LINEAR PROGRAMMING

[V. CH3]: DEGENERACY

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DEFINITION / EXAMPLES OF DEGENERACY

PERTURBATION METHOD

FUNDAMENTAL THEOREM

GEOMETRY

OTHER PIVOT RULES

We say that a dictionary is *degenerate* if $\bar{b}_i = 0$ for some $i \in \mathscr{B}$.

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As an example

$$\begin{array}{rrrr} \zeta = & 5 + & x_3 - & 1x_1 \\ \hline x_2 = & 5 + & 2x_3 - & 3x_1 \\ x_4 = & 7 & - & 4x_1 \\ x_5 = & 0 & + & x_1 \end{array}$$

It is clear that the problem is unbounded and therefore no more pivots were required.

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$\rightsquigarrow$ A degenerate dictionary could cause difficulties for the simplex method. Problems arise, when a degenerate dictionary produces *degenerate pivots*.

We say that a pivot is a degenerate pivot if one of the ratios in the calculation of the leaving variable is 0; i.e., if the numerator is zero and the denominator is positive, We say that a pivot is a degenerate pivot if one of the ratios in the calculation of the leaving variable is 0; i.e., if the numerator is zero and the denominator is positive, like

$$\begin{array}{rrrrr} \zeta = & 3 - & 0.5x_1 + & \boxed{2}\, x_2 - & 1.5w_1 \\ \hline x_3 = & 1 - & 0.5x_1 & - & 0.5w_1 \\ w_2 = & 0 + & x_1 - & 1.0x_2 + & w_1 \end{array}$$

For this dictionary, the entering variable is x_2 and the ratio computed to determine the leaving variable is

$$\left\{\, i \in \mathscr{B} : \frac{\bar{b}_i}{\bar{a}_{i2}} \text{ with } \bar{a}_{i2} > 0 \,\right\} = \left\{\, \frac{0}{1.0} \,\right\}$$

Hence, the leaving variable is w_2.

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Hence, the leaving variable is w_2.

⇝ The fact that the ratio is zero means that *as soon as x_2 is increased from zero to a positive value, w_2 will go negative.*

Nonetheless, it can be *reclassified* from *nonbasic to basic* (with w_2 going the other way). Look at the result of this degenerate pivot:

$$\begin{array}{rrrrr}
\zeta = & 3 + & \boxed{1.5}\, x_1 - & 2w_2 + & \boxed{0.5}\, w_1 \\
\hline
x_3 = & 1 - & 0.5x_1 & - & 0.5w_1 \\
x_2 = & & x_1 - & w_2 + & w_1
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Note that

→ $\bar{\zeta}$ remains unchanged at 3. Hence, this degenerate pivot has not produced any increase in the objective function value.

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- → Furthermore, the values of the variables have not even changed: both before and after this degenerate pivot, they are

$$(x_1, x_2, x_3, w_1, w_2) = (0, 0, 1, 0, 0)$$

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$$(x_1, x_2, x_3, w_1, w_2) = (0, 0, 1, 0, 0)$$

But we are just representing the solution in a new way.

The entering variable for the next iteration is x_1 and the leaving variable is x_3, producing a *nondegenerate* pivot that leads to

$$\begin{array}{rrrrr}
\zeta = & 6 - & 3x_3 - & 2w_2 - & w_1 \\
\hline
x_1 = & 2 - & 2x_3 & - & w_1 \\
x_2 = & 2 - & 2x_3 - & w_2 &
\end{array}$$

The entering variable for the next iteration is x_1 and the leaving variable is x_3, producing a *nondegenerate* pivot that leads to

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\end{array}$$

While it is typical for some pivot to “break away” from the degeneracy, the real danger is that

> *the simplex method will make a sequence of degenerate pivots and eventually return to a dictionary that has appeared before, in which case the simplex method enters an infinite loop and never finds an optimal solution.*

This behavior is called *cycling*.

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> *the simplex method will make a sequence of degenerate pivots and eventually return to a dictionary that has appeared before, in which case the simplex method enters an infinite loop and never finds an optimal solution.*

This behavior is called *cycling*.

⇝ Cycling is rare! A program that generates random 2×4 fully degenerate problems was run more than one billion times and did not find one example!

Unfortunately, under certain pivoting rules, cycling is possible.

In fact, cycling is possible even when using one of the most popular pivoting rules:

- Entering variable: Choose the entering variable as the one with the largest coefficient in the ζ-row of the dictionary.
- Leaving variable: When two or more variables compete for leaving the basis, reading left to right, pick the first leaving-variable candidate from the list:

$$x_1, x_2, \cdots, x_n, w_1, w_2, \cdots, w_m$$

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$$x_1, x_2, \cdots, x_n, w_1, w_2, \cdots, w_m$$

Here is an example that cycles:

$$\begin{array}{rrrrrr}
\zeta = & + & x_1 - & 2x_2 & - & 2x_4 \\ \hline
w_1 = & - & 0.5x_1 + & 3.5x_2 + & 2x_3 - & 4x_4 \\
w_2 = & - & 0.5x_1 + & x_2 + & 0.5x_3 - & 0.5x_4 \\
w_3 = & 1 - & x_1 & & &
\end{array}$$

$$
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\zeta = & + & 1\,x_1 - & 2x_2 & - & 2x_4 \\
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w_2 = & - & 0.5x_1 + & x_2 + & 0.5x_3 - & 0.5x_4 \\
w_3 = & 1 - & x_1 & & &
\end{array}
$$

For the first pivot, x_1 enters and w_1 leaves bringing us to:

$$
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w_1 = & - & 0.5x_1 + & 3.5x_2 + & 2x_3 - & 4x_4 \\
w_2 = & - & 0.5x_1 + & x_2 + & 0.5x_3 - & 0.5x_4 \\
w_3 = & 1 - & x_1 & & &
\end{array}
$$

For the first pivot, x_1 enters and w_1 leaves bringing us to:

$$
\begin{array}{rrrrrr}
\zeta = & - & 2w_1 + & 5\,x_2 + & 4\,x_3 - & 10x_4 \\
\hline
x_1 = & - & 2w_1 + & 7x_2 + & 4x_3 - & 8x_4 \\
w_2 = & + & w_1 - & 2.5x_2 - & 1.5x_3 + & 3.5x_4 \\
w_3 = & 1 + & 2w_1 - & 7x_2 - & 4x_3 + & 8x_4
\end{array}
$$

For the second iteration, x_2 enters and w_2 leaves bringing us to:

$$
\begin{array}{rrrrrr}
\zeta = & + & 1\,x_1 - & 2x_2 & - & 2x_4 \\
\hline
w_1 = & - & 0.5x_1 + & 3.5x_2 + & 2x_3 - & 4x_4 \\
w_2 = & - & 0.5x_1 + & x_2 + & 0.5x_3 - & 0.5x_4 \\
w_3 = & 1 - & x_1 & & &
\end{array}
$$

For the first pivot, x_1 enters and w_1 leaves bringing us to:

$$
\begin{array}{rrrrrr}
\zeta = & - & 2w_1 + & 5\,x_2 + & 4\,x_3 - & 10x_4 \\
\hline
x_1 = & - & 2w_1 + & 7x_2 + & 4x_3 - & 8x_4 \\
w_2 = & + & w_1 - & 2.5x_2 - & 1.5x_3 + & 3.5x_4 \\
w_3 = & 1 + & 2w_1 - & 7x_2 - & 4x_3 + & 8x_4
\end{array}
$$

For the second iteration, x_2 enters and w_2 leaves bringing us to:

$$
\begin{array}{rrrrrr}
\zeta = & & - & 2w_2 + & 1\,x_3 - & 3x_4 \\
\hline
x_1 = & + & 0.8w_1 - & 2.8w_2 - & 0.2x_3 + & 1.8x_4 \\
x_2 = & + & 0.4w_1 - & 0.4w_2 - & 0.6x_3 + & 1.4x_4 \\
w_3 = & 1 - & 0.8w_1 + & 2.8w_2 + & 0.2x_3 - & 1.8x_4
\end{array}
$$

For the third iteration, x_3 enters and x_1 leaves:

$$\begin{array}{rrrrrr}
\zeta = & & +\ 4w_1 & -\ 16w_2 & -\ 5x_1 & +\ 6x_4 \\ \hline
x_3 = & & +\ 4w_1 & -\ 14w_2 & -\ 5x_1 & +\ 9x_4 \\
x_2 = & & -\ 2w_1 & +\ 8w_2 & +\ 3x_1 & -\ 4x_4 \\
w_3 = & 1 & & & -\ x_1 &
\end{array}$$

For the fourth iteration, x_4 enters and x_2 leaves:

For the third iteration, x_3 enters and x_1 leaves:

$$\begin{array}{rrrrrr}
\zeta = & & +\ 4w_1 - & 16w_2 - & 5x_1 + & 6x_4 \\
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x_3 = & & +\ 4w_1 - & 14w_2 - & 5x_1 + & 9x_4 \\
x_2 = & & -\ 2w_1 + & 8w_2 + & 3x_1 - & 4x_4 \\
w_3 = & 1 & & - & x_1 &
\end{array}$$

For the fourth iteration, x_4 enters and x_2 leaves:

$$\begin{array}{rrrrrr}
\zeta = & & +\ 1w_1 - & 4w_2 - & 0.5x_1 - & 1.5x_2 \\
\hline
x_3 = & & -\ 0.5w_1 + & 4w_2 + & 1.75x_1 - & 2.25x_2 \\
x_4 = & & -\ 0.5w_1 + & 2w_2 + & 0.75x_1 - & 0.25x_2 \\
w_3 = & 1 & & - & x_1 &
\end{array}$$

In the fifth iteration, w_1 enters and x_3 leaves:

For the third iteration, x_3 enters and x_1 leaves:

$$
\begin{array}{rrrrrr}
\zeta = & & + \; 4w_1 & - \; 16w_2 & - \; 5x_1 & + \; 6x_4 \\
\hline
x_3 = & & + \; 4w_1 & - \; 14w_2 & - \; 5x_1 & + \; 9x_4 \\
x_2 = & & - \; 2w_1 & + \; 8w_2 & + \; 3x_1 & - \; 4x_4 \\
w_3 = & 1 & & & - \; x_1 &
\end{array}
$$

For the fourth iteration, x_4 enters and x_2 leaves:

$$
\begin{array}{rrrrrr}
\zeta = & & + \; 1w_1 & - \; 4w_2 & - \; 0.5x_1 & - \; 1.5x_2 \\
\hline
x_3 = & & - \; 0.5w_1 & + \; 4w_2 & + \; 1.75x_1 & - \; 2.25x_2 \\
x_4 = & & - \; 0.5w_1 & + \; 2w_2 & + \; 0.75x_1 & - \; 0.25x_2 \\
w_3 = & 1 & & & - \; x_1 &
\end{array}
$$

In the fifth iteration, w_1 enters and x_3 leaves:

$$
\begin{array}{rrrrrr}
\zeta = & & - \; 2x_3 & + \; 4w_2 & + \; 3x_1 & - \; 6x_2 \\
\hline
w_1 = & & - \; 2x_3 & + \; 8w_2 & + \; 3.5x_1 & - \; 4.5x_2 \\
x_4 = & & x_3 & - \; 2w_2 & - \; x_1 & + \; 2x_2 \\
w_3 = & 1 & & & - \; x_1 &
\end{array}
$$

Lastly, for the sixth iteration, w_2 enters and x_4 leaves:

$$
\begin{array}{rcrrrr}
\zeta = & & - & 2x_4 + & 1x_1 - & 2x_2 \\
\hline
w_1 = & + & 2x_3 - & 4x_4 - & 0.5x_1 + & 3.5x_2 \\
w_2 = & + & 0.5x_3 - & 0.5x_4 - & 0.5x_1 + & x_2 \\
w_3 = & 1 & & - & x_1 &
\end{array}
$$

Lastly, for the sixth iteration, w_2 enters and x_4 leaves:

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\begin{array}{rrrrrr}
\zeta = & & - & 2x_4 + & 1x_1 - & 2x_2 \\ \hline
w_1 = & + & 2x_3 - & 4x_4 - & 0.5x_1 + & 3.5x_2 \\
w_2 = & + & 0.5x_3 - & 0.5x_4 - & 0.5x_1 + & x_2 \\
w_3 = & 1 & & - & x_1 &
\end{array}
$$

Note that we have come back to the original dictionary:

$$
\begin{array}{rrrrrr}
\zeta = & + & x_1 - & 2x_2 & - & 2x_4 \\ \hline
w_1 = & - & 0.5x_1 + & 3.5x_2 + & 2x_3 - & 4x_4 \\
w_2 = & - & 0.5x_1 + & x_2 + & 0.5x_3 - & 0.5x_4 \\
w_3 = & 1 - & x_1 & & &
\end{array}
$$

⇝ from here on the simplex method simply cycles through these six dictionaries and never makes any further progress toward an optimal solution.

As bad as cycling is, the following theorem tells us that nothing worse can happen:

THEOREM

If the simplex method fails to terminate, then it must cycle.

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PROOF.

A dictionary is completely determined by specifying which variables are basic and which are nonbasic. There are only

$$\binom{n+m}{m} = \frac{(n+m)!}{n!m!}$$

different possibilities. This number is big, but it is finite. If the simplex method fails to terminate, it must visit some of these dictionaries more than once. Hence, the algorithm cycles. □

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Whenever a vanishing ″rhs″, (zero $\bar{b}_i$), appears *perturb* it.

- set : $\bar{b}_i = \bar{b}_i + \epsilon_i$

If there are lots of them, say k, perturb them all. Make the perturbations at different *scales*:

$$0 < \epsilon_k \ll \cdots \ll \epsilon_2 \ll \epsilon_1 \ll \text{other nonzero data}$$

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$$0 < \epsilon_k \ll \cdots \ll \epsilon_2 \ll \epsilon_1 \ll \text{other nonzero data}$$

Try to keep in mind that:

- no linear combination of the ϵ_i ’s using coefficients that might arise in the course of the simplex method can ever produce a number whose size is of the same order as the data in the problem.
- each of the “lower down” ϵ_i ’s can never “escalate” to a higher level.

Hence, *cancellations* can only occur on a given scale.

First example:

EXAMPLE

$$\begin{array}{rrrrr} \zeta = & 0 & + & 6x_1 + & 4x_2 \\ \hline w_1 = & 0 & + & 9x_1 + & 4x_2 \\ w_2 = & 0 & - & 4x_1 - & 2x_2 \\ w_3 = & 1 & & - & x_2 \end{array}$$

The first step is to introduce symbolic parameters

$$0 < \epsilon_3 \ll \epsilon_2 \ll \epsilon_1$$

to get a perturbed problem:

$$\begin{array}{rlllllll}
\zeta = & 0 & & & + & 6\,x_1 & + & 4\,x_2 \\
\hline
w_1 = & 0 + & \epsilon_1 & & + & 9x_1 & + & 4x_2 \\
w_2 = & 0 & + \quad \epsilon_2 & & - & 4x_1 & - & 2x_2 \\
w_3 = & 1 & & + \quad \epsilon_3 & & & - & x_2
\end{array}$$

This dictionary is *not degenerate*. The entering variable is x_1 and the leaving variable is unambiguously w_2.

The next dictionary is

$$\begin{array}{rlllllll}
\zeta = & 0 & & + & 1.5\epsilon_2 & & - \ 1.5w_2 + & 1\,x_2 \\ \hline
w_1 = & 0 + & \epsilon_1 & + & 2.25\epsilon_2 & & - \ 2.25w_2 - & 0.5x_2 \\
x_1 = & 0 & & + & 0.25\epsilon_2 & & - \ 0.25w_2 - & 0.5x_2 \\
w_3 = & 1 & & & & + \ \epsilon_3 & \qquad\quad - & x_2
\end{array}$$

The next dictionary is

$$
\begin{array}{rcllllll}
\zeta & = & 0 & & +\ 1.5\epsilon_2 & & -\ 1.5w_2 & +\ 1\,x_2 \\
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w_1 & = & 0 & +\ \epsilon_1 & +\ 2.25\epsilon_2 & & -\ 2.25w_2 & -\ 0.5x_2 \\
x_1 & = & 0 & & +\ 0.25\epsilon_2 & & -\ 0.25w_2 & -\ 0.5x_2 \\
w_3 & = & 1 & & & +\ \epsilon_3 & & -\ x_2
\end{array}
$$

For the next pivot, the entering variable is x_2 and, using the fact that $\epsilon_2 \ll \epsilon_1$, we see that the leaving variable is x_1. The new dictionary is

$$
\begin{array}{rcllllll}
\zeta & = & 0 & & +\ 2\epsilon_2 & & -\ 2w_2 & -\ 2x_1 \\
\hline
w_1 & = & 0 & +\ \epsilon_1 & +\ 2\epsilon_2 & & -\ 2w_2 & +\ x_1 \\
x_2 & = & 0 & & +\ 0.5\epsilon_2 & & -\ 0.5w_2 & -\ 2x_1 \\
w_3 & = & 1 & & -\ 0.5\epsilon_2 & +\ \epsilon_3 & +\ 0.5w_2 & +\ 2x_1
\end{array}
$$

This last dictionary is optimal. At this point, we simply *drop* the symbolic ϵ_i parameters and get an optimal dictionary for the *unperturbed* problem.

$$\begin{array}{rrrr}
\zeta = & 0 - & 2w_2 - & 2x_1 \\
\hline
w_1 = & 0 - & 2w_2 + & x_1 \\
x_2 = & 0 - & 0.5w_2 - & 2x_1 \\
w_3 = & 1 + & 0.5w_2 + & 2x_1
\end{array}$$

When treating the ϵ_i 's as symbols, the method is called the *lexicographic* method.

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When treating the ϵ_i's as symbols, the method is called the *lexicographic* method.

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When treating the ϵ_i 's as symbols, the method is called the *lexicographic* method.

⇝ The lexicographic method does not affect the choice of entering variable but does amount to a precise prescription for the choice of leaving variable.

⇝ The lexicographic method produces a variant of the simplex method that never cycles:

THEOREM

The simplex method always terminates provided that the leaving variable is selected by the lexicographic rule.

Proof.

It suffices to show that no degenerate dictionary is ever produced.

As we have discussed before, the ϵ_i 's operate on different scales and hence cannot cancel with each other. Therefore, can think of the ϵ_i 's as a collection of independent variables.

PROOF.

It suffices to show that no degenerate dictionary is ever produced.

As we have discussed before, the ϵ_i 's operate on different scales and hence cannot cancel with each other. Therefore, can think of the ϵ_i 's as a collection of independent variables.

Extracting the ϵ terms from the first dictionary, we see that we start with the following pattern:

$$\begin{matrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_m \end{matrix}$$

□

PROOF CONT.

And, after several pivots, the ϵ terms will form a system of linear combinations, say,

$$\begin{array}{cccc}
r_{11}\epsilon_1 + & r_{12}\epsilon_2 + & \cdots & r_{1m}\epsilon_m \\
r_{21}\epsilon_1 + & r_{22}\epsilon_2 + & \cdots & r_{2m}\epsilon_m \\
\vdots & \vdots & \ddots & \vdots \\
r_{m1}\epsilon_1 + & r_{m2}\epsilon_2 + & \cdots & r_{mm}\epsilon_m
\end{array}$$

PROOF CONT.

And, after several pivots, the ϵ terms will form a system of linear combinations, say,

$$\begin{array}{rrcr} r_{11}\epsilon_1 + & r_{12}\epsilon_2 + & \cdots & r_{1m}\epsilon_m \\ r_{21}\epsilon_1 + & r_{22}\epsilon_2 + & \cdots & r_{2m}\epsilon_m \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1}\epsilon_1 + & r_{m2}\epsilon_2 + & \cdots & r_{mm}\epsilon_m \end{array}$$

This system of linear combinations is obtained from the original system by pivot operations and, since pivot operations are reversible, it follows that the rank of the two systems must be the same.

PROOF CONT.

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This system of linear combinations is obtained from the original system by pivot operations and, since pivot operations are reversible, it follows that the rank of the two systems must be the same.

Since the original system had rank m, we see that every subsequent system must have rank m.

→ There must be at least one nonzero r_{ij} in every row i, which of course implies that

→ None of the rows can be degenerate.

Hence, no dictionary can be degenerate. ☐

Second example:

EXAMPLE

$$\begin{array}{rcrrr}
\zeta = & 0 & + & 2x_1 + & 4x_2 \\
\hline
w_1 = & 0 & + & x_1 - & x_2 \\
w_2 = & 0 & + & 3x_1 - & x_2 \\
w_3 = & 0 & - & 4x_1 + & x_2
\end{array}$$

Perturb vanishing rhs

$$\begin{array}{rllllll}
\zeta = & 0.0 & & & + & 2x_1 + & 4x_2 \\
\hline
w_1 = & 0.0 + & \epsilon_1 & & + & x_1 - & x_2 \\
w_2 = & 0.0 & + \ \epsilon_2 & & + & 3x_1 - & x_2 \\
w_3 = & 0.0 & & + \ \epsilon_3 & - & 4x_1 + & x_2
\end{array}$$

Perturb vanishing rhs

$$\begin{array}{rlrrrrr}
\zeta = & 0.0 & & & + & 2x_1 + & 4x_2 \\
\hline
w_1 = & 0.0 + & \epsilon_1 & & + & x_1 - & x_2 \\
w_2 = & 0.0 & + & \epsilon_2 & + & 3x_1 - & x_2 \\
w_3 = & 0.0 & & + & \epsilon_3 - & 4x_1 + & x_2
\end{array}$$

or equivalently

$$\begin{array}{rrrrrrr}
\zeta = & 0.0 + & 0.0\epsilon_1 + & 0.0\epsilon_2 + & 0.0\epsilon_3 + & \boxed{2}\, x_1 + & \boxed{4}\, x_2 \\
\hline
w_1 = & 0.0 + & 1.0\epsilon_1 + & 0.0\epsilon_2 + & 0.0\epsilon_3 + & x_1 - & x_2 \\
w_2 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 0.0\epsilon_3 + & 32x_1 - & x_2 \\
w_3 = & 0.0 + & 0.0\epsilon_1 + & 0.0\epsilon_2 + & 1.0\epsilon_3 - & 4x_1 + & x_2
\end{array}$$

x_2 enters and as $\epsilon_2 \ll \epsilon_1, w_2$ leaves.

Perturb vanishing rhs

$$\begin{array}{rlllll}
\zeta = & 0.0 & & & + & 2x_1 + 4x_2 \\
\hline
w_1 = & 0.0 + & \epsilon_1 & & + & x_1 - x_2 \\
w_2 = & 0.0 & + \epsilon_2 & & + & 3x_1 - x_2 \\
w_3 = & 0.0 & & + \epsilon_3 & - & 4x_1 + x_2
\end{array}$$

or equivalently

$$\begin{array}{rllllll}
\zeta = & 0.0 + & 0.0\epsilon_1 + & 0.0\epsilon_2 + & 0.0\epsilon_3 + & 2\,x_1 + & 4\,x_2 \\
\hline
w_1 = & 0.0 + & 1.0\epsilon_1 + & 0.0\epsilon_2 + & 0.0\epsilon_3 + & x_1 - & x_2 \\
w_2 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 0.0\epsilon_3 + & 32x_1 - & x_2 \\
w_3 = & 0.0 + & 0.0\epsilon_1 + & 0.0\epsilon_2 + & 1.0\epsilon_3 - & 4x_1 + & x_2
\end{array}$$

x_2 enters and as $\epsilon_2 \ll \epsilon_1, w_2$ leaves.

$\rightsquigarrow$ Note that, as ϵ_i 's have different scale, the will not cancel each other.

We get

$$
\begin{array}{rrrrrrr}
\zeta = & 0.0 + & 0.0\epsilon_1 + & 4.0\epsilon_2 + & 0.0\epsilon_3 + & 14\,x_1 - & 4w_2 \\
\hline
w_1 = & 0.0 + & 1.0\epsilon_1 - & 1.0\epsilon_2 + & 0.0\epsilon_3 - & 2x_1 + & w_2 \\
x_2 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 0.0\epsilon_3 + & 3x_1 - & w_2 \\
w_3 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 1.0\epsilon_3 - & x_1 - & w_2
\end{array}
$$

We get

$$\begin{array}{rrrrrrr}
\zeta = & 0.0 + & 0.0\epsilon_1 + & 4.0\epsilon_2 + & 0.0\epsilon_3 + & 14\,x_1 - & 4w_2 \\
\hline
w_1 = & 0.0 + & 1.0\epsilon_1 - & 1.0\epsilon_2 + & 0.0\epsilon_3 - & 2x_1 + & w_2 \\
x_2 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 0.0\epsilon_3 + & 3x_1 - & w_2 \\
w_3 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 1.0\epsilon_3 - & x_1 - & w_2
\end{array}$$

x_1 enters and as $\epsilon_2 + \epsilon_3 \ll \epsilon_1 - \epsilon_2$, w_3 leaves. We get

$$\begin{array}{rrrrrrr}
\zeta = & 0.0 + & 0.0\epsilon_1 + & 18\epsilon_2 + & 14\epsilon_3 - & 14w_3 - & 18w_2 \\
\hline
w_1 = & 0.0 + & 1.0\epsilon_1 - & 3.0\epsilon_2 - & 2.0\epsilon_3 + & 2w_3 + & 3w_2 \\
x_2 = & 0.0 + & 0.0\epsilon_1 + & 4.0\epsilon_2 + & 3.0\epsilon_3 - & 3w_3 - & 4w_2 \\
x_1 = & 0.0 + & 0.0\epsilon_1 + & 1.0\epsilon_2 + & 1.0\epsilon_3 - & w_3 - & w_2
\end{array}$$

Done!

Now that we have a Phase I algorithm and a variant of the simplex method that is guaranteed to terminate, we can summarize the main points of this chapter in the following theorem:

THEOREM

For an arbitrary linear program in standard form, the following statements are true:

- *If there is no optimal solution, then the problem is either infeasible or unbounded.*
- *If a feasible solution exists, then a basic feasible solution exists.*
- *If an optimal solution exists, then a basic optimal solution exists.*

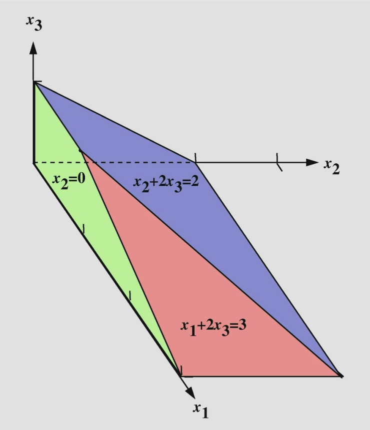
DEFINITION / EXAMPLES OF DEGENERACY

PERTURBATION METHOD

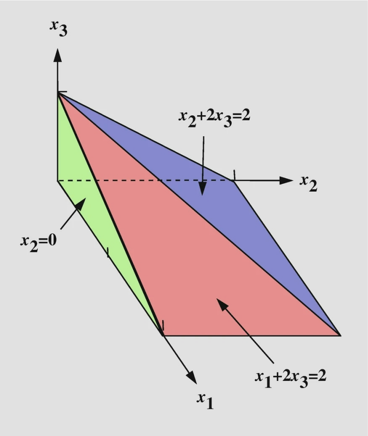
FUNDAMENTAL THEOREM

GEOMETRY

OTHER PIVOT RULES



$$
\begin{array}{rllll}
\max\limits_{x} & x_1 + & 2x_2 + & 3x_3 & \\
\text{s.t.} & x_1 & \quad + & 2x_3 \le 3 & \\
 & & x_2 + & 2x_3 \le 2 & \\
 & x_1, & x_2, & x_3 \ge 0 &
\end{array}
$$



$$\begin{array}{llllll} \max\limits_{x} & x_1 + & 2x_2 + & 3x_3 & & \\ \text{s.t.} & x_1 & + & 2x_3 & \leq & 2 \\ & & x_2 + & 2x_3 & \leq & 2 \\ & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

DEFINITION / EXAMPLES OF DEGENERACY

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OTHER PIVOT RULES

The second pivoting rule we consider is called Bland's rule or smallest index rule.

It stipulates that:

> *Both the entering and the leaving variable be selected from their respective sets of choices by choosing the variable* x_k *with the smallest index* k*.*

THEOREM

The simplex method always terminates provided that both the entering and the leaving variable are chosen according to Bland's rule.

Other rule called random selection rule.

Select at random from the set of possibilities.

Other rule called greatest increase rule.

Pick the entering/leaving pair so as to maximize the increase of the objective function over all other possibilities.

Too much computations.

OTHER PIVOT RULES

OTHER PIVOT RULES