Linear Programming

[V. ch8]: Problems in General Form

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Recap: Dual Simplex on Dictionaries

Problems in General Form

In the last exercise, questions regarding the entering variable selection for dual simplex steps (on the primal dictionary) arose.

In primal simplex, we need a primally feasible dictionary, and we can choose an entering variable among those that are not dually feasible.

In dual simplex, we can choose a leaving variable among those that are not primally feasible. The question was how to select the corresponding entering variable, only looking at the primal dictionary. For this, recall the negative transpose property and let's look at an example.

$$\begin{array}{rrrr} \zeta = & 0 - & 1x_1 - & 1x_2 \\ \hline w_1 = & 4 + & 2x_1 + & x_2 \\ w_2 = & -8 + & 2x_1 - & 4x_2 \\ w_3 = & -7 + & 1x_1 - & 3x_2 \end{array} \qquad \begin{array}{rrrr} -\xi = & 0 - & 4y_1 - & (-8)\,y_2 - & (-7)\,y_3 \\ \hline z_1 = & 1 - & 2y_1 - & 2y_2 - & 1y_3 \\ z_2 = & 1 - & 1y_1 + & 4y_2 + & 3y_3 \end{array}$$

With this in mind, we can derive an analog rule for dual simplex. In a dually feasible dictionary, the objective has the form $\zeta = \zeta^* - (z_N^*)^T x_N$, with $z_N^* \geq 0$.
Rule: Scan the positive entries a_{ij} of the leaving row i for the one minimizing z_j^*/a_{ij}.

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Very often, in practice, we have problems as follows (allowing for infinite bounds).

$$\max c^T x \text{ s.t.}$$

$$a \leq Ax \leq b$$

$$\ell \leq x \leq u$$

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Can we extend Simplex to handle such problems directly?

Example

$$
\begin{array}{lrcrcrcr}
\text{maximize} & & & 3x_1 & - & x_2 & & \\
\text{subject to} & 1 & \leq & -x_1 & + & x_2 & \leq & 5 \\
& 2 & \leq & -3x_1 & + & 2x_2 & \leq & 10 \\
& -\infty & \leq & 2x_1 & - & x_2 & \leq & 0 \\
& -2 & \leq & x_1 & & & \leq & \infty \\
& 0 & \leq & & & x_2 & \leq & 6
\end{array}
$$

Some notes:

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Some notes:

- This is often the model professional LP-solvers handle; some parts of their interfaces refer to this type of model.
- We do not have a general $x \geq 0$ constraint; 0 is no longer special. Instead of a fixed lower bound of zero, we have different lower and upper bounds.
- In the general case, we will have infinities as some lower or upper bounds. We let $\infty \cdot x = \infty$ for $x > 0$, $\infty \cdot x = 0$ for $x = 0$ and $\infty \cdot x = -\infty$ for $x < 0$.

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The slack w_1 for $1 \leq -x_1 + x_2 \leq 5$ is now simply $w_1 = -x_1 + x_2$ with bounds $1 \leq w_1 \leq 5$.

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Result has only equality constraints and variables with upper and lower bounds.

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\begin{array}{lrrrrr}
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\text{subject to} & w_1 = & -x_1 + & x_2 & & \\
 & w_2 = & -3x_1 + & 2x_2 & & \\
 & w_3 = & 2x_1 - & x_2 & & \\
 & -2 \leq & x_1 & & \leq & \infty \\
 & 0 \leq & & x_2 \leq & & 6 \\
 & 1 \leq & w_1 & & \leq & 5 \\
 & -\infty \leq & w_2 & & \leq & 10 \\
 & 0 \leq & w_3 & & \leq & 0
\end{array}
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General Dictionaries

General dictionaries look different because we need more information. The overall idea is still to describe basic variables and the objective in terms of non-basic ones.
Can we still simply set non-basic variables to zero to obtain a basic (dictionary) solution?

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	u			∞		6	
		ζ	$=$	$3x_1$	$-$	x_2	$=-6$
1	5	w_1	$=$	$-x_1$	$+$	x_2	$=2$
2	10	w_2	$=$	$-3x_1$	$+$	$2x_2$	$=6$
$-\infty$	0	w_3	$=$	$2x_1$	$-$	x_2	$=-4$

Is this dictionary feasible?

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How far can we increase x_1? $w_1 \geq 1 \Rightarrow x_1 \leq -1$, $w_2 \geq 2 \Rightarrow x_1 \leq -\frac{2}{3}$, $w_3 \leq 0 \Rightarrow x_1 \leq 0$.
When x_1 is increased to -1, w_1 hits its lower bound (becomes non-basic); w_1 is leaving variable!

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Is this now optimal? No! We could increase x_2 to improve ζ! How far?
x_1: no limit, $w_2 \geq 2 \Rightarrow x_2 \leq 1$, $w_3 \leq 0 \Rightarrow x_2 \leq 2$; w_2 is leaving variable!

Another General Pivot

We say that w_2 becomes *non-basic at its lower bound*.
Result of pivoting out w_2 in favor of x_2:

ℓ	u						
				1^*		2^*	
				5		10	
		ζ	$=$	$3w_1$	$-$	$2w_2$	$=-1$
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Is this now optimal? No! We can increase w_1 from its lower bound! How far?
The increase is limited to at most 1 unit due to w_3 hitting its upper bound. w_3 becomes non-basic at its upper bound.

Next General Pivot

Result of pivoting out w_3 in favor of w_1:

ℓ	u						
				$-\infty$		2^*	
				0^*		10	
		ζ	$=$	$3w_3$	$+$	w_2	$= 2$
-2	∞	x_1	$=$	$2w_3$	$+$	w_2	$= 2$
0	6	x_2	$=$	$3w_3$	$+$	$2w_2$	$= 4$
1	5	w_1	$=$	w_3	$+$	w_2	$= 2$

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Is this now optimal?

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Is this now optimal? No! We cannot increase w_3, but we can increase w_2 from its lower bound!

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Result of pivoting out w_3 in favor of w_1:

ℓ				$-\infty$		2^*	
	u			0^*		10	
		ζ	$=$	$3w_3$	$+$	w_2	$= 2$
-2	∞	x_1	$=$	$2w_3$	$+$	w_2	$= 2$
0	6	x_2	$=$	$3w_3$	$+$	$2w_2$	$= 4$
1	5	w_1	$=$	w_3	$+$	w_2	$= 2$

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Result of pivoting out x_2 in favor of w_2:

ℓ				$-\infty$		0	
	u			0^*		6^*	
		ζ	$=$	$1.5w_3$	$+$	$0.5x_2$	$= 3$
-2	∞	x_1	$=$	$0.5w_3$	$+$	$0.5x_2$	$= 3$
0	6	w_2	$=$	$-1.5w_3$	$+$	$0.5x_2$	$= 3$
1	5	w_1	$=$	$-0.5w_3$	$+$	$0.5x_2$	$= 3$

Next General Pivot

Result of pivoting out w_3 in favor of w_1:

ℓ	u						
				$-\infty$		2^*	
				0^*		10	
		ζ	$=$	$3w_3$	$+$	w_2	$= 2$
-2	∞	x_1	$=$	$2w_3$	$+$	w_2	$= 2$
0	6	x_2	$=$	$3w_3$	$+$	$2w_2$	$= 4$
1	5	w_1	$=$	w_3	$+$	w_2	$= 2$

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Result of pivoting out x_2 in favor of w_2:

ℓ	u						
				$-\infty$		0	
				0^*		6^*	
		ζ	$=$	$1.5w_3$	$+$	$0.5x_2$	$= 3$
-2	∞	x_1	$=$	$0.5w_3$	$+$	$0.5x_2$	$= 3$
0	6	w_2	$=$	$-1.5w_3$	$+$	$0.5x_2$	$= 3$
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Is this now optimal?

Next General Pivot

Result of pivoting out w_3 in favor of w_1:

ℓ	u			$-\infty$ / 0^*		2^* / 10	
		ζ	$=$	$3w_3$	$+$	w_2	$= 2$
-2	∞	x_1	$=$	$2w_3$	$+$	w_2	$= 2$
0	6	x_2	$=$	$3w_3$	$+$	$2w_2$	$= 4$
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1	5	w_1	$=$	$-0.5w_3$	$+$	$0.5x_2$	$= 3$

Is this now optimal? Yes! Objective coefficients positive, both variables at their upper bound!

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The algorithm outlined on the example straightforwardly generalizes into primal Simplex for problems in general form.

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As stated before, from an interface standpoint, most professional solvers implement this type of interface, where any linear expression can be given a lower and upper bound simultaneously without needing two matrix rows.
If one can query which variables are basic, one will notice that basic variables need not be 0, but can be at one of their bounds.

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Furthermore, a basis usually consists of a mixture of variables and constraints (we now have a more direct correspondence between constraints and their “slack” variables).

What about Phase I/Dual Simplex?

We will present both at the same time (with a modified objective, dual feasibility is easy to obtain and we can use dual Simplex to find a feasible solution).
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Complementarity: $f_i v_i = 0, p_i q_i = 0, t_j s_j = 0, g_j h_j = 0$ at optimality.

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W.l.o.g. also complementary: $v_i q_i = 0, s_j h_j = 0$!
Note: Very similar to making a free variable from two non-negative ones, but with different objective coefficients!

Preliminaries for Dual Simplex

$$\text{minimize } b^T v - a^T q + u^T s - \ell^T h \text{ subject to}$$
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For some real variable ξ, let $\xi^+ = \max\{\xi, 0\}, \xi^- = \max\{-\xi, 0\}$.
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This is no longer linear, only (a special type of) piecewise linear!

PRELIMINARIES FOR DUAL SIMPLEX

$$\text{minimize } b^T v - a^T q + u^T s - \ell^T h \text{ subject to}$$
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Rewrite using complementarity $v = y^+, q = y^-, h = z^+, s = z^-$:

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This is no longer linear, only (a special type of) piecewise linear!
Our Dual Simplex for general problems will solve this type of problem.

General Dual Simplex Example

$$
\begin{array}{lrrrrr}
\text{maximize} & & 2x_1 - & x_2 & & \\
\text{subject to} & 0 \leq & x_1 + & x_2 \leq & 6 \\
& 2 \leq & -x_1 + & 2x_2 \leq & 10 \\
& -\infty \leq & x_1 - & x_2 \leq & 0 \\
& -2 \leq & x_1 & \leq & \infty \\
& 1 \leq & & x_2 \leq & 5
\end{array}
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The dual is

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The dual is

$$\text{minimize } \xi = 6y_1^+ + 10y_2^+ + 2z_1^+ - z_2^+ - 2y_2^- + \infty y_3^- + \infty z_1^- + 5z_2^- \text{ s.t.}$$

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$$
y_1 - y_2 + y_3 - z_1 = 2
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Note: Infinities in the objective! We use our conventions. $-\infty$ indicates infeasibility!

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\begin{gathered}
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y_1 - y_2 + y_3 - z_1 = 2 \\
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\end{gathered}
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Note: Infinities in the objective! We use our conventions. $-\infty$ indicates infeasibility!
Also, we cannot use row operations on the objective. But we can use them on the constraints!

$$-\xi = -6y_1^+ - 10y_2^+ - 2z_1^+ + z_2^+ + 2y_2^- - \infty y_3^- - \infty z_1^- - 5z_2^-$$

We have the following dictionary (with no objective):

$$\begin{aligned} z_1 &= -2 + y_1 - y_2 + y_3 \\ z_2 &= 1 + y_1 + 2y_2 - y_3 \end{aligned}$$

$$-\xi = -6y_1^+ - 10y_2^+ - 2z_1^+ + z_2^+ + 2y_2^- - \infty y_3^- - \infty z_1^- - 5z_2^-$$

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For a dictionary solution, we set non-basic variables to 0 again (where the objective changes slope). Therefore, we have $z_1 = -2, z_2 = 1$, so $z_1^+ = 0, z_1^- = 2, z_2^+ = 1, z_2^- = 0$. Unfortunately, the objective is $-\infty$, because $z_1^- > 0$; this dictionary is infeasible!

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If we change the primal objective to $\eta = -2x_1 - x_2$, this will not happen! We then start with $z_1 = 2, z_2 = 1$, which is feasible.

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If we change the primal objective to $\eta = -2x_1 - x_2$, this will not happen! We then start with $z_1 = 2, z_2 = 1$, which is feasible.

We need to check whether we can improve the objective by increasing or decreasing one of y_1, y_2, y_3. To find out whether an increase or decrease improves the objective, we look locally (in the environment of our solution).

$$-\xi = -6y_1^+ - 10y_2^+ - 2z_1^+ + z_2^+ + 2y_2^- - \infty y_3^- - \infty z_1^- - 5z_2^-$$

We have the following dictionary (with no objective):

$$\begin{aligned} z_1 &= -2 + y_1 - y_2 + y_3 \\ z_2 &= 1 + y_1 + 2y_2 - y_3 \end{aligned}$$

For a dictionary solution, we set non-basic variables to 0 again (where the objective changes slope). Therefore, we have $z_1 = -2, z_2 = 1$, so $z_1^+ = 0, z_1^- = 2, z_2^+ = 1, z_2^- = 0$. Unfortunately, the objective is $-\infty$, because $z_1^- > 0$; this dictionary is infeasible!

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At the solution $z_1 = 2, z_2 = 1$, we have $-\xi = -6y_1^+ - 10y_2^+ - 2z_1 + z_2 + 2y_2^- - \infty y_3^-$. We can take left and right partial derivatives of $-\xi$ to look for improvements; note that z_1, z_2 are functions of y_1, y_2, y_3 here!

Initial Primal Dictionary

$$z_1 = 2 + y_1 - y_2 + y_3$$
$$z_2 = 1 + y_1 + 2y_2 - y_3$$

How does our initial primal dictionary look?

- Original problem gives matrix and bounds.
- How do we know which non-basic variable is at its upper, and which at its lower bounds?

The last question is the only difficult part, but complementarity helps here, as well.
$z_1 > 0 \Rightarrow z_1^+ > 0 \Rightarrow h_1 > 0 \Rightarrow g_1 = 0 \Rightarrow x_1 = \ell_1$ (at lower bound),
$z_2 > 0 \Rightarrow z_2^+ > 0 \Rightarrow h_2 > 0 \Rightarrow g_2 = 0 \Rightarrow x_2 = \ell_2$ (at lower bound).

ℓ	u						
				-2^*		1^*	
				∞		5	
		η	$=$	$-2x_1$	$-$	x_2	$= 3$
1	5	w_1	$=$	x_1	$+$	x_2	$= -1$
2	10	w_2	$=$	$-x_1$	$+$	$2x_2$	$= 4$
$-\infty$	0	w_3	$=$	x_1	$-$	x_2	$= 3$

Dual Pivot

$$-\xi = -6y_1^+ - 10y_2^+ - 2z_1 + z_2 + 2y_2^- - \infty y_3^-$$

$$\begin{aligned} z_1 &= 2 + y_1 - y_2 + y_3 \\ z_2 &= 1 + y_1 + 2y_2 - y_3 \end{aligned}$$

Derivative for increasing y_1:
Derivative for decreasing y_1:
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Derivative for decreasing y_2:
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Derivative for decreasing y_3:

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Derivative for increasing y_1: $-6 - 2 \cdot 1 + 1 \cdot 1 = -7 < 0$.
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Derivative for decreasing y_2: $2 - (-2 \cdot (-1) + 1 \cdot 2) = -2 < 0$
Derivative for increasing y_3:
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Which variable hits 0 first?

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Which variable hits 0 first? Only z_2 moves towards 0, and hits 0 for $y_1 = -1$.

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$$\begin{aligned} z_1 &= 1 + z_2 - 3y_2 + 2y_3 \\ y_1 &= -1 + z_2 - 2y_2 - y_3 \end{aligned}$$

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Analyzing derivatives shows that this is actually optimal. The primal dictionary is updated as follows: w_1 leaves, x_2 enters. $y_1^- > 0 \Rightarrow q_1 > 0 \Rightarrow p_1 = 0 \Rightarrow w_1 = a_1$.