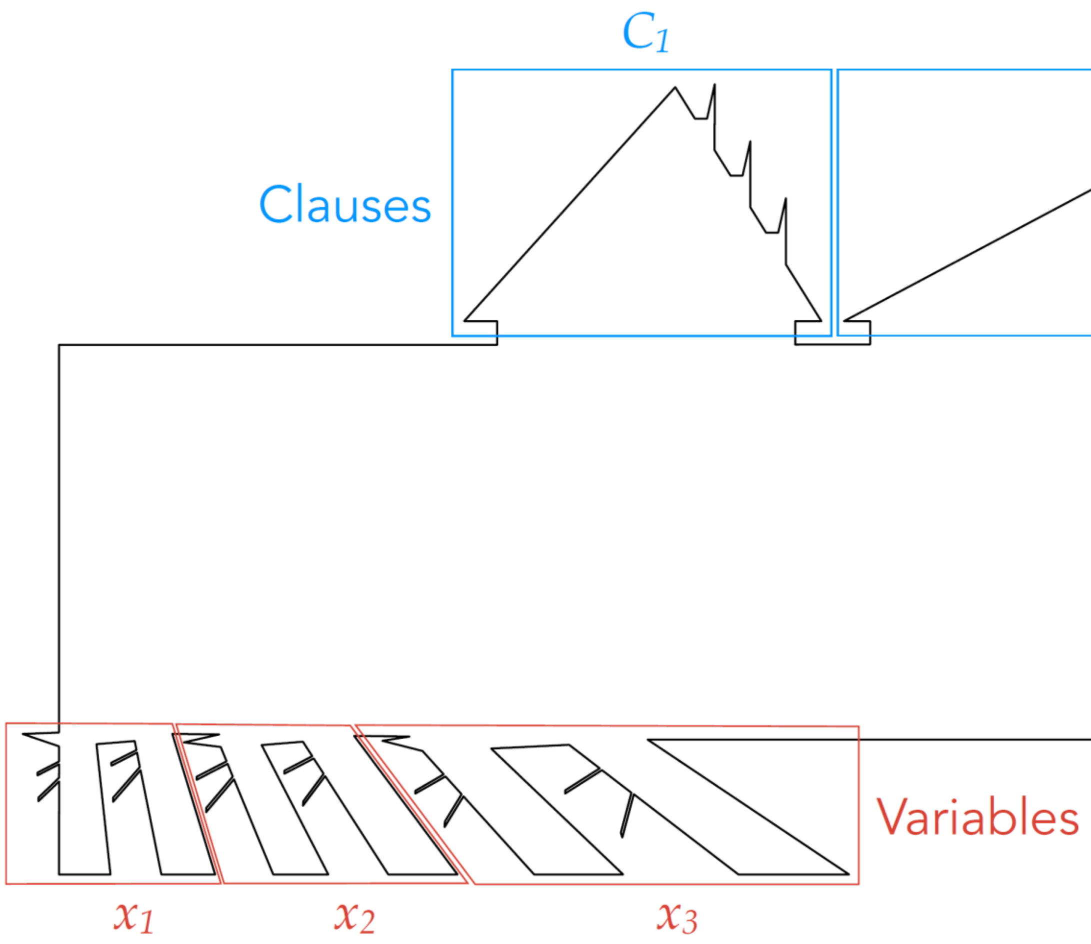
Technische Universität Braunschweig

Computational Geometry – The Art Gallery Problem

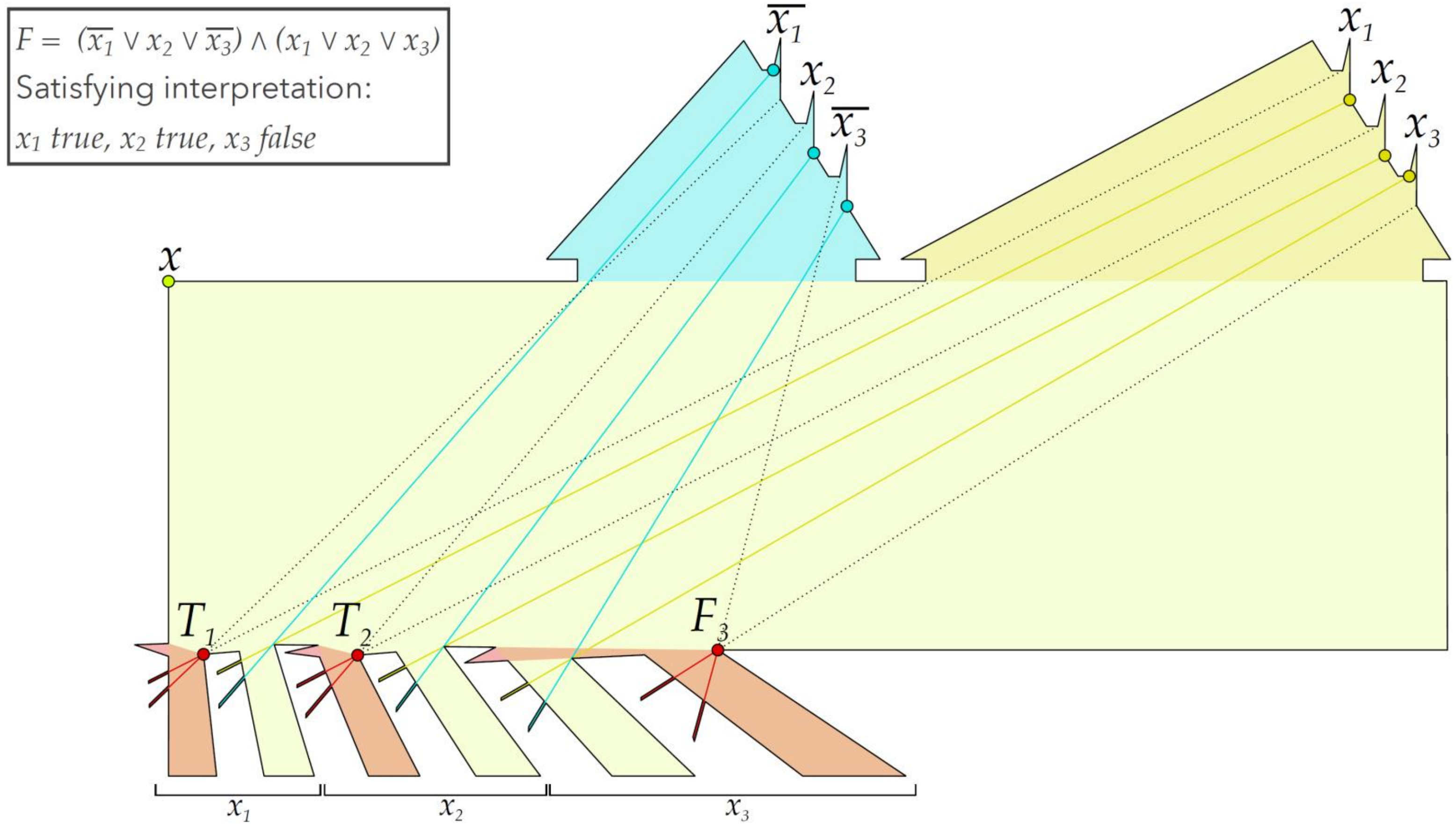
January 26th, 2023

Art Gallery Problem – NP-hardness

Example. $F = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$

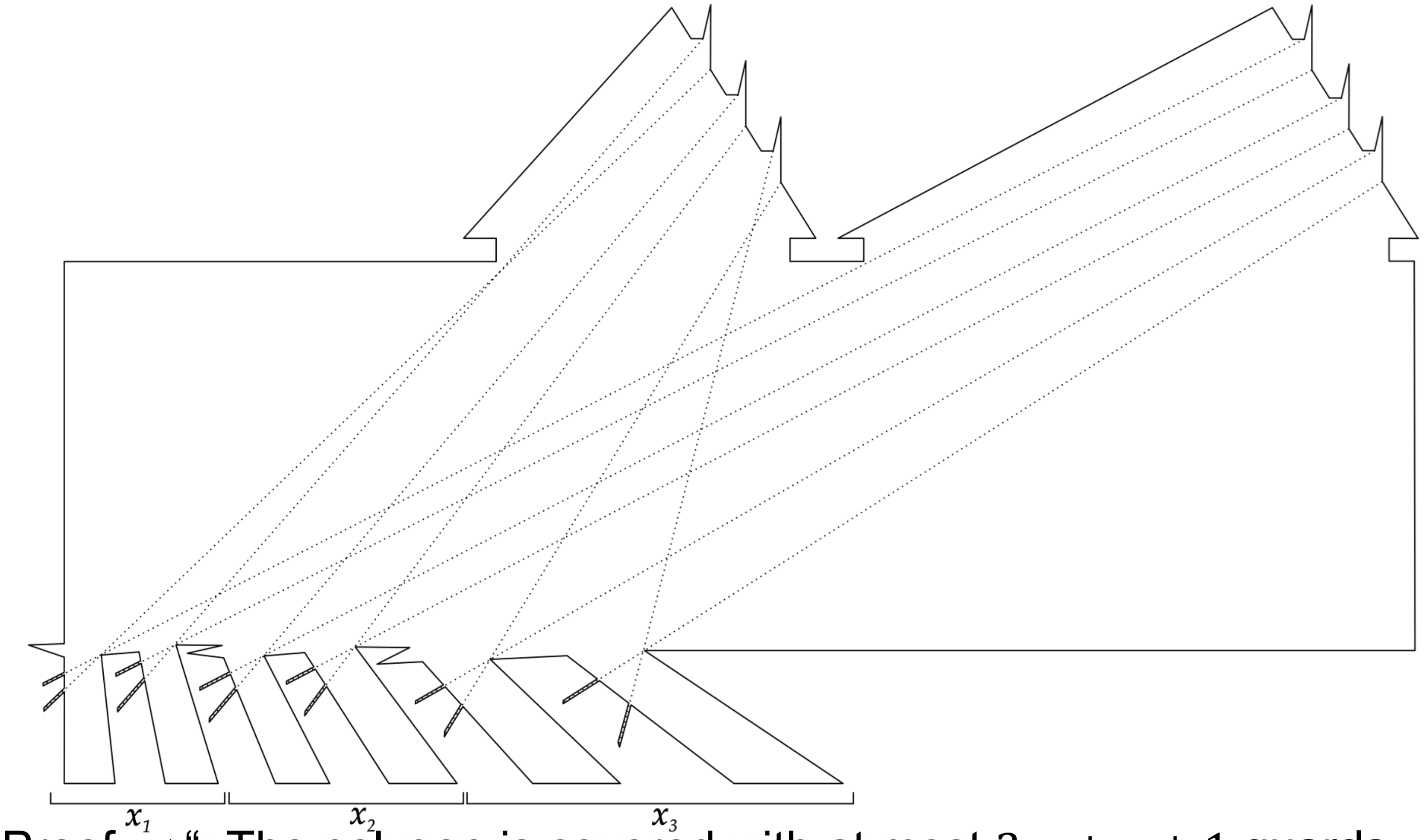


Art Gallery Problem – NP-hardness



Proof „⇐“: Consider a satisfying assignment of F

Art Gallery Problem – NP-hardness



Proof „⇒“: The polygon is covered with at most $3m + n + 1$ guards

Art Gallery Problem – Irrational guards

Indeed, Sándor Fekete posed at MIT in 2010 and at Dagstuhl in 2011 an open problem, asking whether there are polygons requiring irrational coordinates in an optimal guard set [1, 17]. The question has been raised again by Günter Rote at EuroCG 2011 [26]. It has also been mentioned by Rezende *et al.* [13]: "it remains an open question whether there are polygons given by rational coordinates that require optimal guard positions with irrational coordinates". A similar question has been raised by Friedrichs *et al.* [19]: "[. . .] it is a long-standing open problem for the more general Art Gallery Problem (AGP): For the AGP it is not known whether the coordinates of an optimal guard cover can be represented with a polynomial number of bits".

Our results. We answer the open question of Sándor Fekete, by proving the following main result of our paper. Recall that a polygon $\mathcal{P}$ is called *monotone* if there exists a line l such that every line orthogonal to l intersects $\mathcal{P}$ at most twice.

Theorem 1. *There is a simple monotone polygon $\mathcal{P}$ with integer coordinates of the vertices such that*

(i) $\mathcal{P}$ can be guarded by 3 guards placed at points with irrational coordinates, and

(ii) an optimal guard set of $\mathcal{P}$ with guards at points with rational coordinates has size 4.

Irrational Guards are Sometimes Needed*

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Abstract

In this paper we study the *art gallery problem*, which is one of the fundamental problems in computational geometry. The objective is to place a minimum number of guards inside a simple polygon so that the guards together can see the whole polygon. We say that a guard at position x sees a point y if the line segment xy is contained in the polygon.

Despite an extensive study of the art gallery problem, it remained an open question whether there are polygons given by integer coordinates that require guard positions with irrational coordinates in any optimal solution. We give a positive answer to this question by constructing a *monotone* polygon with integer coordinates that can be guarded by three guards only when we allow to place the guards at points with irrational coordinates. Otherwise, four guards are needed. By extending this example, we show that for every n, there is a polygon which can be guarded by $3n$ guards with irrational coordinates but needs $4n$ guards if the coordinates have to be rational. Subsequently, we show that there are rectilinear polygons given by integer coordinates that require guards with irrational coordinates in any optimal solution.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems

Keywords and phrases art gallery problem, computational geometry, irrational numbers

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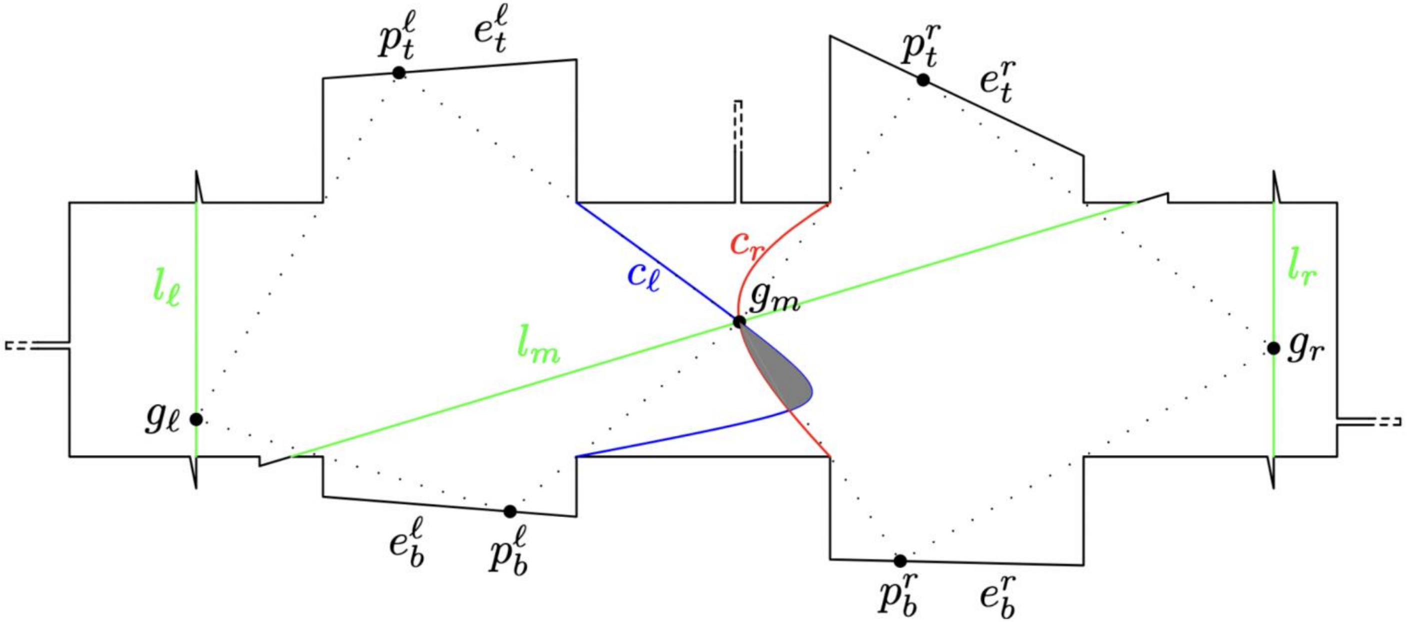

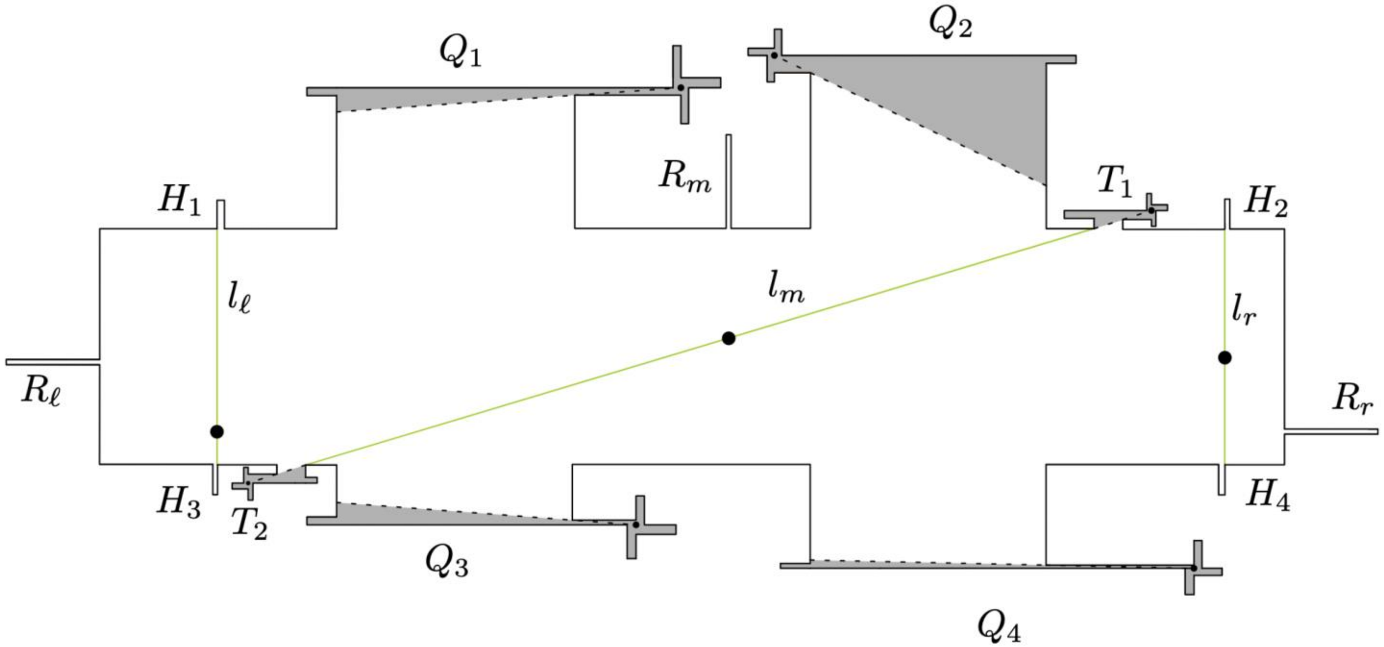

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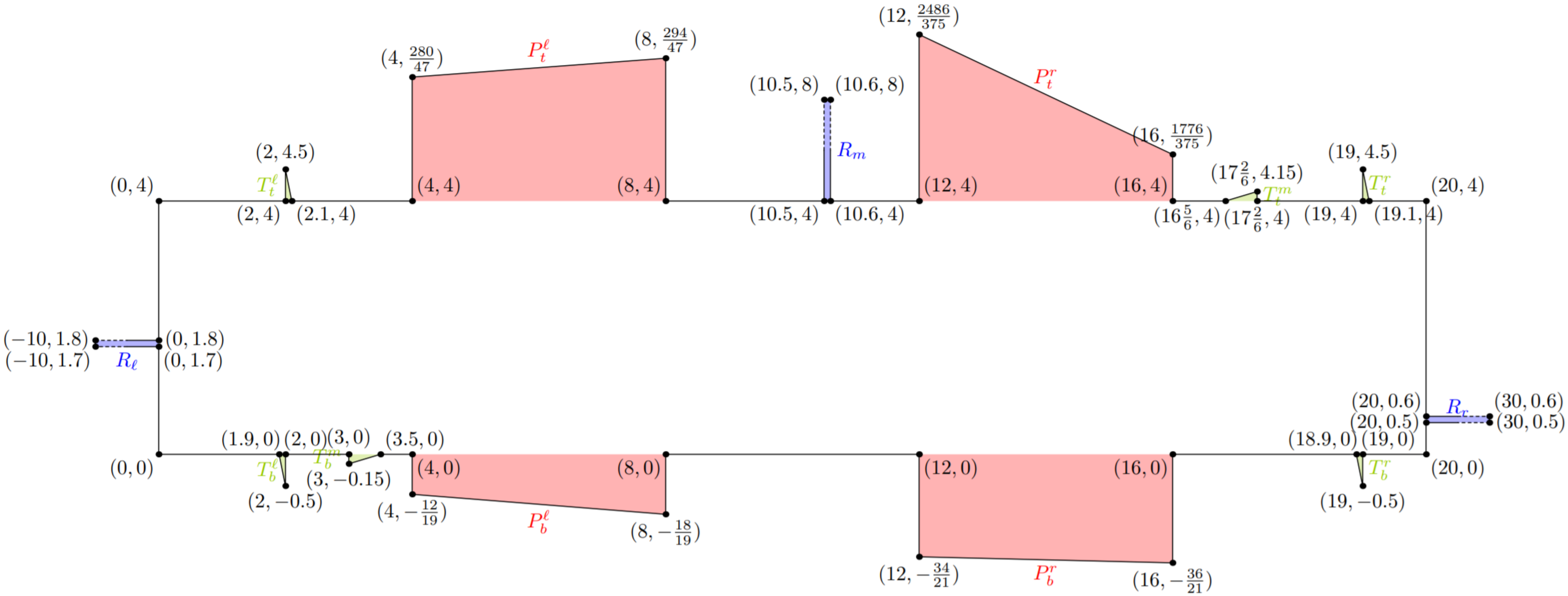
$(12, \frac{2486}{375})$
$(4, \frac{280}{47})$
P_t^ℓ
$(8, \frac{294}{47})$
$(10.5, 8)$
$(10.6, 8)$
P_t^r
R_m
$(16, \frac{1776}{375})$
$(2, 4.5)$
$(17\frac{2}{6}, 4.15)$
$(19, 4.5)$
$(0, 4)$
T_t^ℓ
$(4, 4)$
$(8, 4)$
$(12, 4)$
$(16, 4)$
T_t^m
T_t^r
$(20, 4)$
$(2, 4)$
$(2.1, 4)$
$(10.5, 4)$
$(10.6, 4)$
$(16\frac{5}{6}, 4)$
$(17\frac{2}{6}, 4)$
$(19, 4)$
$(19.1, 4)$
$(-10, 1.8)$
$(0, 1.8)$
$(-10, 1.7)$
R_ℓ
$(0, 1.7)$
$(20, 0.6)$
R_r
$(30, 0.6)$
$(20, 0.5)$
$(30, 0.5)$
$(1.9, 0)$
$(2, 0)$
$(3, 0)$
$(3.5, 0)$
$(18.9, 0)$
$(19, 0)$
$(0, 0)$
T_b^ℓ
T_b^m
$(4, 0)$
$(8, 0)$
$(12, 0)$
$(16, 0)$
T_b^r
$(20, 0)$
$(3, -0.15)$
$(2, -0.5)$
$(19, -0.5)$
$(4, -\frac{12}{19})$
P_b^ℓ
$(8, -\frac{18}{19})$
$(12, -\frac{34}{21})$
P_b^r
$(16, -\frac{36}{21})$

Art Gallery Problem – $\exists\mathbb{R}$-completeness

The Art Gallery Problem is $\exists\mathbb{R}$-complete

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The *Art Gallery Problem* (AGP) is a classic problem in computational geometry, introduced in 1973 by Victor Klee. Given a simple polygon $\mathcal{P}$ and an integer k, the goal is to decide if there exists a set G of k *guards* within $\mathcal{P}$ such that every point $p \in \mathcal{P}$ is seen by at least one guard $g \in G$. Each guard corresponds to a point in the polygon $\mathcal{P}$, and we say that a guard g *sees* a point p if the line segment pg is contained in $\mathcal{P}$.

We prove that the AGP is $\exists\mathbb{R}$-complete, implying that (1) any system of polynomial equations over the real numbers can be encoded as an instance of the AGP, and (2) the AGP is not in the complexity class NP unless NP $= \exists\mathbb{R}$. As a corollary of our construction, we prove that for any real algebraic number α, there is an instance of the AGP where one of the coordinates of the guards equals α in any guard set of minimum cardinality. That rules out many natural geometric approaches to the problem, as it shows that any approach based on constructing a finite set of candidate points for placing guards has to include points with coordinates being roots of polynomials with arbitrary degree. As an illustration of our techniques, we show that for every compact semi-algebraic set $S \subseteq [0,1]^2$, there exists a polygon with corners at rational coordinates such that for every $p \in [0,1]^2$, there is a set of guards of minimum cardinality containing p if and only if $p \in S$.

In the $\exists\mathbb{R}$-hardness proof for the AGP, we introduce a new $\exists\mathbb{R}$-complete problem ETR-INV. We believe that this problem is of independent interest, as it has already been used to obtain $\exists\mathbb{R}$-hardness proofs for other problems.

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CCS Concepts: • **Theory of computation → Computational geometry**; **Problems, reductions and completeness**; **Complexity classes;**

Additional Key Words and Phrases: Art gallery problem, existential theory of the reals

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Other complete problems for the existential theory of the reals include:

- the art gallery problem of finding the smallest number of points from which all points of a given polygon are visible.[22]
- the packing problem of deciding whether a given set of polygons can fit in a given square container.[23]
- recognition of unit distance graphs, and testing whether the dimension or Euclidean dimension of a graph is at most a given value.[9]
- stretchability of pseudolines (that is, given a family of curves in the plane, determining whether they are homeomorphic to a line arrangement);[4][24][25]
- both weak and strong satisfiability of geometric quantum logic in any fixed dimension >2;[26]
- Model checking interval Markov chains with respect to unambiguous automata[27]
- the algorithmic Steinitz problem (given a lattice, determine whether it is the face lattice of a convex polytope), even when restricted to 4-dimensional polytopes;[28][29]
- realization spaces of arrangements of certain convex bodies[30]
- various properties of Nash equilibria of multi-player games[31][32][33]
- embedding a given abstract complex of triangles and quadrilaterals into three-dimensional Euclidean space;[17]
- embedding multiple graphs on a shared vertex set into the plane so that all the graphs are drawn without crossings;[17]
- recognizing the visibility graphs of planar point sets;[17]
- (projective or non-trivial affine) satisfiability of an equation between two terms over the cross product;[34]
- determining the minimum slope number of a non-crossing drawing of a planar graph;[35]
- recognizing graphs that can be drawn with all crossings at right angles;[36]
- the partial evaluation problem for the MATLANG+eigen matrix query language.[37]
- the low-rank matrix completion problem.[38]

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Problem Variant – Dispersive Art Gallery Problem

The Dispersive Art Gallery Problem

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Abstract

We introduce a new variant of the art gallery problem that comes from safety issues. In this variant we are not interested in guard sets of smallest cardinality, but in guard sets with largest possible distances between these guards. To the best of our knowledge, this variant has not been considered before. We call it the DISPERSIVE ART GALLERY PROBLEM. In particular, in the dispersive art gallery problem we are given a polygon $\mathcal{P}$ and a real number ℓ, and want to decide whether $\mathcal{P}$ has a guard set such that every pair of guards in this set is at least a distance of ℓ apart.

In this paper, we study the vertex guard variant of this problem for the class of polyominoes. We consider rectangular visibility and distances as geodesics in the L_1-metric. Our results are as follows. We give a (simple) thin polyomino such that every guard set has minimum pairwise distances of at most 3. On the positive side, we describe an algorithm that computes guard sets for simple polyominoes that match this upper bound, i.e., the algorithm constructs worst-case optimal solutions. We also study the computational complexity of computing guard sets that maximize the smallest distance between all pairs of guards within the guard sets. We prove that deciding whether there exists a guard set realizing a minimum pairwise distance for all pairs of guards of at least 5 in a given polyomino is NP-complete.

Idea: Maintain a minimal geodesic distance between any two guards

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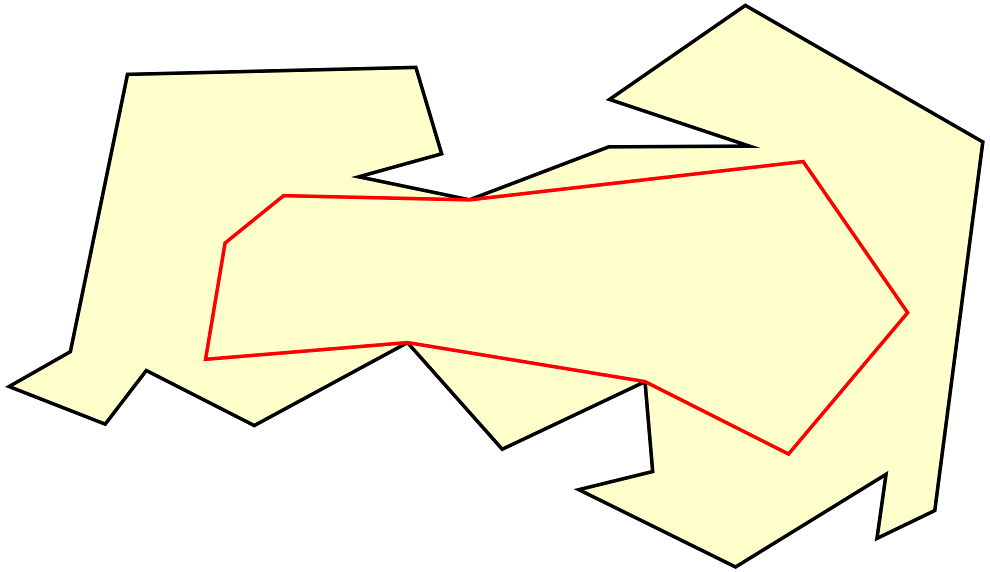
Related Problems – Watchman Route Problem

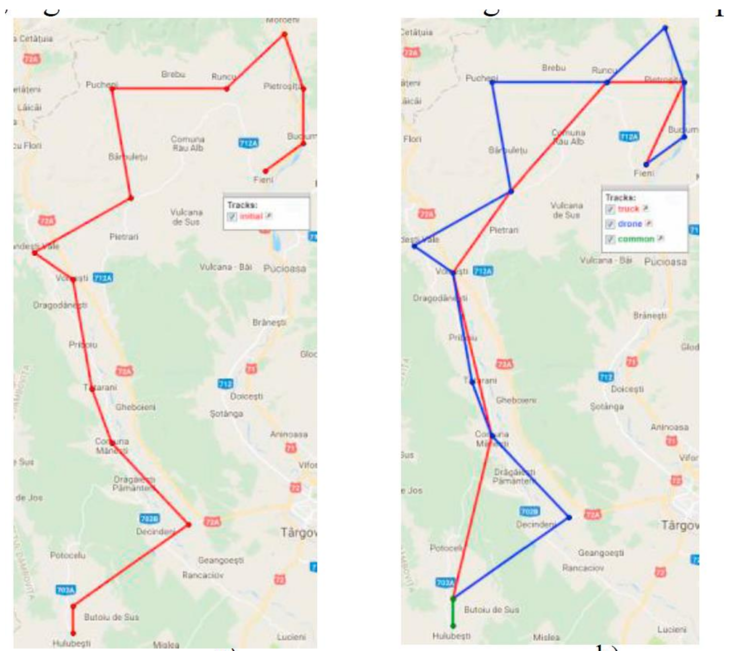
- **Given:** A (simple) polygon P
- **Wanted:** A shortest route within P such that all of P is visible from some point of the route.



- Further variants exist – multiple guards, points of interest, etc. (see lecture!)

Related Problems – “Drone Delivery” / Vehicle Routing Problem

- **Given:** A set of deliveries and a single (or fixed number of) drones with limited power on a truck
- **Wanted:** A shortest route for the truck such that all drone-based deliveries can be made.



a) b)