Mathematical Methods of Algorithms **Winter 2022/23**

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Due: 23.11.2021
Discussion: 23.11.2021

Homework 1

If you are interested in presenting your solution, please come to the tutorial on the discussion date. Please note that there may be other students who also want to present their solution; we will decide randomly between students, unless there is enough time in the tutorial session and the solutions are sufficiently different. If you feel like you have a good reason why you cannot attend the tutorial at all, please contact us via e-mail and we will find a way to deal with this situation so that you can still get your Studienleistung.

Exercise 1 (Two-Phase Simplex): 1 ✓
Solve the following linear program using the two-phase simplex method.

$$\begin{array}{llll} \max & x_1 & + x_2 - x_3 & \\ & -x_1 & & \leq -3 \\ & & -x_2 & \leq -10 \\ & 2x_1 & +3x_2 - 3x_3 & \leq 5 \end{array}$$

$$x_1, x_2, x_3 \geq 0$$

Does it have an optimal solution, or is it infeasible or unbounded?

Exercise 2 (Entering Variables): 1 ✓
Give an example linear program in which a variable that becomes a basic variable in a pivot step immediately leaves the basis in the next pivot step of the simplex algorithm. You may pick an arbitrary method to choose which variable with positive coefficient in the objective function becomes basic.

Exercise 3 (Leaving Variables): 1 ✓
Show that a variable that becomes non-basic in a pivot step cannot become basic again in the next pivot step.

Exercise 4 (Solving an Abstract Linear Program): 2 ✓
Consider the following linear program.

$$\max \sum_{j=1}^{n} p_j x_j$$

$$\text{s.t. } \sum_{j=1}^{n} q_j x_j \leq \beta$$

$$0 \leq x_j \leq 1, j \in \{1, \dots, n\}.$$

In this program, let the coefficients p_j and q_j all be strictly positive. Furthermore, assume that they satisfy

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n},$$

and let $\beta > 0$.

Let $t \geq 1$ be the smallest index with

$$\sum_{j=t}^{n} q_j \leq \beta,$$

and assume that such an index exists. Prove that the optimal solution is

$$x_j = 1, \text{ for } t \leq j \leq n,$$

$$x_{t-1} = \frac{\beta - \sum\limits_{j=t}^{n} q_j}{q_{t-1}}, \text{ and}$$

$$x_j = 0, \text{ for } 1 \leq j < t-1.$$