Linear Programming

[V. ch5]: Duality Theory

Phillip Keldenich   Ahmad Moradi

Department of Computer Science
Algorithms Department
TU Braunschweig

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Associated with every linear program is another called its *dual*. The dual of this dual linear program is the *original linear program* (which is then referred to as the *primal* linear program).

- ⇝ linear programs come in primal/dual pairs.
- ⇝ every feasible solution for one of these two linear programs gives a bound on the optimal value for the other.

These ideas are important and form a subject called *duality theory*, the topic of this chapter.

We begin with an example:

$$\begin{array}{rrrrl} \max\limits_{x} & 4x_1 + & x_2 + & 3x_3 & \\ \text{subject to} & x_1 + & 4x_2 & & \leq 1 \\ & 3x_1 - & x_2 + & x_3 & \leq 3 \\ & x_1, & x_2, & x_3 & \geq 0 \end{array}$$

Observe that

→ every feasible solution to this LP provides a *lower bound* on the optimal value, ζ^*.

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Is it close to the optimal value?

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But, *How good is this solution?*

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To answer:

we need to give upper bounds

$$
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\end{array}
$$

We can find a bound as follows:

$$\begin{aligned} \max_x \quad & 4x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

We can find a bound as follows:

→ Multiply the first constraint by 2 and *add* that to 3 times the second constraint

$$\begin{array}{rrrrl} & 2 \times (& x_1 + 4x_2 & \leq 1 &) \\ + & 3 \times (& 3x_1 - x_2 + x_3 & \leq 3 &) \\ \hline & & 11x_1 + 5x_2 + 3x_3 & \leq 11 & \end{array}$$

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Since each variable is nonnegative, we can compare the sum against the objective function

$$\zeta = 4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11$$

$$\Rightarrow \quad \zeta^* \leq 11$$

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These bounds leave a gap, $9 \leq \zeta^* \leq 11$. Now,

better insight on the quality of feasible solutions!

Now, we ask:
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→ Multiply the first constraint by $y_1 (\geq 0)$ and *add* that to $y_2 (\geq 0)$ times the second constraint

$$
\begin{array}{rrrrl}
 & y_1 \times (1x_1 + & 4x_2 & & \leq 1) \\
+ & y_2 \times (3x_1 - & x_2 + & x_3 & \leq 3) \\
\hline
 & (y_1 + 3y_2)x_1 + & (4y_1 - y_2)x_2 + & (y_2)x_3 & \leq y_1 + 3y_2
\end{array}
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→ Enforce that each of the coefficients of the x_i 's be *at least as large as* the corresponding coefficient in the objective function, i.e.

$$\begin{array}{rrl} y_1 & + 3y_2 & \geq 4 \\ 4y_1 & - y_2 & \geq 1 \\ & y_2 & \geq 3 \end{array}$$

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→ then we can compare the objective function against this sum (and its bound).

$$\zeta = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + (y_2)x_3 \leq y_1 + 3y_2$$

We now have an upper bound, $y_1 + 3y_2$, which we should *minimize* in our effort to obtain the *best possible upper bound*.

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We are naturally led to the following optimization problem:

$$\begin{aligned} \min_{y} \quad & y_1 + 3y_2 \\ & y_1 + 3y_2 \geq 4 \\ & 4y_1 - y_2 \geq 1 \\ & y_2 \geq 3 \\ & y_1, \ y_2 \geq 0 \end{aligned}$$

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\min_{y} \quad & y_1 + 3y_2 \\
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\end{aligned}
$$

⇝ This problem is called the *dual* LP associated with the given LP.

GMPL Code 1: Primal model.

```
var x1 >=0;
var x2 >=0;
var x3 >=0;

subject to con1:   x1 + 4*x2       <= 1;
subject to con2: 3*x1 -   x2 + x3 <= 3;

maximize z: 4*x1 + x2 + 3*x3;

solve;

display x1.val, x2.val, x3.val, z.val;

end;
```

```
GLPSOL: GLPK LP/MIP Solver, v4.65
Reading model section from ex1_inSlides.mod...
.
.
.
Model has been successfully generated
GLPK Simplex Optimizer, v4.65
.
.
.
OPTIMAL LP SOLUTION FOUND
Time used:   0.0 secs
Memory used: 0.1 Mb (102265 bytes)
Display statement at line 12
x1.val = 0
x2.val = 0.25
x3.val = 3.25
z.val = 10
Model has been successfully processed
```

GMPL Code 2: Dual model.

```
var y1 >=0;
var y2 >=0;

subject to con1:   y1 + 3*y2 >= 4  ;
subject to con2: 4*y1 -    y2 >= 1  ;
subject to con3:           y2 >= 3 ;

minimize z: y1 + 3*y2 ;

solve;

display y1.val, y2.val, z.val;

end;
```

```
GLPSOL: GLPK LP/MIP Solver, v4.65
Reading model section from ex1_dual_inSlides.mod...
.
.
.
Model has been successfully generated
GLPK Simplex Optimizer, v4.65
.
.
.
OPTIMAL SOLUTION FOUND BY LP PREPROCESSOR
Time used:   0.0 secs
Memory used: 0.1 Mb (94214 bytes)
Display statement at line 12
y1.val = 1
y2.val = 3
z.val = 10
Model has been successfully processed
```

As a another example, consider the Resource Allocation Problem.

Recall that

$$\begin{aligned} \max_{x} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \cdots, x_n \geq 0 \end{aligned}$$

where

c_j = profit per unit of product j produced

b_i = unit of raw material i on hand

a_{ij} = units raw material i required to produce one unit of product j

If we produce one unit less of product j, then:

for each i, we free up a_{ij} units of raw material i.

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Selling these unused raw materials for $y_1, y_2, \cdots, y_m$ dollars/unit yields:

$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m$ dollars.

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Only interested if this revenue exceeds lost profit on each product j:

$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j, \quad j = 1, 2, \cdots, n.$.

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$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j, \quad j = 1, 2, \cdots, n..$$

Now, consider a buyer offering to purchase our entire inventory. Subject to above constraints, buyer wants to minimize cost:

$$\min_{y} \quad b_1y_1 + b_2y_2 + \cdots + b_my_m$$

And the following linear program needs to be solved

$$
\begin{aligned}
\min_{y} \quad & b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\text{subject to} \quad & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\
& \vdots \\
& a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_n \geq c_n \\
& y_1, y_2, \cdots, y_m \geq 0
\end{aligned}
$$

$\rightsquigarrow$ This problem is called the *dual* LP associated with the given LP.

Given a linear programming problem in standard form,

$$\begin{aligned}
\max_{x} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \cdots, m \\
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the associated *dual* linear program is given by

$$\begin{aligned} \min_y \quad & \sum_{i=1}^{m} b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, \cdots, n \\ & y_i \geq 0, \quad i = 1, 2, \cdots, m \end{aligned}$$

Primal Problem

$$\begin{aligned}\max_x \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \cdots, m \\ & x_j \geq 0, \quad j = 1, 2, \cdots, n\end{aligned}$$

⇝ Original problem is called the primal problem.

Dual in "Standard" Form

$$\begin{aligned}-\max_y \quad & \sum_{i=1}^{m} -b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} -a_{ij} y_i \leq -c_j, \quad j = 1, 2, \cdots, n \\ & y_i \geq 0, \quad i = 1, 2, \cdots, m\end{aligned}$$

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⇝ *Dual is "negative transpose" of primal.*

Primal Problem

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⇝ *Dual is "negative transpose" of primal.*

Theorem *Dual of dual is primal.*

THEOREM (WEAK DUALITY)

if $(x_1, \cdots, x_n)$ *is feasible for the primal problem and* $(y_1, \cdots, y_m)$ *is feasible for the dual problem, then*

$$\sum_j c_j x_j \leq \sum_i b_i y_i$$

PROOF.

$$\begin{aligned}
\sum_j c_j x_j &\leq \sum_j (\sum_i y_i a_{ij}) x_j \\
&= \sum_i \sum_j y_i a_{ij} x_j \\
&= \sum_i (\sum_j a_{ij} x_j) yi \\
&\leq \sum_i b_i y_i
\end{aligned}$$

□

An important question:
Is there a gap between the largest primal value and the smallest dual value?

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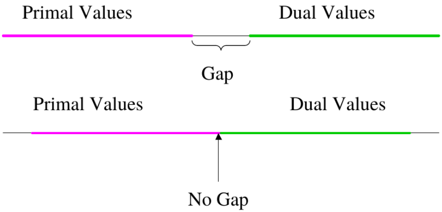

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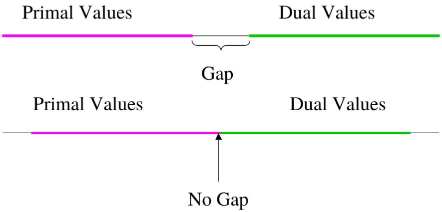

$\rightsquigarrow$ Answer is provided by the Strong Duality Theorem (coming later).

Consider the exmple:
Initial primal and dual dictionaries are provided.

Primal

$$\begin{array}{rrrrr} \zeta = & 0 - & 3x_1 + & 2\,x_2 + & 1\,x_3 \\ \hline w_1 = & 0 - & 0x_1 + & x_2 - & 2x_3 \\ w_2 = & 3 + & 3x_1 - & 4x_2 - & x_3 \end{array}$$

Dual

$$\begin{array}{rrrr} -\xi = & 0 + & 0y_1 - & 3y_2 \\ \hline z_1 = & 3 - & 0y_1 - & 3y_2 \\ z_2 = & -2 - & 1y_1 + & 4y_2 \\ z_3 = & -1 + & 2y_1 + & 1y_2 \end{array}$$

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Primal

$$
\begin{array}{rrrrr}
\zeta = & 0 - & 3x_1 + & \boxed{2}\, x_2 + & \boxed{1}\, x_3 \\
\hline
w_1 = & 0 - & 0x_1 + & x_2 - & 2x_3 \\
w_2 = & 3 + & 3x_1 - & 4x_2 - & x_3
\end{array}
$$

⇝ *Dual is negative transpose of primal.*

⇝ *Primal is feasible, dual is not.*

Dual

$$
\begin{array}{rrrr}
-\xi = & 0 + & 0y_1 - & 3y_2 \\
\hline
z_1 = & 3 - & 0y_1 - & 3y_2 \\
z_2 = & \boxed{-2} - & 1y_1 + & 4y_2 \\
z_3 = & \boxed{-1} + & 2y_1 + & 1y_2
\end{array}
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Initial primal and dual dictionaries are provided.

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⇝ *Dual is negative transpose of primal.*

⇝ *Primal is feasible, dual is not.*

⇝ *Use primal to choose pivot: x_2 enters, w_2 leaves.*

Dual

$$\begin{array}{rrrr} -\xi = & 0 + & 0y_1 - & 3y_2 \\ \hline z_1 = & 3 - & 0y_1 - & 3y_2 \\ z_2 = & -2 - & 1y_1 + & 4y_2 \\ z_3 = & -1 + & 2y_1 + & 1y_2 \end{array}$$

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⇝ *Dual is negative transpose of primal.*

⇝ *Primal is feasible, dual is not.*

⇝ *Use primal to choose pivot: x_2 enters, w_2 leaves.*

⇝ *Make analogous pivot in dual: z_2 leaves, y_2 enters.*

After first pivot

Primal

$$\begin{array}{rrrrr} \zeta = & \frac{3}{2} - & \frac{3}{2}x_1 - & \frac{1}{2}w_2 + & \frac{1}{2}x_3 \\ \hline w_1 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{9}{4}x_3 \\ x_2 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{1}{4}x_3 \end{array}$$

Dual

$$\begin{array}{rrrr} -\xi = & -\frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\ \hline z_1 = & \frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\ y_2 = & \frac{1}{2} + & \frac{1}{4}y_1 + & \frac{1}{4}z_2 \\ z_3 = & -\frac{1}{2} + & \frac{9}{4}y_1 + & \frac{1}{4}z_2 \end{array}$$

After first pivot

Primal

$$
\begin{array}{rcrrrr}
\zeta & = & \frac{3}{2} - & \frac{3}{2}x_1 - & \frac{1}{2}w_2 + & \frac{1}{2}x_3 \\
\hline
w_1 & = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{9}{4}x_3 \\
x_2 & = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{1}{4}x_3
\end{array}
$$

$\rightsquigarrow$ *negative transpose property intact.*

Dual

$$
\begin{array}{rcrrr}
-\xi & = & -\frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
\hline
z_1 & = & \frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
y_2 & = & \frac{1}{2} + & \frac{1}{4}y_1 + & \frac{1}{4}z_2 \\
z_3 & = & -\frac{1}{2} + & \frac{9}{4}y_1 + & \frac{1}{4}z_2
\end{array}
$$

After first pivot

Primal

$$\begin{array}{rrrrr} \zeta = & \frac{3}{2} - & \frac{3}{2}x_1 - & \frac{1}{2}w_2 + & \frac{1}{2}x_3 \\ \hline w_1 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{9}{4}x_3 \\ x_2 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{1}{4}x_3 \end{array}$$

⇝ *negative transpose property intact.*

⇝ *Primal is feasible, dual is not.*

Dual

$$\begin{array}{rrrr} -\xi = & -\frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\ \hline z_1 = & \frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\ y_2 = & \frac{1}{2} + & \frac{1}{4}y_1 + & \frac{1}{4}z_2 \\ z_3 = & -\frac{1}{2} + & \frac{9}{4}y_1 + & \frac{1}{4}z_2 \end{array}$$

After first pivot

Primal

$$
\begin{array}{rrrrr}
\zeta = & \frac{3}{2} - & \frac{3}{2}x_1 - & \frac{1}{2}w_2 + & \frac{1}{2}x_3 \\
\hline
w_1 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{9}{4}x_3 \\
x_2 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{1}{4}x_3
\end{array}
$$

Dual

$$
\begin{array}{rrrr}
-\xi = & -\frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
\hline
z_1 = & \frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
y_2 = & \frac{1}{2} + & \frac{1}{4}y_1 + & \frac{1}{4}z_2 \\
z_3 = & -\frac{1}{2} + & \frac{9}{4}y_1 + & \frac{1}{4}z_2
\end{array}
$$

⇝ *negative transpose property intact.*

⇝ *Primal is feasible, dual is not.*

⇝ *Use primal to pick pivot: x_3 enters, w_1 leaves.*

After first pivot

Primal

$$\begin{array}{rcrrrr} \zeta &=& \frac{3}{2} - & \frac{3}{2}x_1 - & \frac{1}{2}w_2 + & \frac{1}{2}x_3 \\ \hline w_1 &=& \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{9}{4}x_3 \\ x_2 &=& \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{1}{4}x_3 \end{array}$$

Dual

$$\begin{array}{rcrrr} -\xi &=& -\frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\ \hline z_1 &=& \frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\ y_2 &=& \frac{1}{2} + & \frac{1}{4}y_1 + & \frac{1}{4}z_2 \\ z_3 &=& -\frac{1}{2} + & \frac{9}{4}y_1 + & \frac{1}{4}z_2 \end{array}$$

⇝ *negative transpose property intact.*

⇝ *Primal is feasible, dual is not.*

⇝ *Use primal to pick pivot: x_3 enters, w_1 leaves.*

⇝ *Make analogous pivot in dual: z_3 leaves, y_1 enters.*

After second pivot

Primal

$$\begin{array}{rcccc} \zeta = & \frac{5}{3} - & \frac{4}{3}x_1 - & \frac{5}{9}w_2 - & \frac{2}{9}w_1 \\ \hline x_3 = & \frac{1}{3} + & \frac{1}{3}x_1 - & \frac{1}{9}w_2 - & \frac{4}{9}w_1 \\ x_2 = & \frac{2}{3} + & \frac{2}{3}x_1 - & \frac{2}{9}w_2 + & \frac{1}{9}w_1 \end{array}$$

Dual

$$\begin{array}{rccc} -\xi = & -\frac{5}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\ \hline z_1 = & \frac{4}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\ y_2 = & \frac{5}{9} + & \frac{1}{9}z_3 + & \frac{2}{9}z_2 \\ y_1 = & \frac{2}{9} + & \frac{4}{9}z_3 - & \frac{1}{9}z_2 \end{array}$$

After second pivot

Primal

$$\begin{array}{rrrrr} \zeta = & \frac{5}{3} - & \frac{4}{3}x_1 - & \frac{5}{9}w_2 - & \frac{2}{9}w_1 \\ \hline x_3 = & \frac{1}{3} + & \frac{1}{3}x_1 - & \frac{1}{9}w_2 - & \frac{4}{9}w_1 \\ x_2 = & \frac{2}{3} + & \frac{2}{3}x_1 - & \frac{2}{9}w_2 + & \frac{1}{9}w_1 \end{array}$$

⇝ *negative transpose property remains intact.*

Dual

$$\begin{array}{rrrr} -\xi = & -\frac{5}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\ \hline z_1 = & \frac{4}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\ y_2 = & \frac{5}{9} + & \frac{1}{9}z_3 + & \frac{2}{9}z_2 \\ y_1 = & \frac{2}{9} + & \frac{4}{9}z_3 - & \frac{1}{9}z_2 \end{array}$$

After second pivot

Primal

$$\begin{array}{rcccc} \zeta = & \frac{5}{3} - & \frac{4}{3}x_1 - & \frac{5}{9}w_2 - & \frac{2}{9}w_1 \\ \hline x_3 = & \frac{1}{3} + & \frac{1}{3}x_1 - & \frac{1}{9}w_2 - & \frac{4}{9}w_1 \\ x_2 = & \frac{2}{3} + & \frac{2}{3}x_1 - & \frac{2}{9}w_2 + & \frac{1}{9}w_1 \end{array}$$

Dual

$$\begin{array}{rccc} -\xi = & -\frac{5}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\ \hline z_1 = & \frac{4}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\ y_2 = & \frac{5}{9} + & \frac{1}{9}z_3 + & \frac{2}{9}z_2 \\ y_1 = & \frac{2}{9} + & \frac{4}{9}z_3 - & \frac{1}{9}z_2 \end{array}$$

⇝ *negative transpose property remains intact.*

⇝ *Primal and dual are both optimal.*

After second pivot

Primal

$$
\begin{array}{rrrrr}
\zeta = & \frac{5}{3} - & \frac{4}{3}x_1 - & \frac{5}{9}w_2 - & \frac{2}{9}w_1 \\
\hline
x_3 = & \frac{1}{3} + & \frac{1}{3}x_1 - & \frac{1}{9}w_2 - & \frac{4}{9}w_1 \\
x_2 = & \frac{2}{3} + & \frac{2}{3}x_1 - & \frac{2}{9}w_2 + & \frac{1}{9}w_1
\end{array}
$$

Dual

$$
\begin{array}{rrrr}
-\xi = & -\frac{5}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\
\hline
z_1 = & \frac{4}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\
y_2 = & \frac{5}{9} + & \frac{1}{9}z_3 + & \frac{2}{9}z_2 \\
y_1 = & \frac{2}{9} + & \frac{4}{9}z_3 - & \frac{1}{9}z_2
\end{array}
$$

⇝ *negative transpose property remains intact.*

⇝ *Primal and dual are both optimal.*

⇝ *Simplex Alg. applied to primal, solves both the primal and the dual.*

THEOREM (STRONG DUALITY)

If the primal problem has an optimal solution,

$$x^* = (x_1^*, x_2^*, \cdots, x_n^*)$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, y_2^*, \cdots, y_m^*)$$

and

$$\sum_j c_j x_j^* = \sum_i c_i y_i^*$$

$\rightsquigarrow$ If primal has an optimal solution, then there is no duality gap.

Theorem (Strong Duality)

If the primal problem has an optimal solution,

$$x^* = (x_1^*, x_2^*, \cdots, x_n^*)$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, y_2^*, \cdots, y_m^*)$$

and

$$\sum_j c_j x_j^* = \sum_i c_i y_i^*$$

⇝ If primal has an optimal solution, then there is no duality gap.

⇝ Let's prove it.