Mathematical Methods of Algorithmics

Chapter 1: Introduction to Linear Programming

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Meet Your Teachers

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INTRODUCTION

MOTIVATION

DEFINITIONS

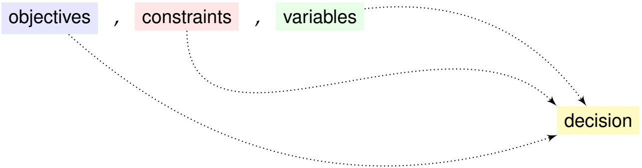
ORGANIZATION

- As usual, the module has a „Prüfungsleistung" and a „Studienleistung".
- The „Prüfungsleistung" will most likely be an oral exam, depending mostly on the number of participants. The „Prüfungsleistung" determines your grade.
- The „Studienleistung" is tied to the homework sheets. We will start homework sheets next week.
- You have two weeks to solve each homework assignment.
- To get the „Studienleistung", each student needs to present a *sufficiently correct and complete* solution to two exercises (in total) in the tutorial.
- The solution presentation for a homework assignment is done in the tutorial two weeks after the homework is published.
- As usual, there is a lecture (one per week) and a tutorial class (one per week, every other week being dedicated to homework discussion). The lecture is where the main content is presented. The tutorial adds additional content, practical stuff, shows applications, examples, and discusses questions related to the content.
- There is a mailing list and a course website. Please refer to that site instead of QIS/StudIP for information. Please sign up for the mailing list; you might miss important announcements otherwise.
  - https://www.ibr.cs.tu-bs.de/courses/ws2223/mma/
  - https://lists.ibr.cs.tu-bs.de/postorius/lists/mma.ibr.cs.tu-bs.de

CONTENT

What is this course about?

The mathematics behind making optimal decisions [1]

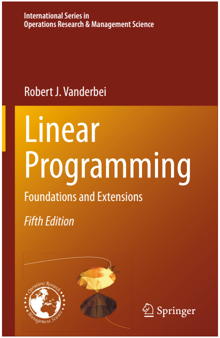


[1] https://stellato.io/downloads/teaching/orf522/01_lecture.pdf

LITERATURE

The main reference for this course:
[V] R. J. Vanderbei. Linear Programming: Foundations and Extensions. Springer Nature (2020).
Can be accessed through SpringerLink from the university network:

https://link.springer.com/book/10.1007/978-3-030-39415-8



INTRODUCTION

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DEFINITIONS

Managing a Production Facility

Consider a production facility which is capable of producing a variety of products, say n products. We enumerate these products as $1, 2, \ldots, n$.

These products are made from certain raw materials. Suppose that there are m different raw materials, which again we simply enumerate as $1, 2, \ldots, m$.

MANAGING A PRODUCTION FACILITY

Further properties:

- The facility has, for each raw material $i = 1, 2, \ldots, m$, a known amount, say b_i, on hand.
- Each raw material has, at this moment in time, a known unit market value. We denote the unit value of the ith raw material by ρ_i.
- Producing one unit of product j requires a certain known amount, say a_{ij} units, of raw material i.
- The jth final product can be sold at the known market price of σ_j dollars per unit.

MANAGING A PRODUCTION FACILITY

Let us assume that the production manager decides to produce one unit of the jth product.

- Revenue of one unit of product j is σ_j
- Cost of producing one unit of j is $\sum_{i=1}^{m} \rho_i a_{ij}$

Therefore, the net revenue associated with the production of one unit of j is the difference between the revenue and the cost.

$$c_j = \sigma_j - \sum_{i=1}^{m} \rho_i a_{ij}, \quad j = 1, 2, \ldots, n$$

For our optimization, we do not really care about the individual material costs; we only need to know the net revenue c_j associated with each product.

Managing a Production Facility

Let us capture the available information up to now:

$$
\begin{array}{c|ccc|c}
 & \text{product } 1 & \cdots & \text{product } n & \\
 & \boxed{c_1 \quad \cdots \quad c_n} & & & \\
\text{raw material } 1 & a_{11} & \cdots & a_{1n} & b_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\text{raw material } m & a_{m1} & \cdots & a_{mn} & b_m
\end{array}
$$

Maximizing Revenue

The problem we wish to consider is the one faced by the companies' production manager.

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→ Her goal is to find values x_j to maximize this quantity.

CONSTRAINTS

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→ She cannot produce more product than she has raw material for. The amount of raw material i consumed by a given production schedule is

$$\sum_{j=1}^{n} a_{ij} x_j,$$

so she must adhere to the following constraints

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m.$$

Objective

$$\max \quad \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad , \quad i = 1, 2, \cdots, m$$

$$x_j \geq 0 \quad , \quad j = 1, 2, \cdots, n$$

$$x_j \quad , \quad j = 1, 2, \cdots, n$$

Constraints

variables

EXAMPLE

Resource allocation in a toy factory. [2]

	toy 1	toy 2	toy 3	toy 4	toy 5	
	\$15	\$30	\$20	\$25	\$25	
1. Red paint	0	1	0	1	3	625
2. Blue paint	3	1	0	1	0	640
3. White paint	2	1	2	0	2	1100
4. Plastic	1	5	2	2	1	875
5. Wood	3	0	3	5	5	2200
6. Glue	1	2	3	2	3	1500

[2] https://www.exceldemy.com/allocating-resources-in-excel-using-solver/

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$$
\begin{array}{lrl}
\max\limits_{x} & 15x_1 + 30x_2 + 20x_3 + 25x_4 + 25x_5 & \\
s.t. & 0x_1 + 1x_2 + 0x_3 + 1x_4 + 3x_5 \leq & 625 \\
 & 3x_1 + 1x_2 + 0x_3 + 1x_4 + 0x_5 \leq & 640 \\
 & 2x_1 + 1x_2 + 2x_3 + 0x_4 + 2x_5 \leq & 1100 \\
 & 1x_1 + 5x_2 + 2x_3 + 2x_4 + 1x_5 \leq & 875 \\
 & 3x_1 + 0x_2 + 3x_3 + 5x_4 + 5x_5 \leq & 2200 \\
 & 1x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 \leq & 1500 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0 &
\end{array}
$$

Blue paint cons.

Wood cons.

(Linear Programming formulation of the problem)

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Linear Program

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→ In the examples, there have been variables whose values are to be decided in some optimal fashion. These variables are referred to as *decision variables*. They are usually denoted as

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Note: No multiplication of decision variables with each other!

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→ An inequality constraint

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

can be converted to an equality constraint by adding a *nonnegative* variable, w, called *slack variable*:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + w = b, \quad w \geq 0.$$

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- A $\geq$-constraint can be transformed to $\leq$ by negating both sides:

$$\sum_i a_ix_i \geq b_i \Leftrightarrow \sum_i -a_ix_i \leq -b_i.$$

STANDARD FORM

There is no a priori preference for how one poses the constraints (as long as they are linear, of course). However, from a mathematical point of view, there is a preferred presentation.

Linear program in *Standard Form* representation:

- Consider a $\max$ problem,
- pose the inequalities in $\leq$-form,
- stipulate that all the decision variables be nonnegative.

$$
\begin{aligned}
\max_{x} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
& \qquad\qquad\qquad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\
& \qquad x_1, x_2, \ldots, x_n \geq 0.
\end{aligned}
$$

SOLUTIONS & FEASIBILITY

A proposal of *specific values* for the decision variables is called a *solution*.

- A solution $(x_1, x_2, \cdots, x_n)$ is called *feasible* if it satisfies all of the constraints.
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Some problems are just simply infeasible. Consider

$$\begin{aligned} \max_{x} \quad & 5x_1 + 4x_2 \\ s.t. \quad & x_1 + x_2 \leq 2 \\ & -2x_1 - 2x_2 \leq -9 \\ & x_1, x_2 \geq 0. \end{aligned}$$

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- If a problem has no feasible solution, then the problem itself is called *infeasible*.

UNBOUNDEDNESS

At the other extreme from infeasible problems, one finds unbounded problems.

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⇝ Set x_2 to zero and let x_1 be arbitrarily large.
⇝ As long as $x_1 \geq 2$, the solution will be feasible, and
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In addition to finding optimal solutions to linear programming problems, we are going to *detect* when a problem is infeasible or unbounded.

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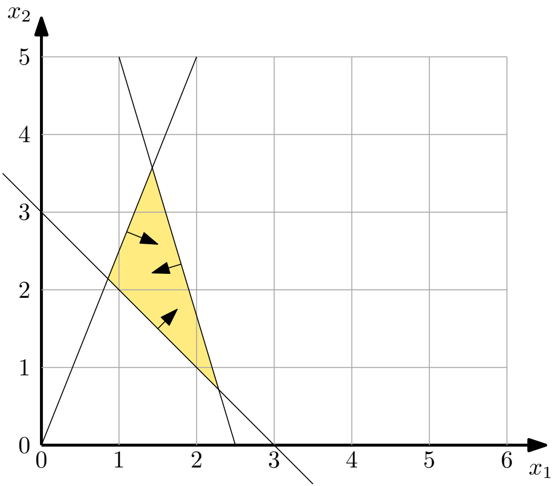
Why do we not allow $<$ and $>$ constraints in linear programs?
Because X would not be closed: $\max x$ s.t. $x < 1$?

GEOMETRY

Find linear inequalities whose intersection makes the yellow region (feasible space).



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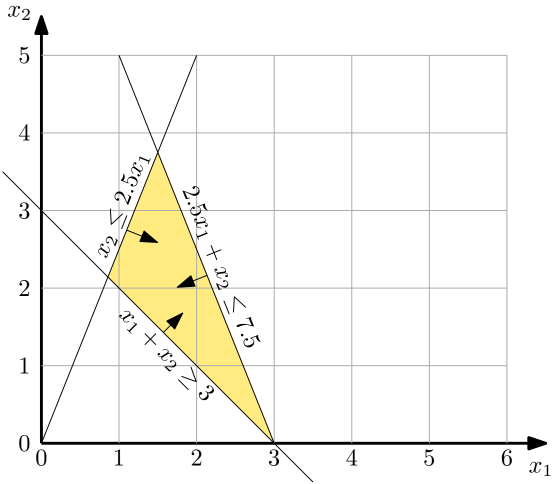

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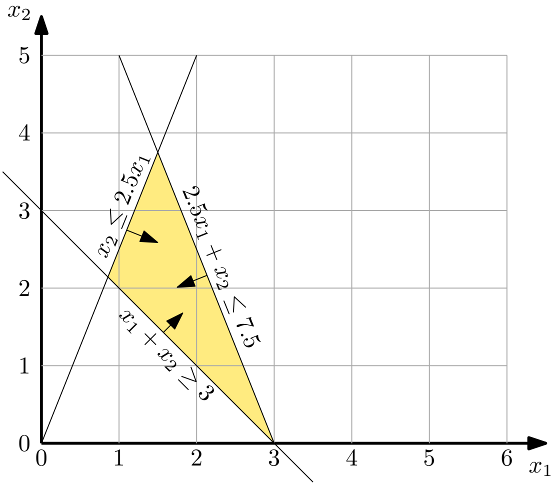

Up next: An algorithm to solve linear programs!