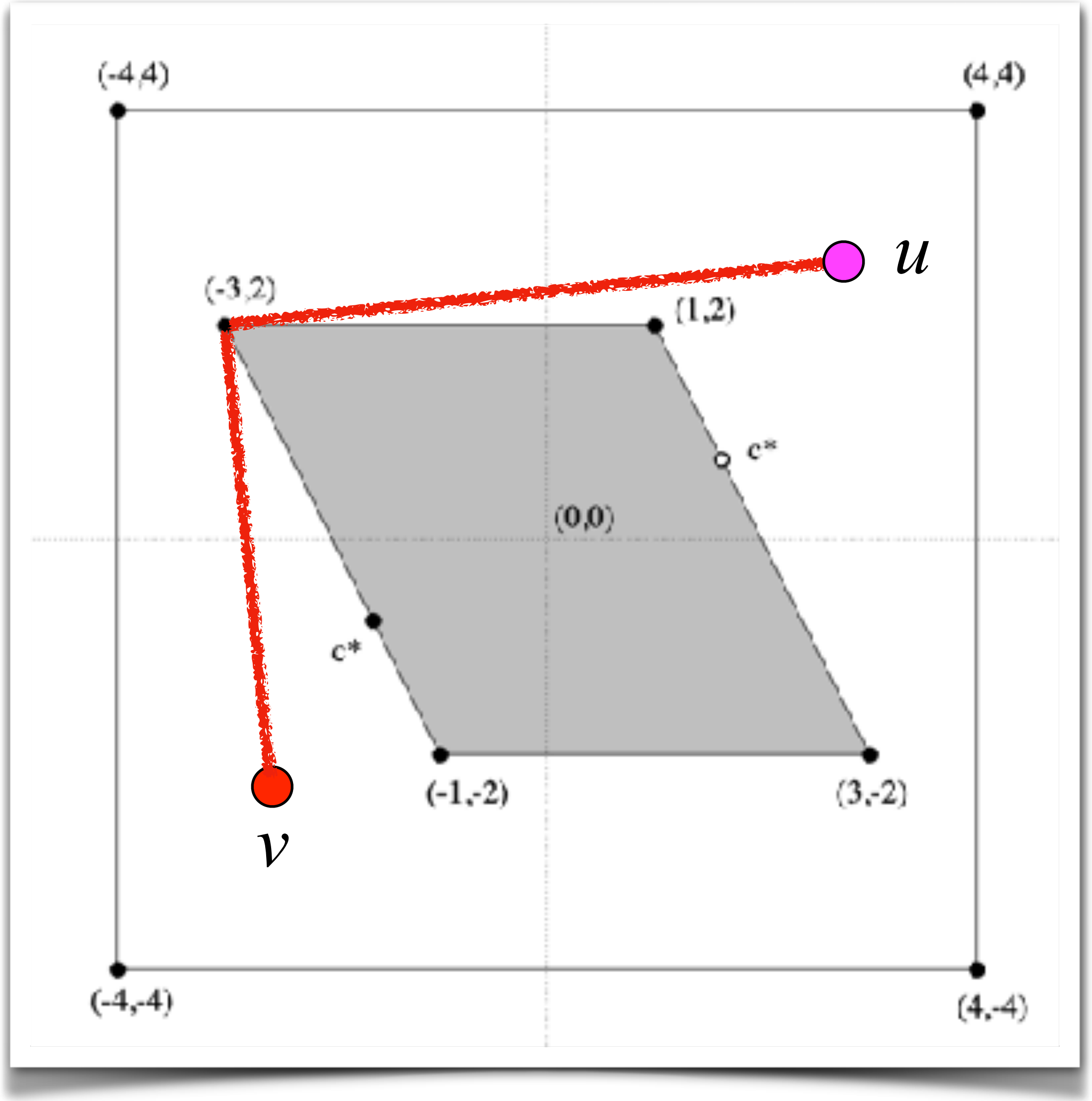


Issues:

- L1-Median may not be feasible.
- Distances may have to deal with obstacles:
  - straight-line distances
  - geodesic distances



Issues:

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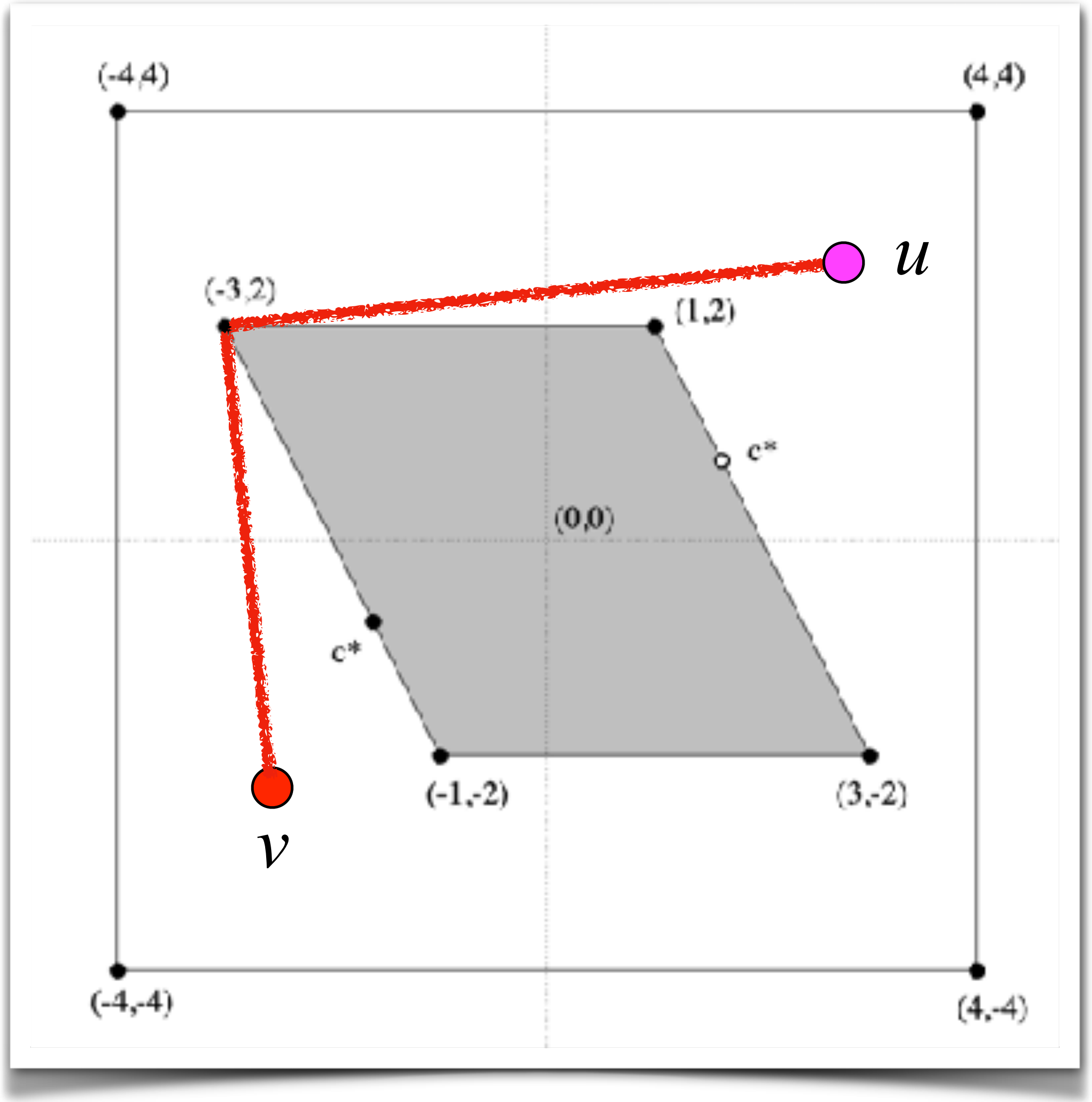
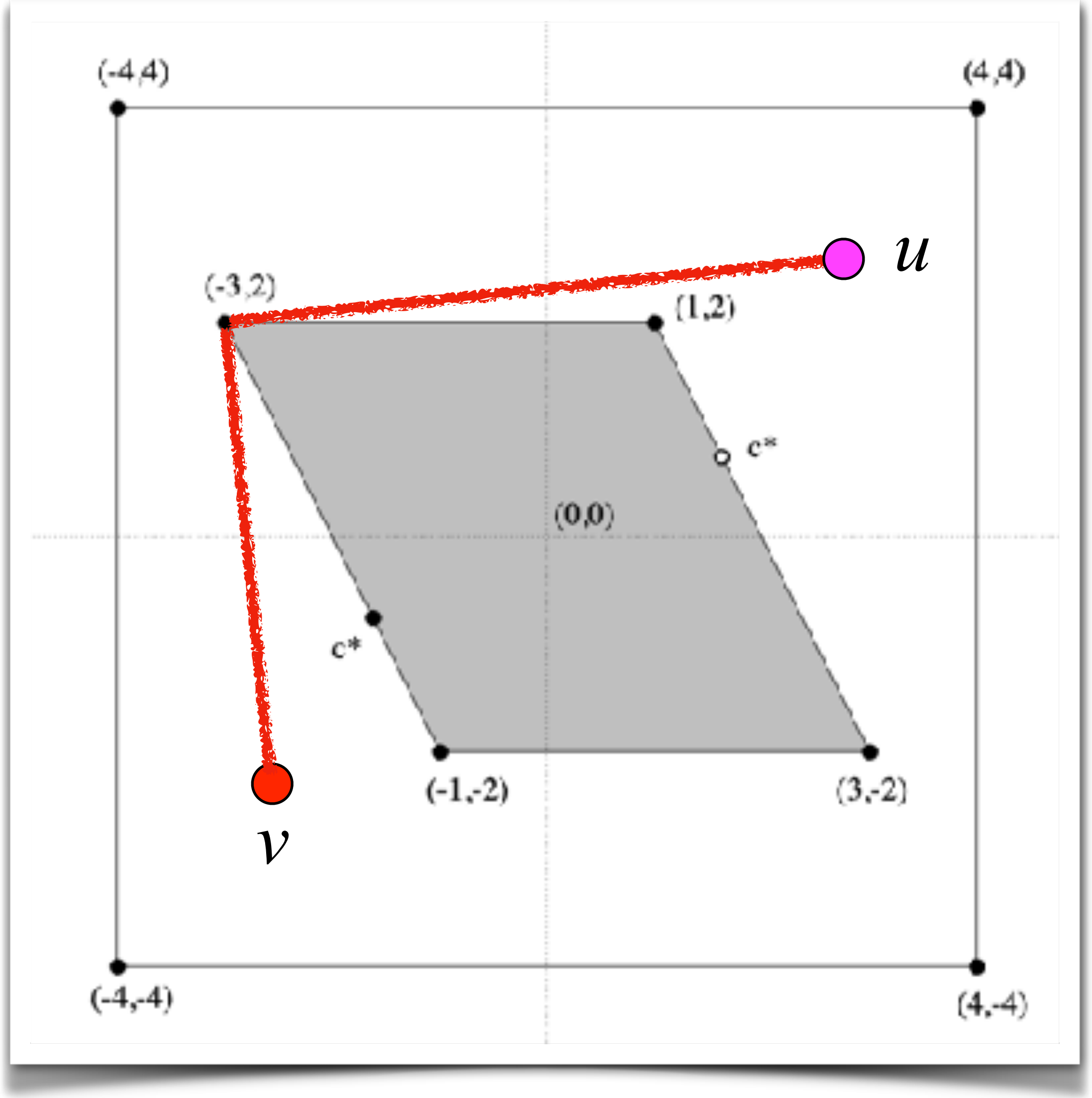
  - geodesic distances
- Demand distribution?



Issues:

- **L1-Median may not be feasible.**
- **Distances may have to deal with obstacles:**
    - **straight-line distances**
    - **geodesic distances**
- **Demand distribution?**

Demand is evenly distributed over the set *D*=*P*.



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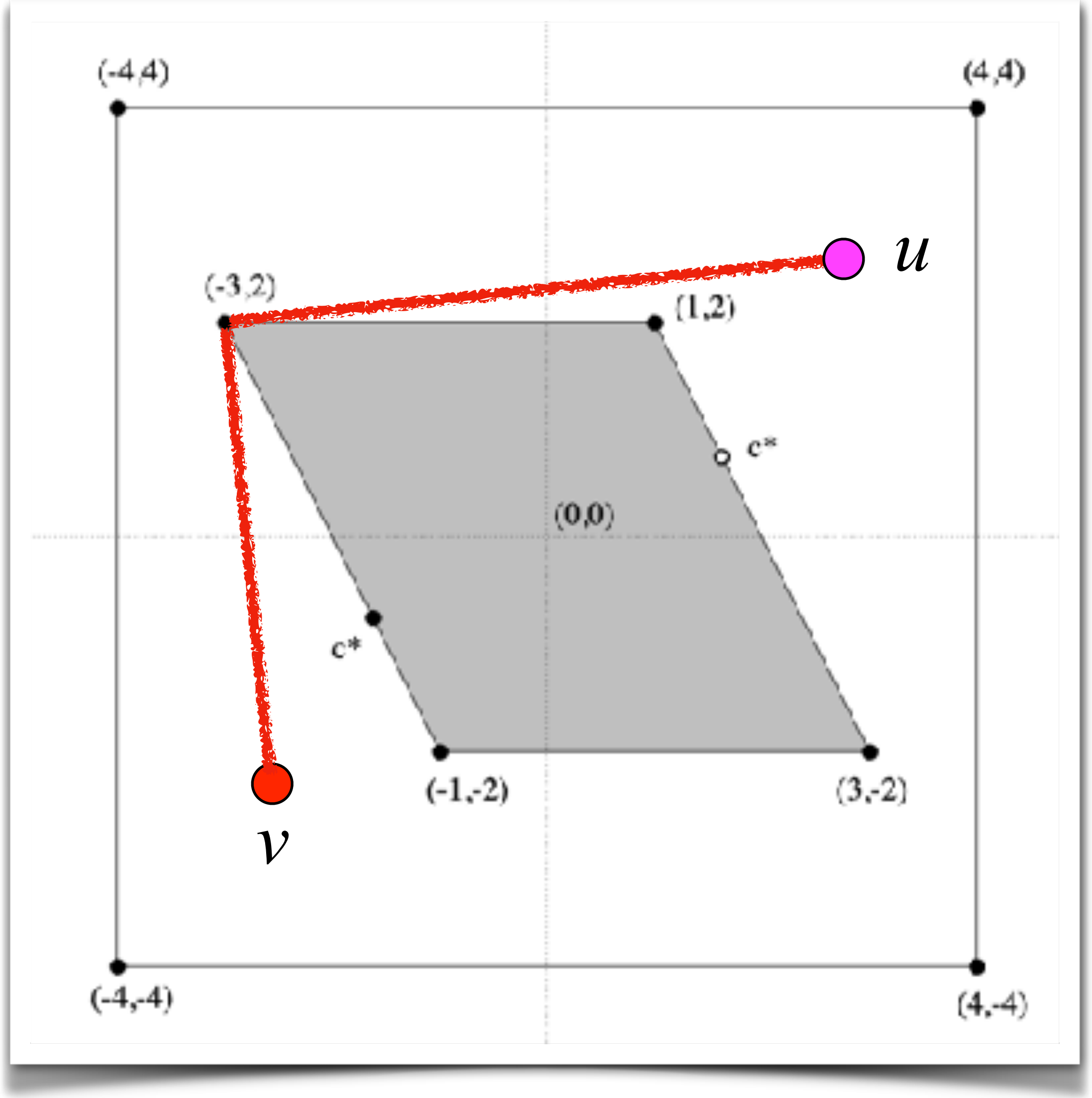
Demand is evenly distributed over the set *D*=*P*.

$$f(Z) = f(x,y) = \frac{1}{\mu}\int\int_{(u,v)\in P} d(Z,(u,v))dudv,$$



Tools

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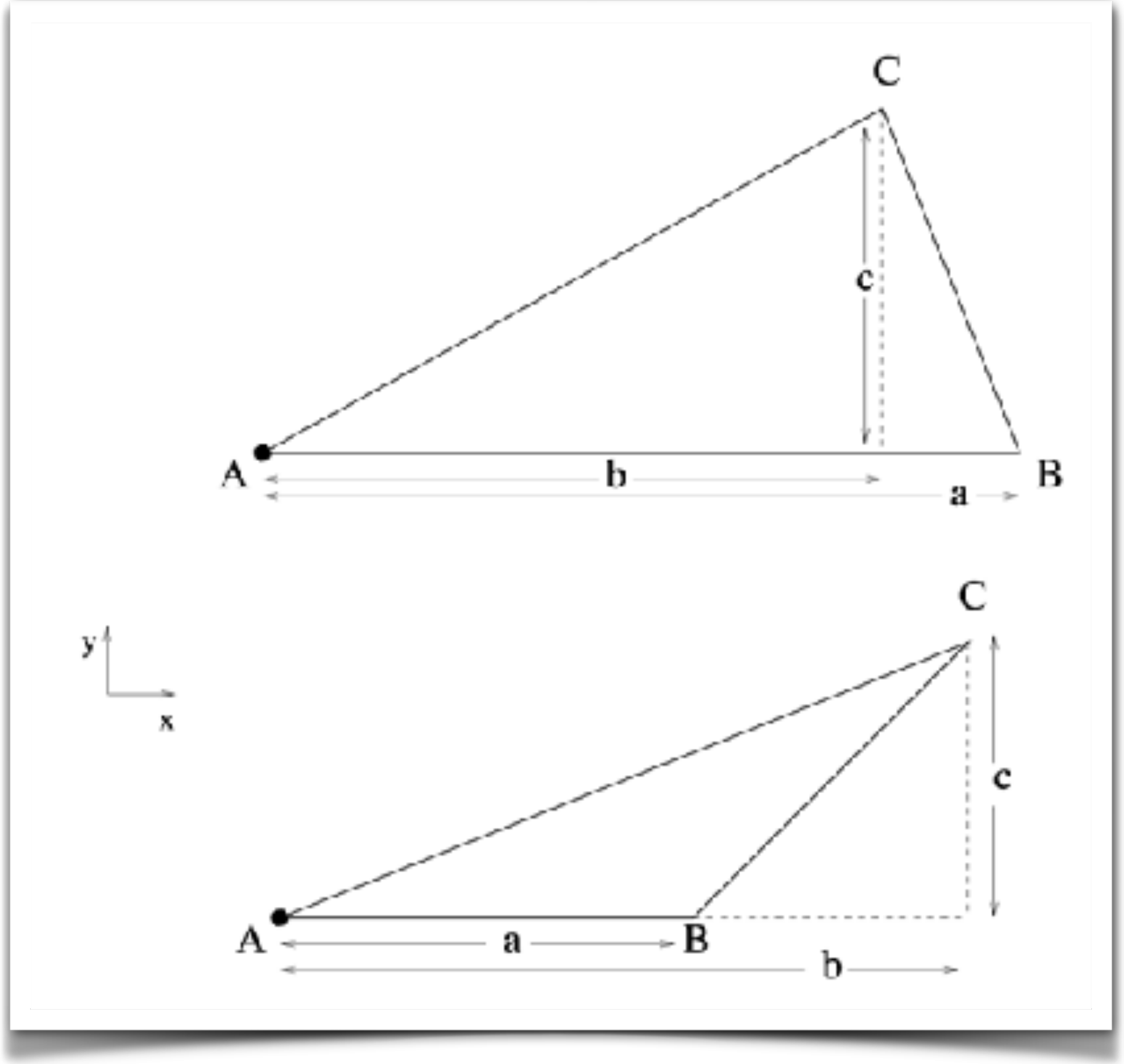
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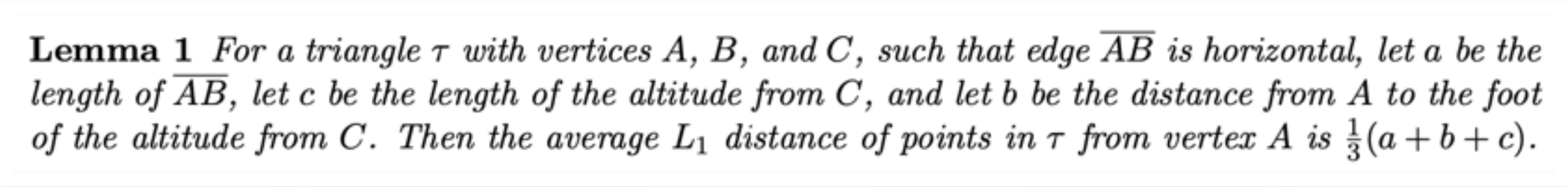
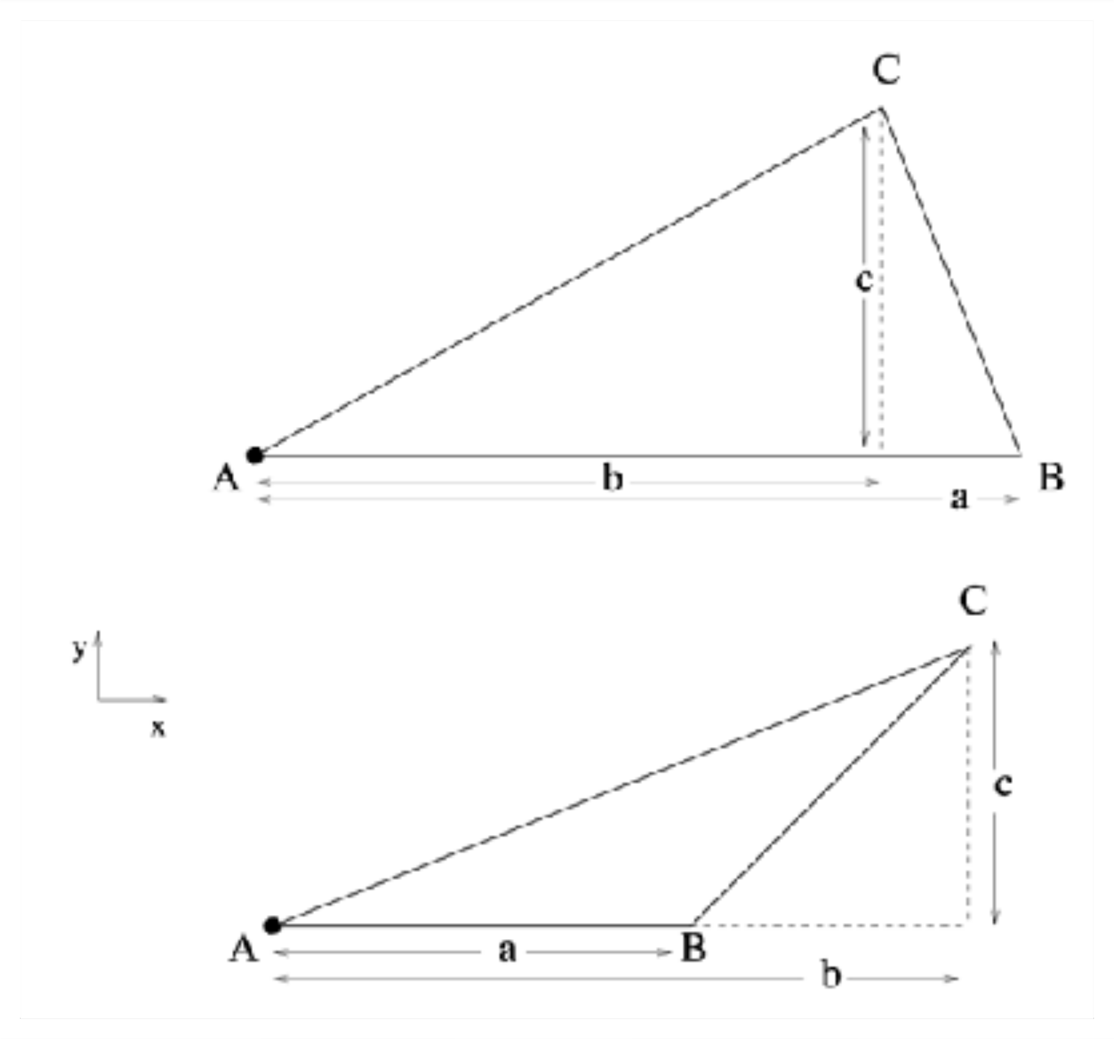
Tools

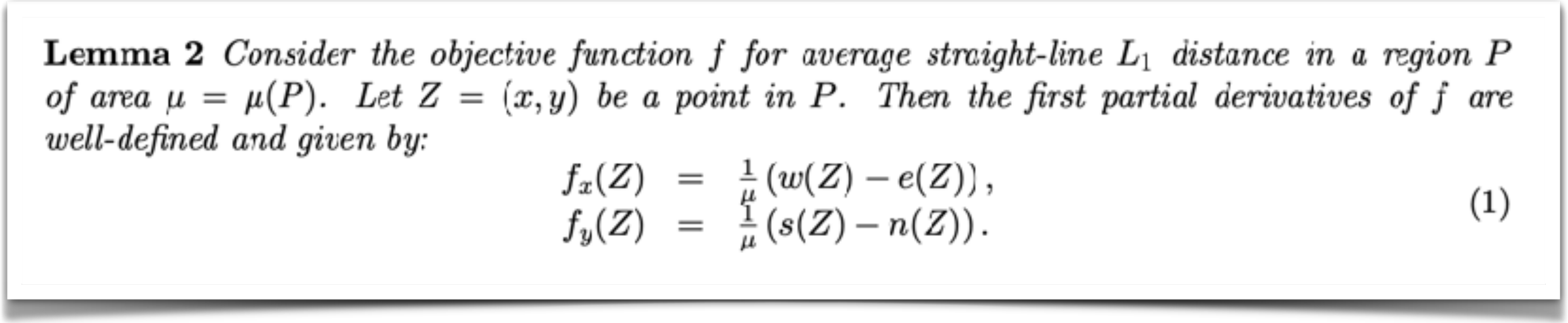


Lemma 1 *For a triangle τ with vertices A, B, and C, such that edge $\overline{AB}$ is horizontal, let a be the length of $\overline{AB}$, let c be the length of the altitude from C, and let b be the distance from A to the foot of the altitude from C. Then the average L_1 distance of points in τ from vertex A is $\frac{1}{3}(a+b+c)$.*



Tools

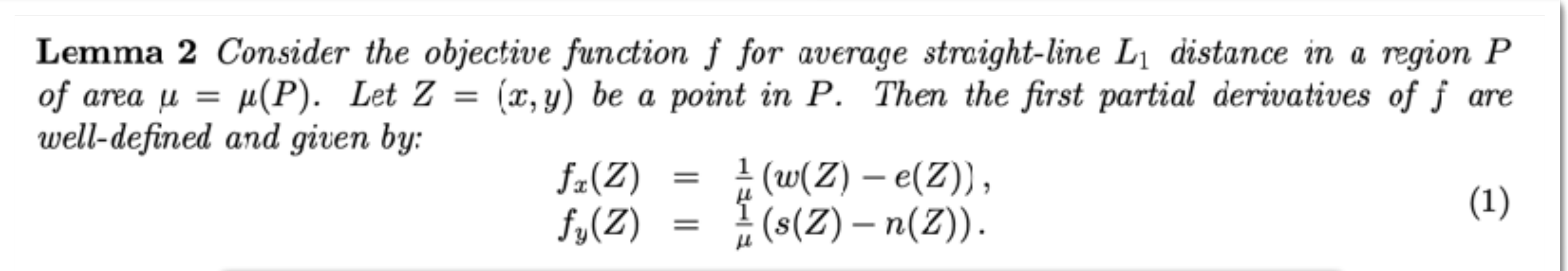
Tools



Lemma 2 *Consider the objective function f for average straight-line L_1 distance in a region P of area $\mu = \mu(P)$. Let $Z = (x, y)$ be a point in P. Then the first partial derivatives of f are well-defined and given by:*

$$\begin{aligned} f_x(Z) &= \tfrac{1}{\mu}\left(w(Z) - e(Z)\right), \\ f_y(Z) &= \tfrac{1}{\mu}\left(s(Z) - n(Z)\right). \end{aligned} \tag{1}$$

Tools



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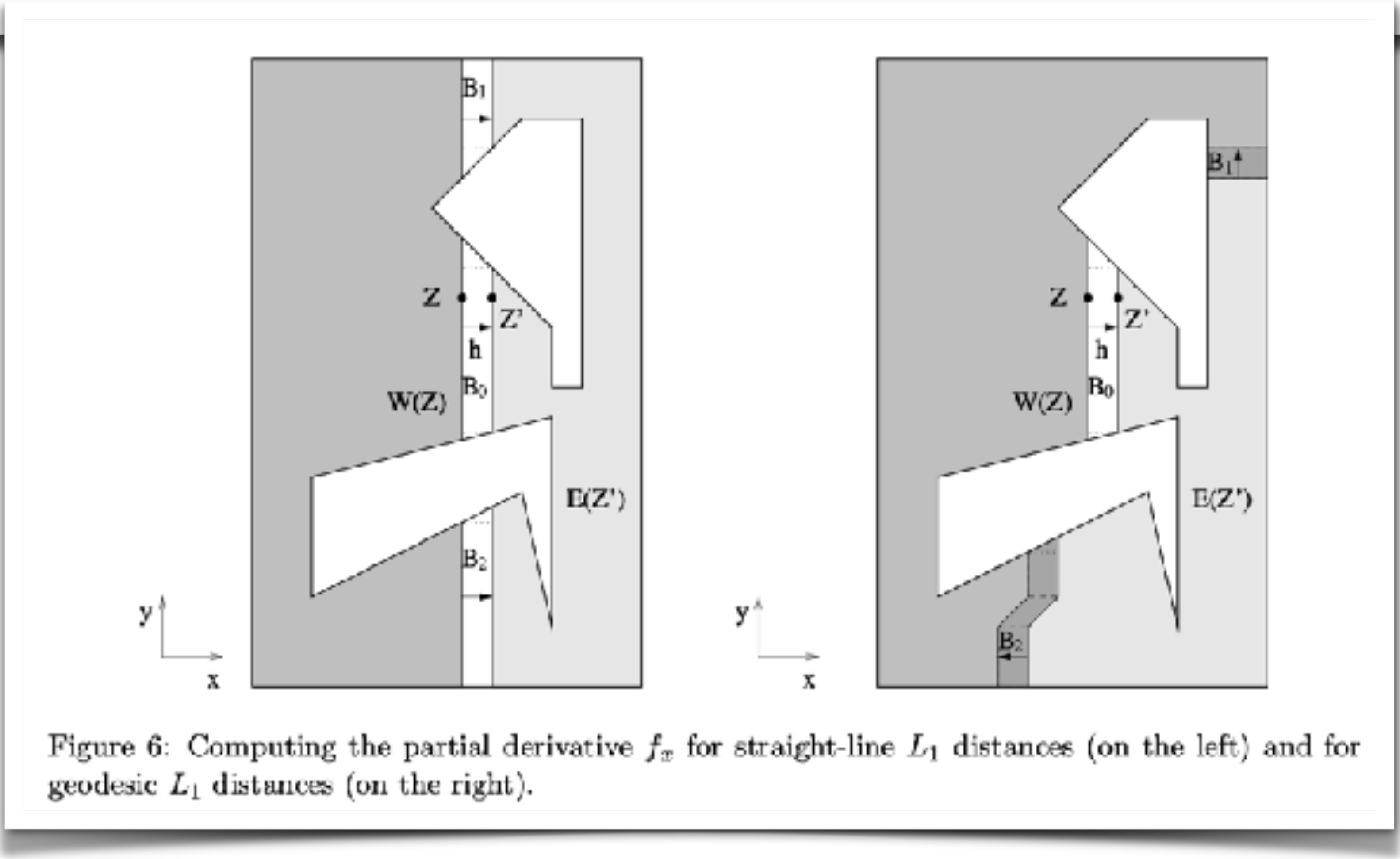
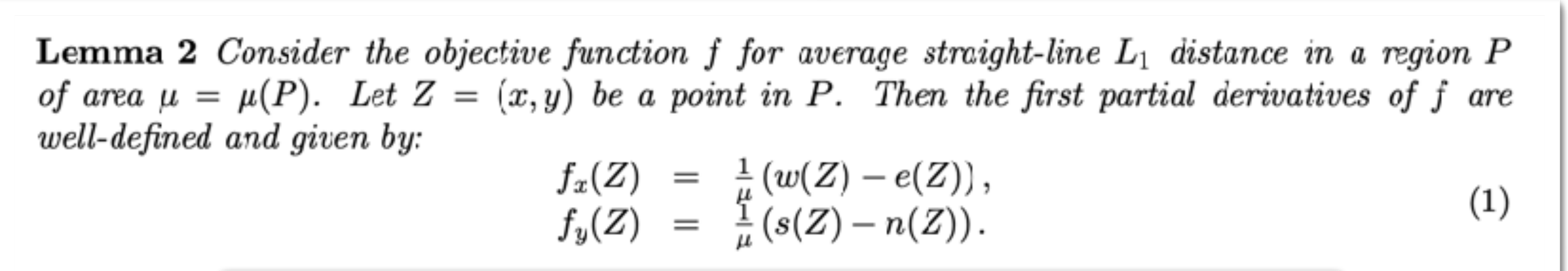


Figure 6: Computing the partial derivative f_x for straight-line L_1 distances (on the left) and for geodesic L_1 distances (on the right).

Tools



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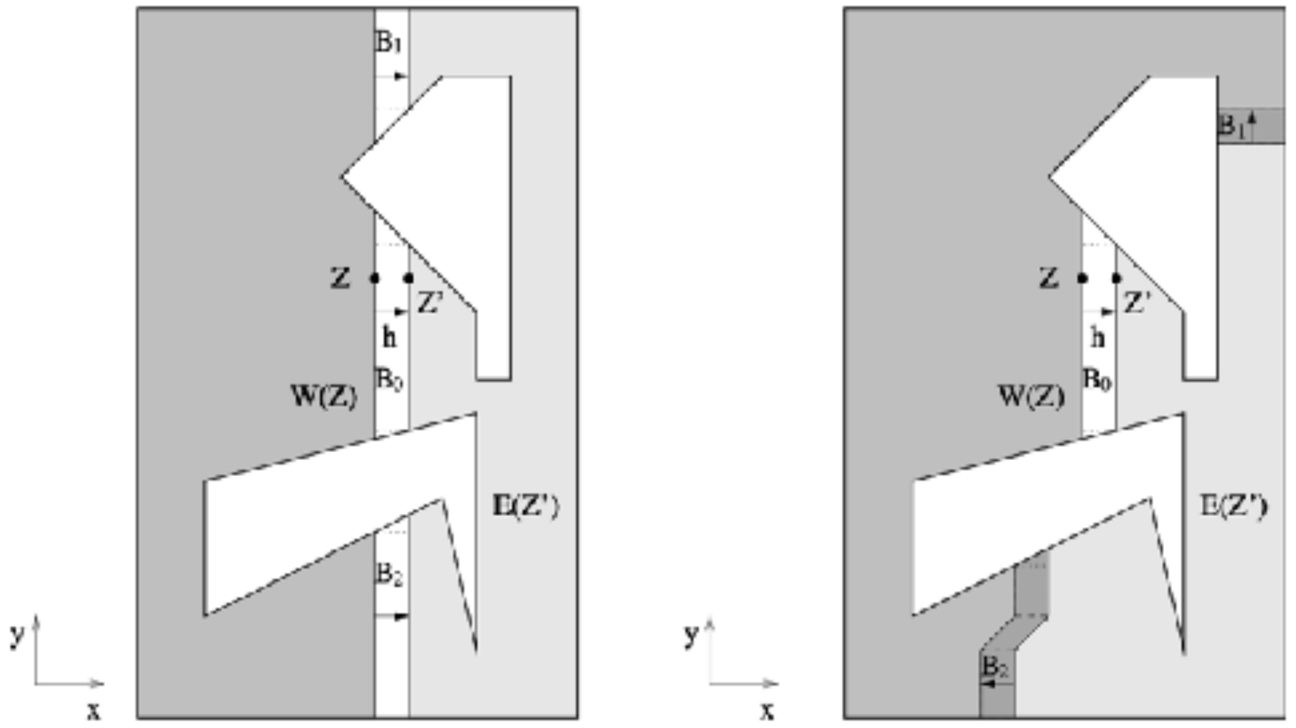
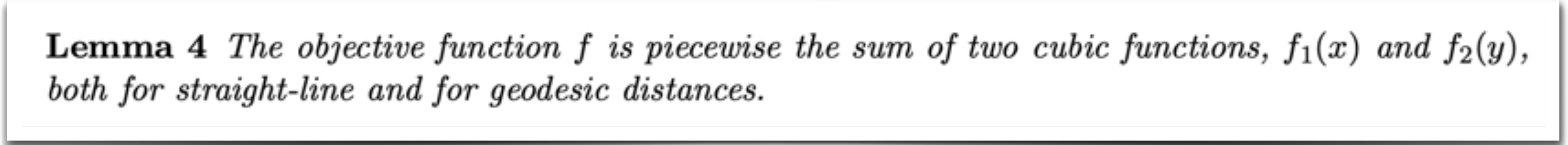
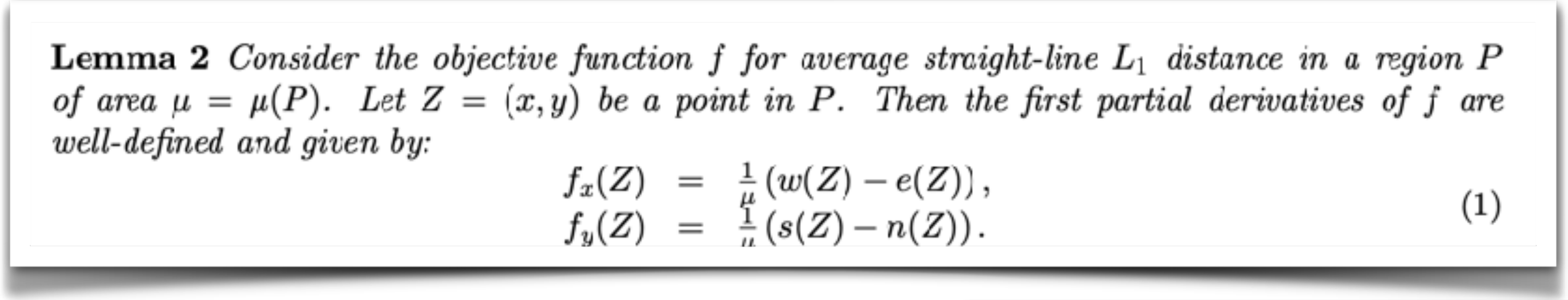


Figure 6: Computing the partial derivative f_x for straight-line L_1 distances (on the left) and for geodesic L_1 distances (on the right).



Lemma 4 *The objective function f is piecewise the sum of two cubic functions, $f_1(x)$ and $f_2(y)$, both for straight-line and for geodesic distances.*

Tools



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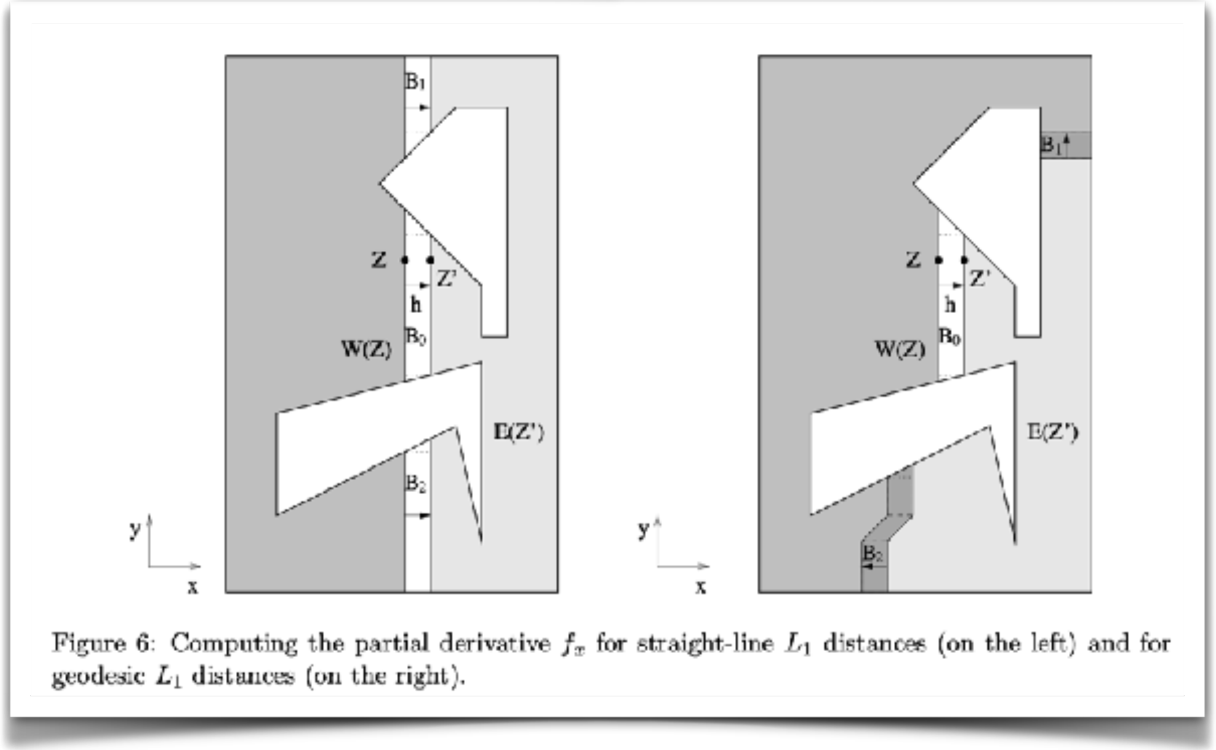
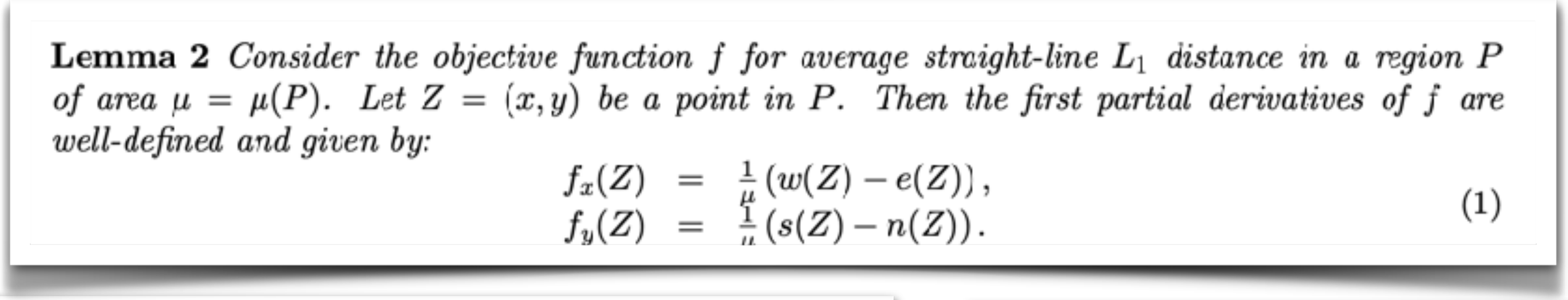


Figure 6: Computing the partial derivative f_x for straight-line L_1 distances (on the left) and for geodesic L_1 distances (on the right).

Tools



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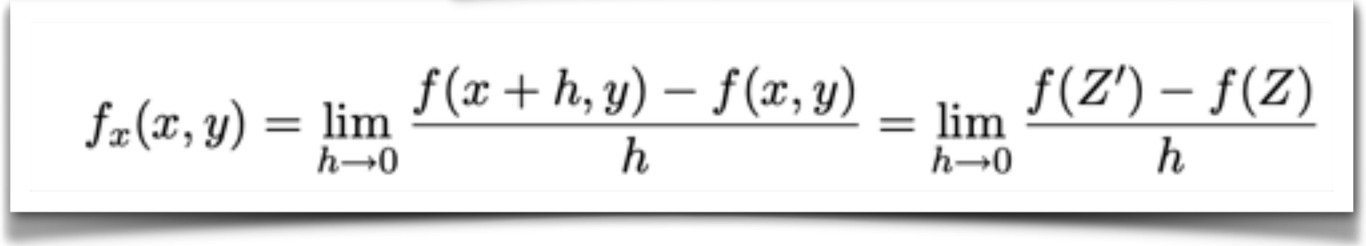


$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{f(Z') - f(Z)}{h}$$

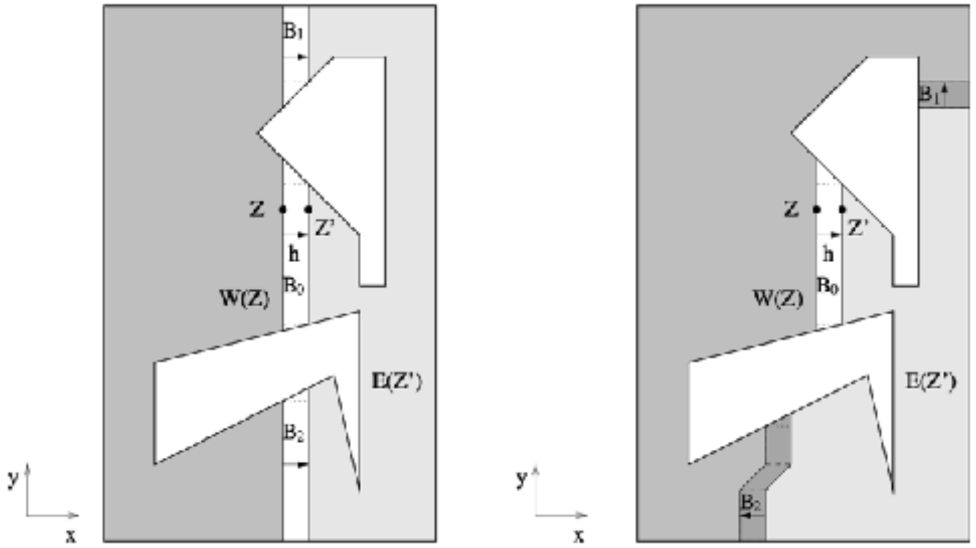
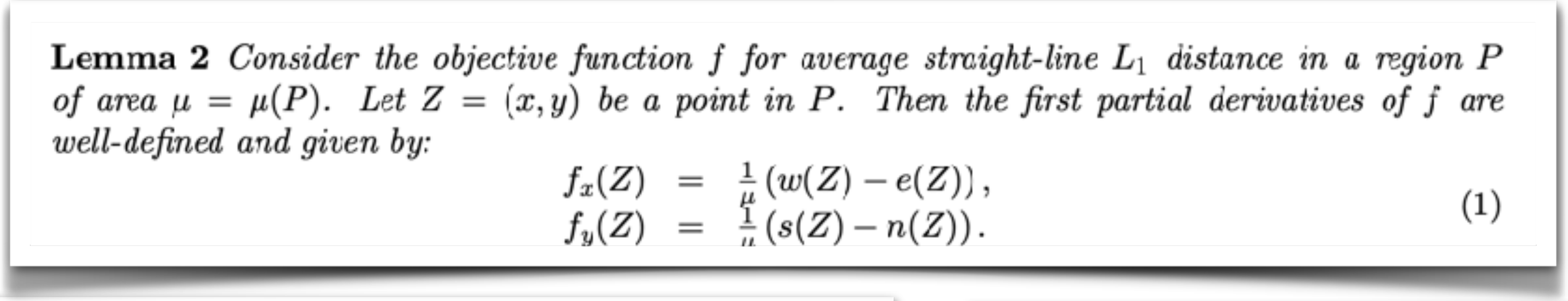


Figure 6: Computing the partial derivative f_x for straight-line L_1 distances (on the left) and for geodesic L_1 distances (on the right).

Tools



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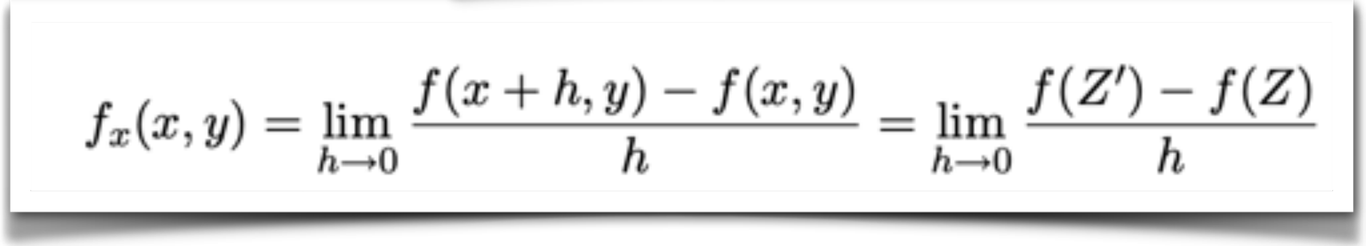

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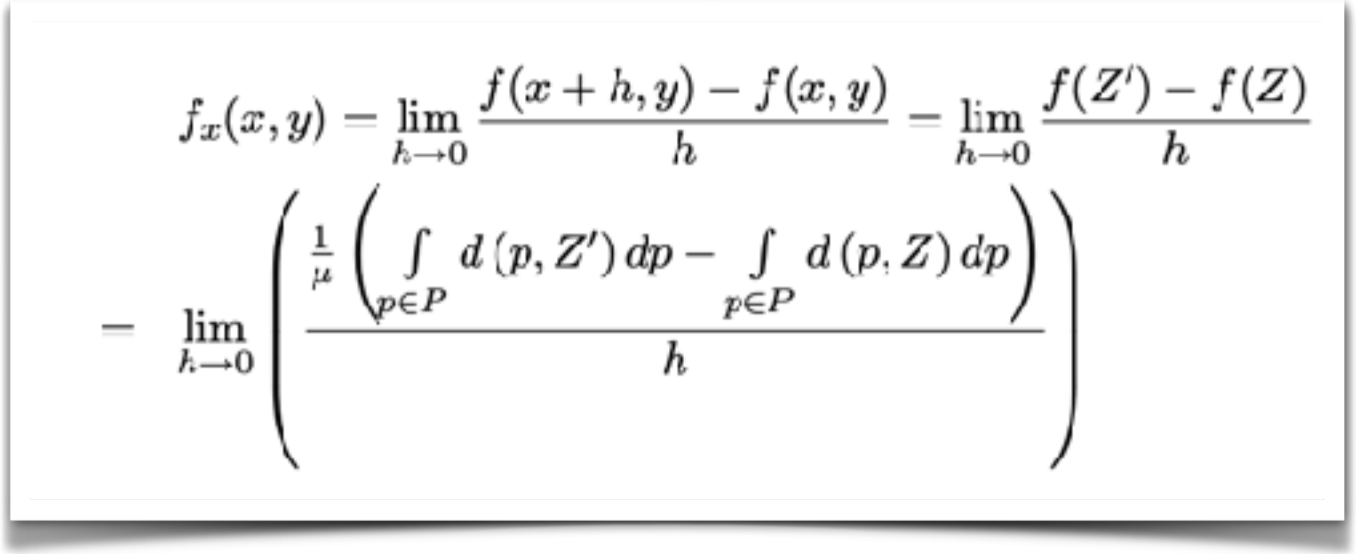

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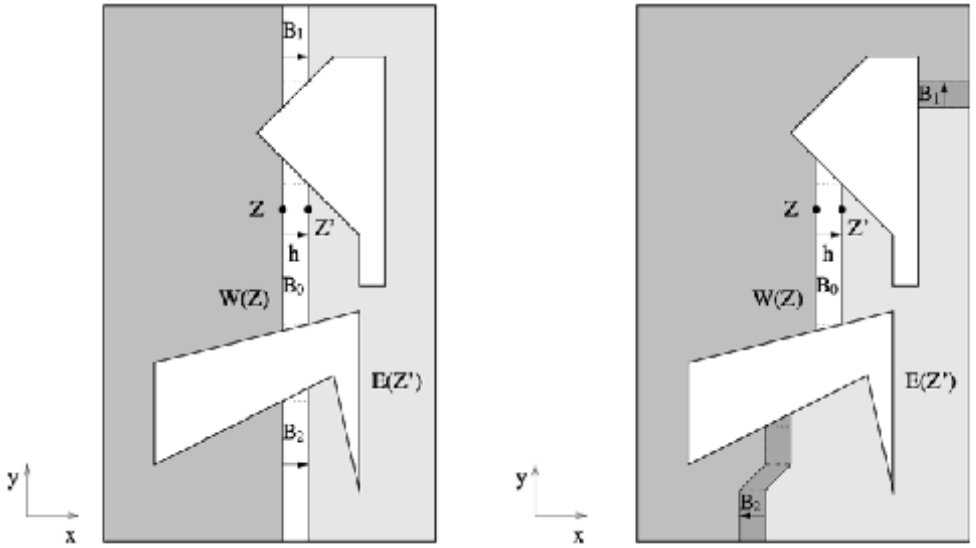
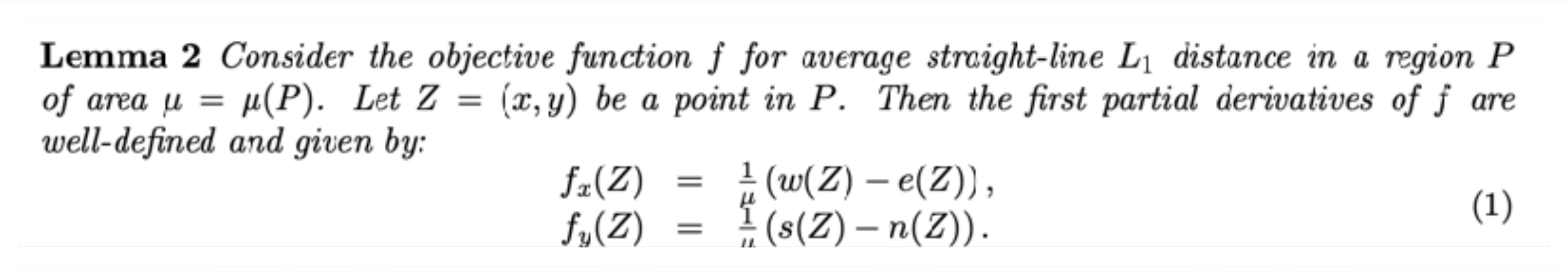


Figure 6: Computing the partial derivative f_x for straight-line L_1 distances (on the left) and for geodesic L_1 distances (on the right).

Tools



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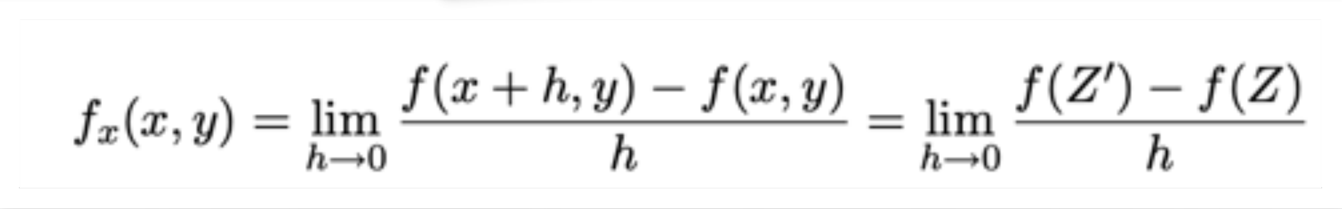

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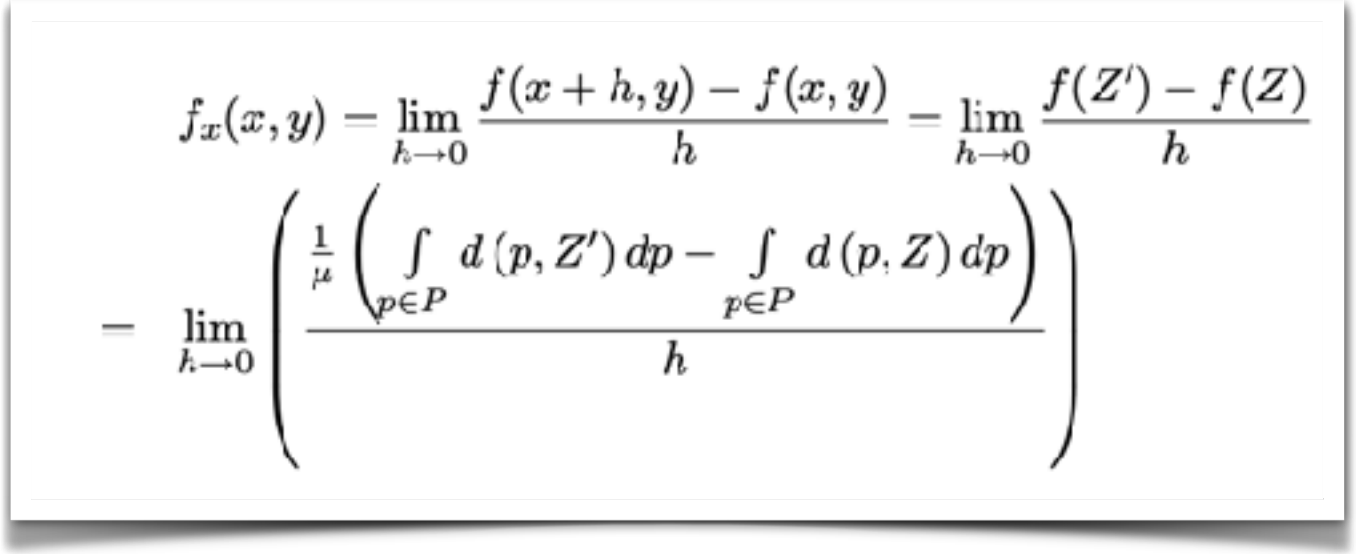

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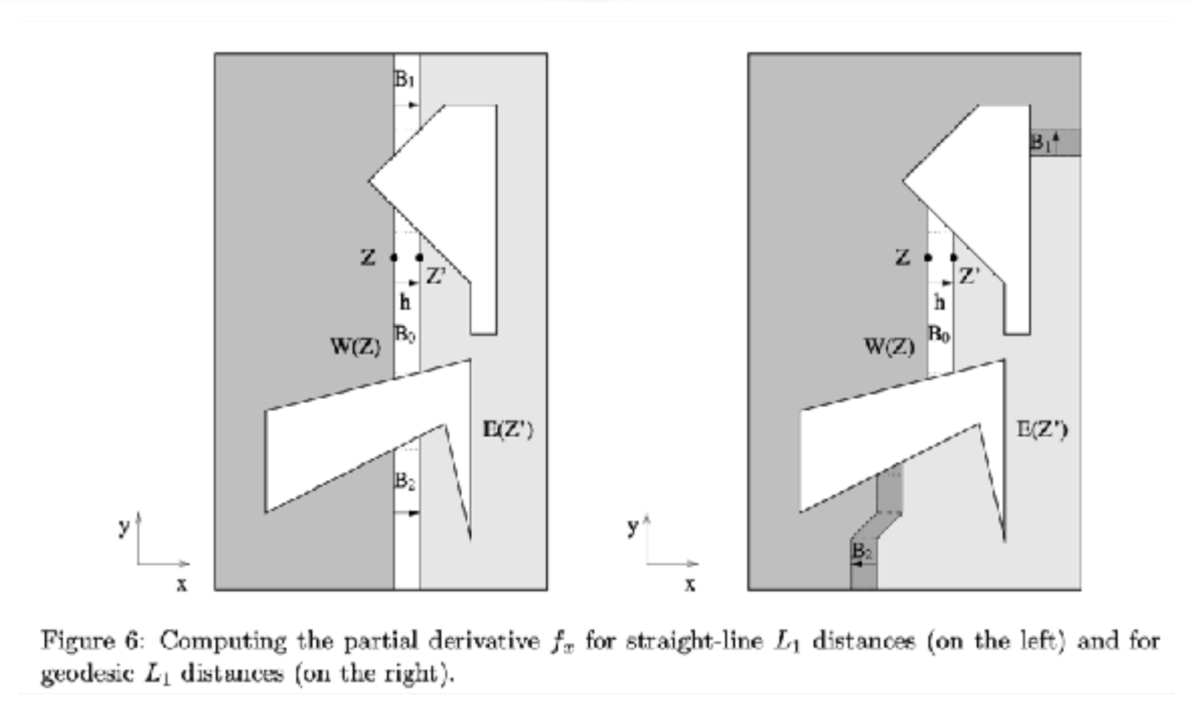
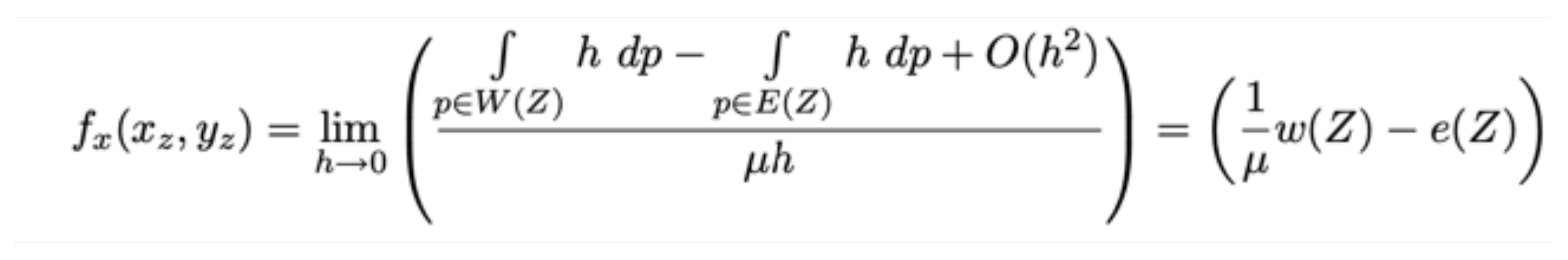


Figure 6: Computing the partial derivative f_x for straight-line L_1 distances (on the left) and for geodesic L_1 distances (on the right).



$$f_x(x_z, y_z) = \lim_{h\to 0} \left(\frac{\int\limits_{p\in W(Z)} h\ dp - \int\limits_{p\in E(Z)} h\ dp + O(h^2)}{\mu h} \right) = \left(\frac{1}{\mu} w(Z) - e(Z) \right)$$

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Local optimality for straight-line distances

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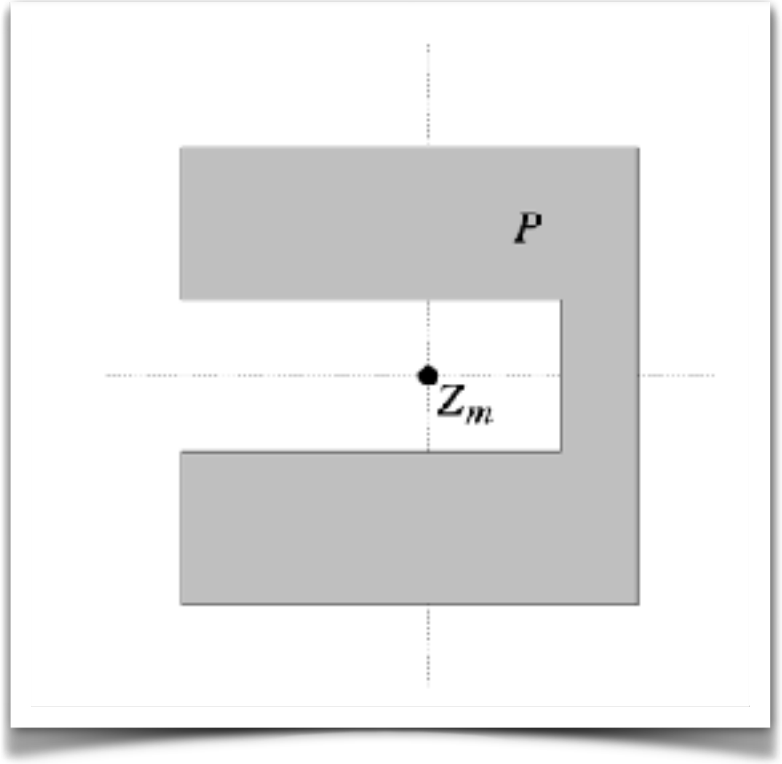
Local optimality for straight-line distances

Insights:

Local optimality for straight-line distances

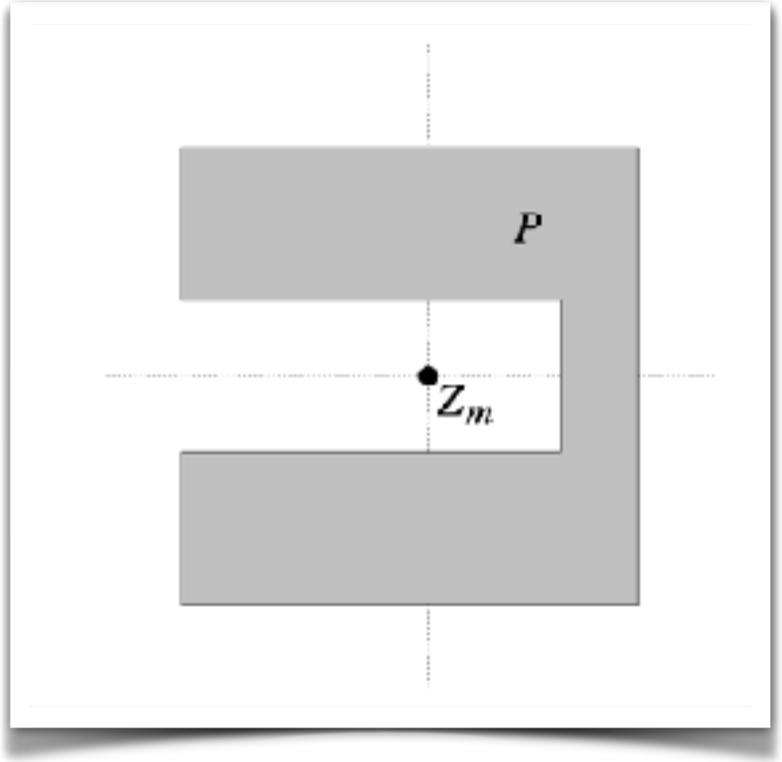
Insights:



Local optimality for straight-line distances

Insights:

a) L1 median if feasible

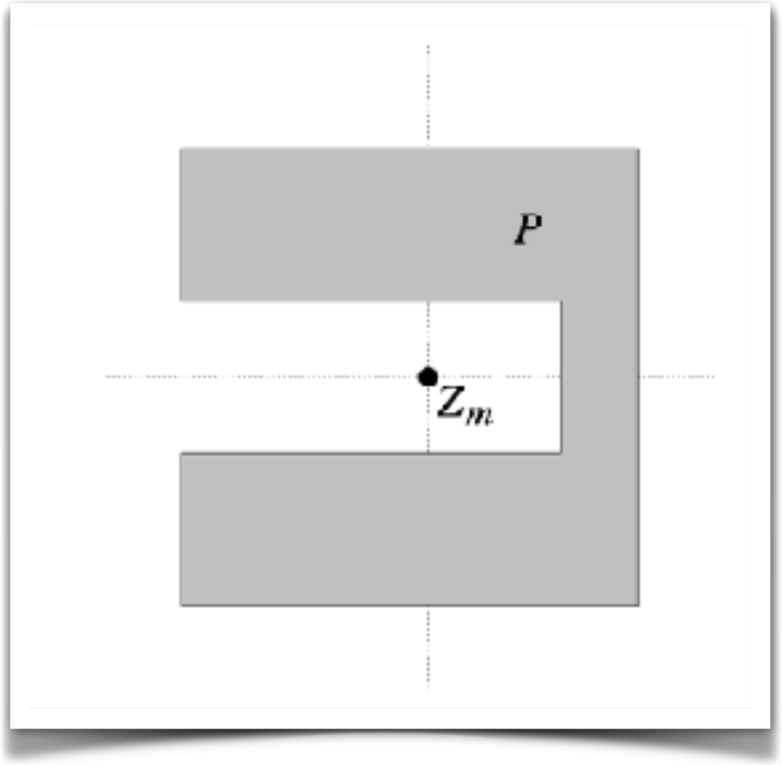


Local optimality for straight-line distances

Insights:

a) L1 median if feasible

b) Otherwise:



Local optimality for straight-line distances

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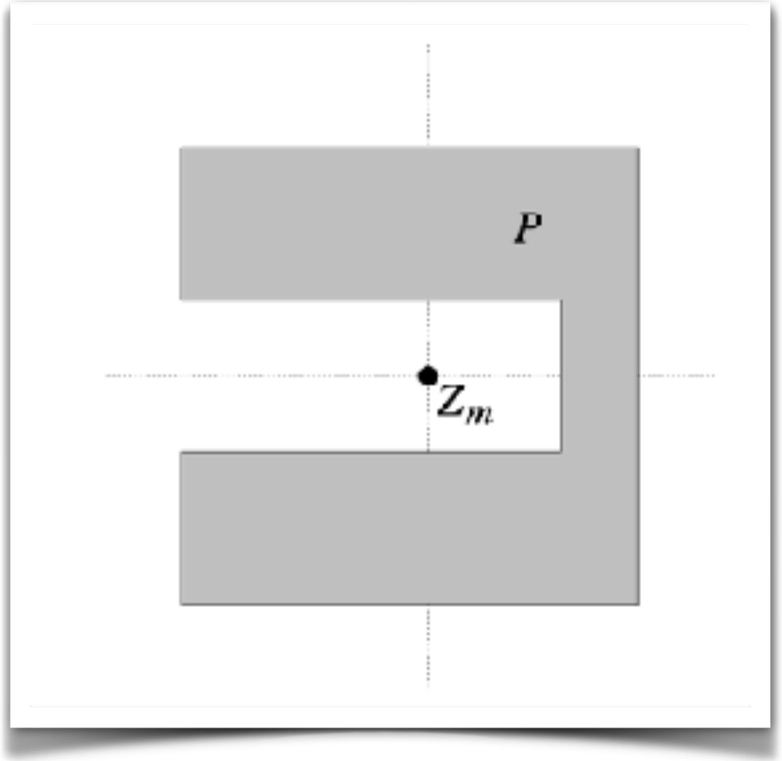
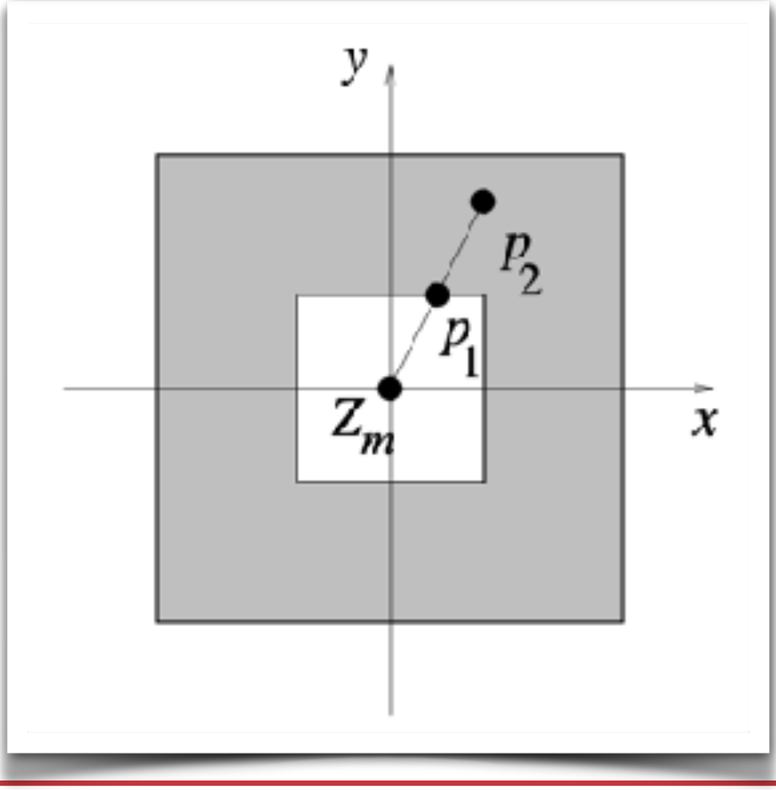


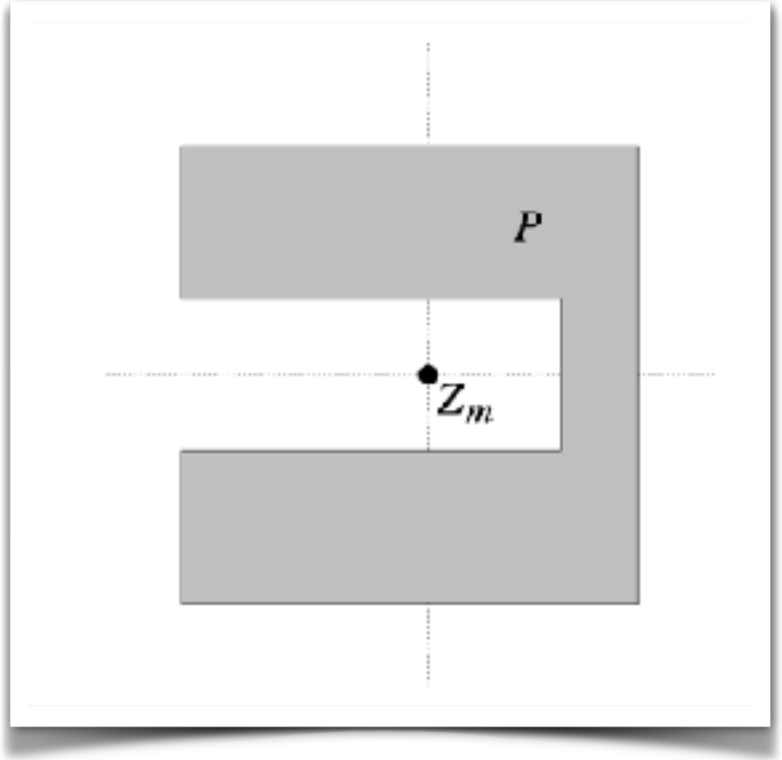


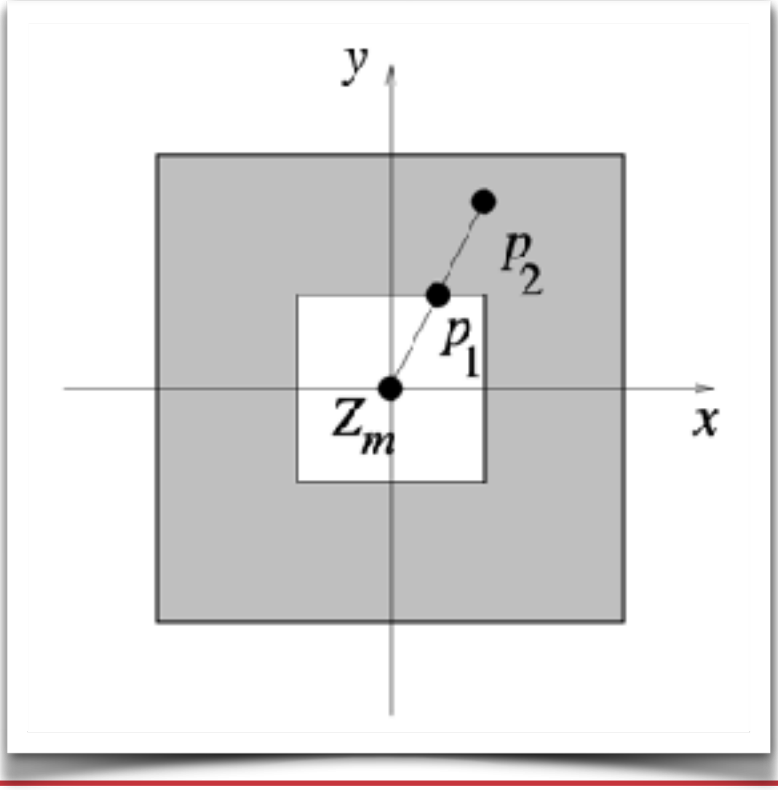
Insights:

a) L1 median if feasible

b) Otherwise:

- Boundary point





Local optimality for straight-line distances

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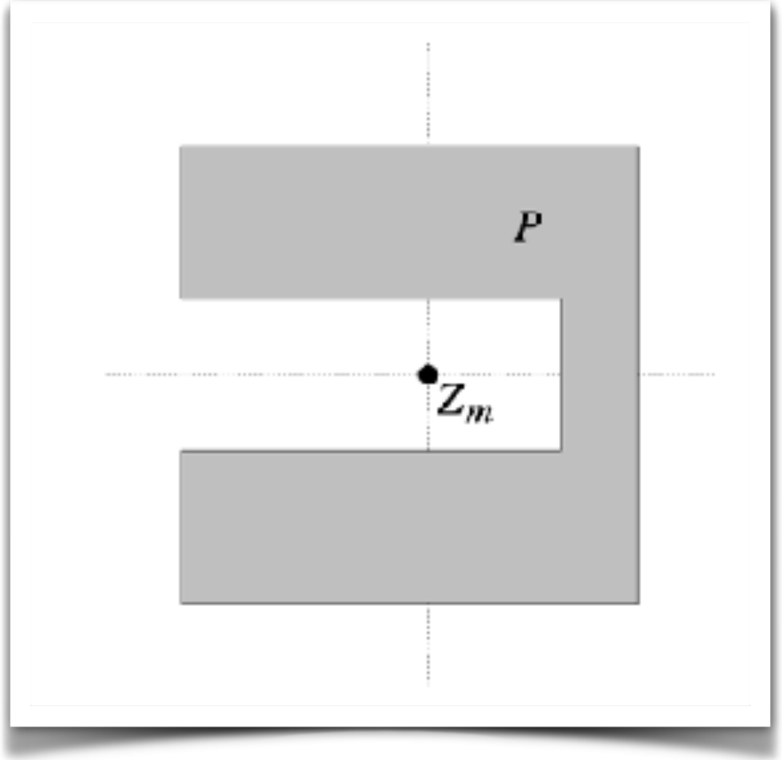
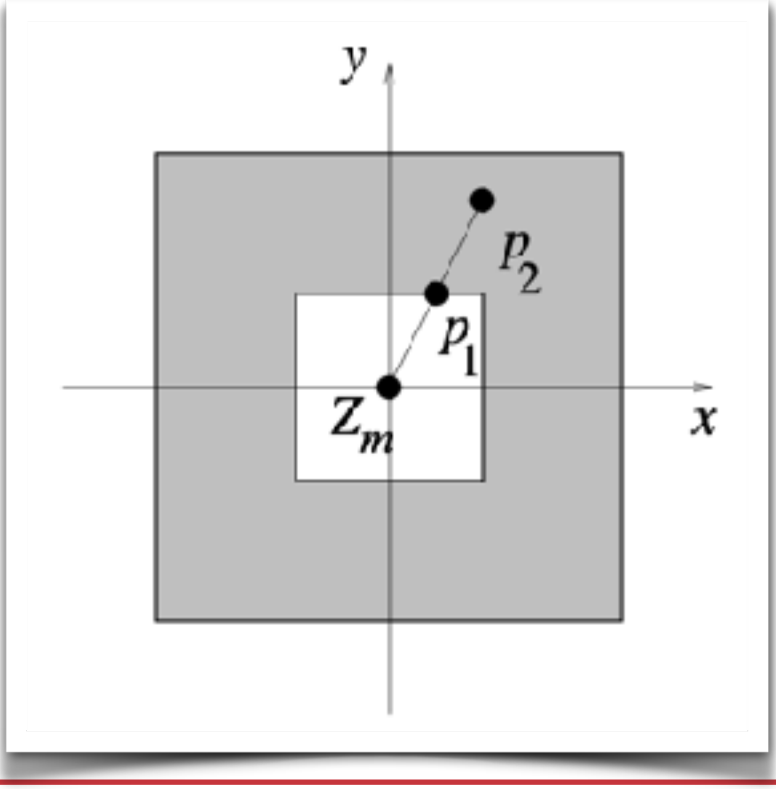
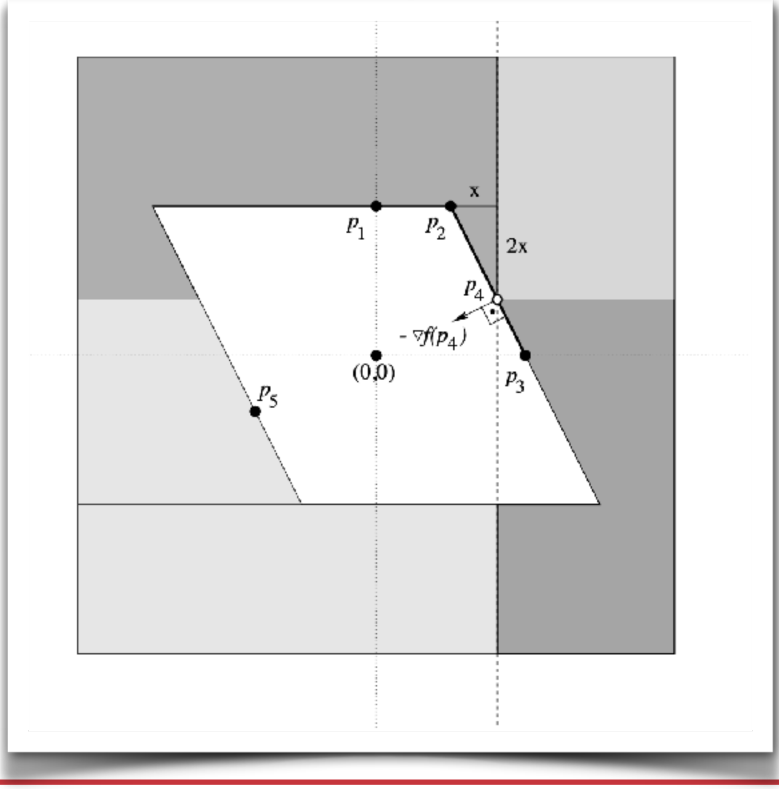






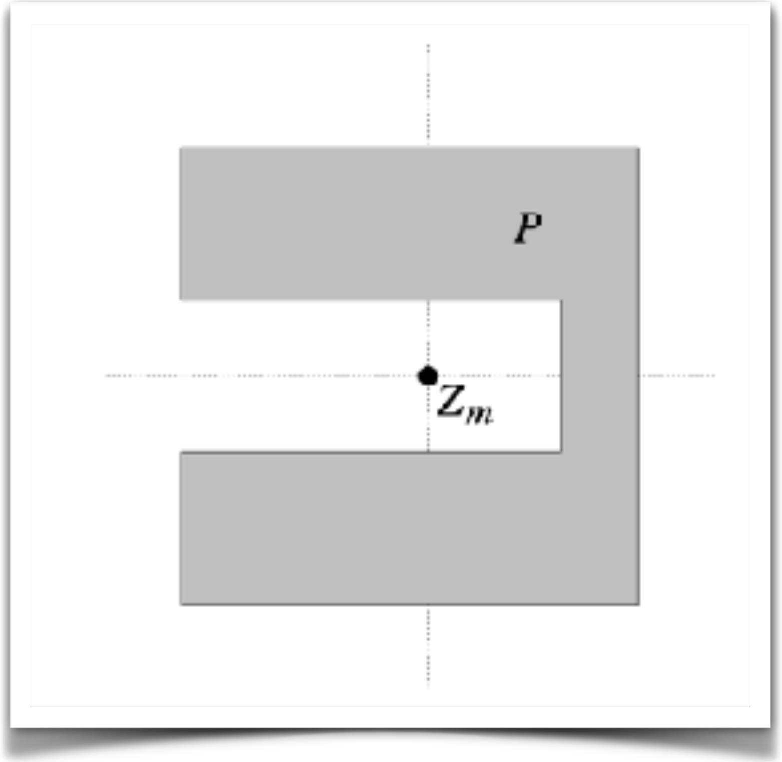
Local optimality for straight-line distances

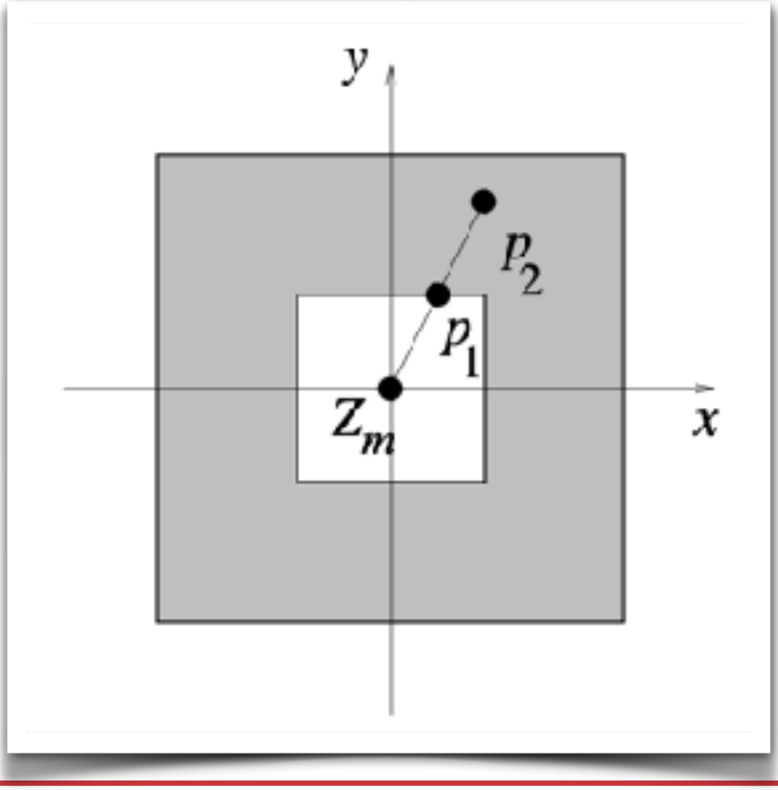
Insights:

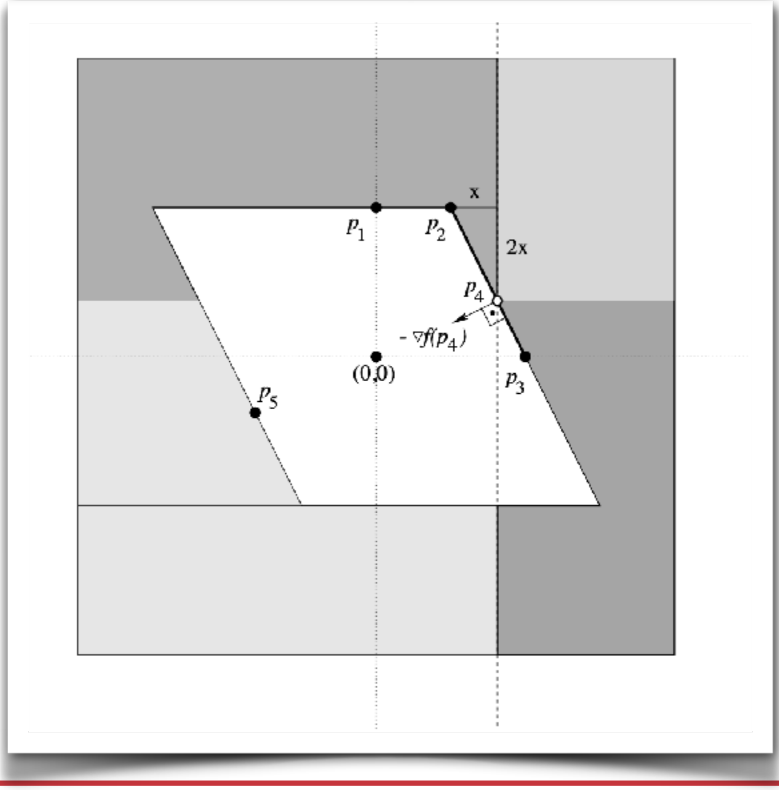
a) L1 median if feasible

b) Otherwise:

- Boundary point
- Gradient must be orthogonal to boundary







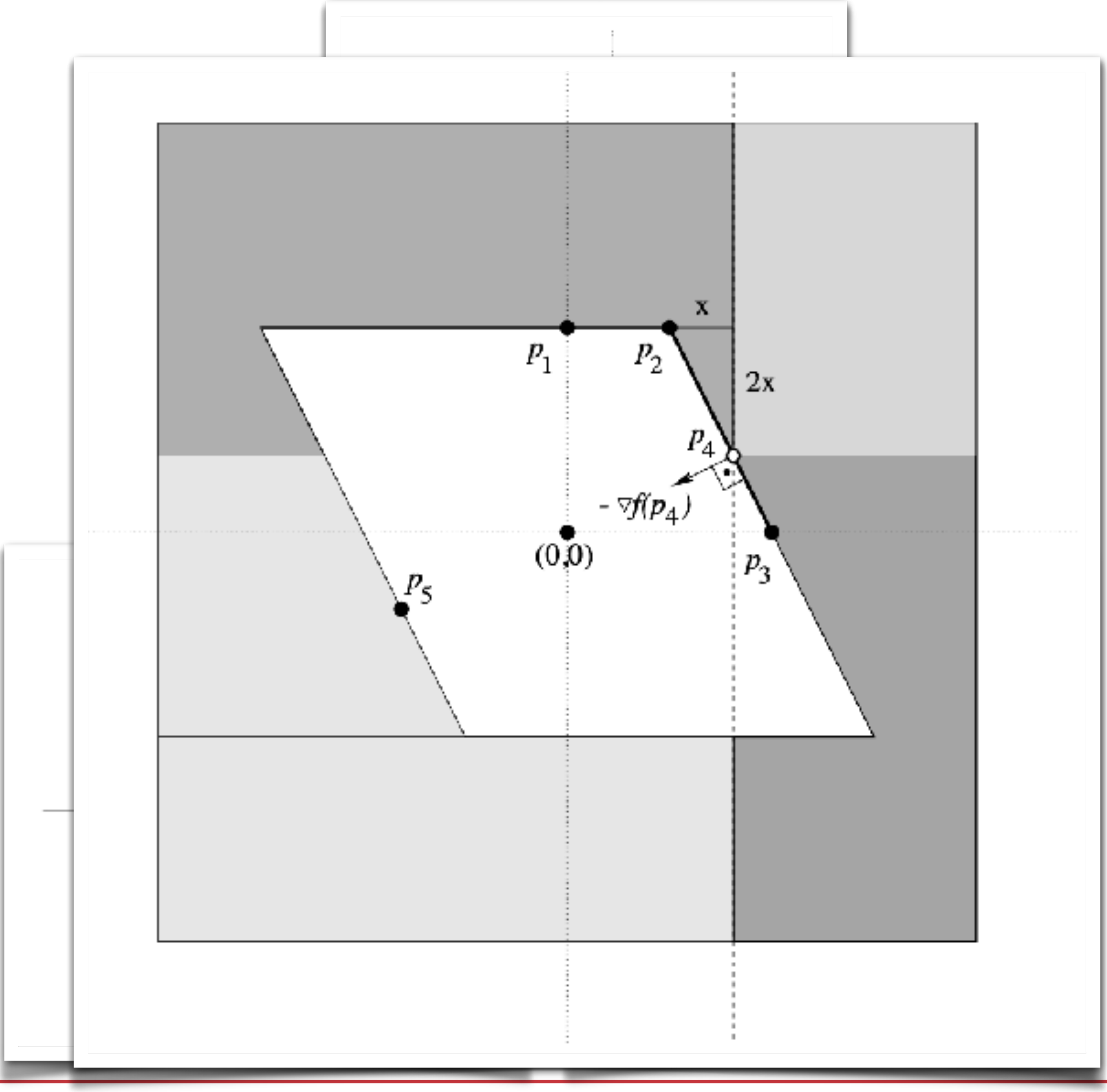
Local optimality for straight-line distances

Insights:

a) L1 median if feasible

b) Otherwise:

- Boundary point
- Gradient must be orthogonal to boundary



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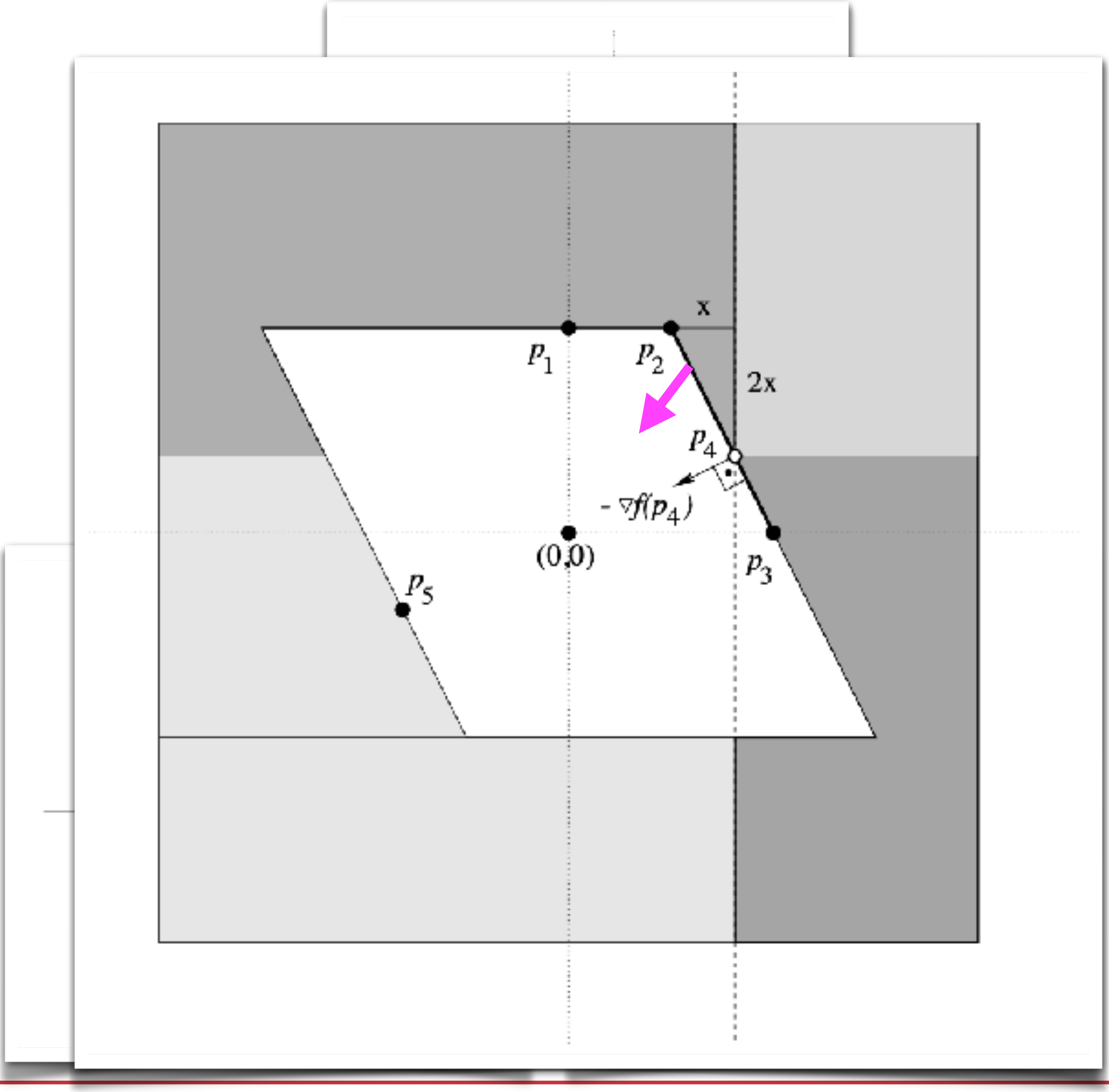

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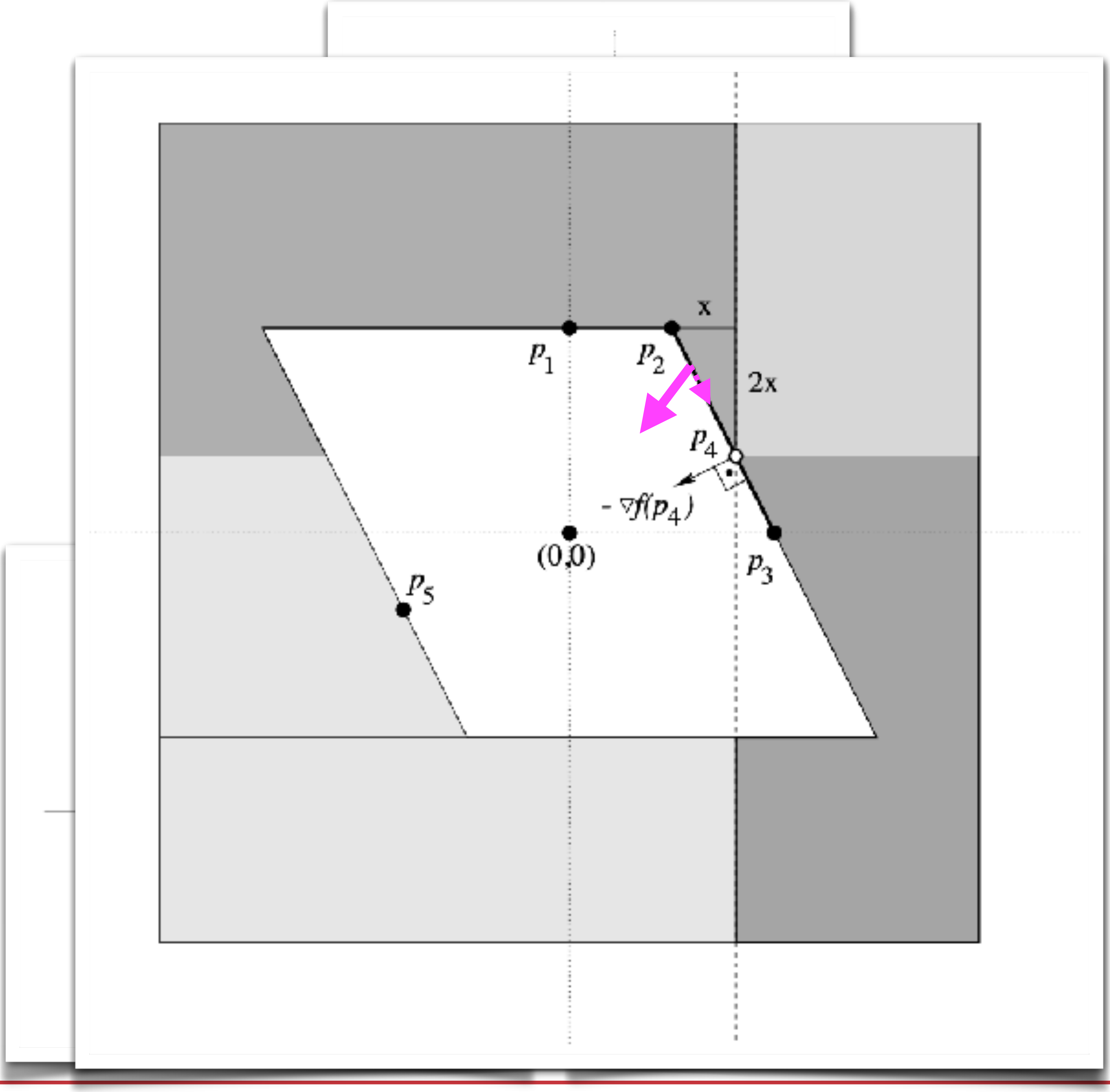

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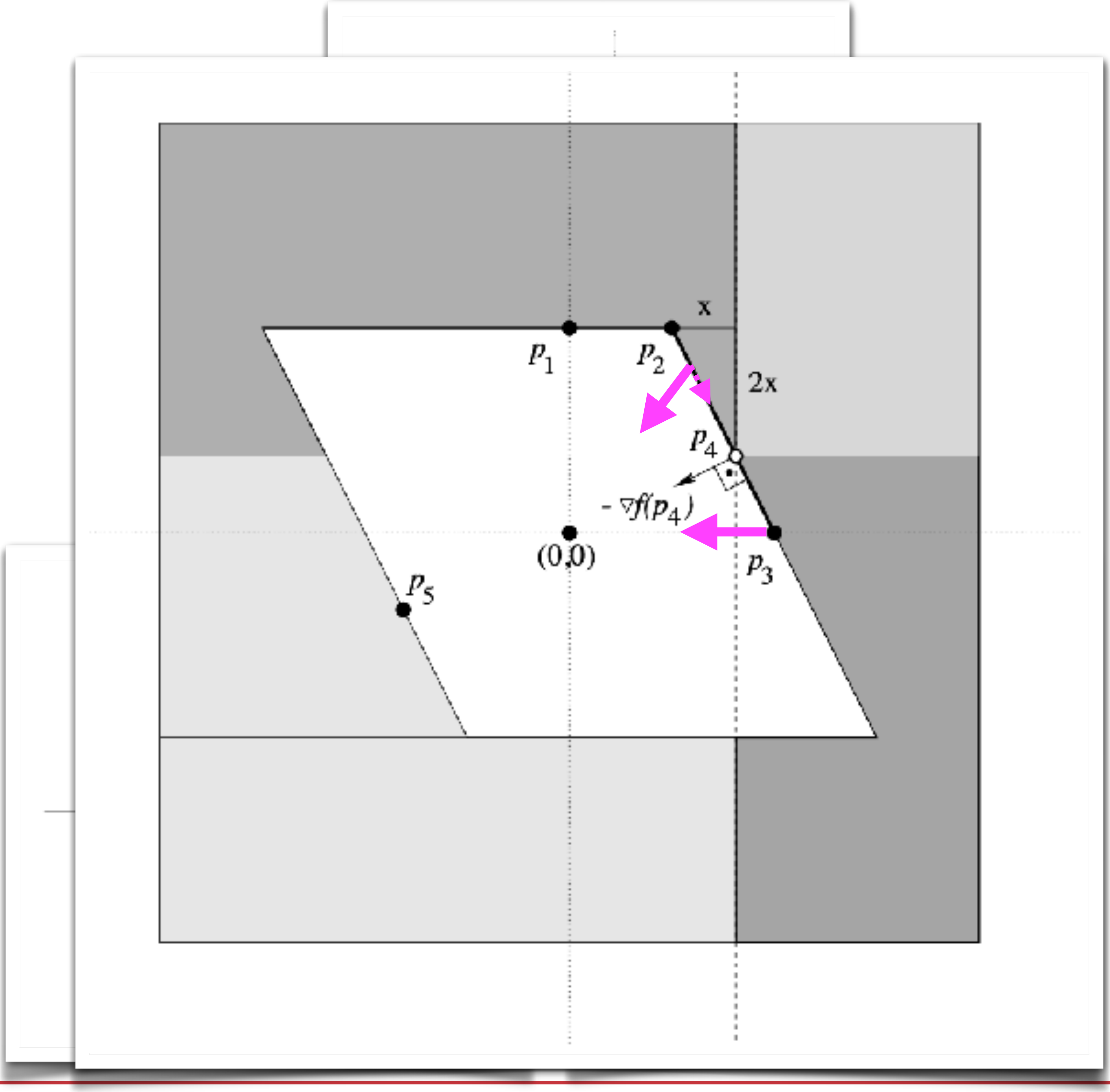

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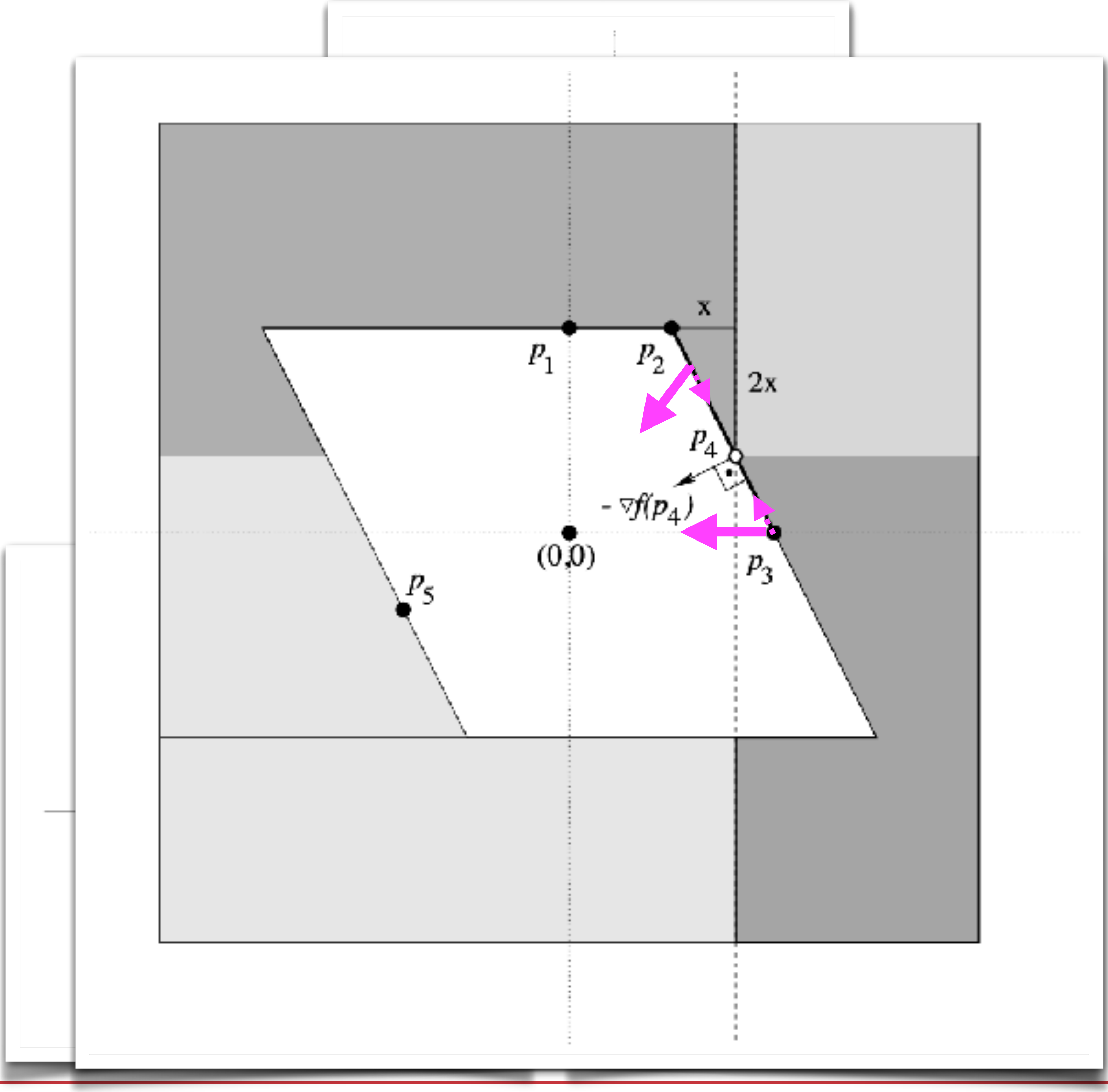

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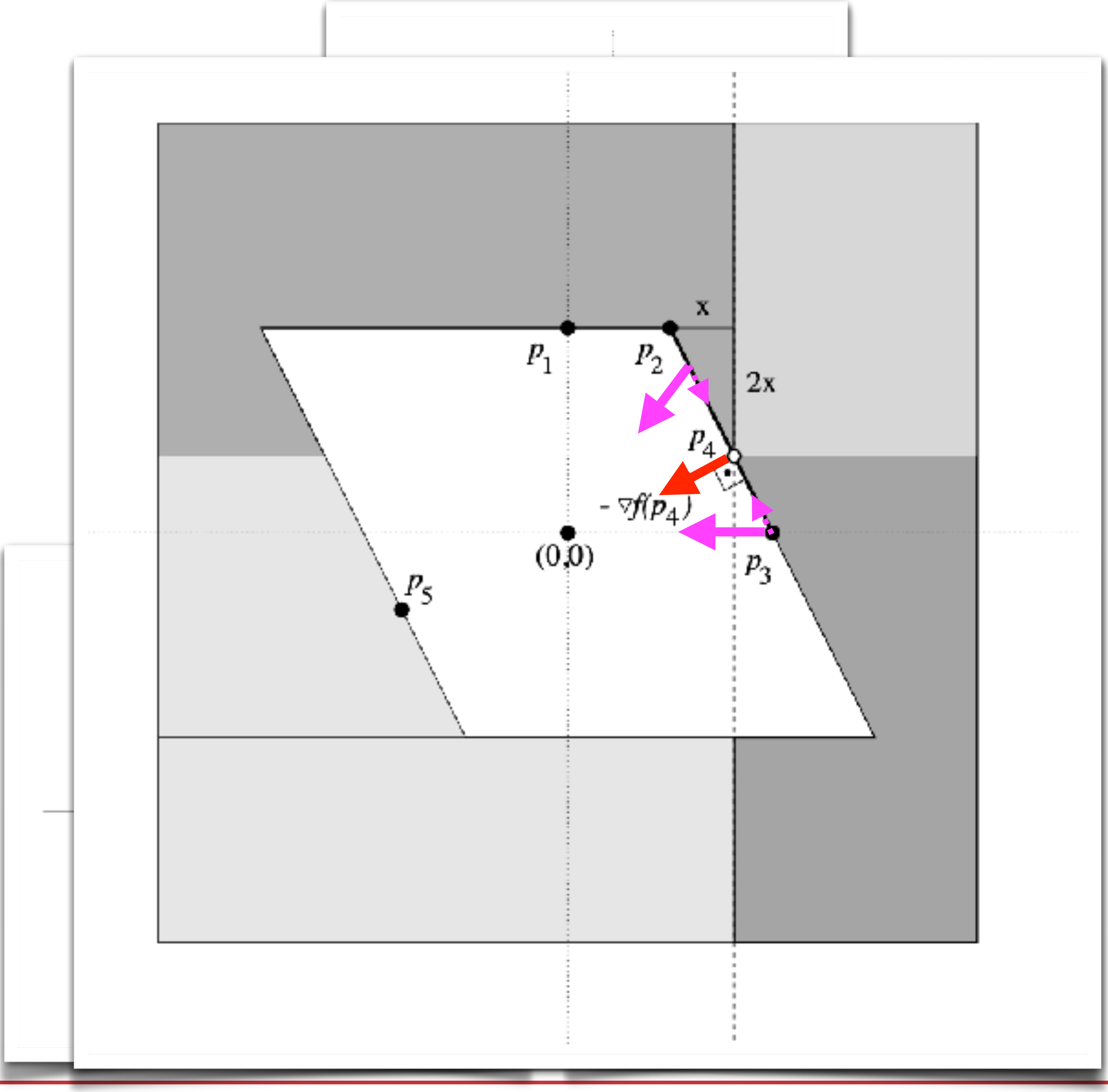

Local optimality for straight-line distances

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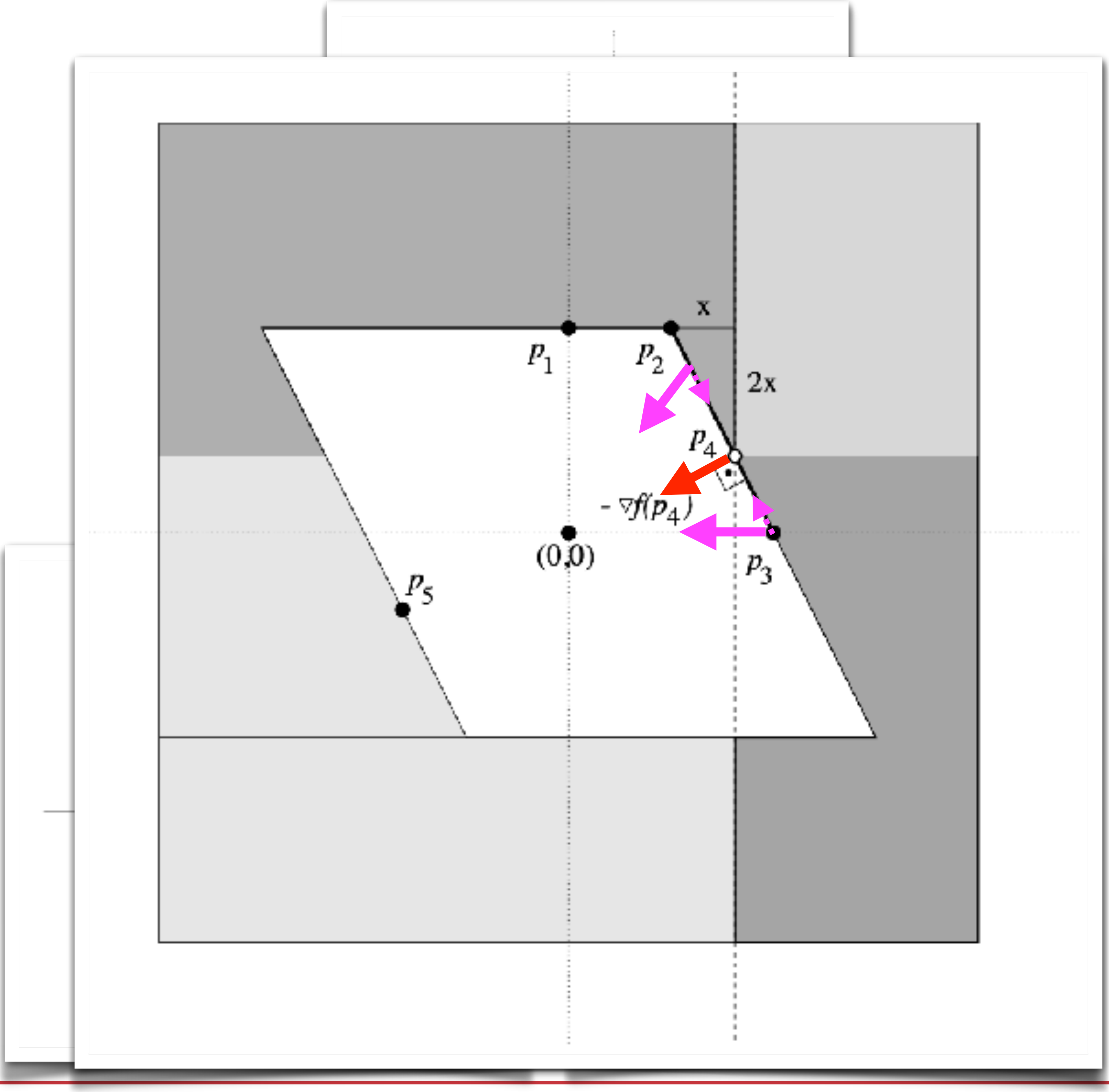
$$\langle \nabla f(p_t), s_j \rangle = 0$$



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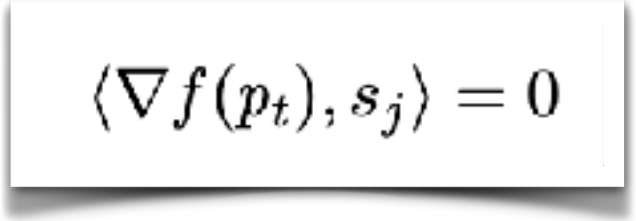

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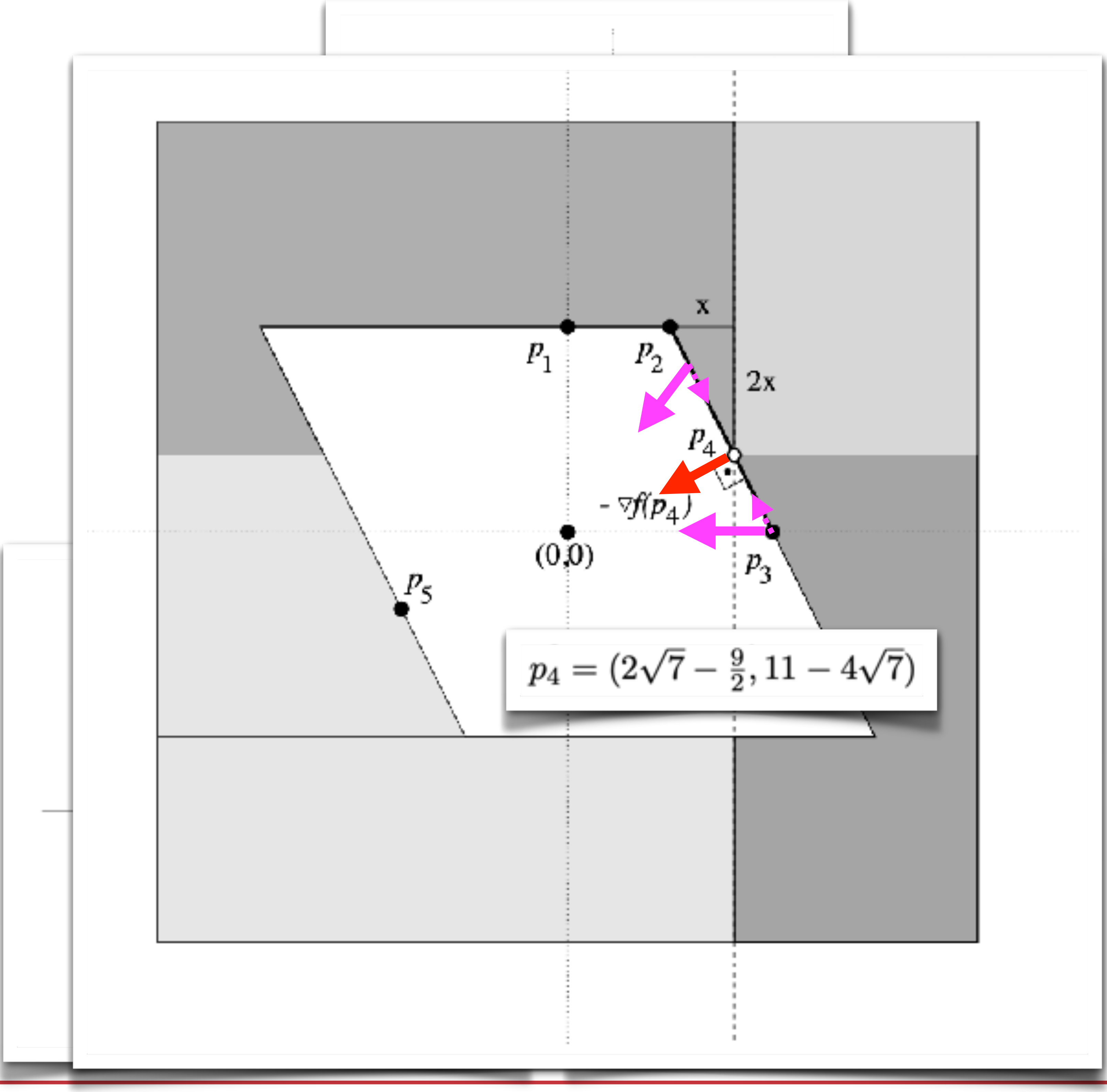
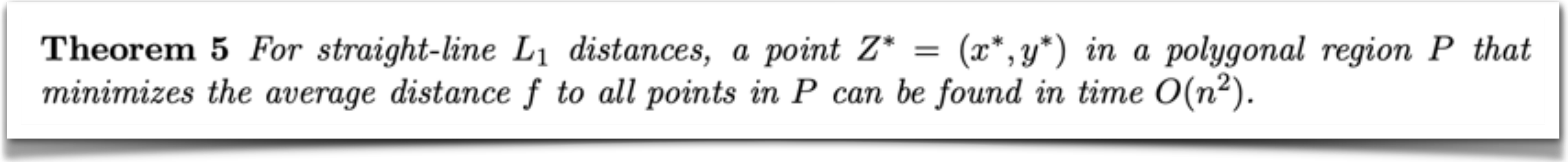


$$p_4 = (2\sqrt{7} - \tfrac{9}{2}, 11 - 4\sqrt{7})$$

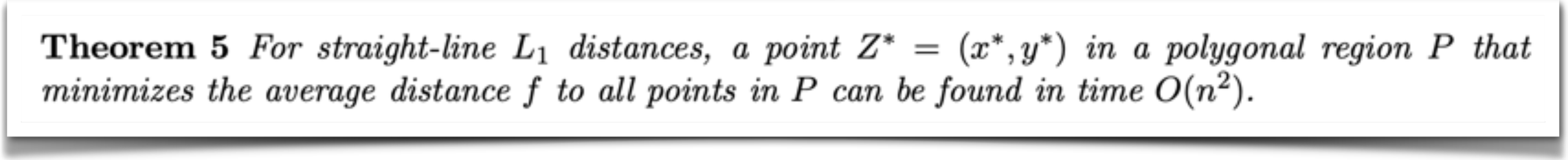
Straight-line distances

Straight-line distances



Theorem 5 *For straight-line L_1 distances, a point $Z^* = (x^*, y^*)$ in a polygonal region P that minimizes the average distance f to all points in P can be found in time $O(n^2)$.*

Straight-line distances



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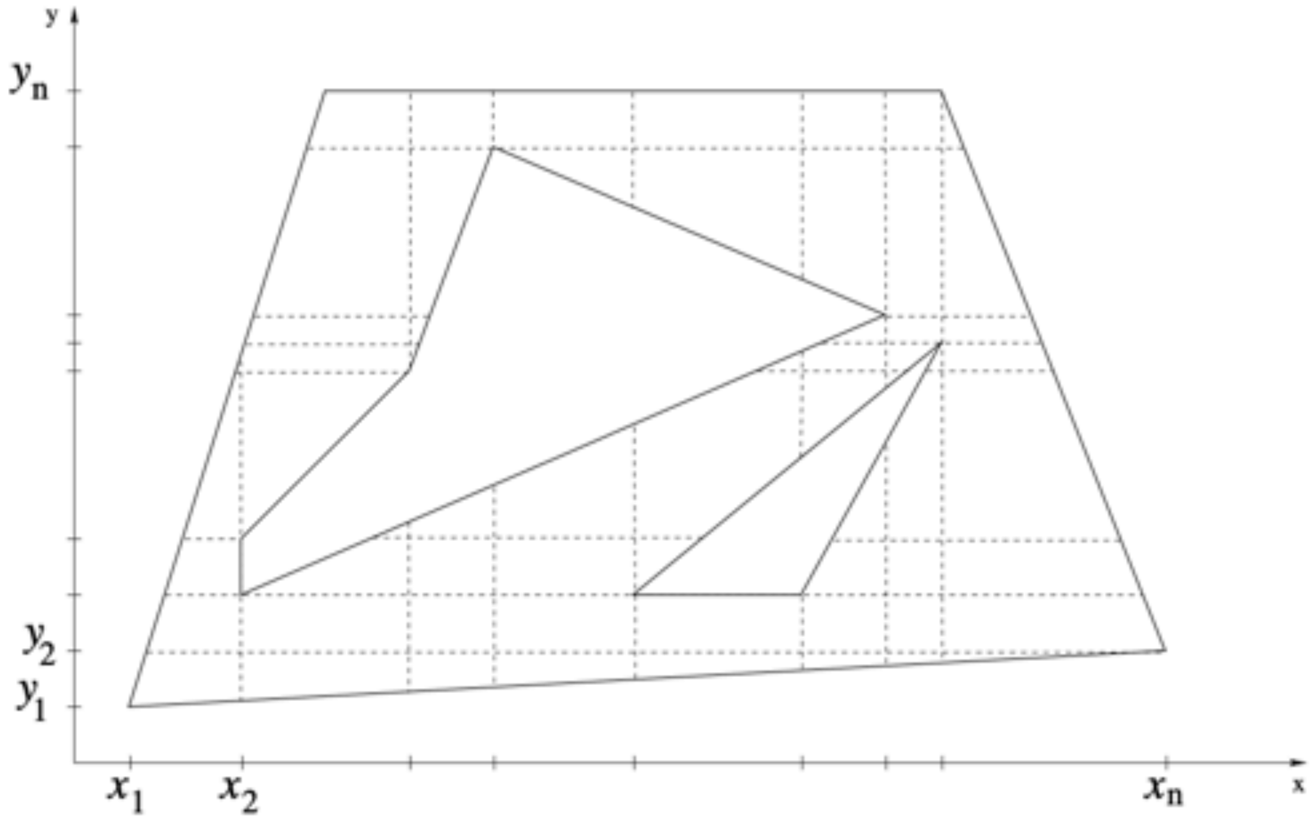
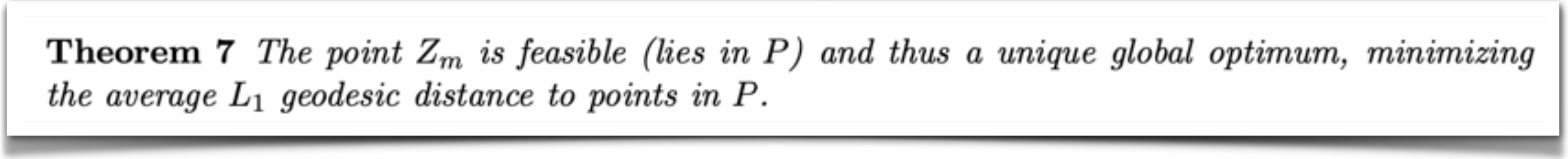


Figure 8: Subdivision of the polygon into cells.

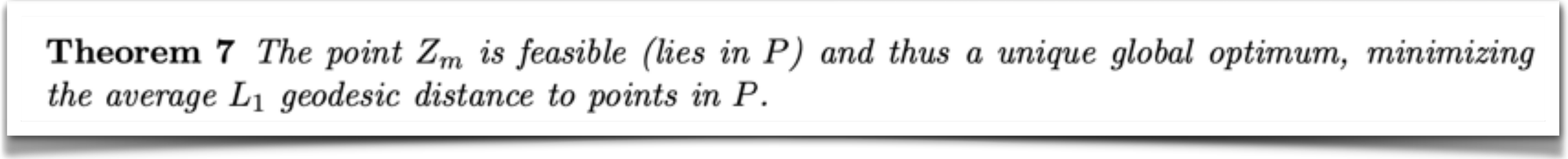
Geodesic distances in simple polygons

Geodesic distances in simple polygons



Theorem 7 *The point Z_m is feasible (lies in P) and thus a unique global optimum, minimizing the average L_1 geodesic distance to points in P.*

Geodesic distances in simple polygons



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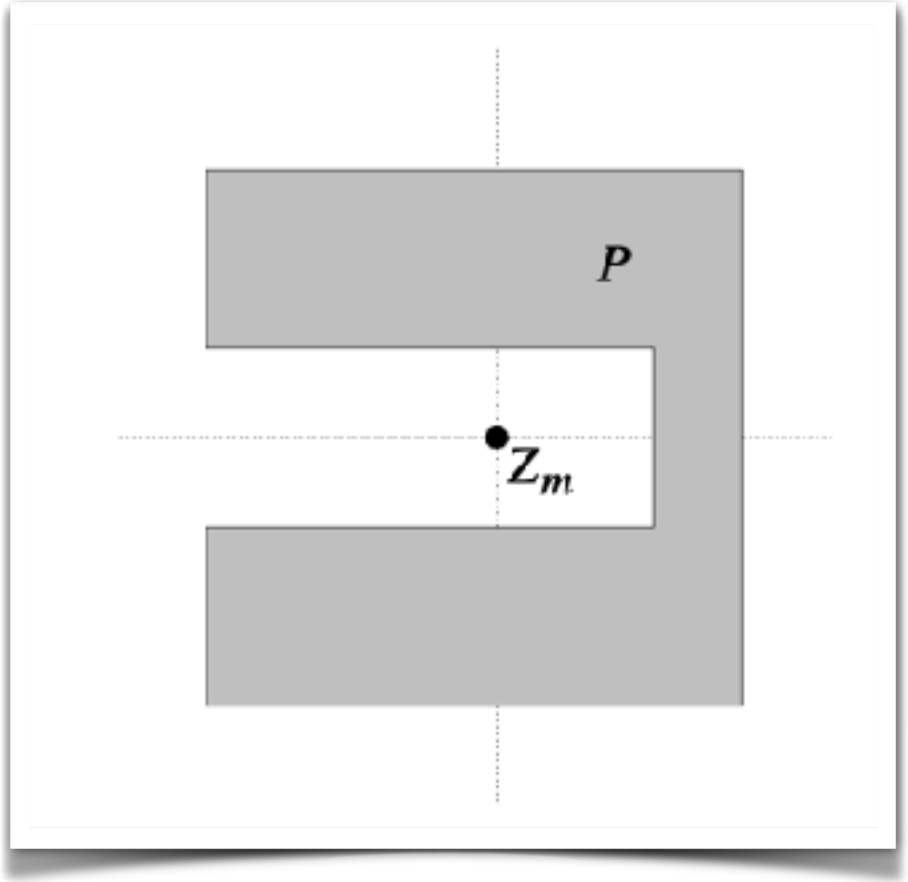
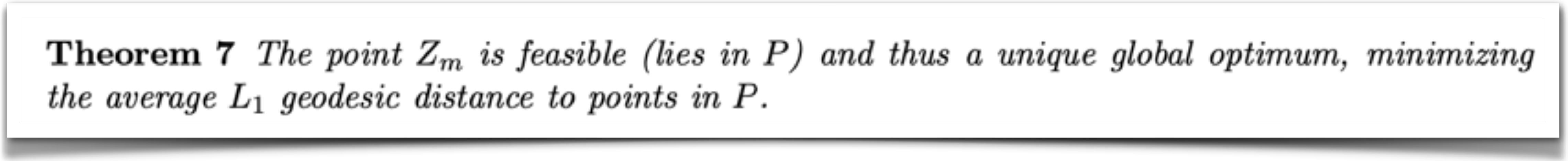


Geodesic distances in simple polygons



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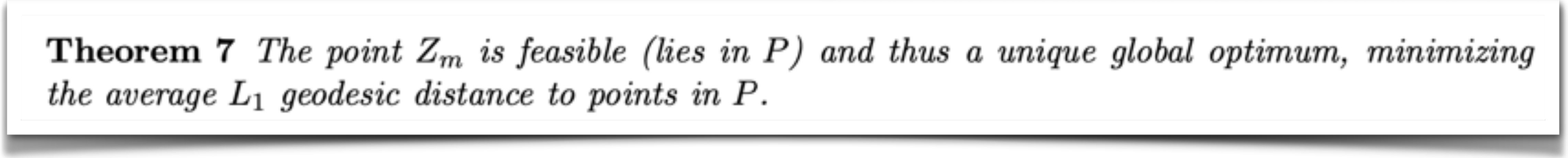
Geodesic distances in simple polygons



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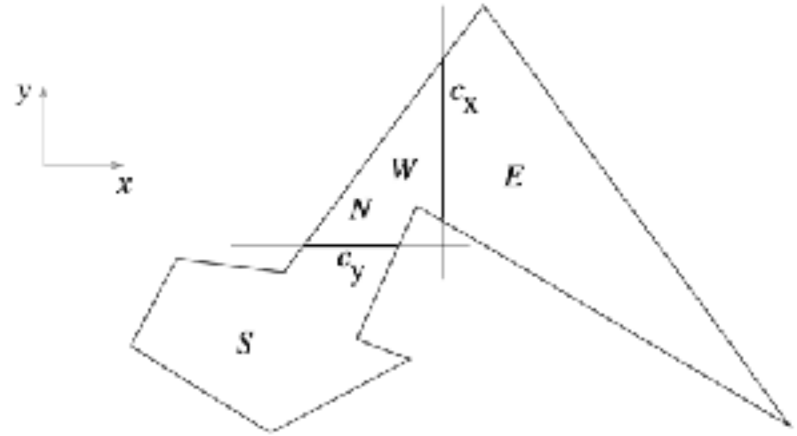

Figure 11: Case 0: Neither of the median chords c_x and c_y is critical.

Geodesic distances in simple polygons

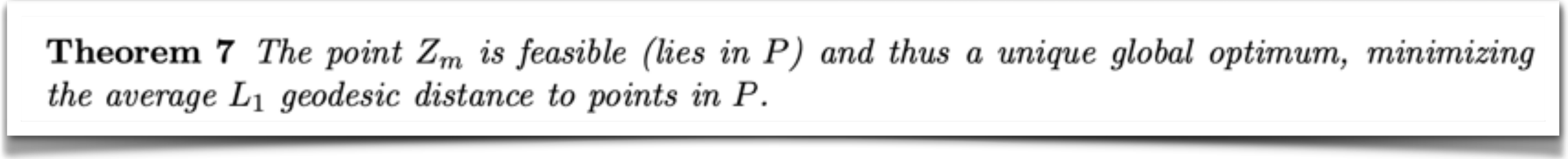

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Figure 11: Case 0: Neither of the me

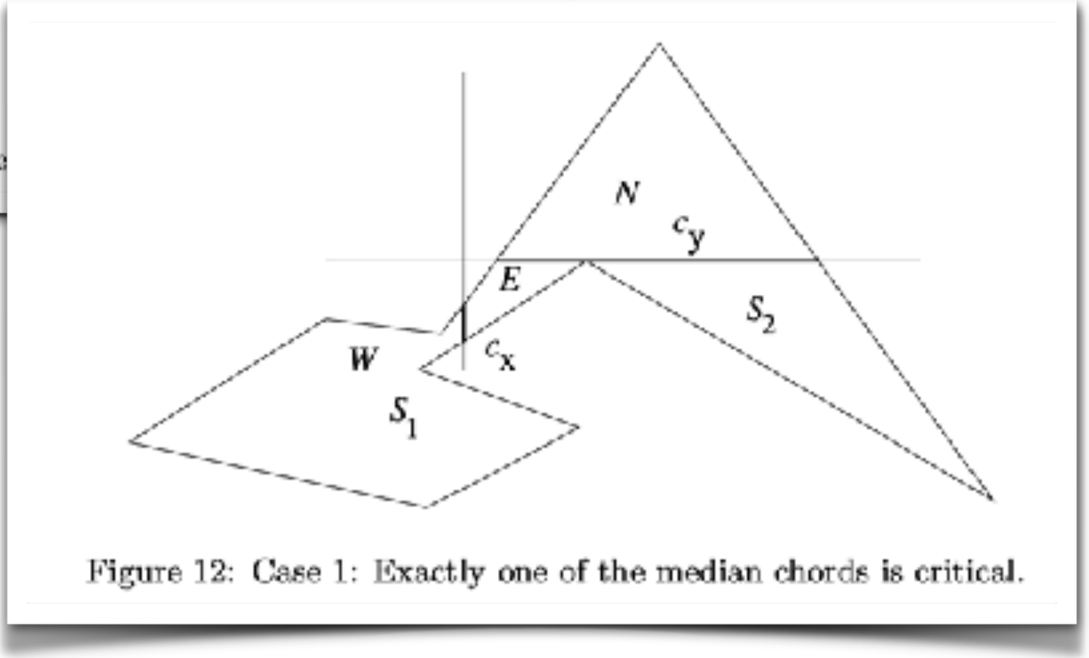
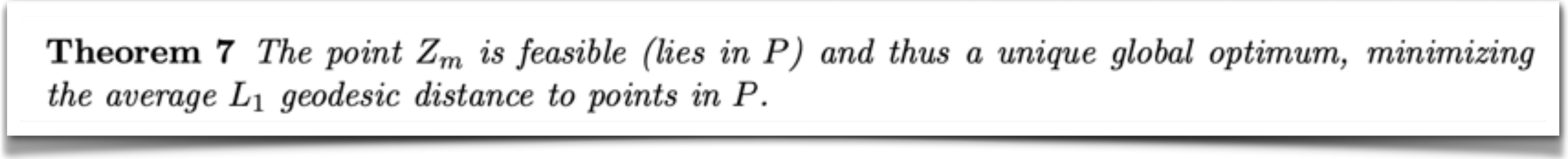


Figure 12: Case 1: Exactly one of the median chords is critical.

Geodesic distances in simple polygons



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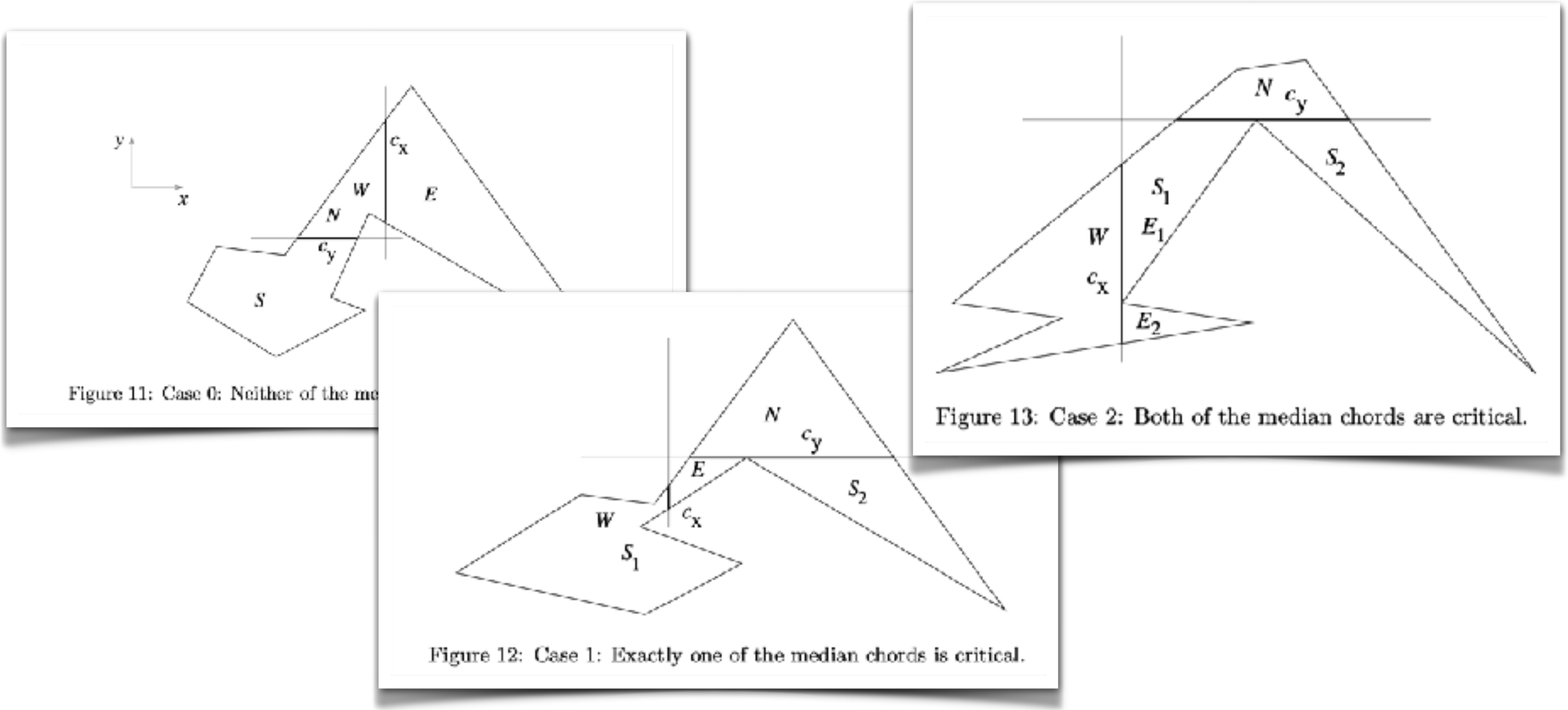


Figure 11: Case 0: Neither of the me

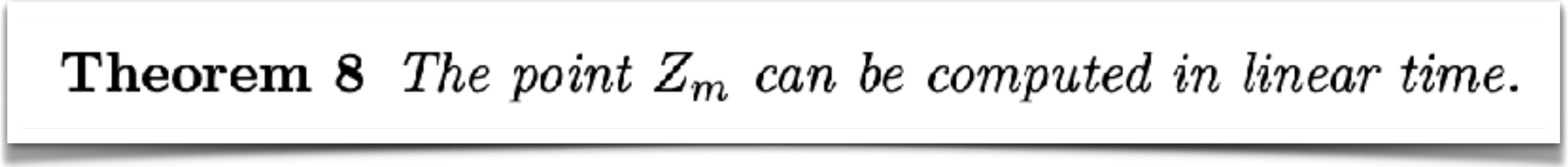
Figure 12: Case 1: Exactly one of the median chords is critical.

Figure 13: Case 2: Both of the median chords are critical.

Technische Universität Braunschweig

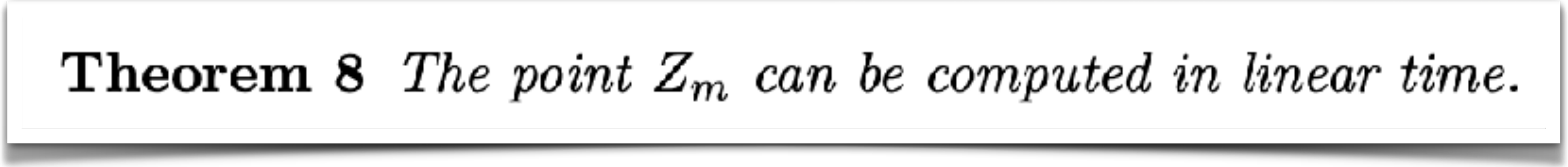
Geodesic distances in simple polygons

Geodesic distances in simple polygons



Theorem 8 *The point Z_m can be computed in linear time.*

Geodesic distances in simple polygons



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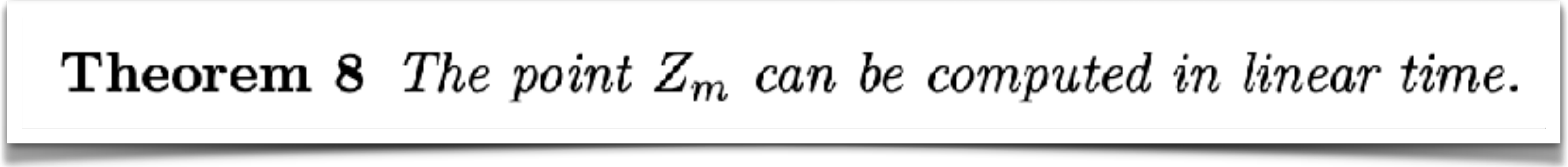
1. **Trapezoidization in linear time**

Geodesic distances in simple polygons



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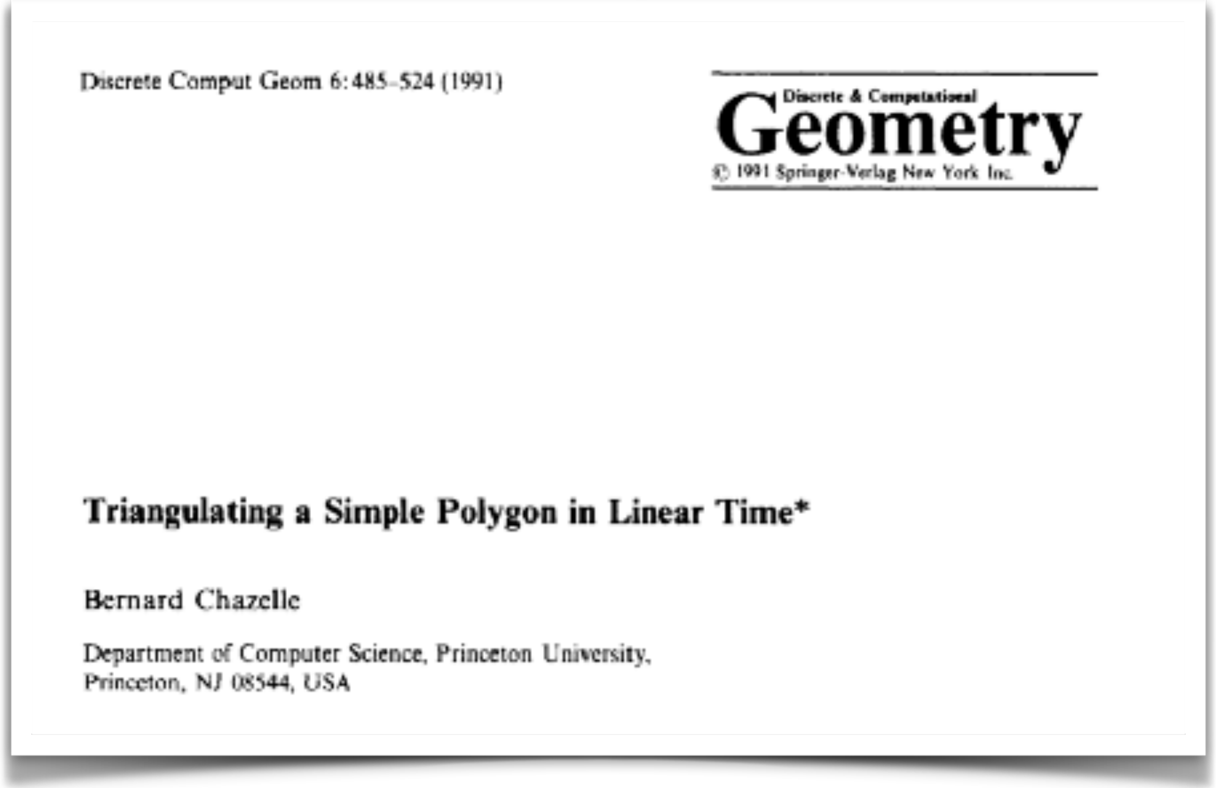

Discrete Comput Geom 6:485–524 (1991)

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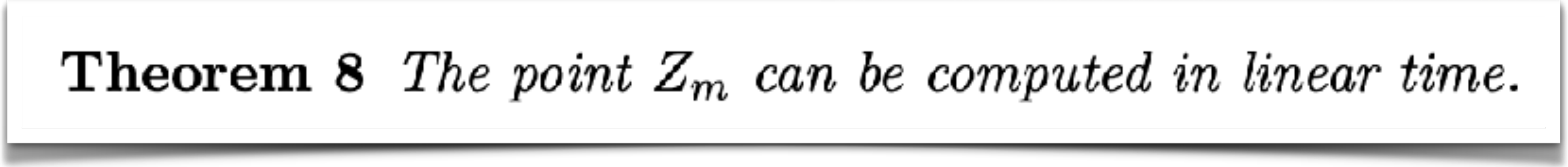
Triangulating a Simple Polygon in Linear Time*

Bernard Chazelle

Department of Computer Science, Princeton University,
Princeton, NJ 08544, USA

Geodesic distances in simple polygons



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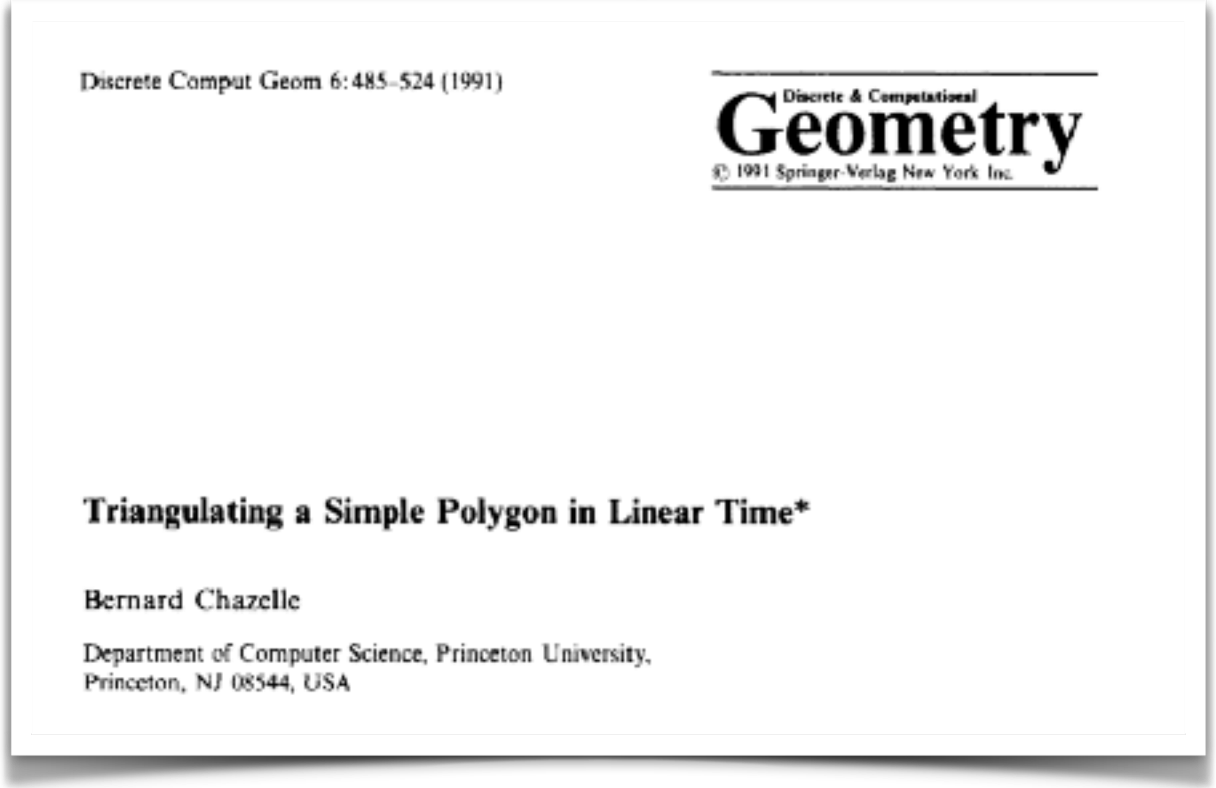
1. **Trapezoidization in linear time**
2. **Compute median in weighted tree**



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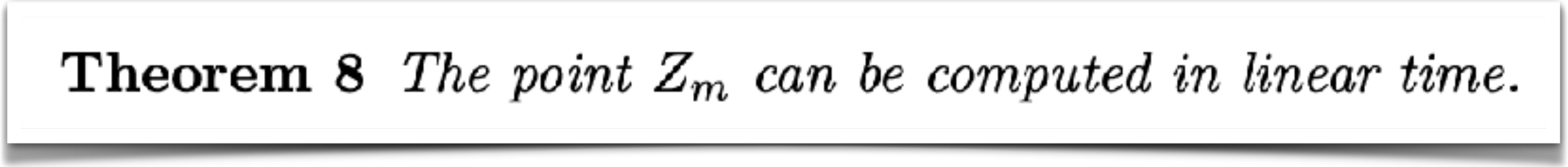

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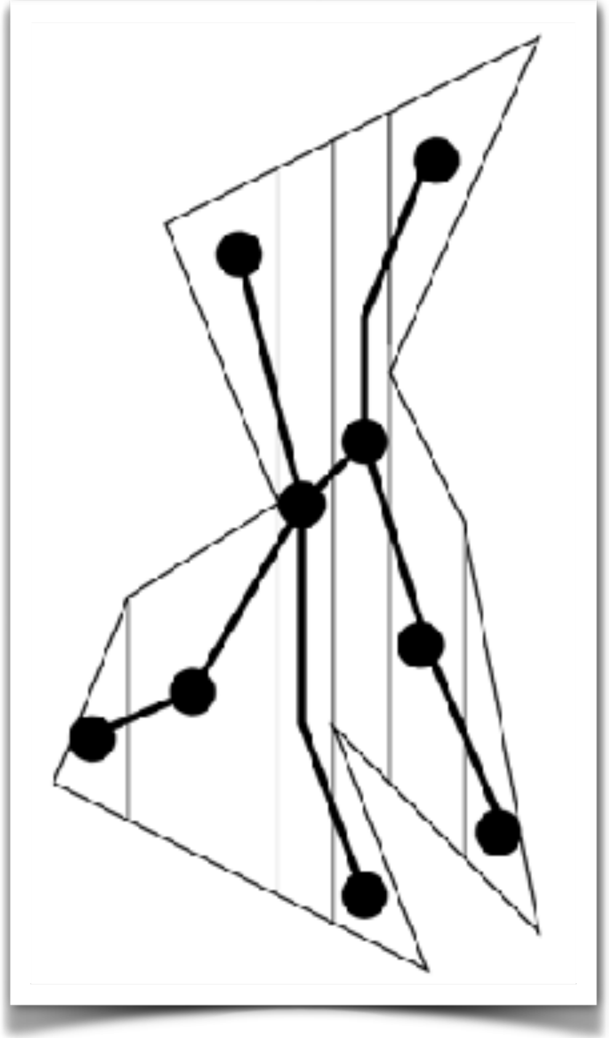
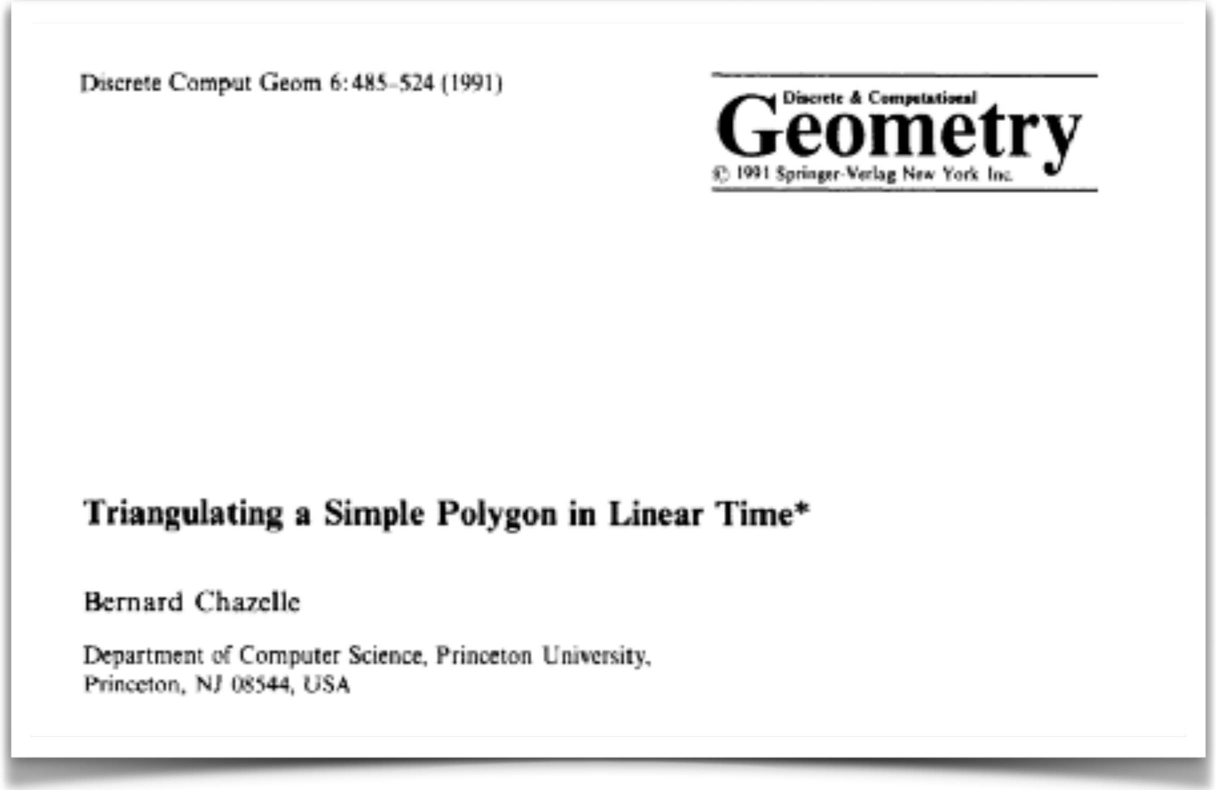
2. Compute median in weighted tree





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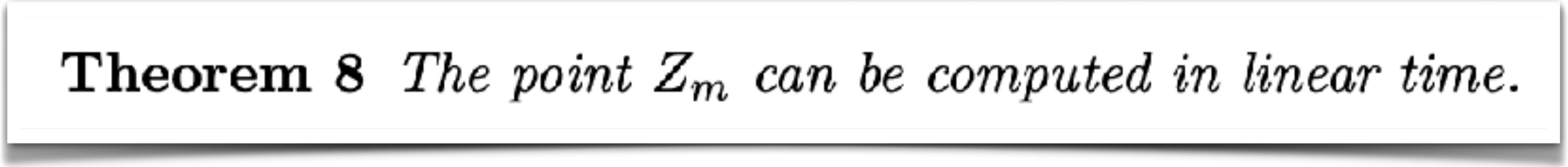

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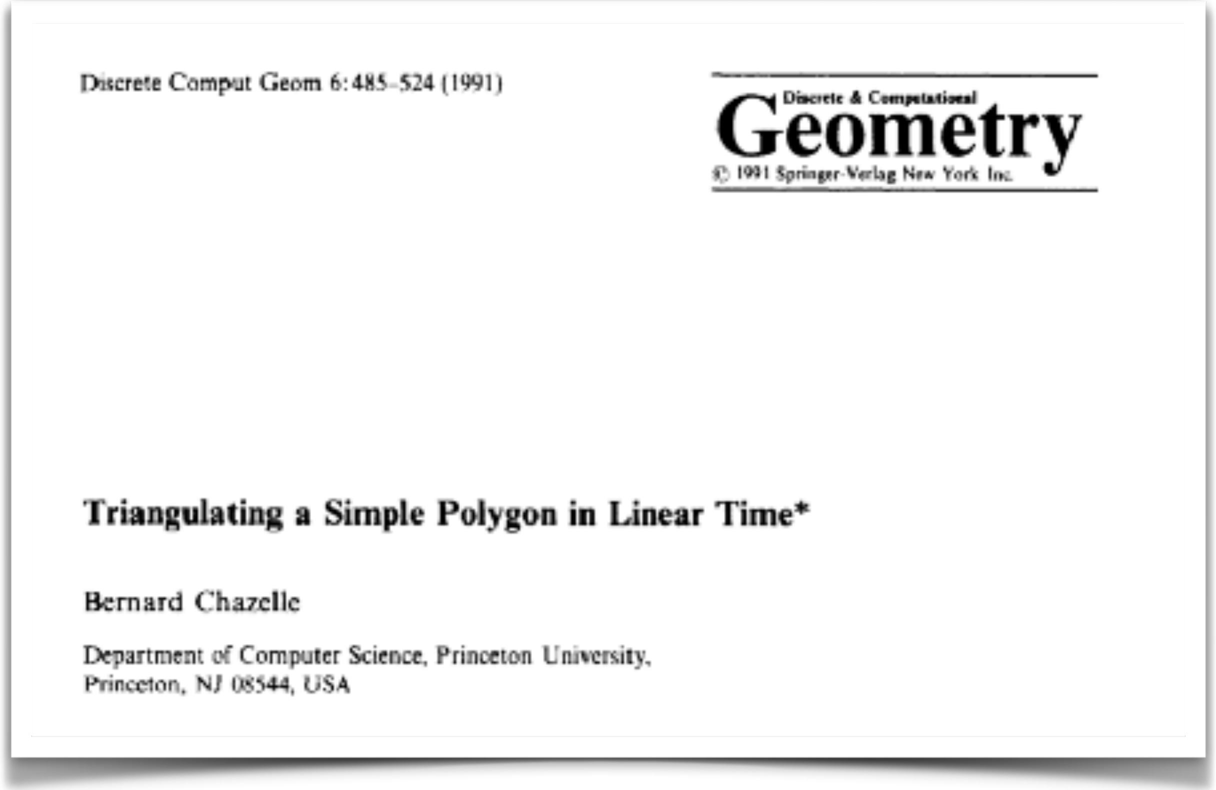
1. Trapezoidization in linear time
2. Compute median in weighted tree
3. Compute point within cell



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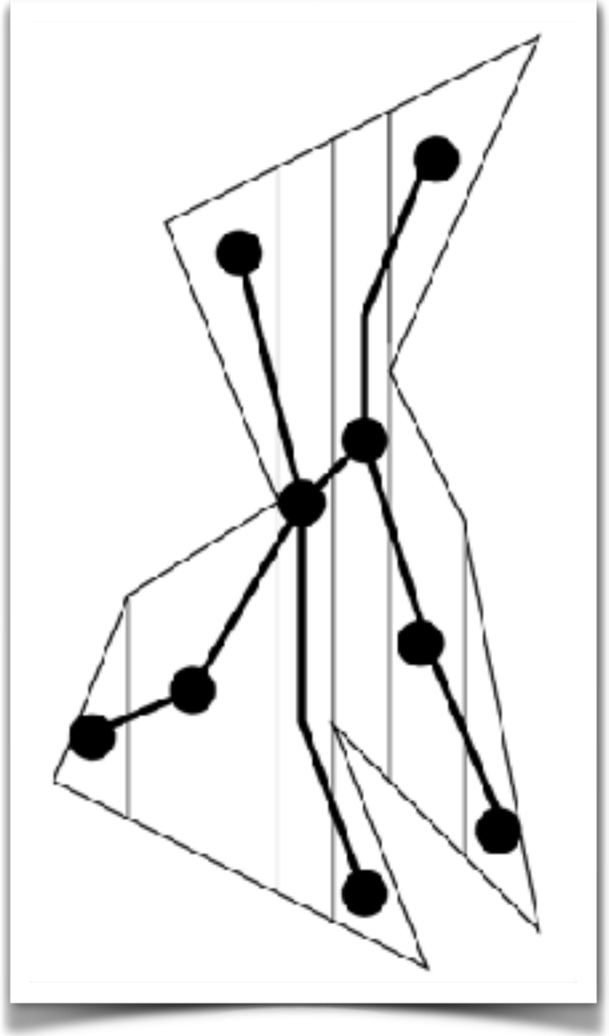

Geodesic distances in non-simple polygons [Mitchell 1992]

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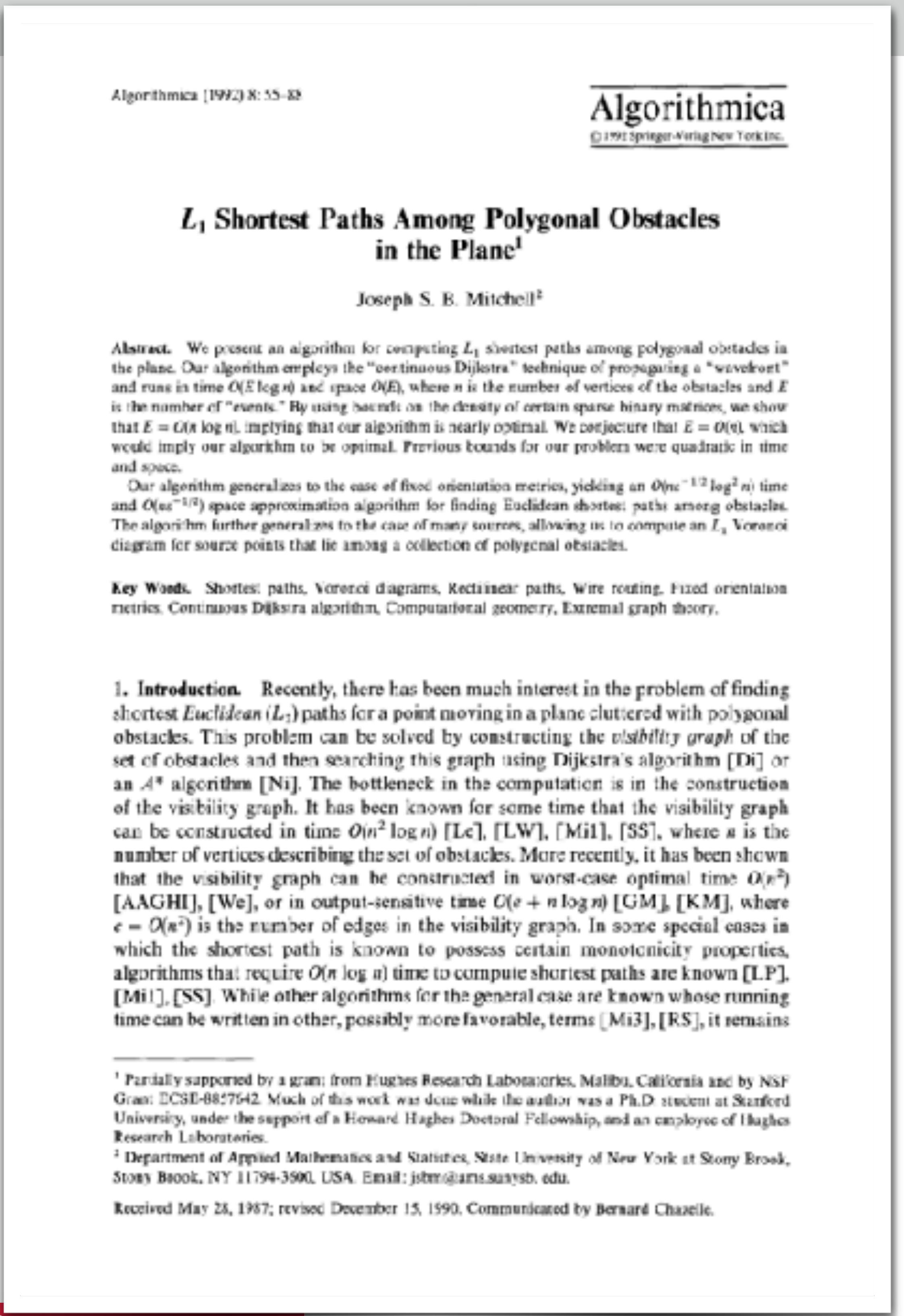

Algorithmica (1992) 8: 55–82

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L_1 Shortest Paths Among Polygonal Obstacles in the Plane[1]

Joseph S. B. Mitchell[2]

Abstract. We present an algorithm for computing L_1 shortest paths among polygonal obstacles in the plane. Our algorithm employs the "continuous Dijkstra" technique of propagating a "wavefront" and runs in time $O(E \log n)$ and space $O(E)$, where n is the number of vertices of the obstacles and E is the number of "events." By using bounds on the density of certain sparse binary matrices, we show that $E = O(n \log n)$, implying that our algorithm is nearly optimal. We conjecture that $E = O(n)$, which would imply our algorithm to be optimal. Previous bounds for our problem were quadratic in time and space.

Our algorithm generalizes to the case of fixed orientation metrics, yielding an $O(n\varepsilon^{-1/2} \log^2 n)$ time and $O(n\varepsilon^{-1/2})$ space approximation algorithm for finding Euclidean shortest paths among obstacles. The algorithm further generalizes to the case of many sources, allowing us to compute an L_1 Voronoi diagram for source points that lie among a collection of polygonal obstacles.

Key Words. Shortest paths, Voronoi diagrams, Rectilinear paths, Wire routing, Fixed orientation metrics, Continuous Dijkstra algorithm, Computational geometry, Extremal graph theory.

1. Introduction. Recently, there has been much interest in the problem of finding shortest *Euclidean* (L_2) paths for a point moving in a plane cluttered with polygonal obstacles. This problem can be solved by constructing the *visibility graph* of the set of obstacles and then searching this graph using Dijkstra's algorithm [Di] or an A^* algorithm [Ni]. The bottleneck in the computation is in the construction of the visibility graph. It has been known for some time that the visibility graph can be constructed in time $O(n^2 \log n)$ [Le], [LW], [Mi1], [SS], where n is the number of vertices describing the set of obstacles. More recently, it has been shown that the visibility graph can be constructed in worst-case optimal time $O(n^2)$ [AAGHI], [We], or in output-sensitive time $O(e + n \log n)$ [GM], [KM], where $e = O(n^2)$ is the number of edges in the visibility graph. In some special cases in which the shortest path is known to possess certain monotonicity properties, algorithms that require $O(n \log n)$ time to compute shortest paths are known [LP], [Mi1], [SS]. While other algorithms for the general case are known whose running time can be written in other, possibly more favorable, terms [Mi3], [RS], it remains

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[2] Department of Applied Mathematics and Statistics, State University of New York at Stony Brook, Stony Brook, NY 11794-3600, USA. Email: jsbm@ams.sunysb.edu.

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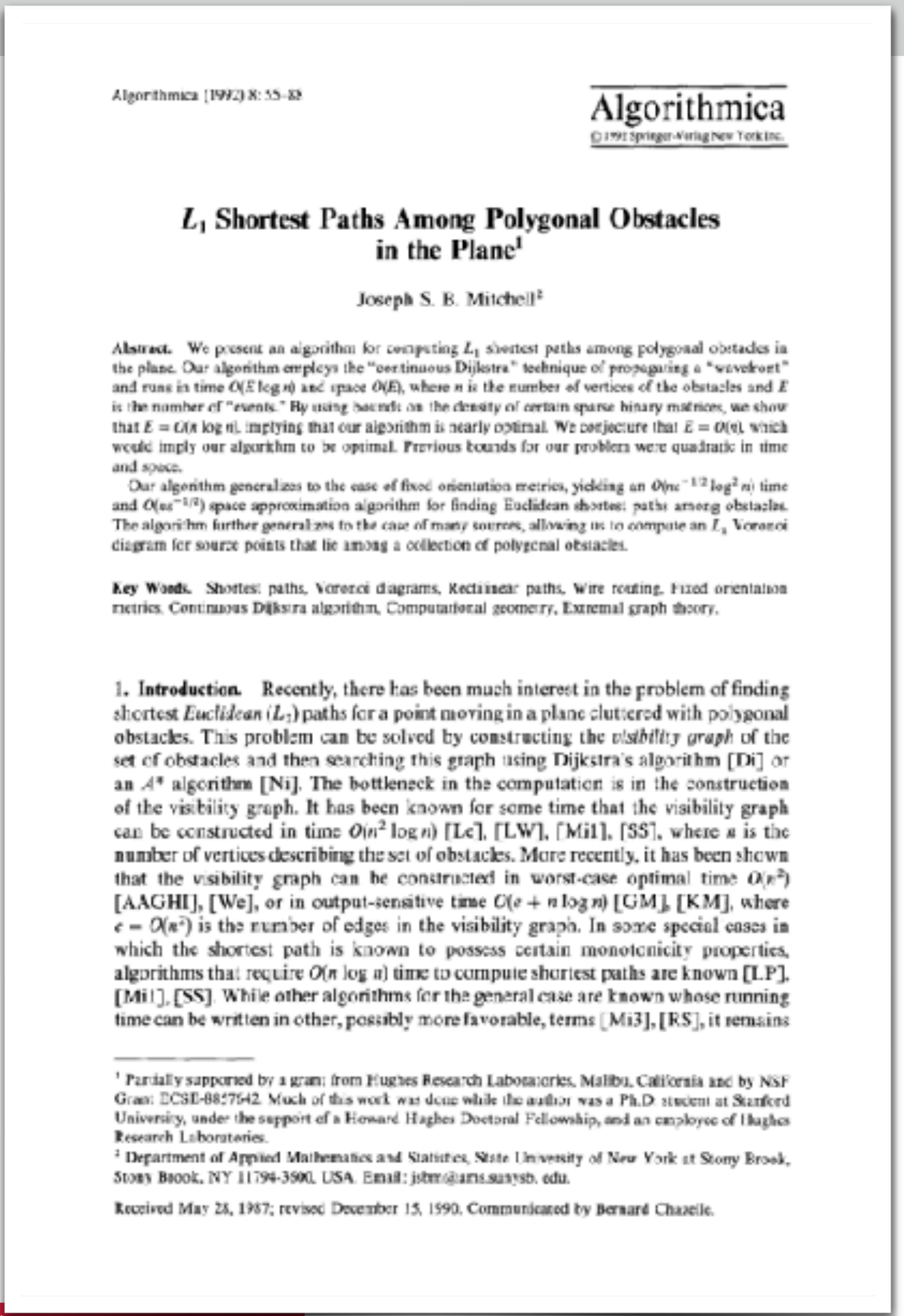

Algorithmica (1992) 8: 55–88

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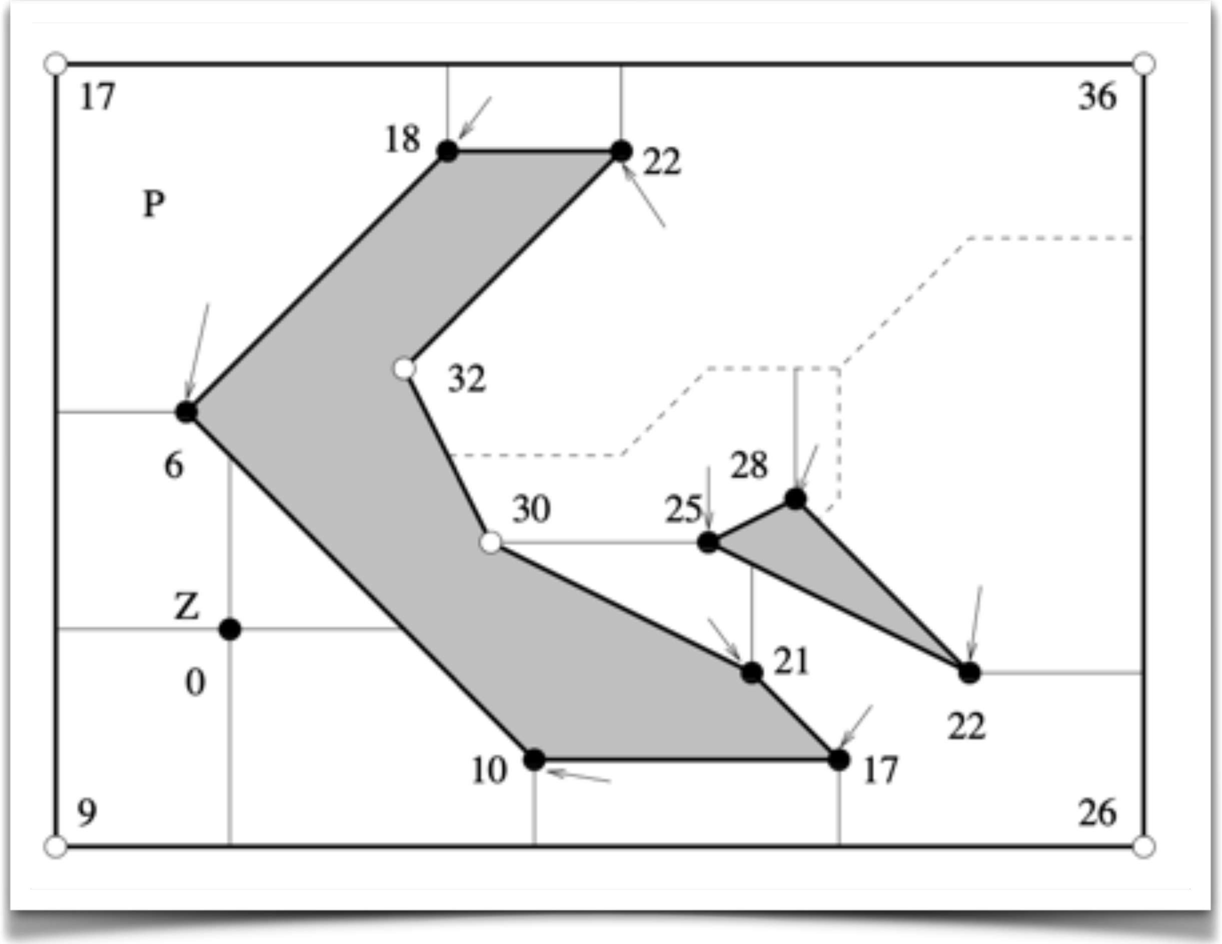

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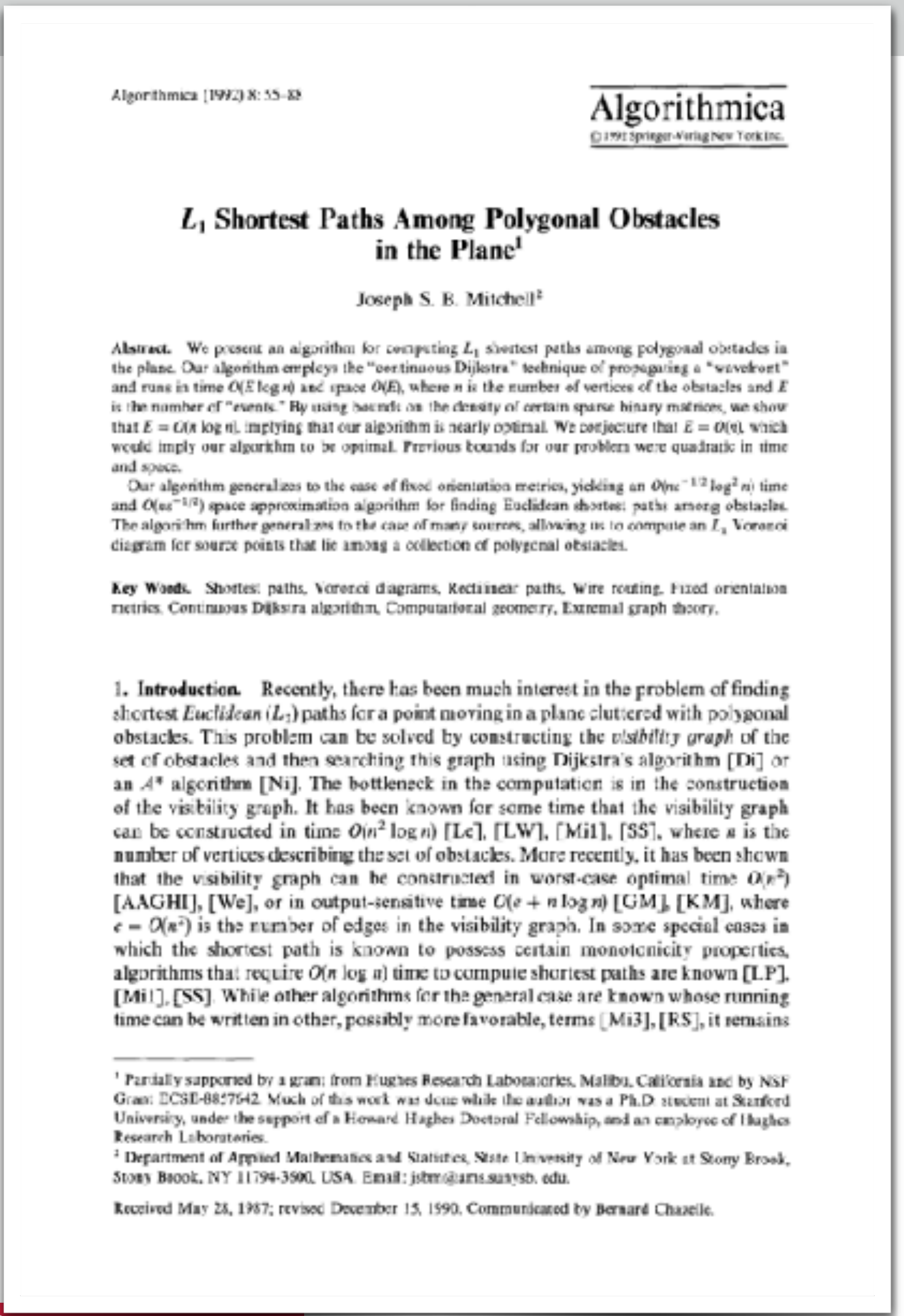

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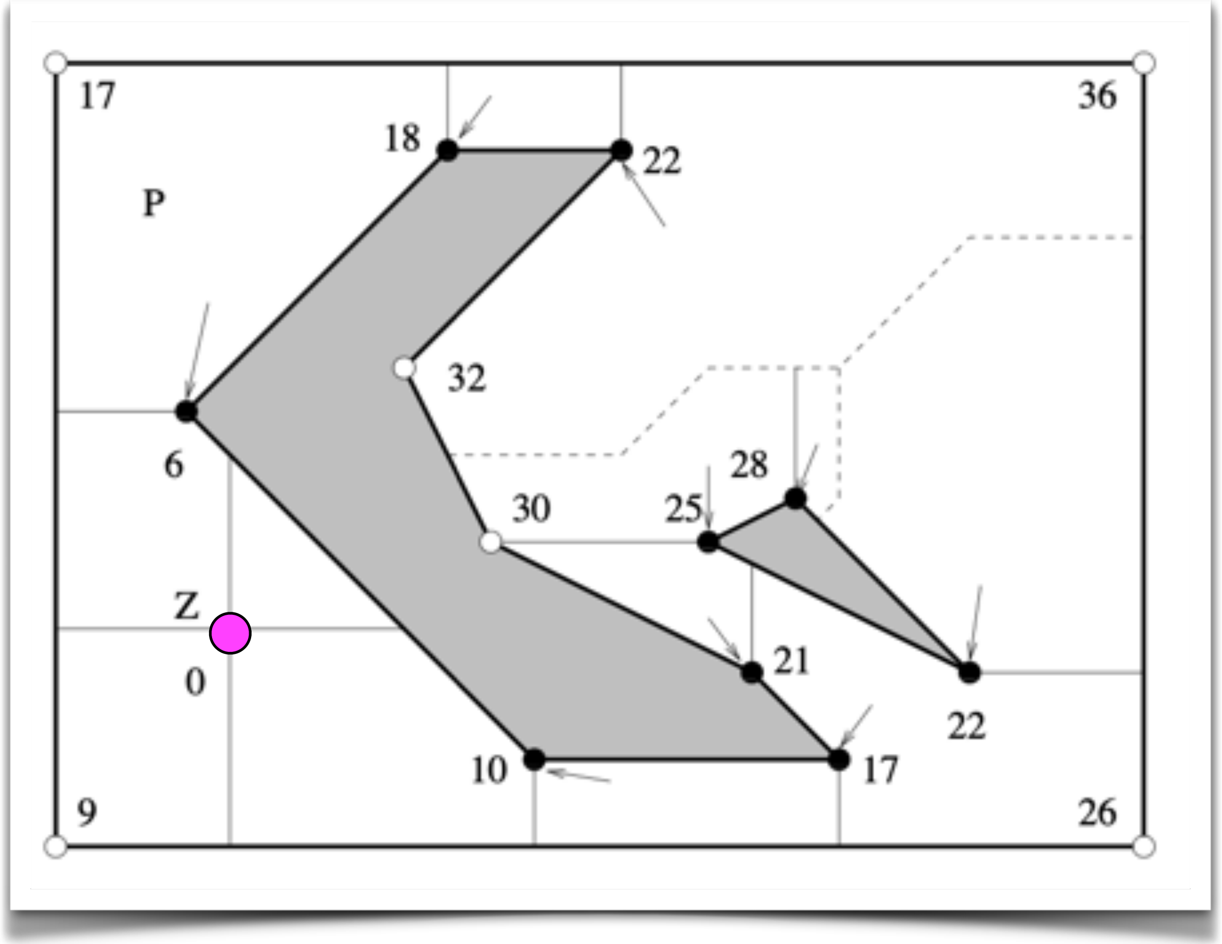

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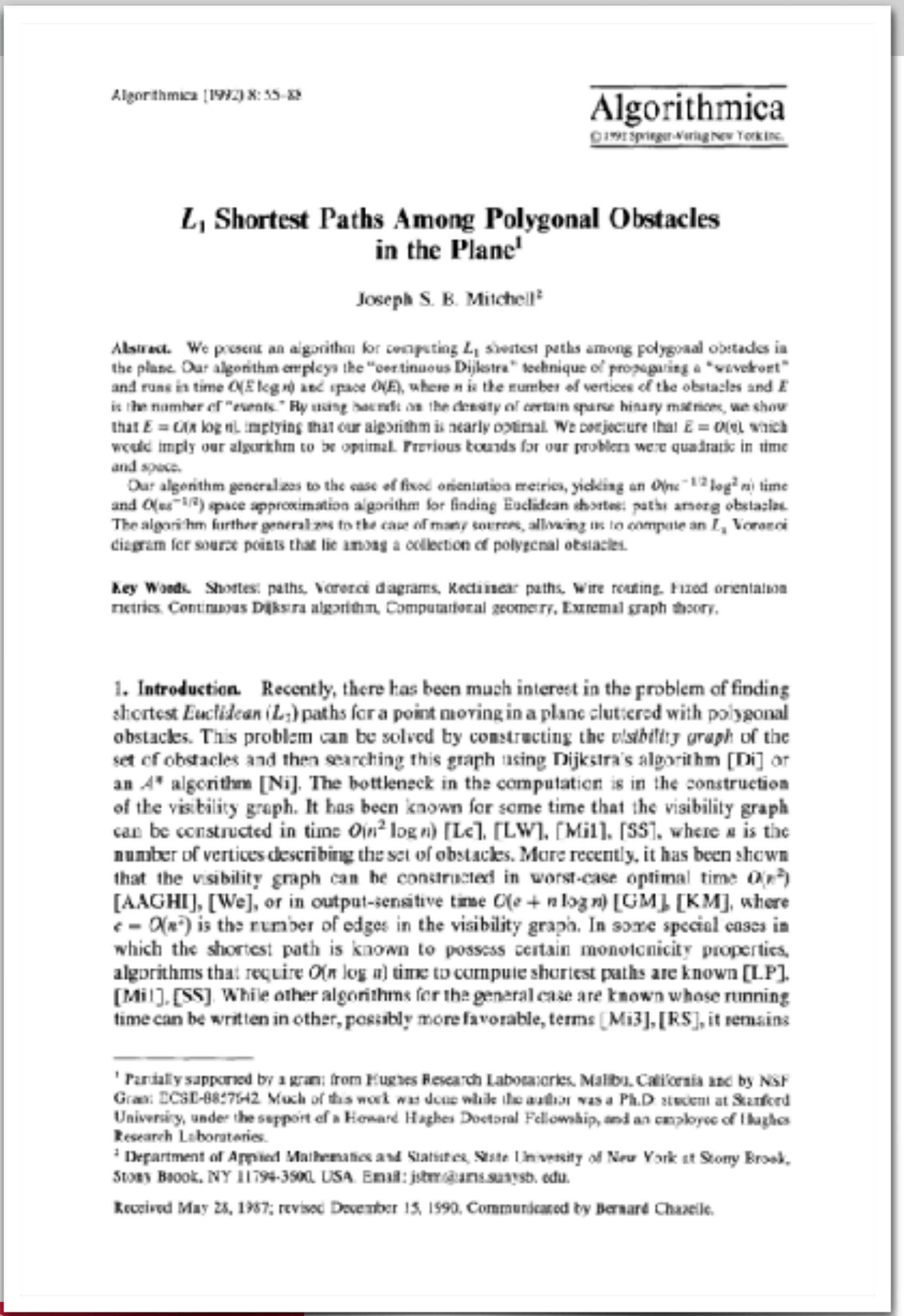

Algorithmica (1992) 8: 55–88

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Joseph S. B. Mitchell[2]

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Key Words. Shortest paths, Voronoi diagrams, Rectilinear paths, Wire routing, Fixed orientation metrics, Continuous Dijkstra algorithm, Computational geometry, Extremal graph theory.

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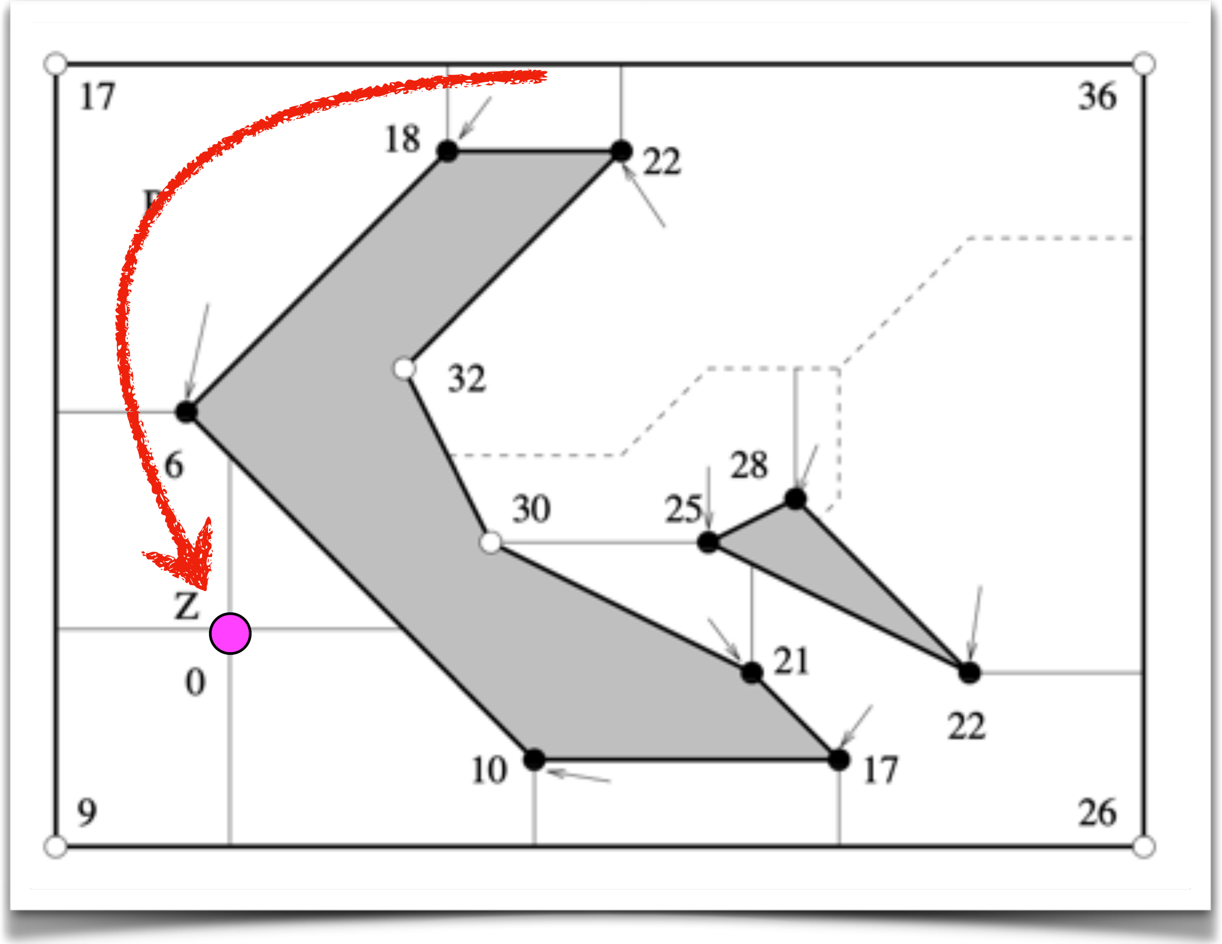

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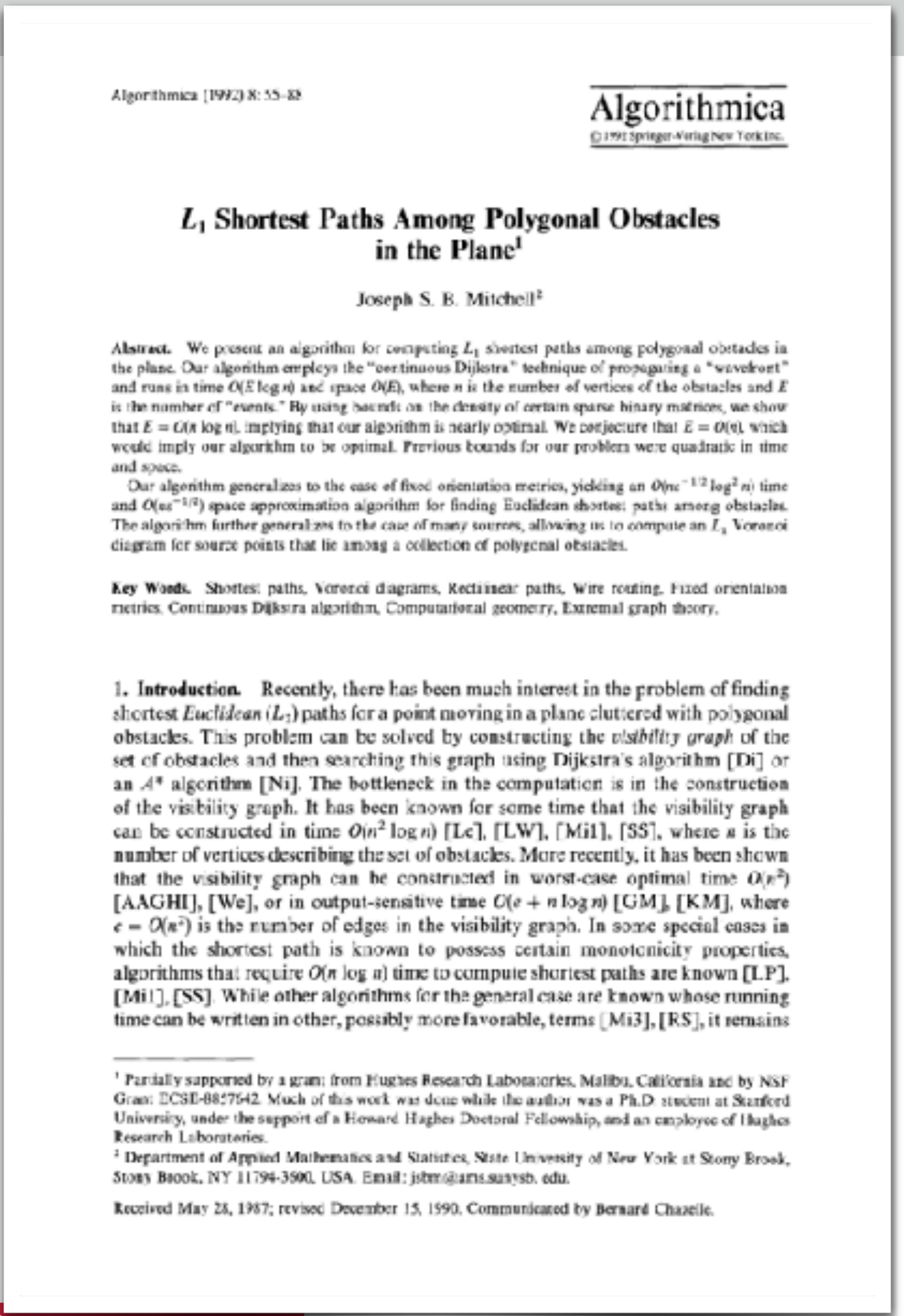

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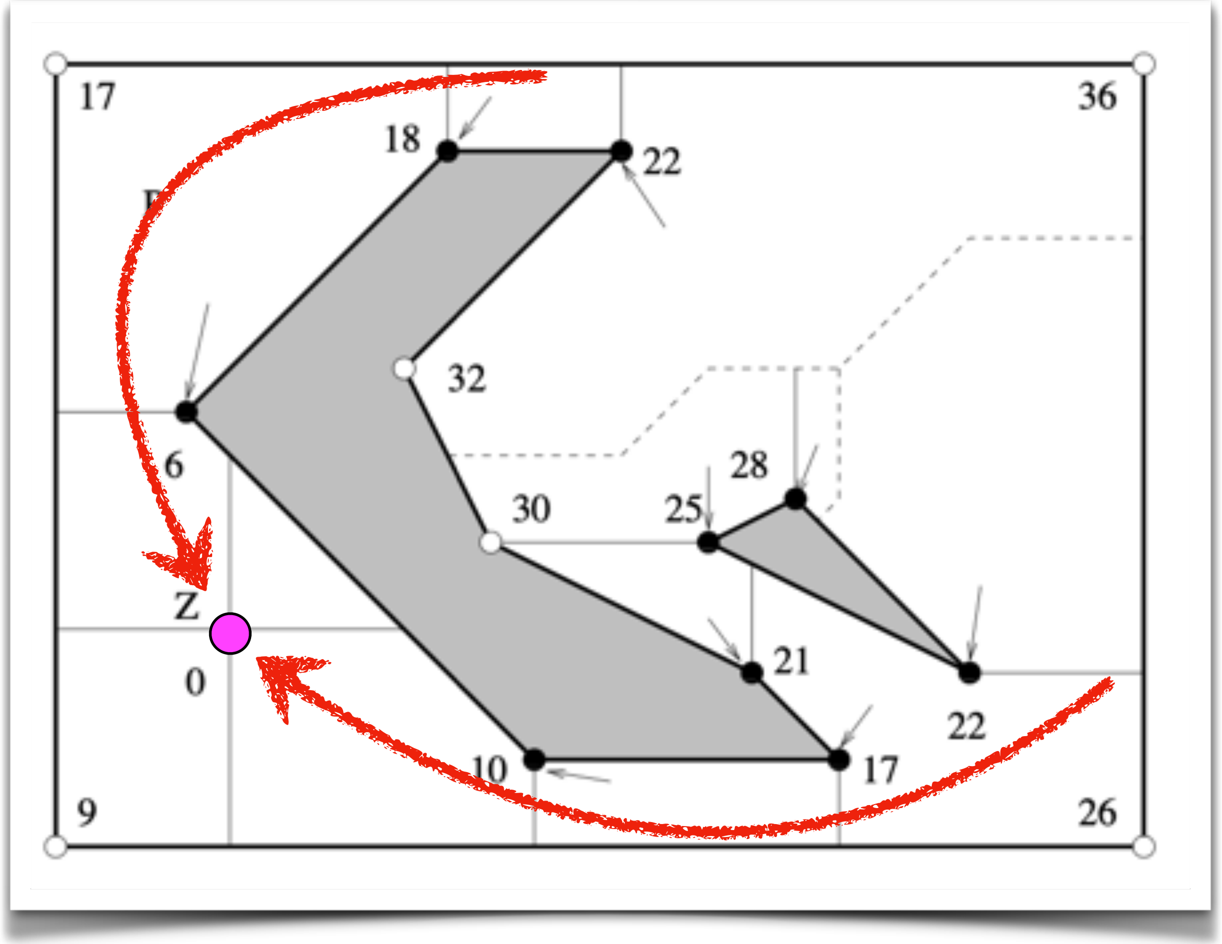

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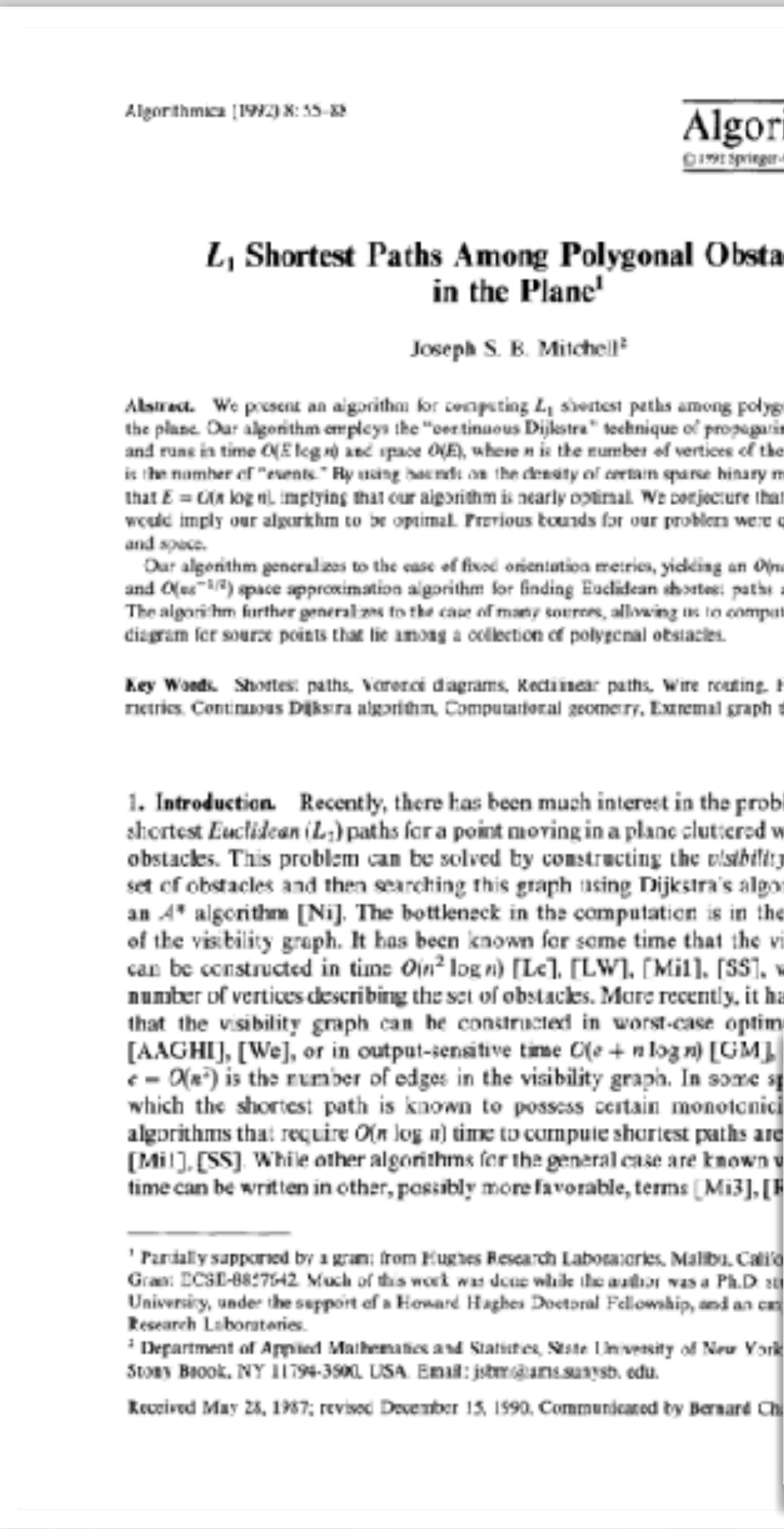

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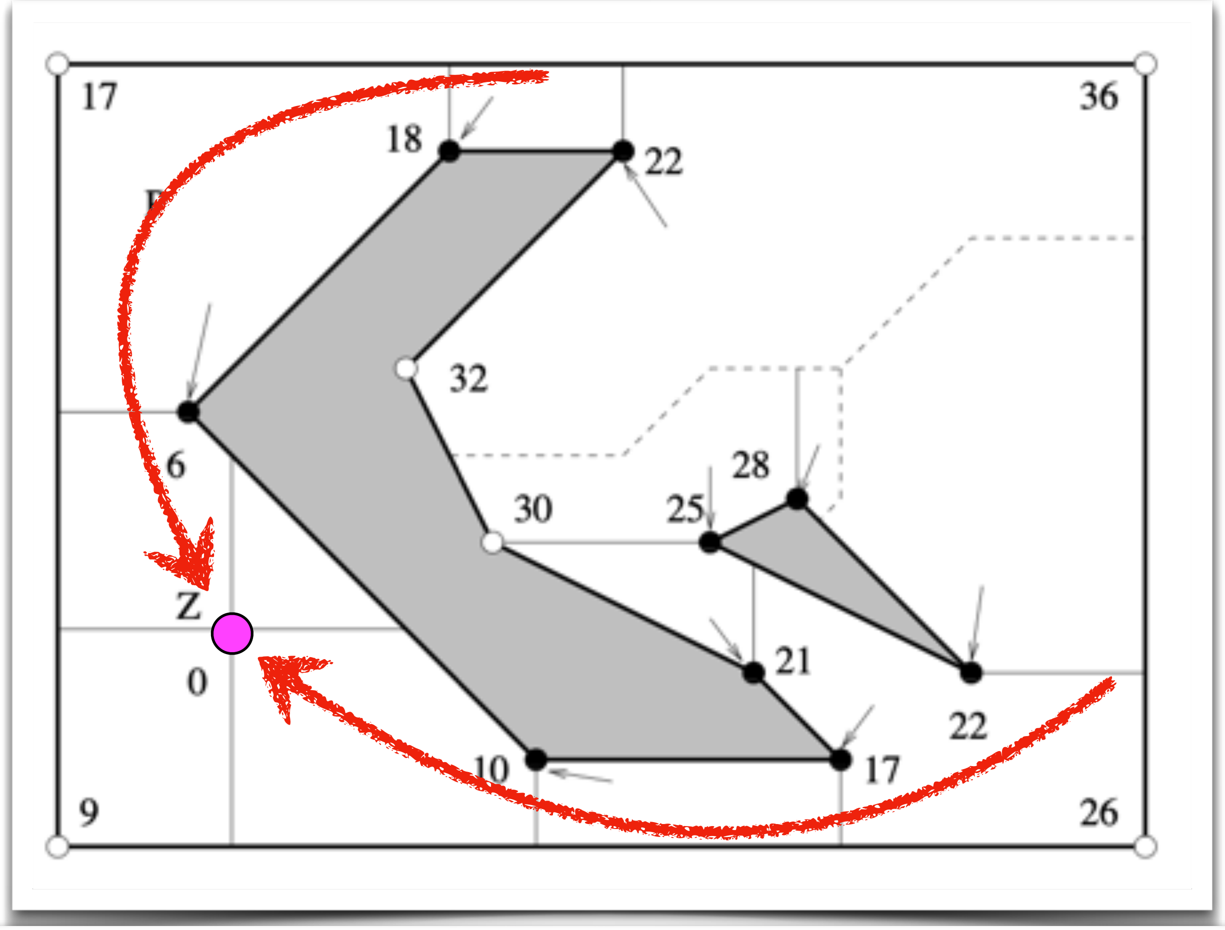

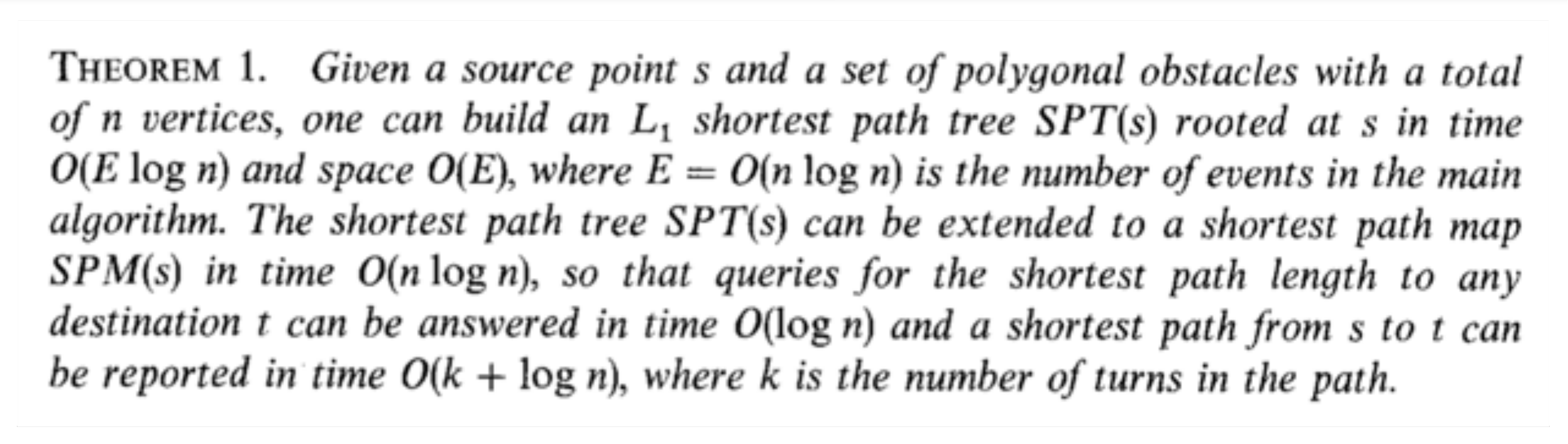

THEOREM 1. *Given a source point s and a set of polygonal obstacles with a total of n vertices, one can build an L_1 shortest path tree $SPT(s)$ rooted at s in time $O(E \log n)$ and space $O(E)$, where $E = O(n \log n)$ is the number of events in the main algorithm. The shortest path tree $SPT(s)$ can be extended to a shortest path map $SPM(s)$ in time $O(n \log n)$, so that queries for the shortest path length to any destination t can be answered in time $O(\log n)$ and a shortest path from s to t can be reported in time $O(k + \log n)$, where k is the number of turns in the path.*

Geodesic distances in polygons with holes

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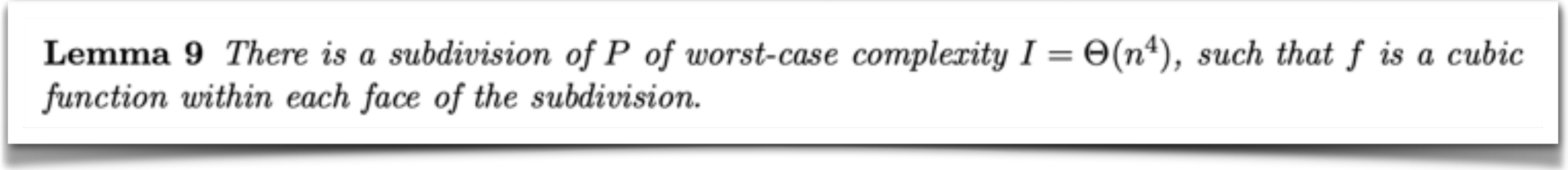

Lemma 9 *There is a subdivision of P of worst-case complexity $I = \Theta(n^4)$, such that f is a cubic function within each face of the subdivision.*

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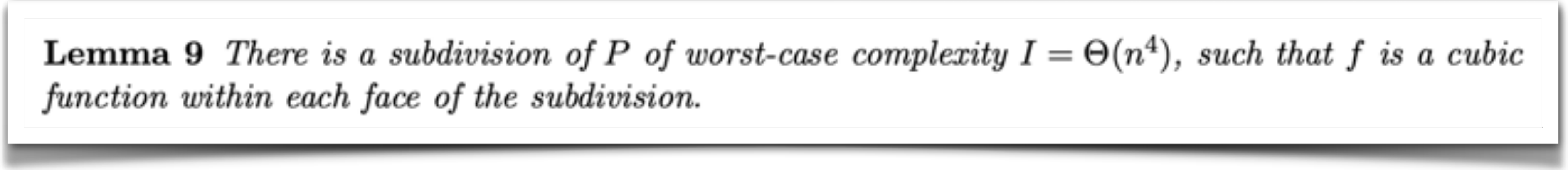

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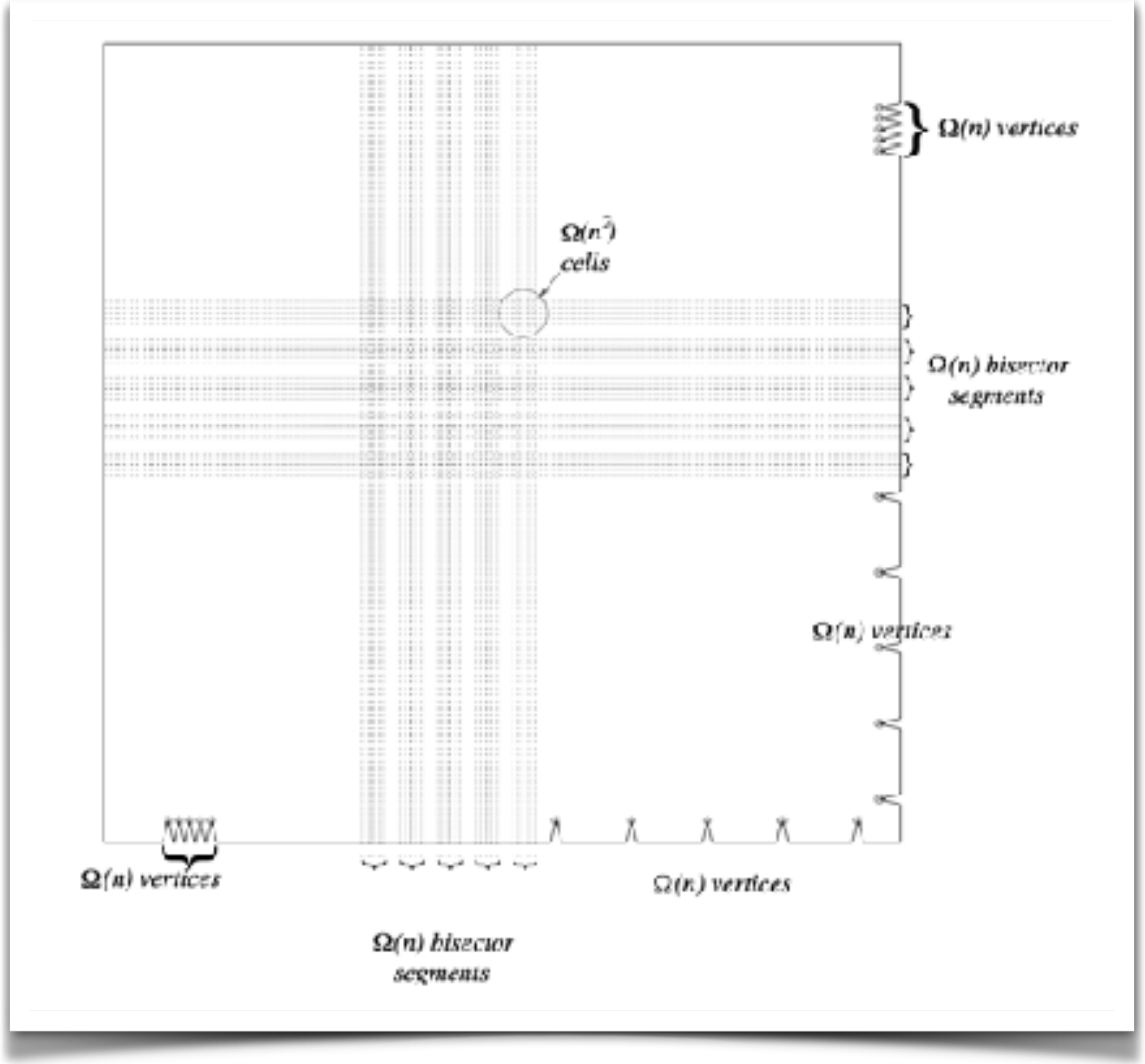

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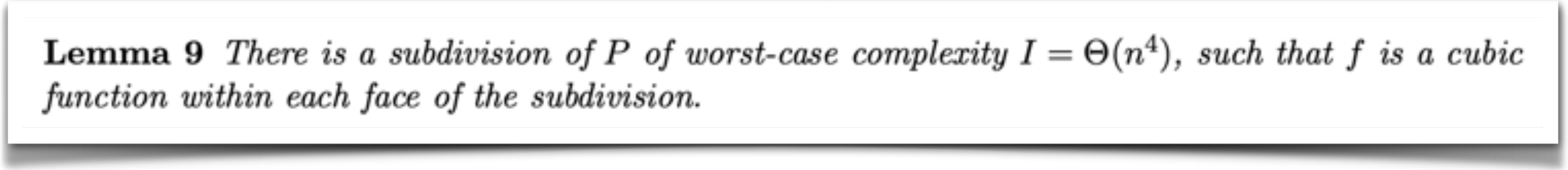
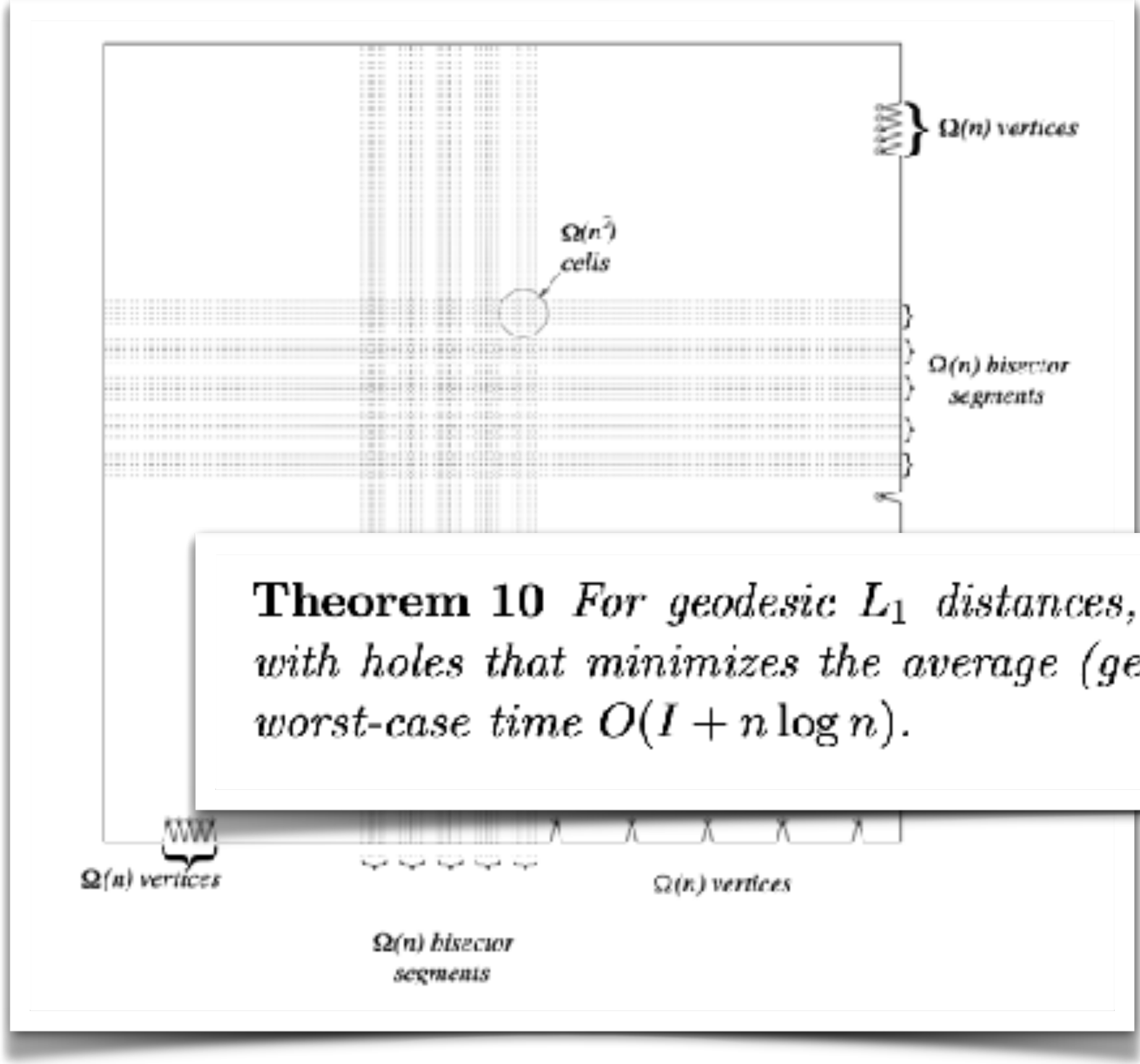


Lemma 9 *There is a subdivision of* P *of worst-case complexity* $I = \Theta(n^4)$*, such that* f *is a cubic function within each face of the subdivision.*



Theorem 10 *For geodesic* L_1 *distances, a feasible point* $Z^* = (x^*, y^*)$ *in a polygonal region* P *with holes that minimizes the average (geodesic* L_1*) distance* f *to all points in* P *can be found in worst-case time* $O(I + n \log n)$.

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Multiple centers

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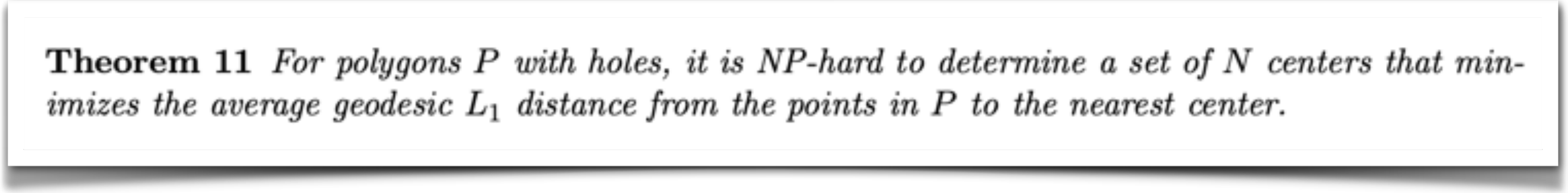
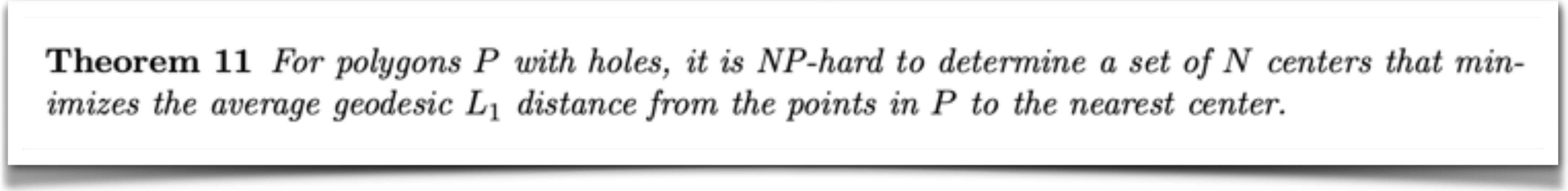
Multiple centers



Theorem 11 *For polygons P with holes, it is NP-hard to determine a set of N centers that minimizes the average geodesic L_1 distance from the points in P to the nearest center.*

Multiple centers



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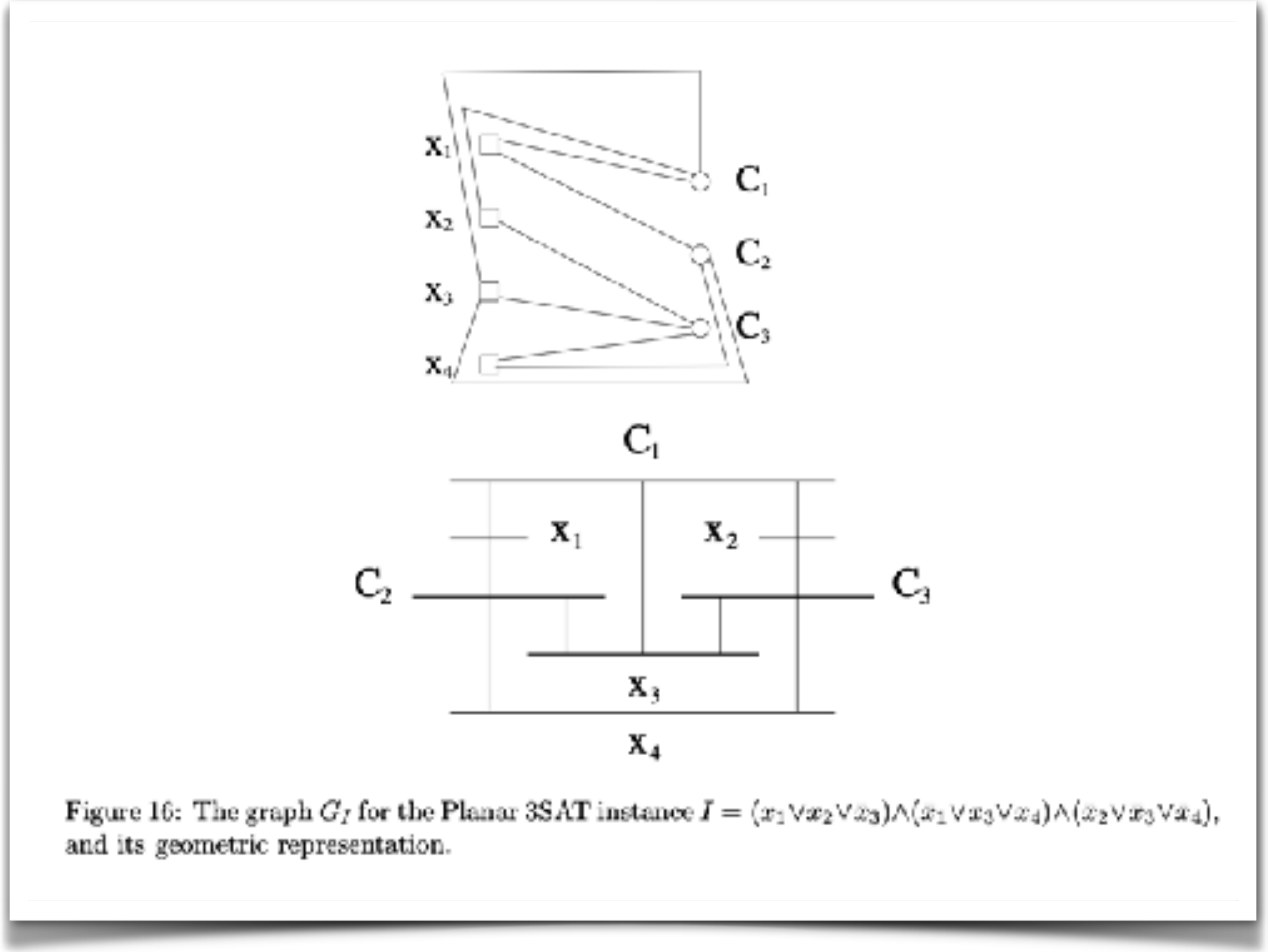

Figure 16: The graph G_I for the Planar 3SAT instance $I = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee x_3 \vee x_4)$, and its geometric representation.

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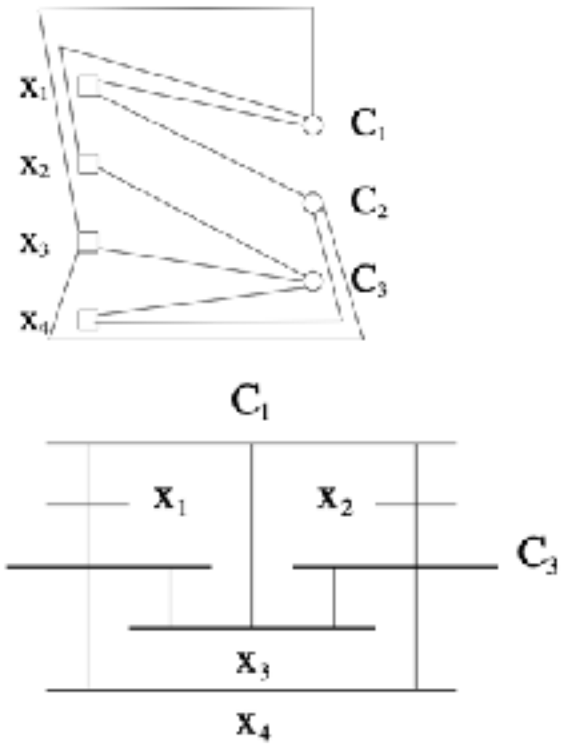

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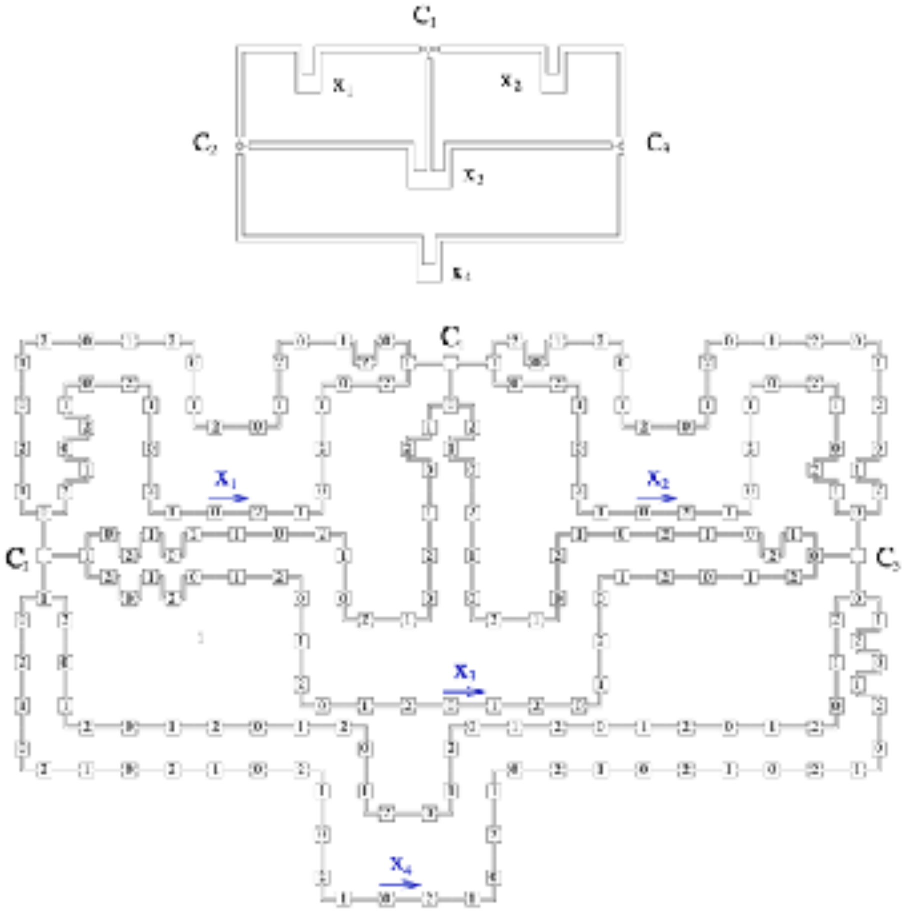

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Thank you for today!