Linear Programming

[V. ch2]: The Simplex Method

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Some examples first

The Simplex Algorithm

Initialization/infeasibility

unboundedness

geometry

Simplex Algorithm

In this chapter, we are going to learn a *method to solve* general linear programs. The method, called *Simplex algorithm*, will be developed for a general linear program (LP) in *standard form*.

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Consider a simple example:

Example

$$\begin{aligned}
\max_x \quad & 5x_1 + 4x_2 + 3x_3 \\
\text{s.t.} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\
& 4x_1 + x_2 + 2x_3 \leq 11 \\
& 3x_1 + 4x_2 + 2x_3 \leq 8 \\
& x_1, \; x_2, \; x_3 \geq 0
\end{aligned}$$

Equalities and Slacks

Start by adding the so-called slack variables and convert *inequality* constraints to *equality* ones.

For each of the less-than inequalities: Introduce a slack variable that represents the difference between the right-hand side and the left-hand side.

⇝ Introducing slack variable w_1

$$2x_1 + 3x_2 + x_3 \leq 5 \iff w_1 = 5 - 2x_1 - 3x_2 - x_3, \quad w_1 \geq 0$$

⇝ Introducing w_2

$$4x_1 + x_2 + 2x_3 \leq 11 \iff w_2 = 11 - 4x_1 - x_2 - 2x_3, \quad w_2 \geq 0$$

⇝ Introducing w_3

$$3x_1 + 4x_2 + 2x_3 \leq 8 \iff w_3 = 8 - 3x_1 - 4x_2 - 2x_3, \quad w_3 \geq 0$$

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We get the following *equivalent* LP

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The simplex method is an iterative process in which:

- ⇝ we start with a less-than-optimal solution $(\dot{x}_1, \dot{x}_2, \cdots, \dot{w}_3)$ that satisfies the equations and non-negativities and then

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- ⇝ we look for a new solution $(\bar{x}_1, \bar{x}_2, \cdots, \bar{w}_3)$, which is better in the sense that it has a larger objective function value:

$$5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 > 5\dot{x}_1 + 4\dot{x}_2 + 3\dot{x}_3$$

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- $\rightsquigarrow$ We continue this process until we arrive at a solution that cannot be improved.

This final solution is then an *optimal* solution.

Initial Solution

Consider our example problem.

$$\begin{aligned} w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \end{aligned}$$

To start the iterative process, we need an *initial feasible solution*.

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Simply set all the *original* variables to *zero*:

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Now, use the equations to determine the slack variables:

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Luckily, we found a *feasible* solution:

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{w}_1, \dot{w}_2, \dot{w}_3) = (0, 0, 0, 5, 11, 8)$$

with objective function value $\zeta = 0$.

Solution Improvement

We now ask whether this solution can be improved.

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Since the coefficient of x_1 in the objective function is *positive*, if we increase the value of x_1 from zero to some positive value, we will increase ζ.

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Observation.
As we change x_1's value, the values of the slack variables will also change. We must make sure that *we do not let any of them go negative.*

Ensuring Non-Negativity

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x_2 and x_3 are currently set to 0, we see that

$$w_1 = 5 - 2x_1,$$

and so keeping w_1 non-negative imposes

$$w_1 \geq 0 \iff 5 - 2x_1 \geq 0 \iff x_1 \leq \frac{5}{2}.$$

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Since *all of these non-negativity conditions* must be met, we see that x_1 cannot be made larger than the smallest of these bounds: $x_1 \leq \frac{5}{2}$.

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Our new solution then is

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (\frac{5}{2}, 0, 0, 0, 1, \frac{1}{2})$$

with objective function value

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We found an improved solution!

Recapitulation

Lets capture what we have done up to now.

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Only this easy because of the special layout!

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What made the first step easy was the fact that we had one group of variables that were initially zero and we had the rest explicitly expressed in terms of these.

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– This special layout is called a dictionary.
In a dictionary, objective and variables on the left are *defined* by variables on the right.

– Dependent variables (on the left) are called basic variables.

– Independent variables (on the right) are called nonbasic variables.

– Setting variables on the right to zero and reading off the values of the variables on the left gives us a dictionary solution.

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Now, use the r.h.s. to describe w_2, w_3 and ζ only with the new set of independent variables: w_1, x_2 and x_3 as

$$\begin{array}{rrrrr} \zeta = & 12.5 - & 2.5w_1 - & 3.5x_2 + & 0.5x_3 \\ \hline x_1 = & 2.5 - & 0.5w_1 - & 1.5x_2 - & 0.5x_3 \\ w_2 = & 1 + & 2w_1 + & 5x_2 & \\ w_3 = & 0.5 + & 1.5w_1 + & 0.5x_2 - & 0.5x_3 \end{array}$$

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Note.
We can recover our current solution by setting the *independent* (non-basic) variables to zero and using the equations to read off the values for the dependent (basic) variables.

Next Improvement

Having the current (dictionary) solution and its corresponding dictionary, we can look for any further improvement.

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Next Improvement

Having the current (dictionary) solution and its corresponding dictionary, we can look for any further improvement.

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x_1 = & 2.5 - & 0.5w_1 - & 1.5x_2 - & 0.5x_3 \\
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$\rightarrow$ x_3 could be increased up to 1.

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Set $x_3 = 1$ and re-compute dependent (basic) variable values according to the defining equations:

$$\begin{aligned} x_1 &= 2.5 - 0.5x_3 \\ w_2 &= 1 \\ w_3 &= 0.5 - 0.5x_3 \end{aligned}$$

we get

$$x_1 = 2, \quad w_2 = 1, \quad w_3 = 0.$$

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with objective function value

$$\zeta = 5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 = 13.$$

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We found an improved solution!

Retaining the Dictionary

In order to retain a dictionary layout for this solution, use w_2's defining equation and re-write it as

$$w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3 \iff x_3 = 1 + 3w_1 + x_2 - 2w_3 .$$

Now, use the right-hand side to describe x_1, w_2 and ζ only with the new set of independent variables: w_1, x_2 and w_3 as

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We got to the equation

$$\zeta = 13 - w_1 - 3x_2 - w_3$$

by equivalence-preserving steps using only the constraints of our linear program!