Linear Programming

[V. ch6]: The Simplex Method in Matrix Notation

Phillip Keldenich   Ahmad Moradi

Department of Computer Science
Algorithms Department
TU Braunschweig

December 14, 2022

Matrix notation

Primal Simplex Method
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As usual, we begin our discussion with the standard-form linear programming problem:

$$\begin{aligned} \max_x \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \cdots, m \\ & x_j \geq 0, \quad j = 1, 2, \cdots, n \end{aligned}$$

It is convenient to introduce slack variables as follows:

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \ldots, m$$

w_i renamed as x_{n+i}

With these slack variables, we now write our problem in matrix form:

$$\begin{aligned} \max_x \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 1 & & & \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & & 1 & & \\ \vdots & \vdots & \ddots & \vdots & & & \ddots & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & & & & 1 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix}$$

Consider an iteration of the Simplex algorithm where $\mathcal{B}$ and $\mathcal{N}$ are the set of basic and non-basic indices

The ith component of Ax can be broken up into a *basic* and a *nonbasic* part

$$\sum_{j=1}^{n+m} a_{ij}x_j = \sum_{j\in\mathcal{B}} a_{ij}x_j + \sum_{j\in\mathcal{N}} a_{ij}x_j$$

to break up the matrix product Ax analogously, let

- B denote an $m \times m$ matrix whose columns are indexed by $\mathcal{B}$. Similarly,
- N denote an $m \times n$ matrix whose columns are indexed by $\mathcal{N}$

Now, one could write A and x in a partitioned-matrix form as:

$$A = [B \;\; N]\,, x = \begin{bmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{bmatrix}$$

This is a rearrangement (basic columns/variables are listed first followed by the nonbasic columns/variables). Equality is not correct.

Now, we can write:

$$Ax = [B \;\; N] \begin{bmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{bmatrix} = Bx_{\mathcal{B}} + Nx_{\mathcal{N}}$$

Similar partitionanting on c gives:

$$c^T x = \begin{bmatrix} c_{\mathcal{B}} \\ c_{\mathcal{N}} \end{bmatrix}^T \begin{bmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{bmatrix} = c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}}$$

As an example, lets first write it in matirx form, then calculate the above values for $\mathcal{B} = \{1, 2\}$

$$\begin{aligned} \max_x \quad & 3x_1 + 4x_2 - 2x_3 \\ \text{subject to} \quad & x_1 + 0.5x_2 - 5x_3 \leq 2 \\ & 2x_1 - x_2 + 3x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Matrix notation

Primal Simplex Method

Primal Simplex Algorithm

A dictionary has the property that the

basic variables are written as functions of the nonbasic variables

In matrix notation, we see that the constraint equations $Ax = b$ can be written as

$$Bx_{\mathcal{B}} + Nx_{\mathcal{N}} = b$$

variables $x_{\mathcal{B}}$ can be written as a function of the nonbasic variables $x_{\mathcal{N}}$ iff the matrix B is *invertible*,

$$x_{\mathcal{B}} = B^{-1}b - B^{-1}Nx_{\mathcal{N}}$$

The fact that B is invertible means that its m column vectors are linearly independent and therefore form a basis for $\mathbb{R}^m$. This is why the basic variables are called *basic*, Similarly, the objective function can be written as

$$\begin{aligned}\zeta &= c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T \left(B^{-1}b - B^{-1}Nx_{\mathcal{N}}\right) + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T B^{-1}b - \left(\left(B^{-1}N\right)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right)^T x_{\mathcal{N}}\end{aligned}$$

Puting all together, we can write the dictionary associated with basis $\mathcal{B}$ as

$$\begin{aligned}\zeta &= c_{\mathcal{B}}^T B^{-1}b - \left(\left(B^{-1}N\right)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= B^{-1}b \quad - B^{-1}Nx_{\mathcal{N}}\end{aligned}$$

$$\begin{aligned} \zeta &= c_{\mathcal{B}}^T B^{-1} b - \left(\left(B^{-1} N \right)^T c_{\mathcal{B}} - c_{\mathcal{N}} \right)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= B^{-1} b \quad - B^{-1} N x_{\mathcal{N}} \end{aligned}$$

Comparing against the component-form notation, we make the following identifications:

$$\begin{aligned} c_{\mathcal{B}}^T B^{-1} b &= \bar{\zeta} \\ c_{\mathcal{N}} - \left(B^{-1} N \right)^T c_{\mathcal{B}} &= [\bar{c}_j] \\ B^{-1} b &= [\bar{b}_i] \\ B^{-1} N &= [\bar{a}_{ij}] \end{aligned}$$

bracketed expressions on the right denote vectors and matrices with the index i running over $\mathcal{B}$ and the index j running over $\mathcal{N}$.

$$\begin{aligned} \zeta &= c_{\mathcal{B}}^T B^{-1} b - \left(\left(B^{-1} N \right)^T c_{\mathcal{B}} - c_{\mathcal{N}} \right)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= B^{-1} b \quad - B^{-1} N x_{\mathcal{N}} \end{aligned}$$

The basic solution associated with this dictionary is obtained by setting $x_{\mathcal{N}}$ equal to zero.

$$x_{\mathcal{N}}^* = 0, \quad x_{\mathcal{B}}^* = B^{-1} b$$

As an example, consider the same LP

$$\begin{aligned} \max_x \quad & 3x_1 + 4x_2 - 2x_3 \\ \text{subject to} \quad & x_1 + 0.5x_2 - 5x_3 \leq 2 \\ & 2x_1 - x_2 + 3x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

write the initial dictionary. Do first pivot. You get a new basis $\mathcal{B} = \{2, 5\}$. Compute different part of this dictionary in matrix form and compare.

$$
\begin{aligned}
\zeta &= c_{\mathcal{B}}^T B^{-1} b - \left(\left(B^{-1} N\right)^T c_{\mathcal{B}} - c_{\mathcal{N}} \right)^T x_{\mathcal{N}} \\
x_{\mathcal{B}} &= B^{-1} b \quad - B^{-1} N x_{\mathcal{N}}
\end{aligned}
$$

To write the associated dual dictionary using the negative transpose property, it is important to correctly associate *complementary pairs of variables*.

Recall that, we have appended the primal slack variables to the end of the original variables:

$$(x_1, \ldots, x_n, w_1, \ldots, w_m) \rightarrow (\, x_1, \ldots, x_n \,,\, x_{n+1}, \ldots, x_{n+m} \,)$$

Recall that,

- dual slacks are complementary to the primal originals, and
- dual originals are complementary to the primal slacks

using similar index for complementary variables,

$$(z_1, \ldots, z_n, y_1, \ldots, y_m) \rightarrow (\, z_1, \ldots, z_n \,,\, z_{n+1}, \ldots, z_{n+m} \,)$$

$$\begin{aligned}\zeta &= c_{\mathcal{B}}^T B^{-1} b - \left(\left(B^{-1}N\right)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= B^{-1} b \quad - B^{-1} N x_{\mathcal{N}}\end{aligned}$$

and the corresponding dual dictionary :

$$\begin{aligned}-\xi &= -c_{\mathcal{B}}^T B^{-1} b \qquad - \left(B^{-1} b\right)^T z_{\mathcal{B}} \\ z_{\mathcal{N}} &= \left(\left(B^{-1}N\right)^T c_{\mathcal{B}} - c_{\mathcal{N}}\right) + B^{-1} N z_{\mathcal{B}}\end{aligned}$$

The dual solution associated with this is obtained by setting $z_{\mathcal{B}}$ equal to zero:

$$z_{\mathcal{B}}^* = 0, \quad z_{\mathcal{N}}^* = \left(B^{-1}N\right)^T c_{\mathcal{B}} - c_{\mathcal{N}}$$

Using the shorthand $\zeta^* = c_{\mathcal{B}}^T B^{-1} b$, we can write
Primal dictionary as

$$\begin{aligned} \zeta &= \zeta^* - \quad \left(z_{\mathcal{N}}^*\right)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= x_{\mathcal{B}}^* - B^{-1} N x_{\mathcal{N}} \end{aligned}$$

and dual dictionary as

$$\begin{aligned} -\xi &= -\zeta^* - \left(x_{\mathcal{B}}^*\right)^T z_{\mathcal{B}} \\ z_{\mathcal{N}} &= z_{\mathcal{N}}^* + \left(B^{-1} N\right)^T z_{\mathcal{B}} \end{aligned}$$

Primal Dictionary

$$\begin{aligned} \zeta &= \zeta^* - (z^*_{\mathcal{N}})^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= x^*_{\mathcal{B}} - B^{-1} N x_{\mathcal{N}} \end{aligned}$$

Dual Dictionary

$$\begin{aligned} -\xi &= -\zeta^* - (x^*_{\mathcal{B}})^T z_{\mathcal{B}} \\ z_{\mathcal{N}} &= z^*_{\mathcal{N}} + \left(B^{-1} N\right)^T z_{\mathcal{B}} \end{aligned}$$

Lets elaborate on details of an iteration of the primal Simplex algorithm.

STEP 1. *Check for optimality*

if $z^*_{\mathcal{N}} \geq 0$ stop
// Primal feasibility and complementary is already maintained, just check dual feasibility.

STEP 2. *Select Entering Variable*

Else: pick $j \in \mathcal{N}$ with $z^*_j < 0$.
// variable x_j is the entering variable.

Primal Dictionary	Dual Dictionary
$\zeta = \zeta^* - (z_{\mathcal{N}}^*)^T x_{\mathcal{N}}$ $x_{\mathcal{B}} = x_{\mathcal{B}}^* - B^{-1} N x_{\mathcal{N}}$	$-\xi = -\zeta^* - (x_{\mathcal{B}}^*)^T z_{\mathcal{B}}$ $z_{\mathcal{N}} = z_{\mathcal{N}}^* + \left(B^{-1}N\right)^T z_{\mathcal{B}}$

STEP 3. *Compute Primal Step Direction* $\Delta x_{\mathcal{B}}$

Having selected the entering variable, we let

$$x_{\mathcal{N}} = te_j$$

where e_j denote the unit vector that is zero in every component except for a one in the position associated with index j,

$$x_{\mathcal{B}} = x_{\mathcal{B}}^* - B^{-1} N t e_j$$

Hence, the step direction $\Delta x_{\mathcal{B}}$ for the primal basic variables is given by

$$\Delta x_{\mathcal{B}} = B^{-1} N e_j$$

STEP 4. *Compute Primal Step Length, t*

We wish to pick the largest $t \geq 0$ for which

$$x^*_{\mathcal{B}} \geq t\Delta x_{\mathcal{B}}.$$

(every component of $x_{\mathcal{B}}$ remains nonnegative)

Since, for each $i \in \mathcal{B}^*, x^*_i \geq 0$ and $t \geq 0$

$$\frac{1}{t} \geq \frac{\Delta x_i}{x^*_i}, \quad \forall i \in \mathcal{B}$$

Hence, the largest t for which all of the inequalities hold is given by

$$t = \left(\max_{i \in \mathcal{B}} \frac{\Delta x_i}{x^*_i} \right)^{-1}$$ // *convention for* $\frac{0}{0}$ *is to set such ratios to 0*

if $t \leq 0$ stop //*primal is unbounded*

STEP 5. *Select Leaving Variable* The leaving variable is chosen as any variable $x_i, i \in \mathcal{B}$, for which the maximum in the calculation of t is obtained.

Primal Dictionary

$$\begin{aligned} \zeta &= \zeta^* - (z^*_{\mathcal{N}})^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= x^*_{\mathcal{B}} - B^{-1} N x_{\mathcal{N}} \end{aligned}$$

Dual Dictionary

$$\begin{aligned} -\xi &= -\zeta^* - (x^*_{\mathcal{B}})^T z_{\mathcal{B}} \\ z_{\mathcal{N}} &= z^*_{\mathcal{N}} + \left(B^{-1} N\right)^T z_{\mathcal{B}} \end{aligned}$$

Essentially all that remains is to explain changes to the objective function.

STEP 6. *Compute Dual Step Direction* $\Delta z_{\mathcal{N}}$
Since in dual dictionary z_i is the entering variable, we see that

$$\Delta z_{\mathcal{N}} = -\left(B^{-1} N\right)^T e_i$$

STEP 7. *Compute Dual Step Length, s*
Since we know that j is the leaving variable in the dual dictionary,

$$s = \frac{z^*_j}{\Delta z_j}$$

Primal Dictionary

$$\begin{aligned} \zeta &= \zeta^* - (z_{\mathcal{N}}^*)^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= x_{\mathcal{B}}^* - B^{-1} N x_{\mathcal{N}} \end{aligned}$$

Dual Dictionary

$$\begin{aligned} -\xi &= -\zeta^* - (x_{\mathcal{B}}^*)^T z_{\mathcal{B}} \\ z_{\mathcal{N}} &= z_{\mathcal{N}}^* + \left(B^{-1} N\right)^T z_{\mathcal{B}} \end{aligned}$$

We now have everything we need to update the data in the dictionary:

Step 8. *Update Current Primal and Dual Solutions*

$$\begin{aligned} x_j^* &\leftarrow t \\ x_{\mathcal{B}}^* &\leftarrow x_{\mathcal{B}}^* - t \Delta x_{\mathcal{B}} \end{aligned}$$

$$\begin{aligned} z_i^* &\leftarrow s \\ z_{\mathcal{N}}^* &\leftarrow z_{\mathcal{N}}^* - s \Delta z_{\mathcal{N}} \end{aligned}$$

Step 9. *Update Basis*

$$\mathcal{B} \leftarrow (\mathcal{B} \setminus \{i\}) \cup \{j\}$$

As an example, Lets solve the following LP

$$
\begin{aligned}
\max_x \quad & 4x_1 + 3x_2 \\
\text{subject to} \quad & x_1 - x_2 \leq 1 \\
& 2x_1 - x_2 \leq 3 \\
& + x_2 \leq 5 \\
& x_1, x_2 \geq 0
\end{aligned}
$$