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Due: 01.02.2023
Discussion: 01.02.2023

Homework 5

If you are interested in presenting your solution, please come to the tutorial on the discussion date. Please note that there may be other students who also want to present their solution; we will decide randomly between students, unless there is enough time in the tutorial session and the solutions are sufficiently different. If you feel like you have a good reason why you cannot attend the tutorial at all, please contact us via e-mail and we will find a way to deal with this situation so that you can still get your Studienleistung.

Exercise 1 (LU-Factorization): $1 + 1 + 2$ ✓

a) Without permuting rows or columns, find an LU-factorization of

$$B = \begin{pmatrix} 2 & 5 & & 6 & \\ 1 & 1 & 3 & 9 & 6 \\ & & 2 & 6 & 4 \\ & & & 4 & 1 \\ & & -1 & -3 & -1 \end{pmatrix}$$

using the method discussed in the lecture. Remember that you can verify your result by matrix multiplication.

b) Using the result from a), solve $B\Delta x_{\mathcal{B}} = a_j$ for $a_j = (0 \;\; 2 \;\; 1 \;\; 3 \;\; 0)^T$.

c) Let $\tilde{B}$ be B, but with its second column replaced by a_j from subtask b). As discussed in the lecture, express this replacement as matrix multiplication by a special matrix E, $\tilde{B} = BE$. Derive E^{-1} using the formula described in the lecture, and use it, together with the LU-factorization from a), to solve $\tilde{B}x = \alpha_j$, where $\alpha_j = (1 \;\; 0 \;\; -1 \;\; 0 \;\; 0)^T$.

Exercise 2 (Minimum Degree Heuristic): 1 ✓

The minimum degree heuristic is a means to reduce fill-in during LU-factorization using Gaussian elimination. It requires keeping track of an independent re-ordering of columns and rows, in our context corresponding to a reordering of variables and constraints, which can be realized by noting, for each row and column, its original index. Eliminated rows and columns are never moved around; on row swaps the kept entries for eliminated non-zeros are moved along with the rest of the row.

Before each eliminated column in the Gaussian elimination scheme, the minimum degree heuristic first swaps the current row with the uneliminated row that has the fewest uneliminated elements. It then swaps the current column with a column that has a non-zero in the (new) current row, choosing a non-zero with minimal number of uneliminated non-zeros in its column.

Compute a LU-factorization of B from Exercise 1a) using this method.

Exercise 3 (Warm Starting): $1+1$ ✓
Consider the problem

$$\max x_1 + 2x_2 + x_3 + x_4 \text{ s.t.}$$
$$2x_1 + x_2 + 5x_3 + x_4 \leq 8$$
$$2x_1 + 2x_2 + 4x_4 \leq 12$$
$$3x_1 + x_2 + 2x_3 \leq 18$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Let's suppose we solved this, arriving at the following optimal dictionary with slacks x_5, x_6, x_7.

$$\begin{array}{rrrrrr} \zeta = & 12.4 - & 1.2x_1 - & 0.2x_5 - & 0.9x_6 - & 2.8x_4 \\ \hline x_2 = & 6 - & 1x_1 & - & 0.5x_6 - & 2x_4 \\ x_3 = & 0.4 - & 0.2x_1 - & 0.2x_5 + & 0.1x_6 + & 0.2x_4 \\ x_7 = & 11.2 - & 1.6x_1 + & 0.4x_5 + & 0.3x_6 + & 1.6x_4 \end{array}$$

a) Re-solve the problem, starting from the previously optimal basis of the dictionary, with the objective updated to $\max 3x_1 + 2x_2 + x_3 + x_4$, and the suitable simplex variant (primal or dual).

b) Re-solve the problem, starting from the previously optimal basis of the dictionary, with the right-hand side of the second constraint updated to 26 (instead of 12).

Exercise 4 (Ranging): $1+2$ ✓
Consider again the problem and the optimal basis from Exercise 3).

a) For each coefficient of the objective function, find the range of values of the coefficient for which the basis of the given dictionary is optimal.

b) For each right-hand side of the constraints, find the range of values of the right-hand side for which the basis of the given dictionary is optimal.