LINEAR PROGRAMMING

[V. CH10]: THE TRAVELING SALESMAN PROBLEM

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January 31, 2023

WHAT? WHY? HOW?

CUTTING PLANES

EXCLUDING EDGES

NUMERICAL ISSUES

WHAT (PRECISELY) IS THE TSP?

In the TSP, we are given a set of n *cities* V with distances $d(i, j)$ between each other.
Our goal is to find a shortest tour that visits each city and returns to the starting point.
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- the distances are symmetric, i.e., $d(i, j) = d(j, i)$, i.e., the graph is undirected,
- we often assume the triangle inequality, i.e., $d(i, k) \leq d(i, j) + d(j, k)$.

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Most important reason for studying it here:
It is one of the main drivers behind the development and improvement of IP solving, and IP solving is exceptionally successful for the TSP. Many ideas that improved (M)IP solvers over the years have been first developed and applied in the context of the TSP.

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- That means we add *violated* subtour elimination constraints as needed while solving.

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  - Without losing provable optimality!
- Numerical issues can cause solutions obtained from our LP solver to not be strictly feasible/optimal. If we really want provable optimality, we have to deal with this (often ignored) problem.

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What about non-integer solutions?

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There are also separation problems for specific types of cutting planes, e.g., for Gomory Cuts, Subtour Elimination Constraints, . . . ; those can sometimes be solved efficiently (and are often only solved heuristically).

Subtour Elimination Constraints — Separation Problem

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Example in an interactive tool: https://www.math.uwaterloo.ca/tsp/app/diy.html

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Usually, we cannot afford to do minimum graph cut computations all the time; we go through the list from the top and use the more expensive methods sparingly.

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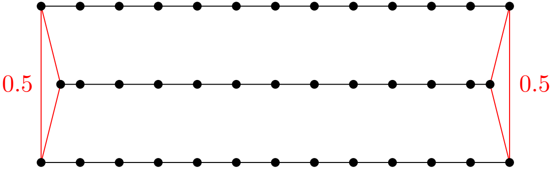

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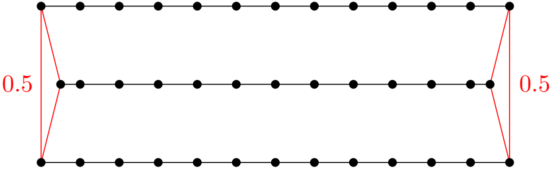

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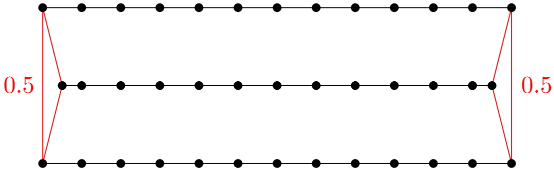

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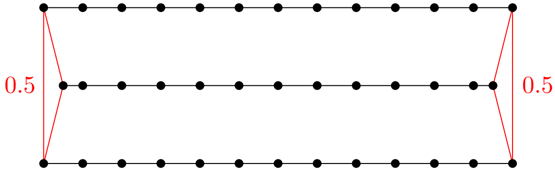

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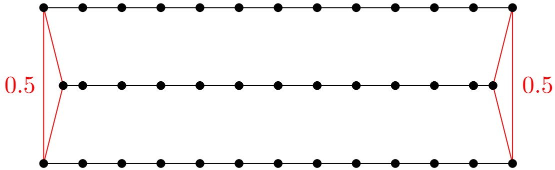

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For generalized versions of the above example, the length of the optimal tour is about $4/3$ of the optimal value of the LP relaxation including all subtours. The $4/3$ conjecture states that this is the worst case, i.e., for all sets of cities, the factor between the optimal integer solution and the optimal fractional solution is at most $4/3$. In general, this factor is called *integrality gap*.

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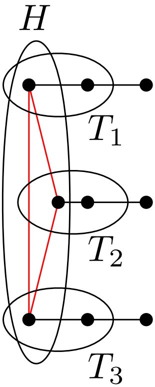

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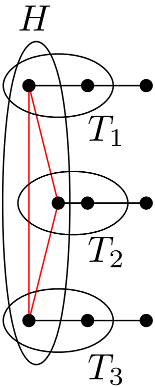

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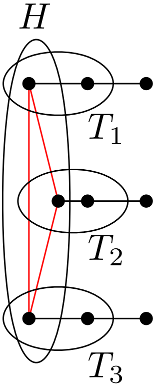

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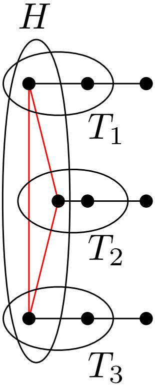

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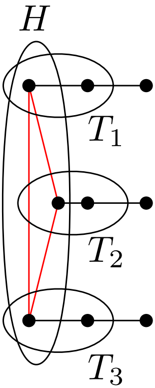

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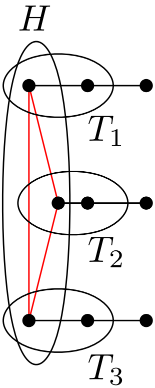

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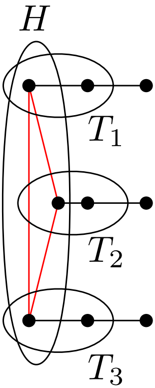

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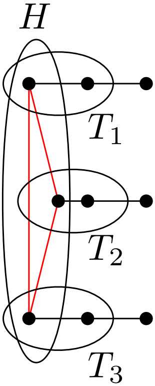

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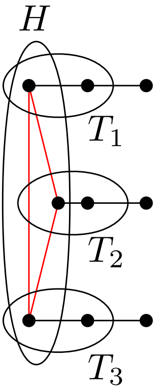

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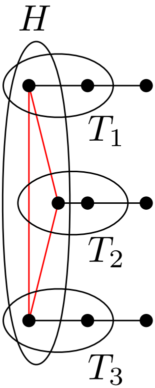

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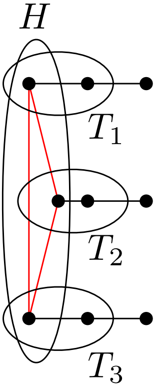

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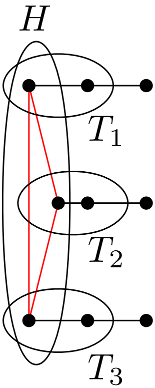

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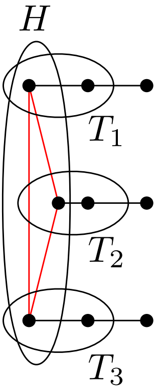

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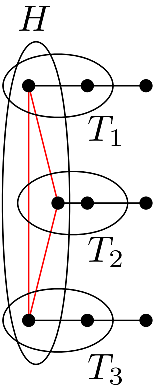

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How can we separate cutting planes of this type?

COMB INEQUALITIES — SEPARATION

For a given (fractional) solution to the linear relaxation of our IP, the separation problem for combs asks for finding a violated comb inequality, i.e., some H and T_i for which $P_1 + P_2 < 3k + 1$, or to find that no such sets exist.

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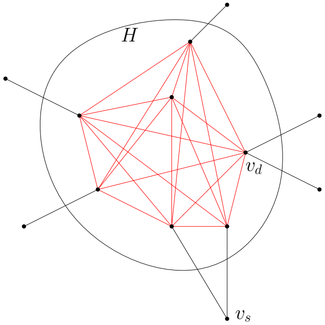
Example in interactive tool: https://www.math.uwaterloo.ca/tsp/app/diy.html

Comb Inequalities — Odd Component Heuristic

The odd component heuristic:



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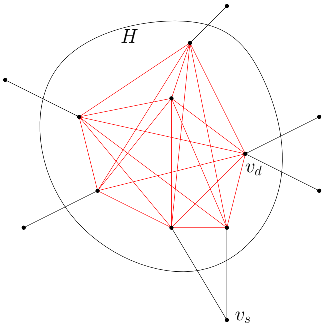

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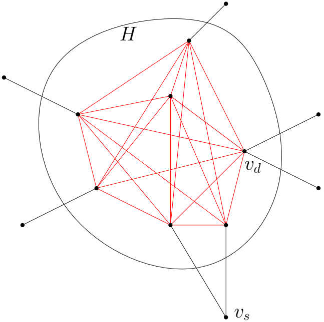

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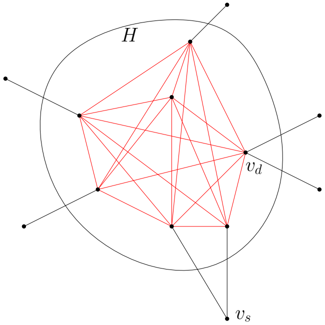

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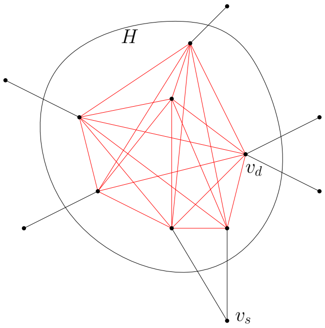

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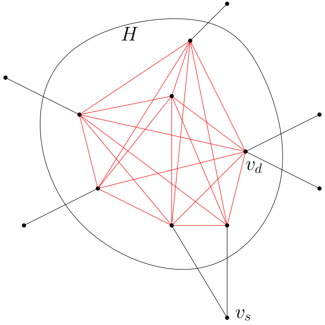

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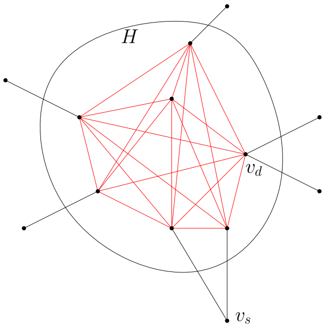

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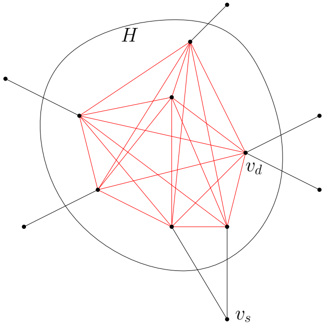

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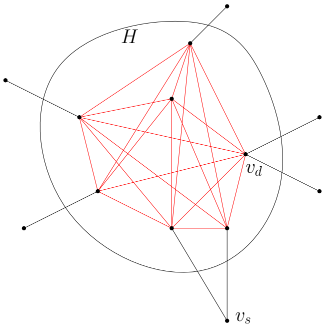

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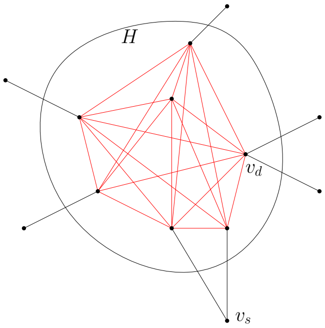

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  - If that number is 3 or higher, we make the crossing edges into teeth T_i, and have likely found a violated comb inequality.

More Cutting Planes

There are more cutting planes that are used by powerful tools like Concorde, for which we do not have the time. A common theme among them is that they always can be translated into the following form:

$$\sum_{S \in \mathcal{F}} \sum_{e \in \delta(S)} x_e \geq \mu,$$

for some family of vertex sets $\mathcal{F}$ and some suitable integral constant μ. This allows efficient storage (only store vertex sets and μ) and also changes to the set of edges.

What? Why? How?

Cutting Planes

Excluding Edges

Numerical Issues

Dynamic Edge Set

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- We can use our formulation to solve the TSP on a smaller edge set. As we discard some potential edges, the solution might be suboptimal if any edges from the optimal solution are removed.

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- Is there a way to exclude lots of edges at once and only consider fewer remaining ones?

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$$\min \sum_{e} c_e x_e \text{ s.t.}$$

$$\forall v \in V : \sum_{e \in \delta(\{v\})} x_e = 2,$$

$$\forall e \in E : x_e \leq 1,$$

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If we can price many variables without explicitly considering each individually, like in the case of geometric instances, the term *batch pricing* is sometimes used.

WHAT? WHY? HOW?

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EXCLUDING EDGES

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- Run regular LP solver, then take the optimal basis and use it as starting point for an exact rational Simplex — still quite expensive!

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- Cutting planes are used purely to speed up the search. Cutting planes that are only violated by a tiny amount are unlikely to be good; if we avoid them, the solver should not consider our cutting planes as non-violated.
- The most problematic issue is accidental dual infeasibility (i.e., non-optimality): it means that our bounds used for pruning are potentially invalid. What can we do?

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- For each dual constraint, check feasibility (in exact rational or fixed-point arithmetic).

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- Start with the dual solution y.
- Turn it into rational (more expensive, better bounds) or fixed-point (less expensive) numbers.
- For each dual constraint, check feasibility (in exact rational or fixed-point arithmetic).
- For each dual infeasibility found, increase y_{vw}, the dual slack variable for $x_{vw} \leq 1$, to make the dual constraint

$$y_v + y_w - y_{vw} + \sum_{(\mathcal{F},\mu_{\mathcal{F}})\in\mathcal{H}} \pi(vw,\mathcal{F})y_{\mathcal{F}} \leq c_{vw}$$

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- Subtract all the infeasibilities from the objective according to the term $-y_{vw}$ in the dual objective.
- The result is a valid (possibly slightly super-optimal) bound.

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- Dynamic Removal of Cutting Planes is performed, but cannot really be controlled.
- Auto-generated cutting planes are usually not accessible.
- Dynamic handling of variables is usually impossible in a (M)IP.
- There are frameworks such as SCIP that allow more flexibility, but usually worse performance.