Computational Geometry
Chapter 6: Point Triangulation

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Universität
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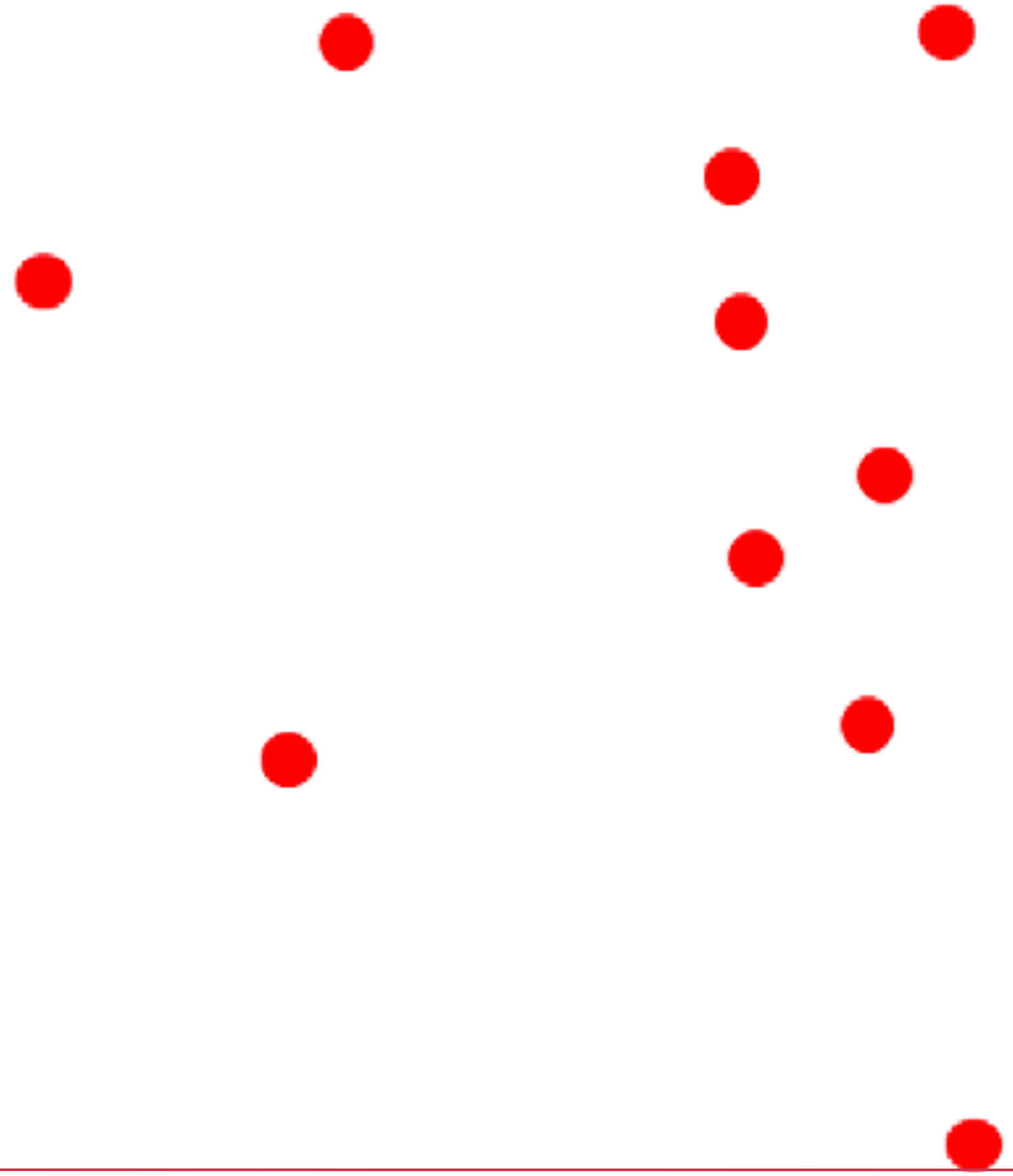
1. Introduction
2. Minimum angle and Delaunay triangulations
3. Minimum-weight triangulations
4. Min-max edge triangulations
5. Max-min edge triangulations

1 Introduction: The Problem

Triangulations

Triangulations



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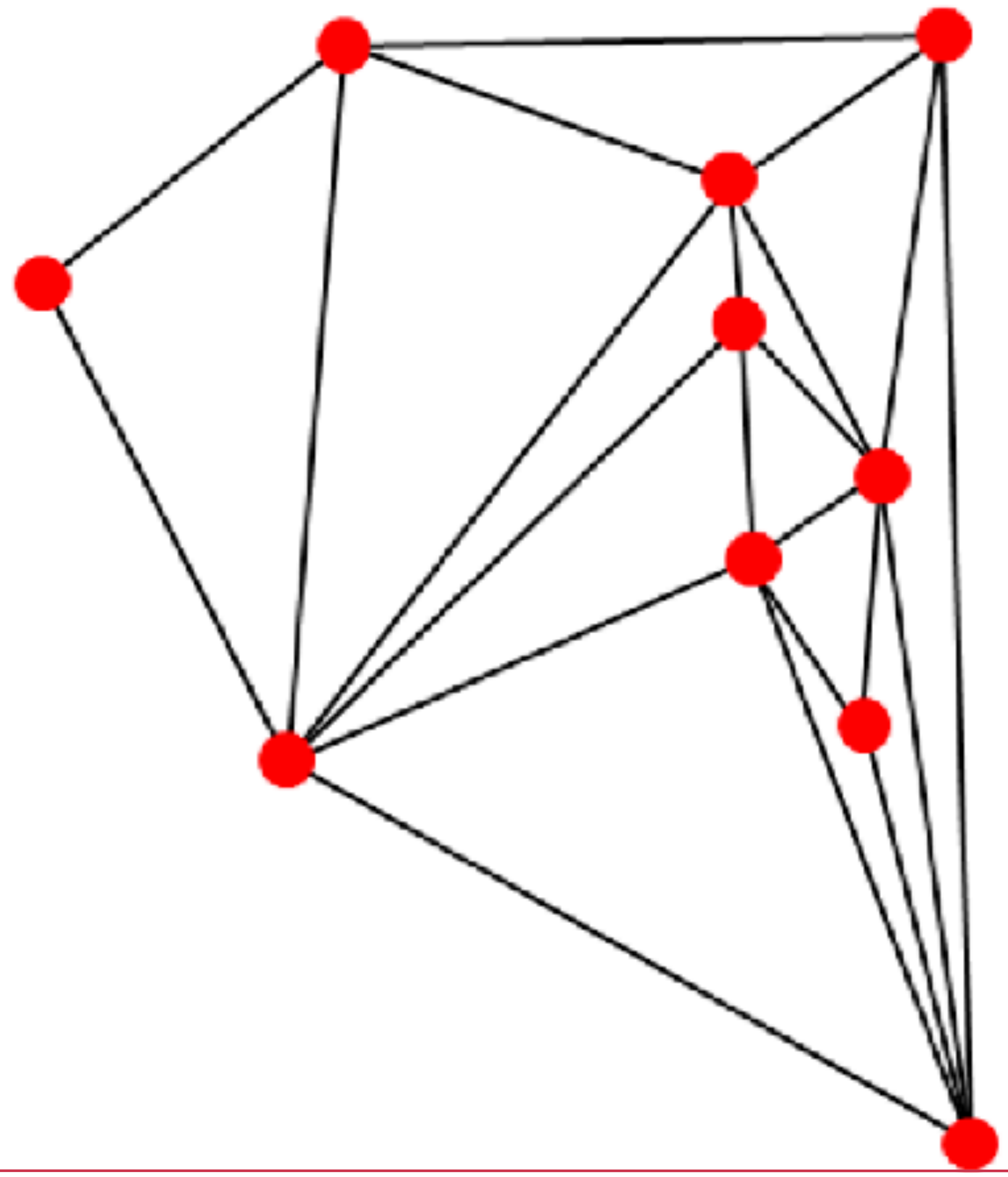

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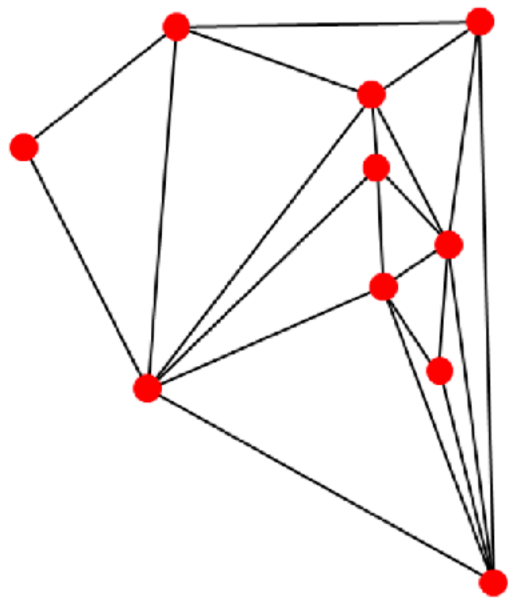
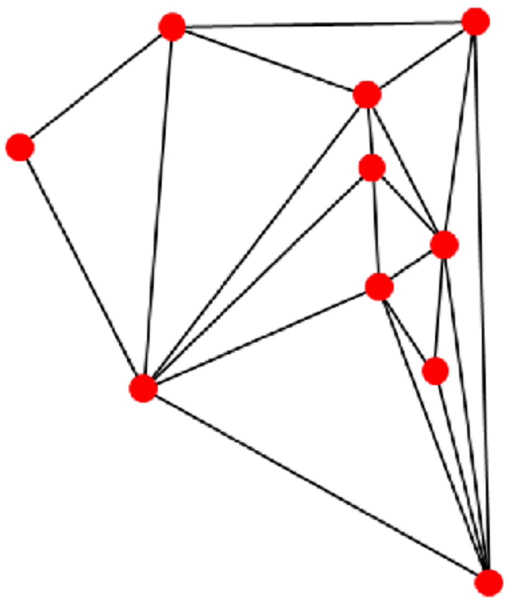


Triangulations

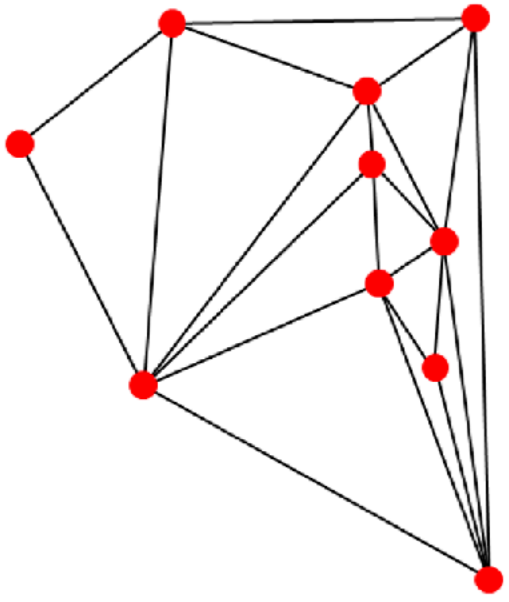
Problem 6.1



Triangulations

Problem 6.1

Given: A set P of points in $\mathbb{R}^2$

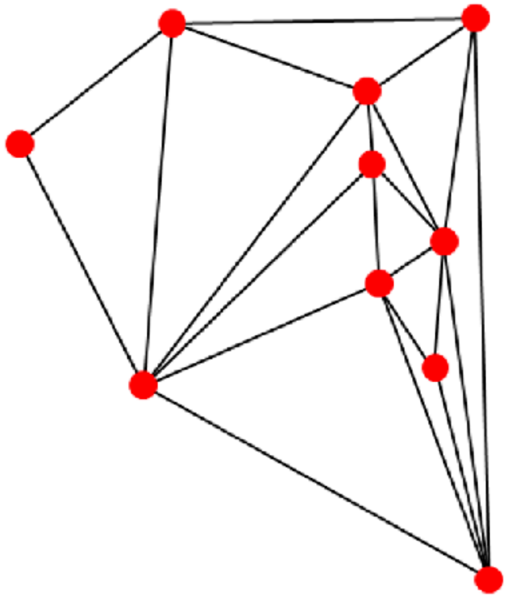


Triangulations

Problem 6.1

Given: A set P of points in $\mathbb{R}^2$

Wanted: Subdivision of $conv(P)$ into triangles with vertices in P and disjoint interior



Lower bound of $\Omega(n \log n)$ on sorting.

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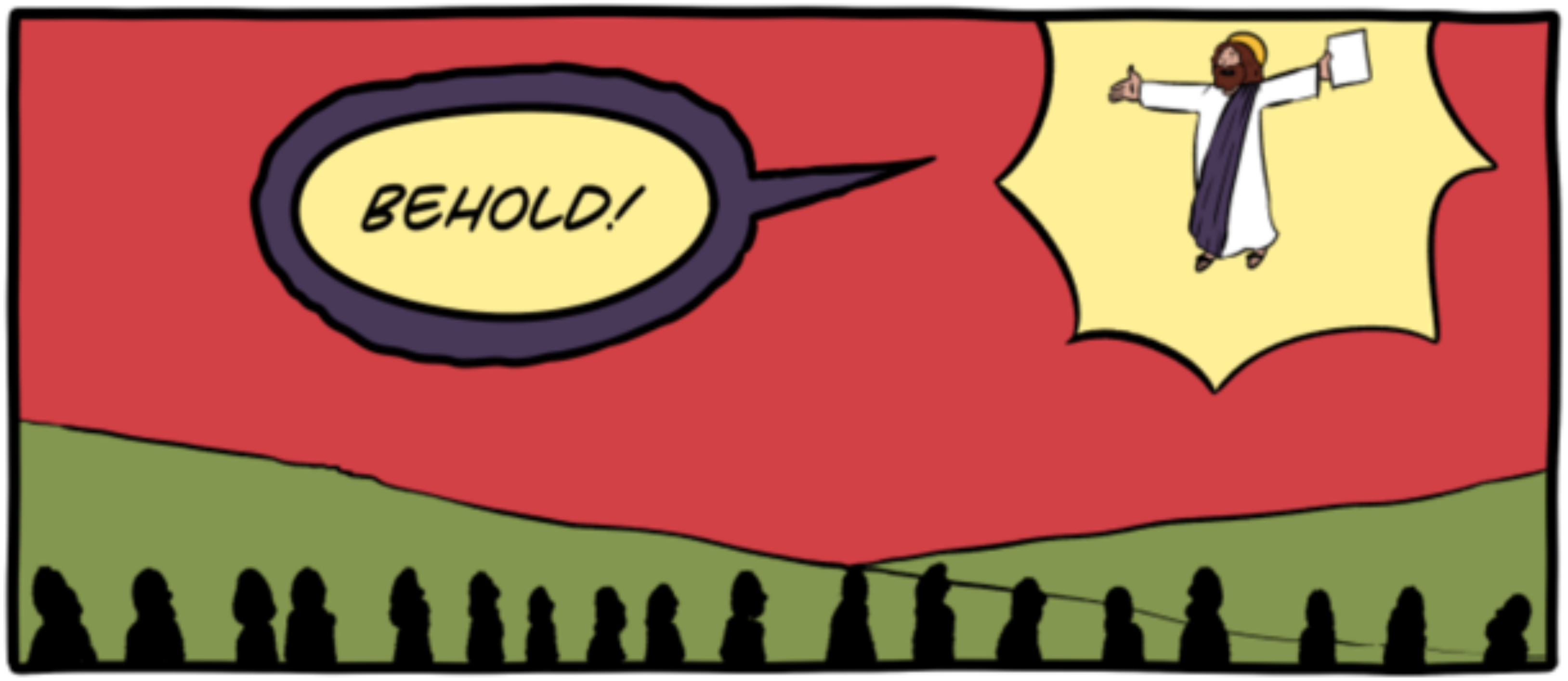

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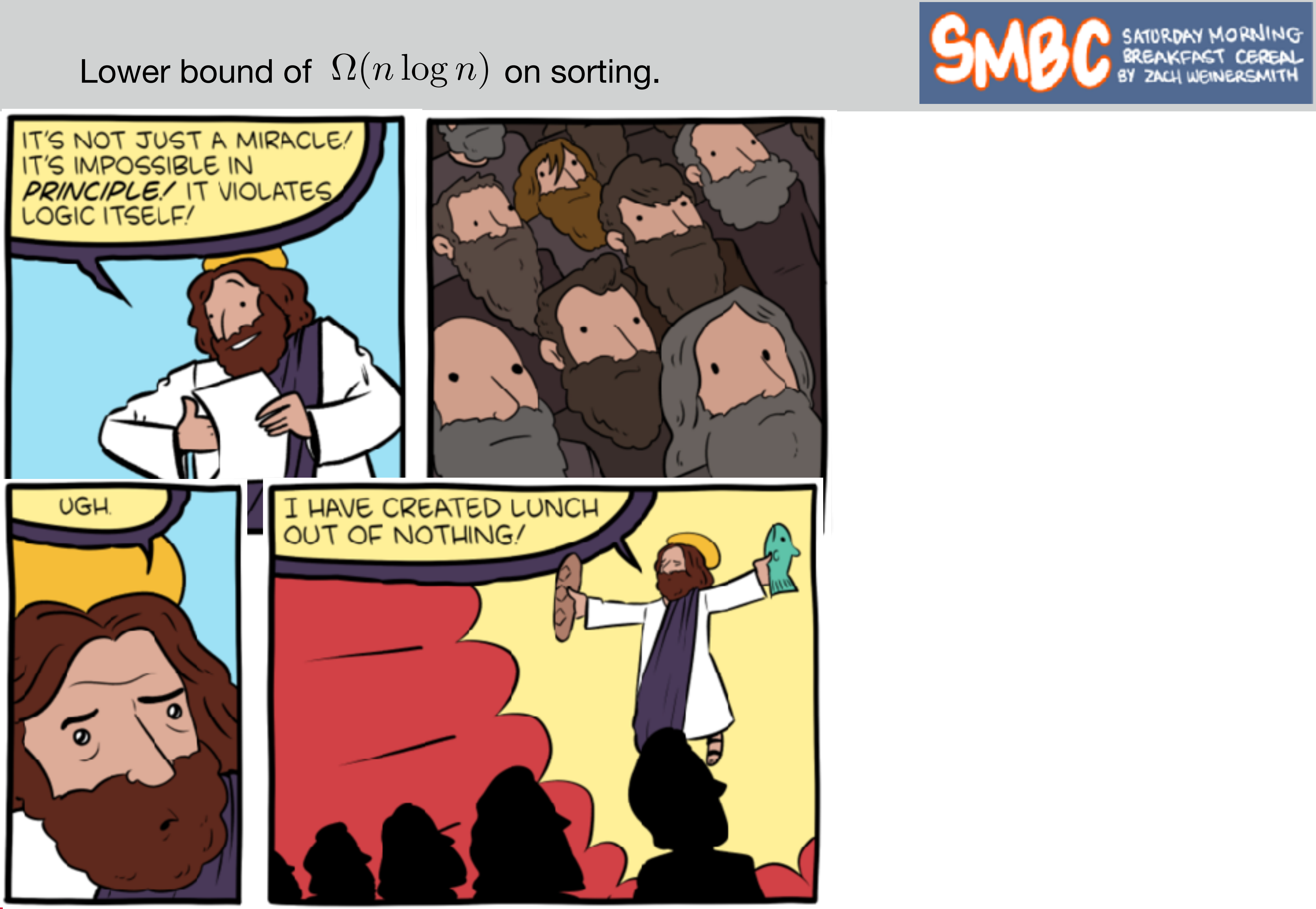

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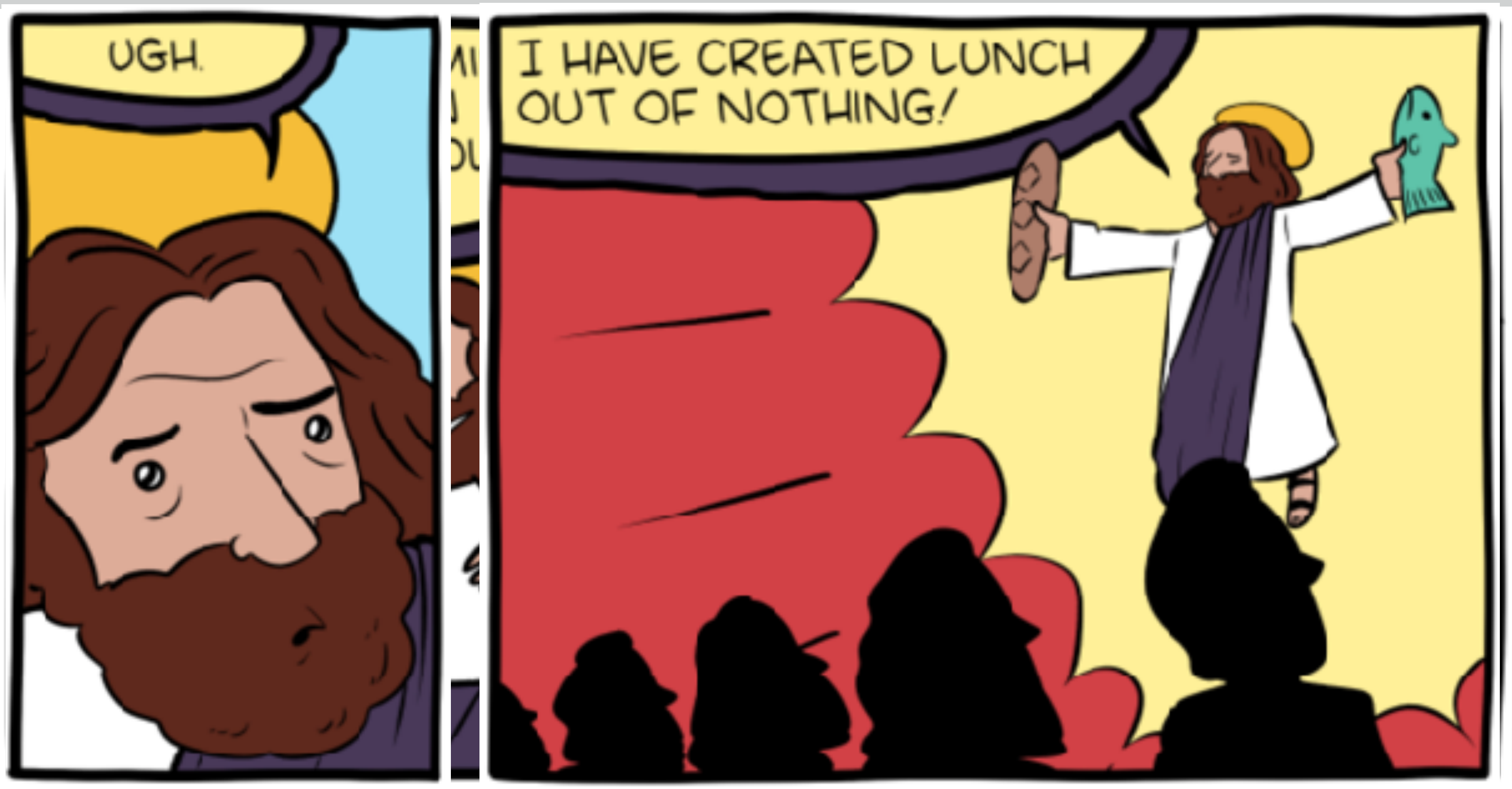

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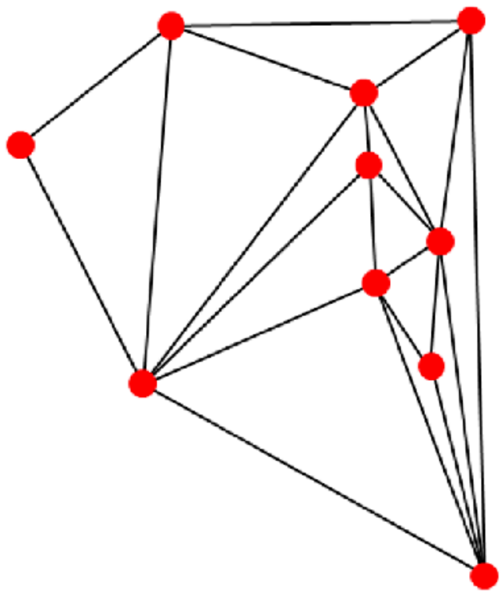
Objective Functions



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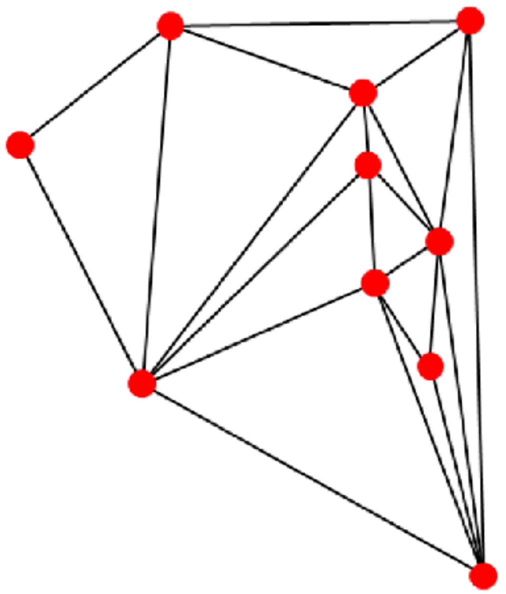
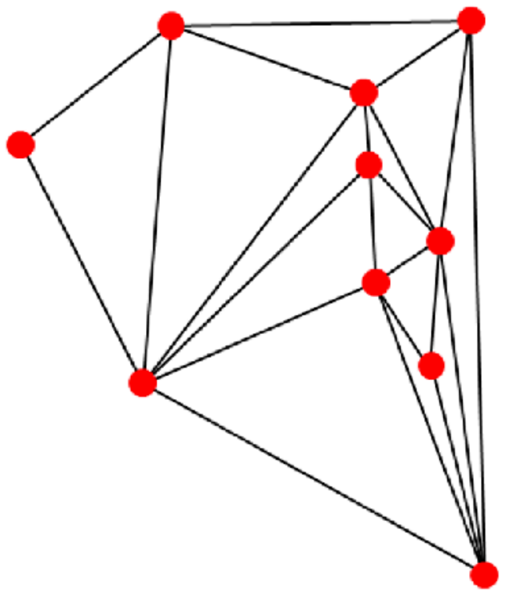


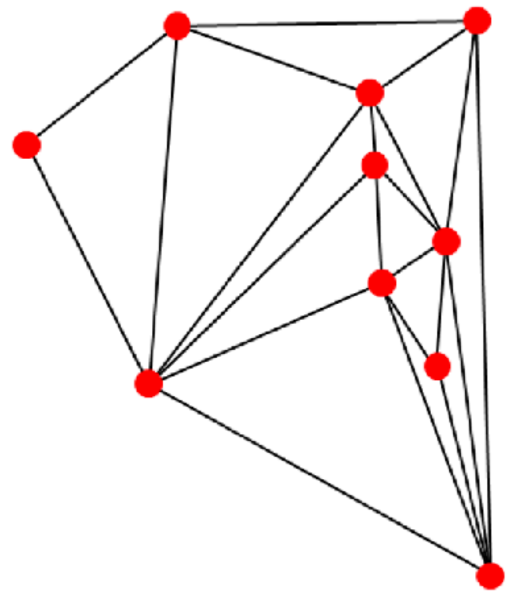
Objective Functions

1. Maximize minimum angle



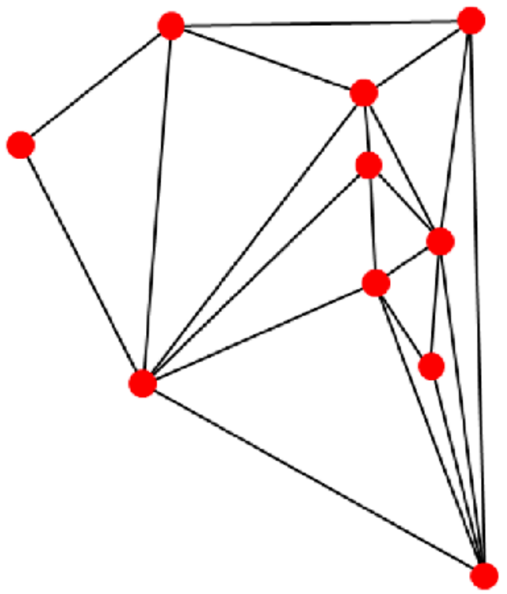
Objective Functions

1. Maximize minimum angle
2. Minimize total edge length



Objective Functions

1. Maximize minimum angle
2. Minimize total edge length
3. Minimize maximum edge length



Objective Functions

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2. Minimize total edge length
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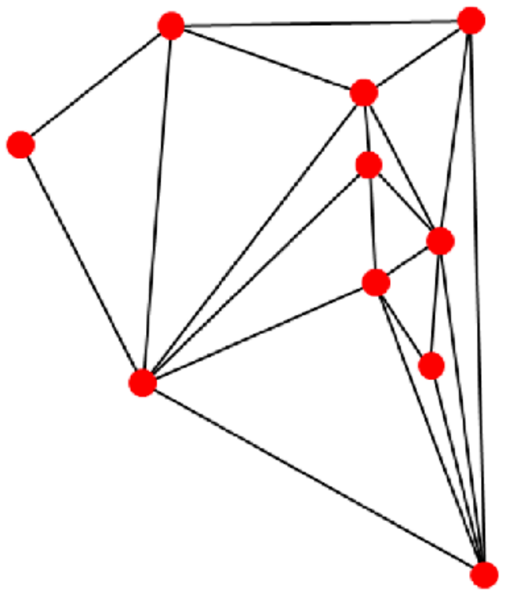
4. Maximize minimum edge length



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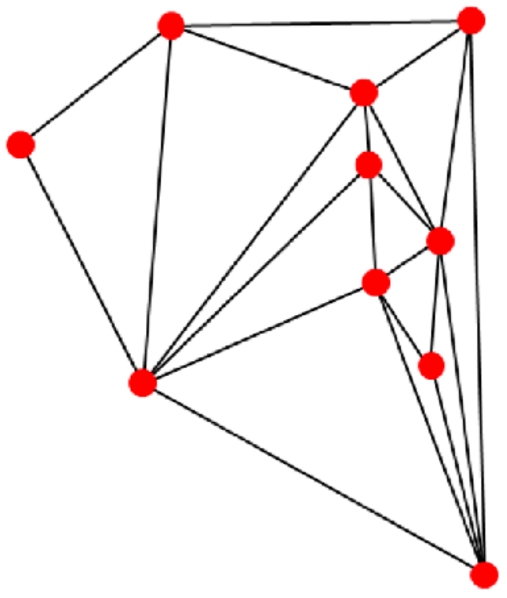
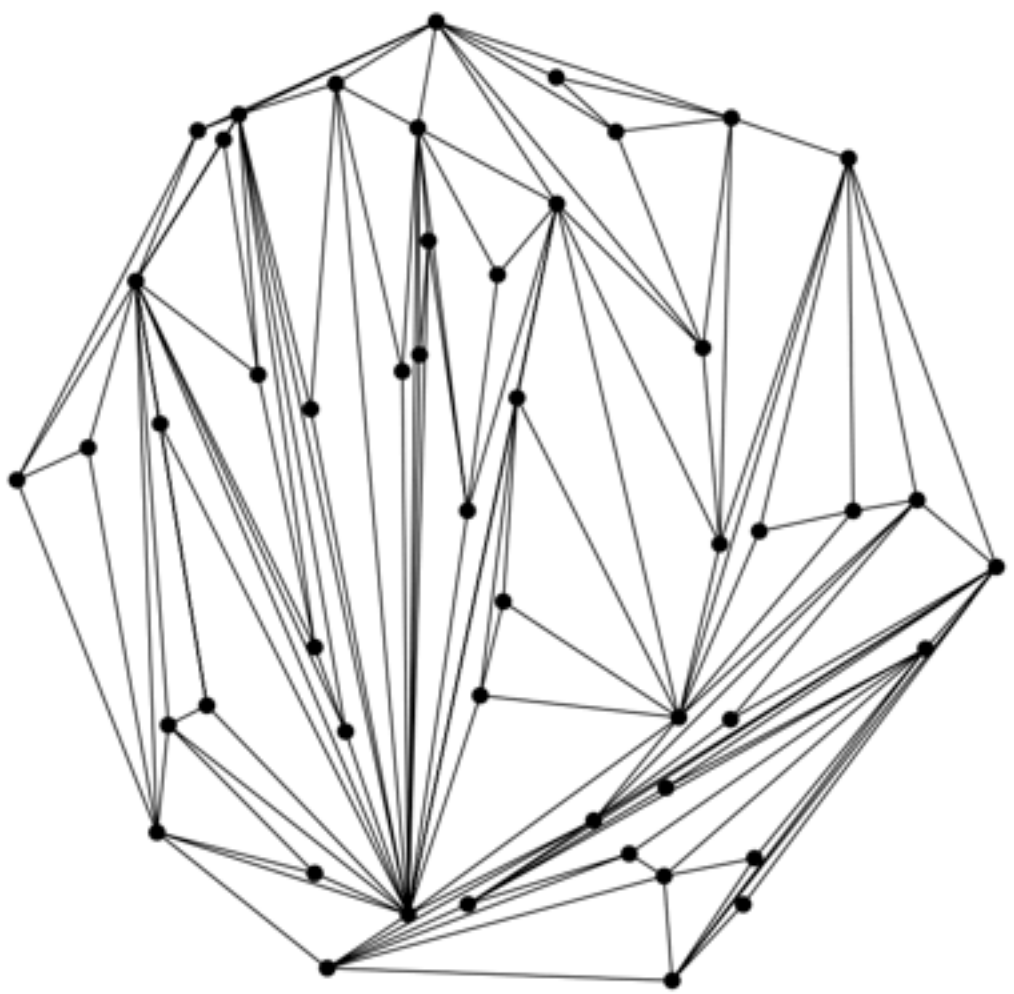
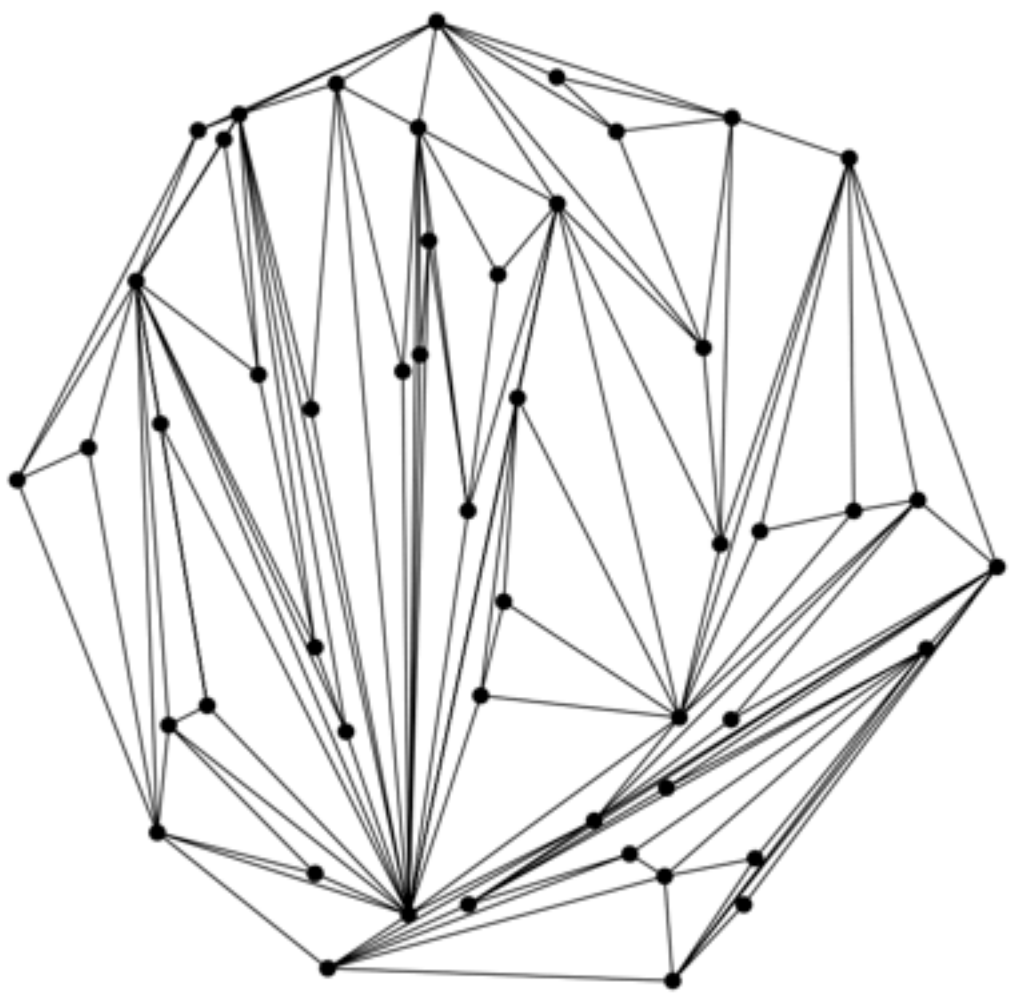


2 Delaunay triangulations



(a) Scan triangulation.



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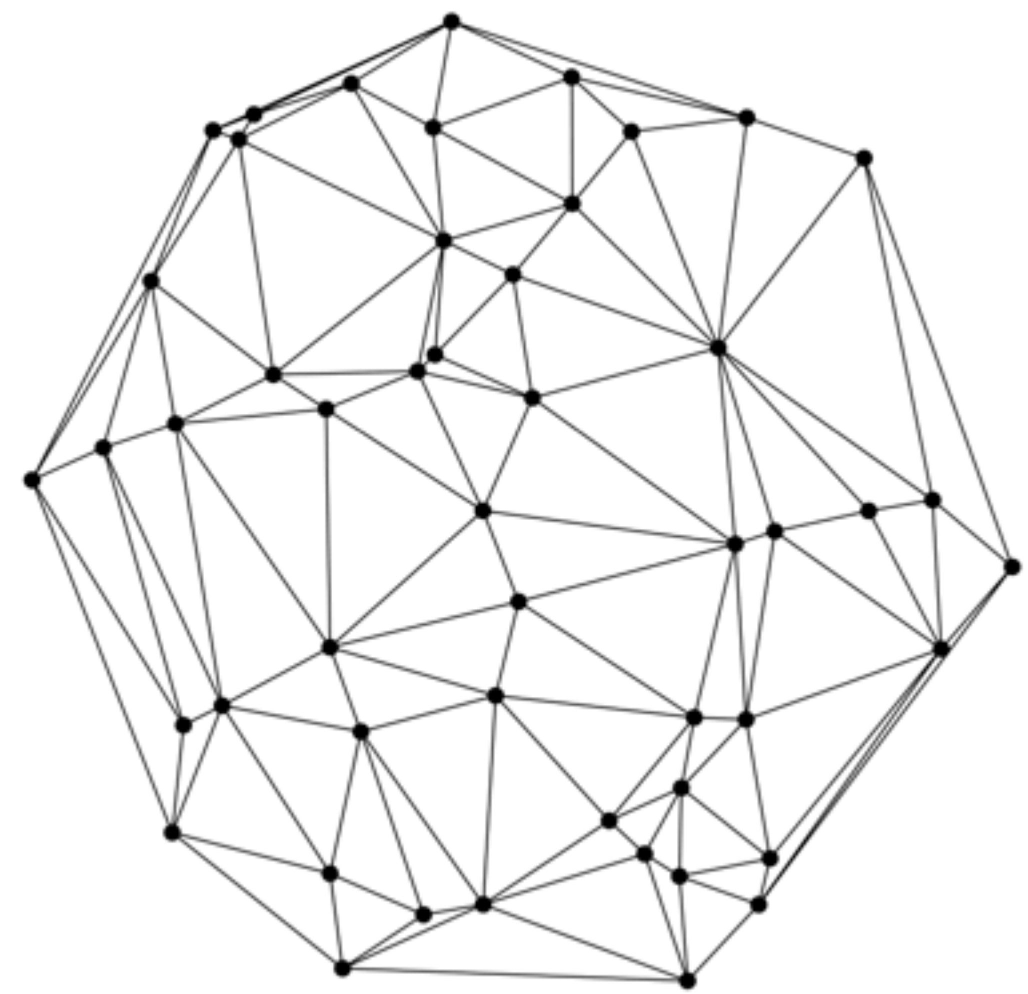

(b) Delaunay triangulation.

Delaunay triangulations [Delaunay, 1934]

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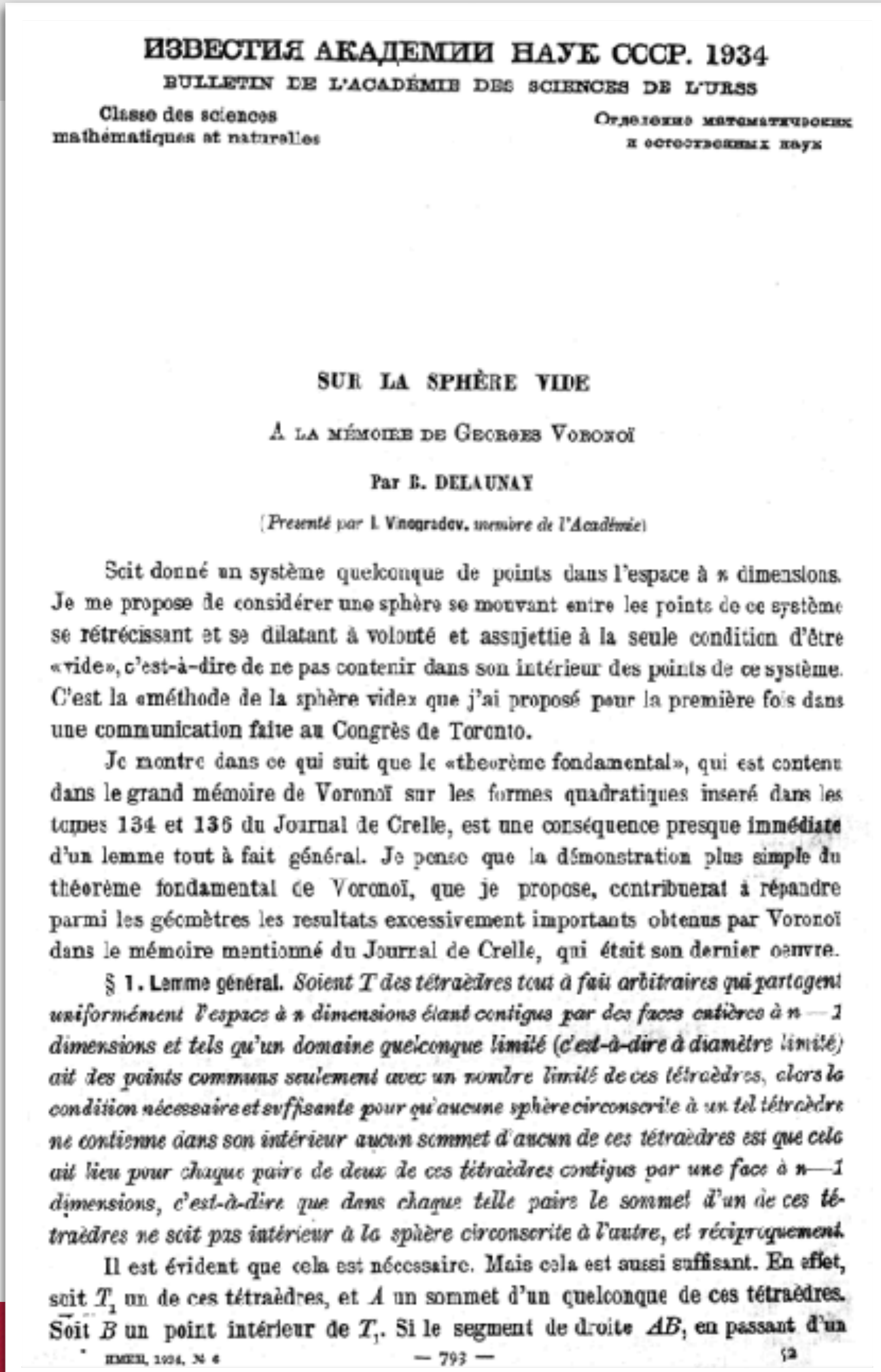

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SUR LA SPHÈRE VIDE

À la mémoire de Georges Voronoï

Par B. DELAUNAY

(Presenté par I. Vinogradov, membre de l'Académie)

Soit donné un système quelconque de points dans l'espace à n dimensions. Je me propose de considérer une sphère se mouvant entre les points de ce système se rétrécissant et se dilatant à volonté et assujettie à la seule condition d'être «vide», c'est-à-dire de ne pas contenir dans son intérieur des points de ce système. C'est la «méthode de la sphère vide» que j'ai proposé pour la première fois dans une communication faite au Congrès de Toronto.

Je montre dans ce qui suit que le «theorème fondamental», qui est contenu dans le grand mémoire de Voronoï sur les formes quadratiques inseré dans les tomes 134 et 136 du Journal de Crelle, est une conséquence presque immédiate d'un lemme tout à fait général. Je pense que la démonstration plus simple du théorème fondamental de Voronoï, que je propose, contribuerait à répandre parmi les géomètres les resultats excessivement importants obtenus par Voronoï dans le mémoire mentionné du Journal de Crelle, qui était son dernier oeuvre.

§ 1. Lemme général. *Soient T des tétraèdres tout à fait arbitraires qui partagent uniformément l'espace à n dimensions étant contigus par des faces entières à $n-1$ dimensions et tels qu'un domaine quelconque limité (c'est-à-dire à diamètre limité) ait des points communs seulement avec un nombre limité de ces tétraèdres, alors la condition nécessaire et suffisante pour qu'aucune sphère circonscrite à un tel tétraèdre ne contienne dans son intérieur aucun sommet d'aucun de ces tétraèdres est que cela ait lieu pour chaque paire de deux de ces tétraèdres contigus par une face à $n-1$ dimensions, c'est-à-dire que dans chaque telle paire le sommet d'un de ces tétraèdres ne soit pas intérieur à la sphère circonscrite à l'autre, et réciproquement.*

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Delaunay triangulations [Delaunay, 1934]

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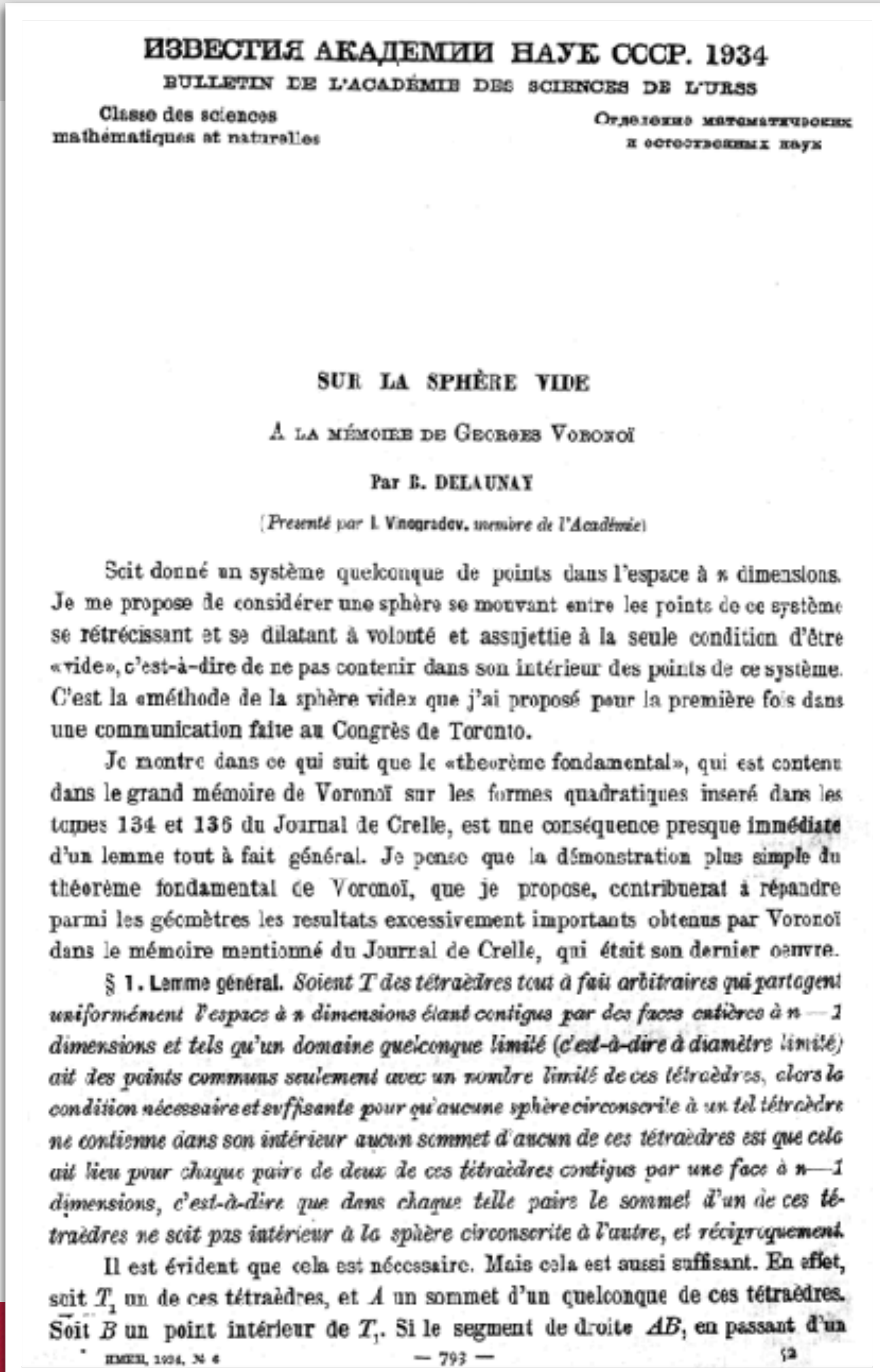
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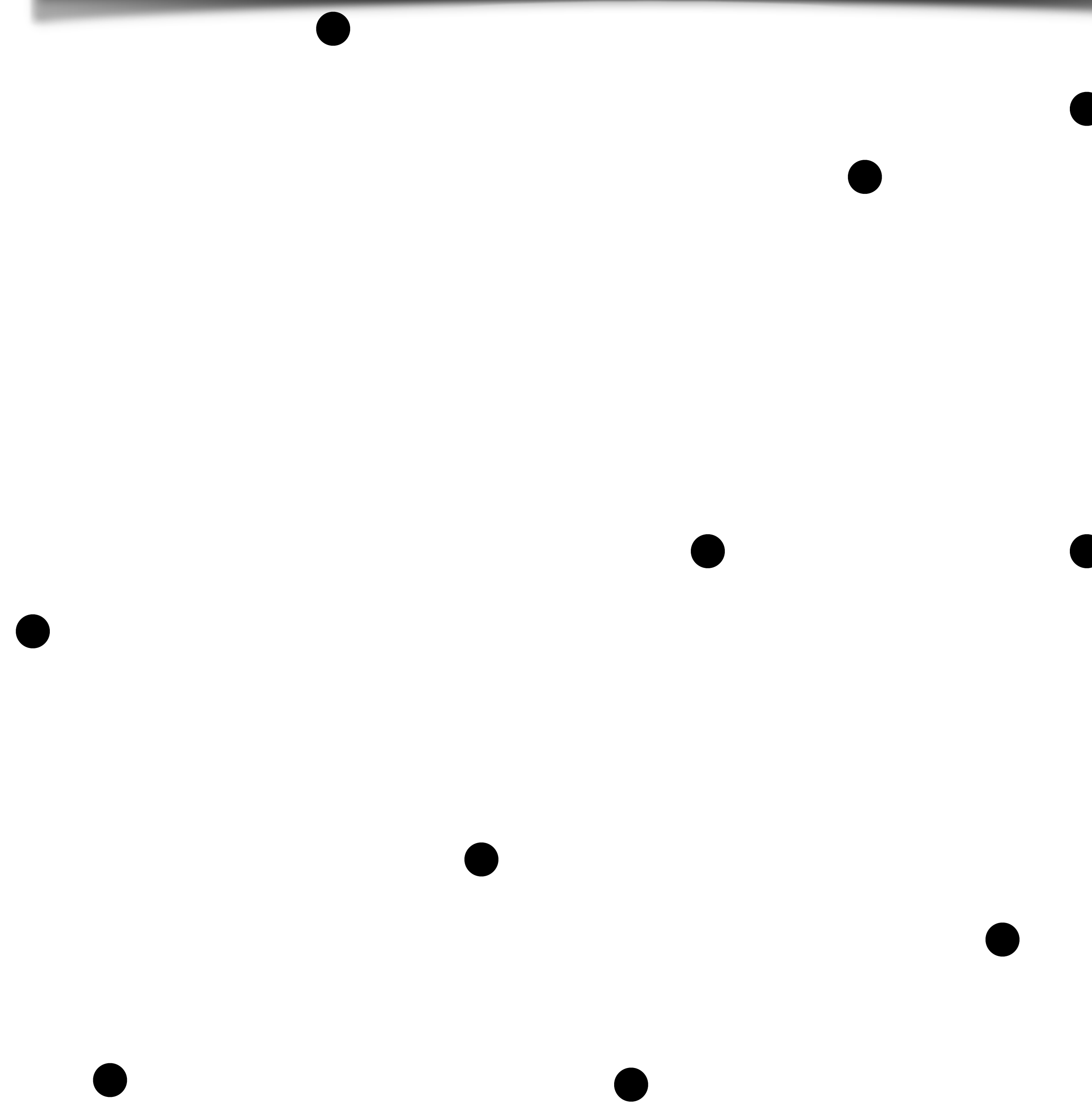

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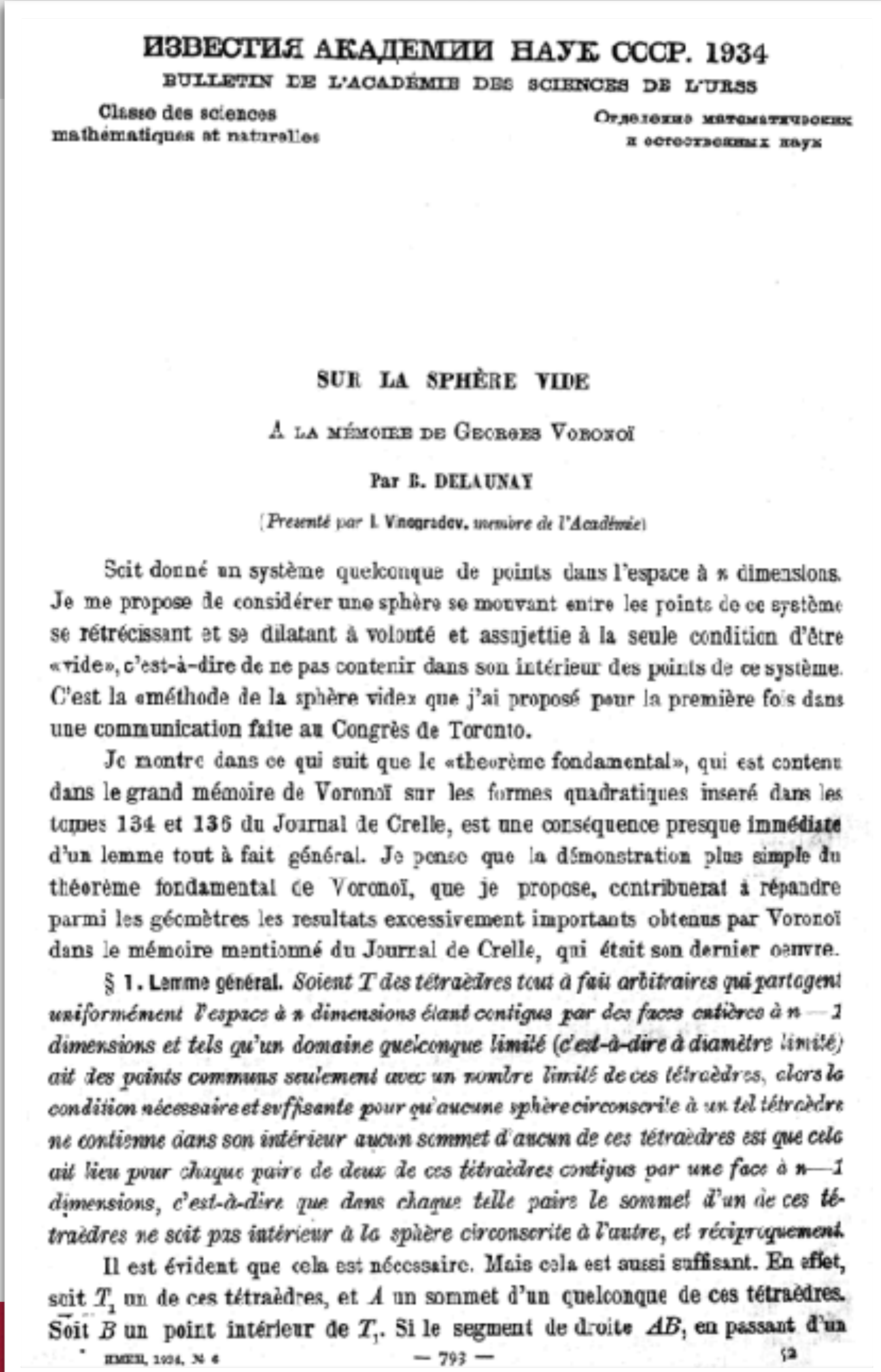

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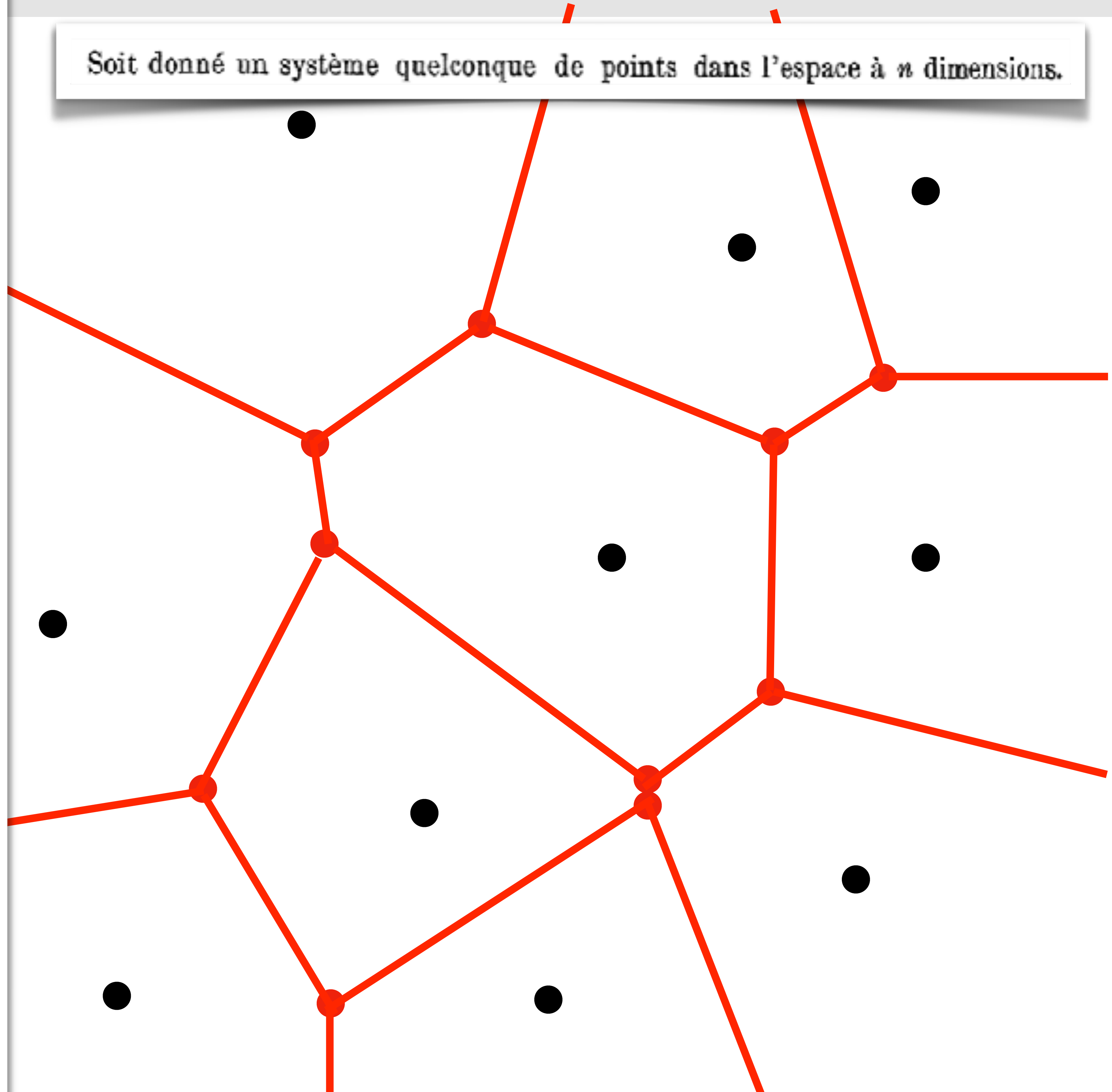

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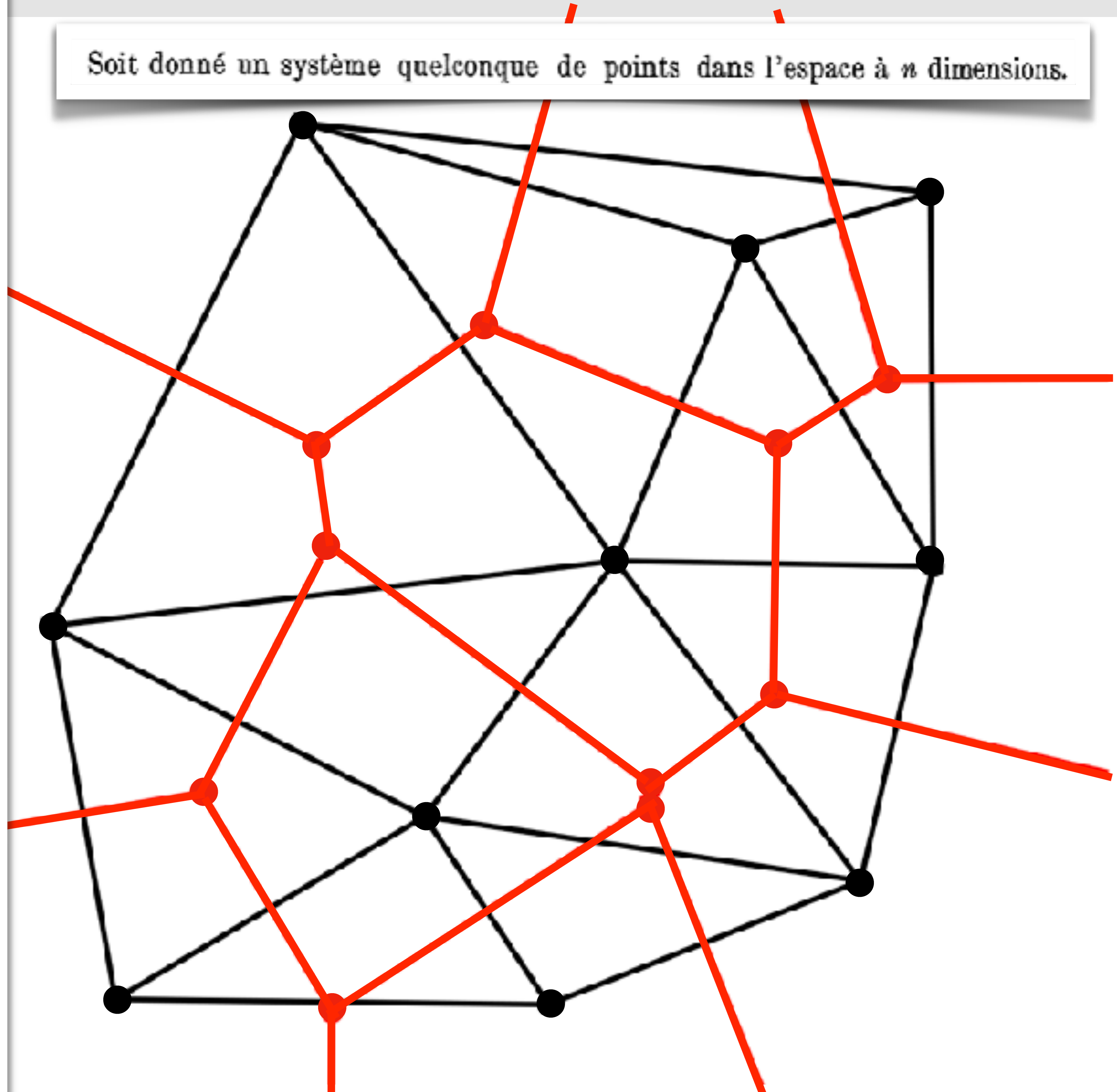

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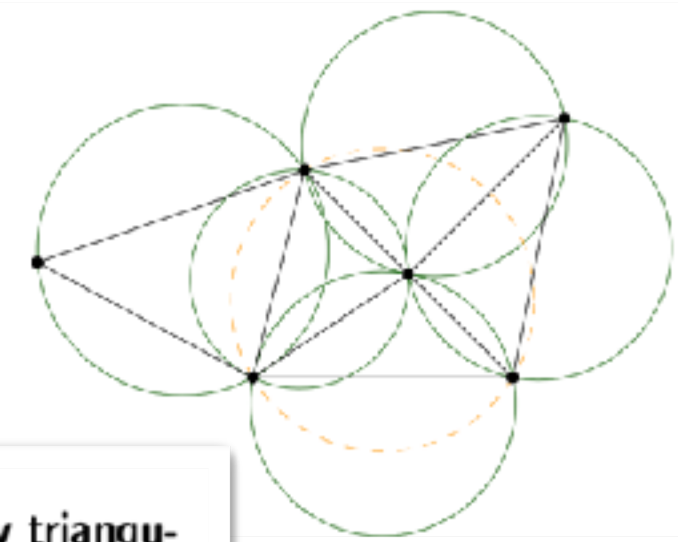

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Theorem 6.18 *Let* $P \subseteq \mathbb{R}^2$ *be a finite set of points in general position (not all collinear and no four cocircular). Let* $\mathcal{D}^*$ *be the unique Delaunay triangulation of* P*, and let* $\mathcal{T}$ *be any triangulation of* P*. Then* $A(\mathcal{T}) \preceq A(\mathcal{D}^*)$.

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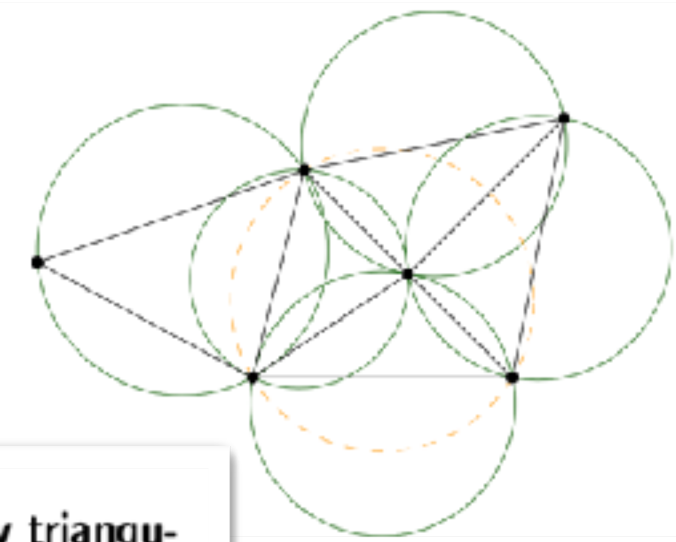

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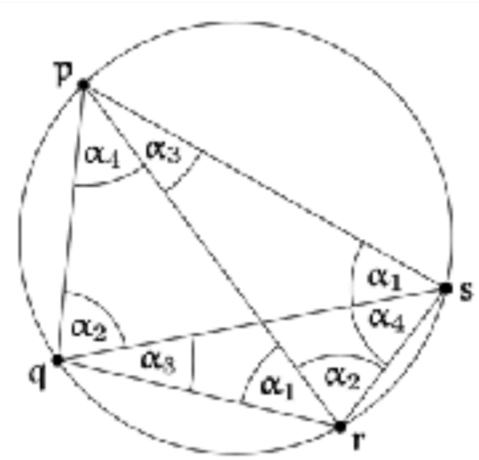
Theorem 6.18 *Let* $P \subseteq \mathbb{R}^2$ *be a finite set of points in general position (not all collinear and no four cocircular). Let* $\mathcal{D}^*$ *be the unique Delaunay triangulation of* P*, and let* $\mathcal{T}$ *be any triangulation of* P*. Then* $A(\mathcal{T}) \leqslant A(\mathcal{D}^*)$.



(a) Four cocircular points and the induced sight angles.

Computational Geometry Lecture Notes HS 2013

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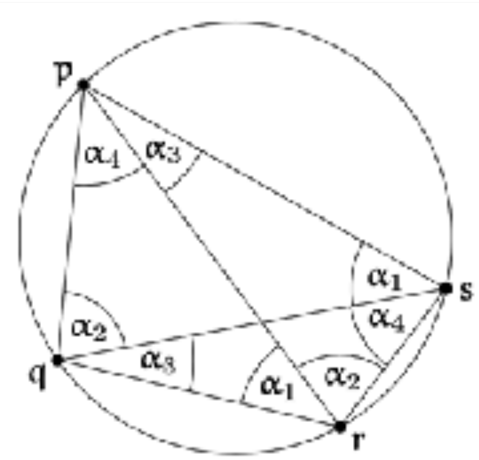

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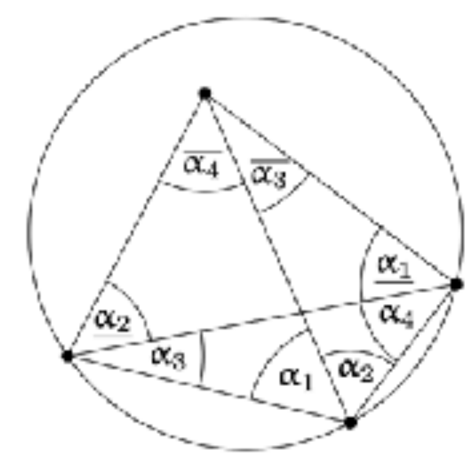
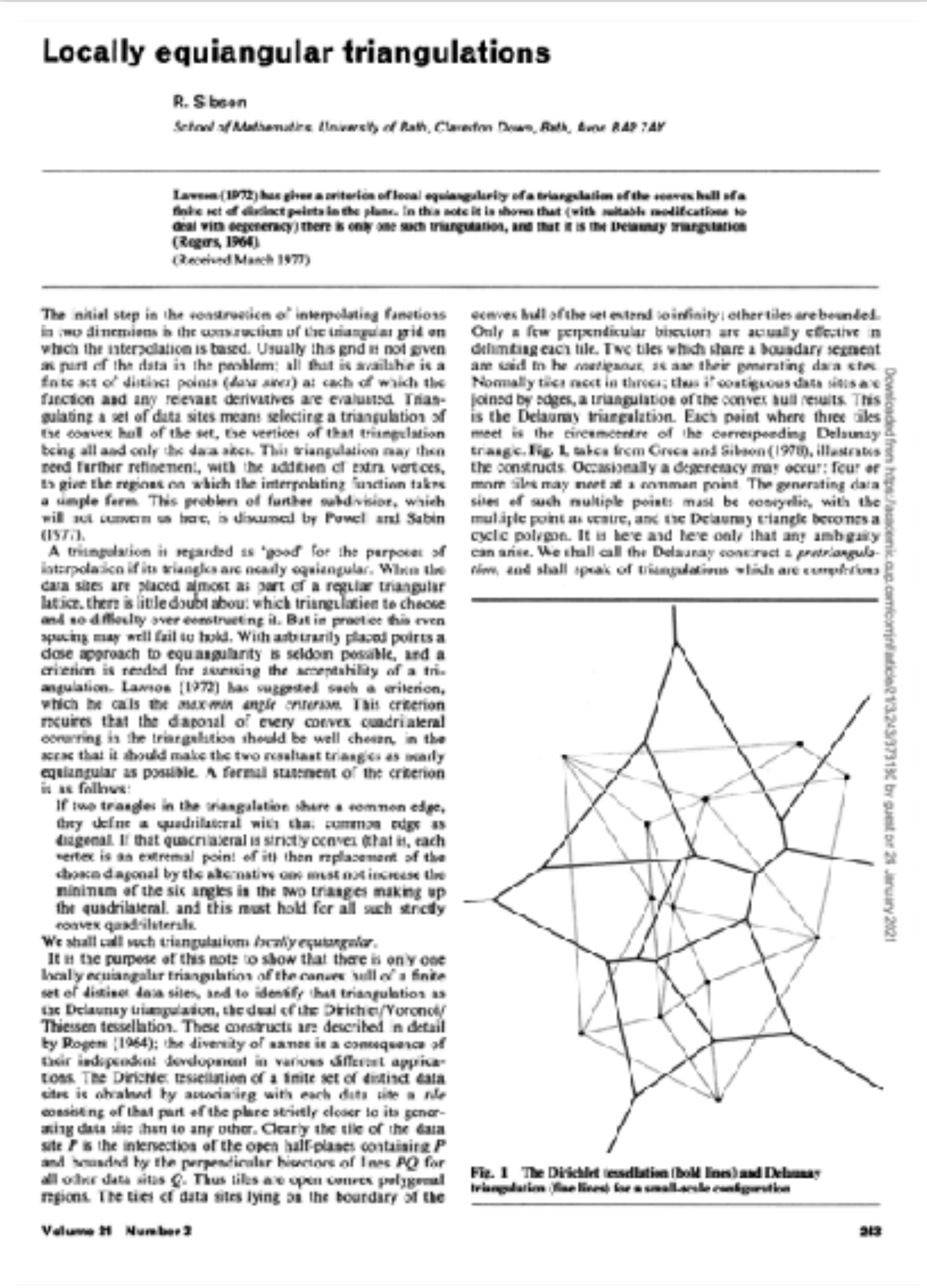


(b) The situation before a flip.

Delaunay triangulation maximizes minimum angle [Sibson 1978]



Locally equiangular triangulations

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School of Mathematics, University of Bath, Claverton Down, Bath, Avon BA2 7AY

Lawson (1972) has given a criterion of local equiangularity of a triangulation of the convex hull of a finite set of distinct points in the plane. In this note it is shown that (with suitable modifications to deal with degeneracy) there is only one such triangulation, and that it is the Delaunay triangulation (Rogers, 1964).

(Received March 1977)

Fig. 1 The Dirichlet tessellation (bold lines) and Delaunay triangulation (fine lines) for a small-scale configuration

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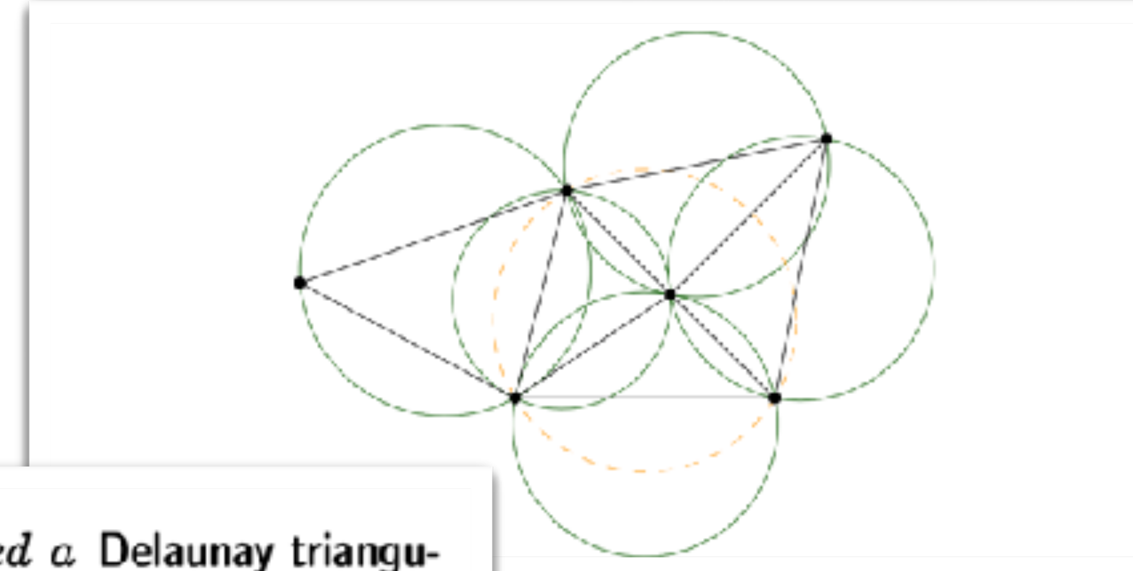

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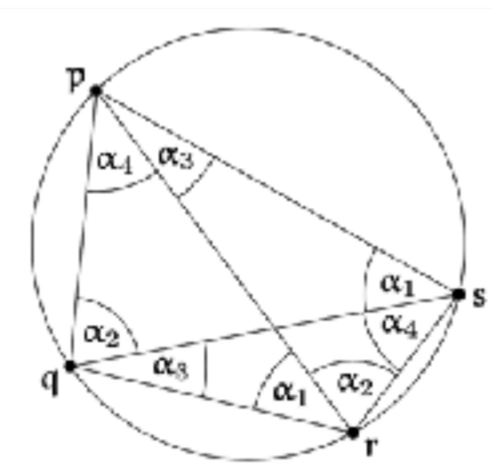

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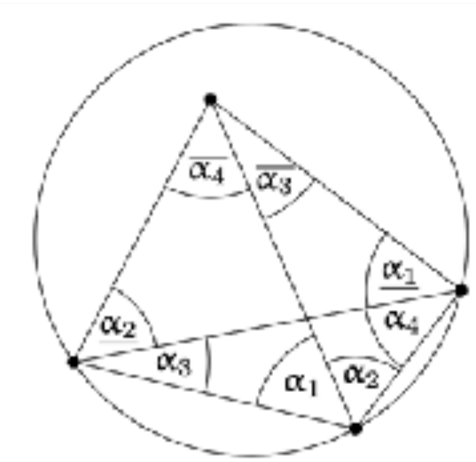

(b) The situation before a flip.

Minimizing maximum angle [Edelsbrunner, Tan, Waupotitsch 1992]

AN $O(n^2 \log n)$ TIME ALGORITHM FOR THE MINMAX ANGLE TRIANGULATION *

HERBERT EDELSBRUNNER†, TIOW SENG TAN† AND ROMAN WAUPOTITSCH†

Abstract. We show that a triangulation of a set of n points in the plane that minimizes the maximum angle can be computed in time $O(n^2 \log n)$ and space $O(n)$. The algorithm is fairly easy to implement and is based on the edge-insertion scheme that iteratively improves an arbitrary initial triangulation. It can be extended to the case where edges are prescribed, and, within the same time- and space-bounds, it can lexicographically minimize the sorted angle vector if the point set is in general position. Experimental results on the efficiency of the algorithm and the quality of the triangulations obtained are included.

Key words. Computational geometry, two dimensions, triangulations, minmax angle criterion, iterative improvement, edge-insertion

AMS(MOS) subject classifications. 68C05, 65M50

Appear in: *SIAM J. Scientific & Statistical Computing*, **13** (4), 994–1008, (1992)

1. Introduction. Let S be a finite set of points in the Euclidean plane. A *triangulation* of S is a maximally connected straight line plane graph whose vertices are the points of S. By maximality, each face is a triangle except for the exterior face which is the complement of the convex hull of S. Occasionally, we will call a triangulation of a finite point set a *general triangulation* in order to distinguish it from a *constrained triangulation* which is a triangulation of a finite point set where some edges are prescribed. A special case of a constrained triangulation is the so-called *polygon triangulation* where S is the set of vertices of a simple polygon and the edges of the polygon are prescribed. In this paper only the triangles *inside* the polygon will be of interest.

For a given set of n points there are, in general, exponentially many triangulations. Among them one can choose those that satisfy certain requirements or optimize certain objective functions. Different properties are desirable for different applications in areas such as finite element analysis [1, 3, 23], computational geometry [21], and surface approximation [12, 18]. The following are some important types of triangulations that optimize certain objective functions.

(i) The *Delaunay triangulation* has the property that the circumcircle of any triangle does not enclose any vertex [5].
(ii) The *constrained Delaunay triangulation* has the same property except that visibility constraints depending on the enforced edges are introduced [13].
(iii) The *minimum weight triangulation* minimizes the total edge length over all possible triangulations of the same set of points and prescribed edges [10, 17].

It is known that the Delaunay triangulation maximizes the minimum angle over all triangulations of the same point set [22]. This result can be extended to a similar statement about the sorted angle vector of the Delaunay triangulation [6] and to the constrained case [13]. The Delaunay triangulation of n points in the plane can be constructed in time $O(n \log n)$ [6, 19], and even if some edges are prescribed its

* Research of the first author was supported by the National Science Foundation under grants CCR-8714565 and CCR-8921421. The second author is on study leave from the Department of Information Systems and Computer Science, National University of Singapore, Republic of Singapore.

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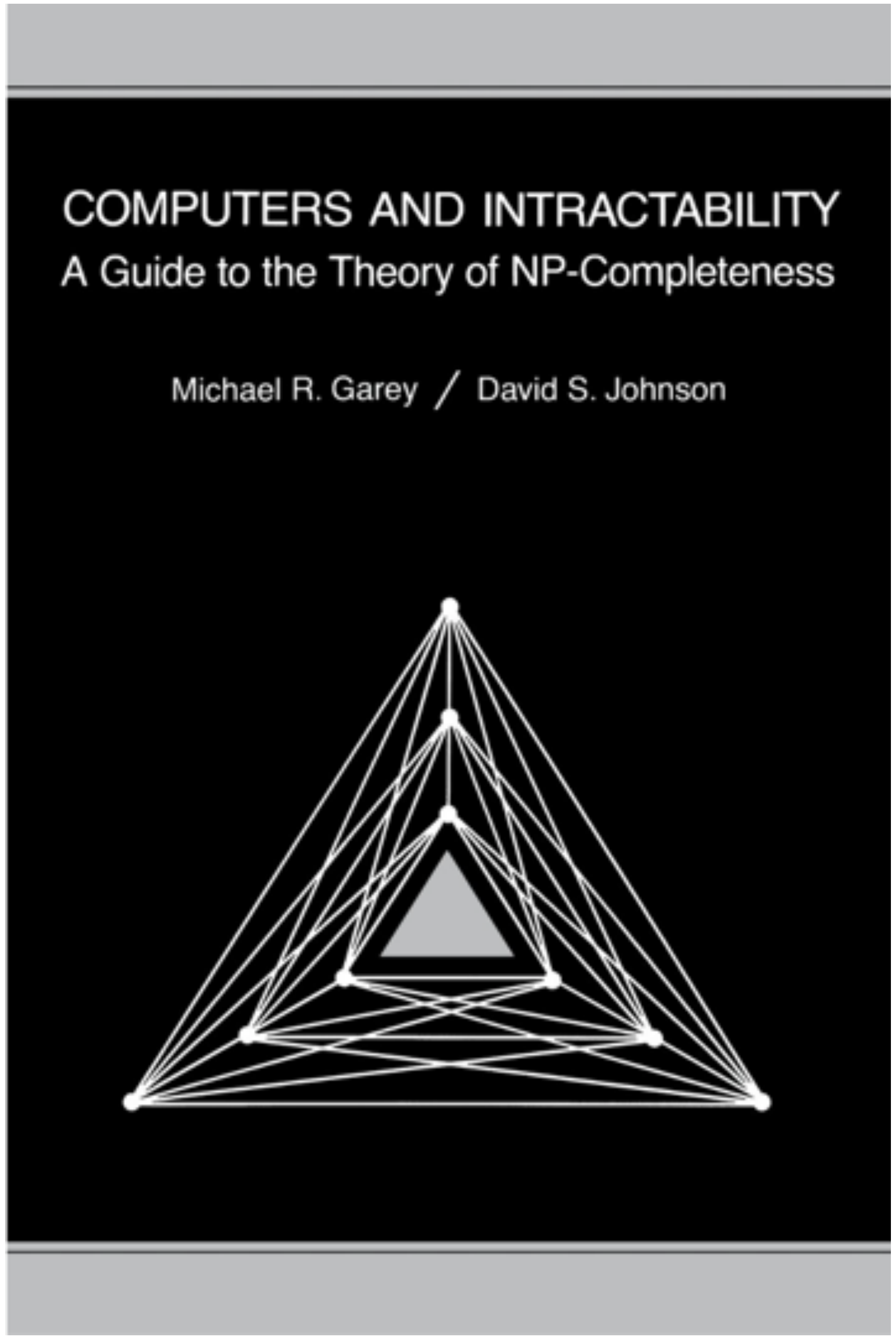
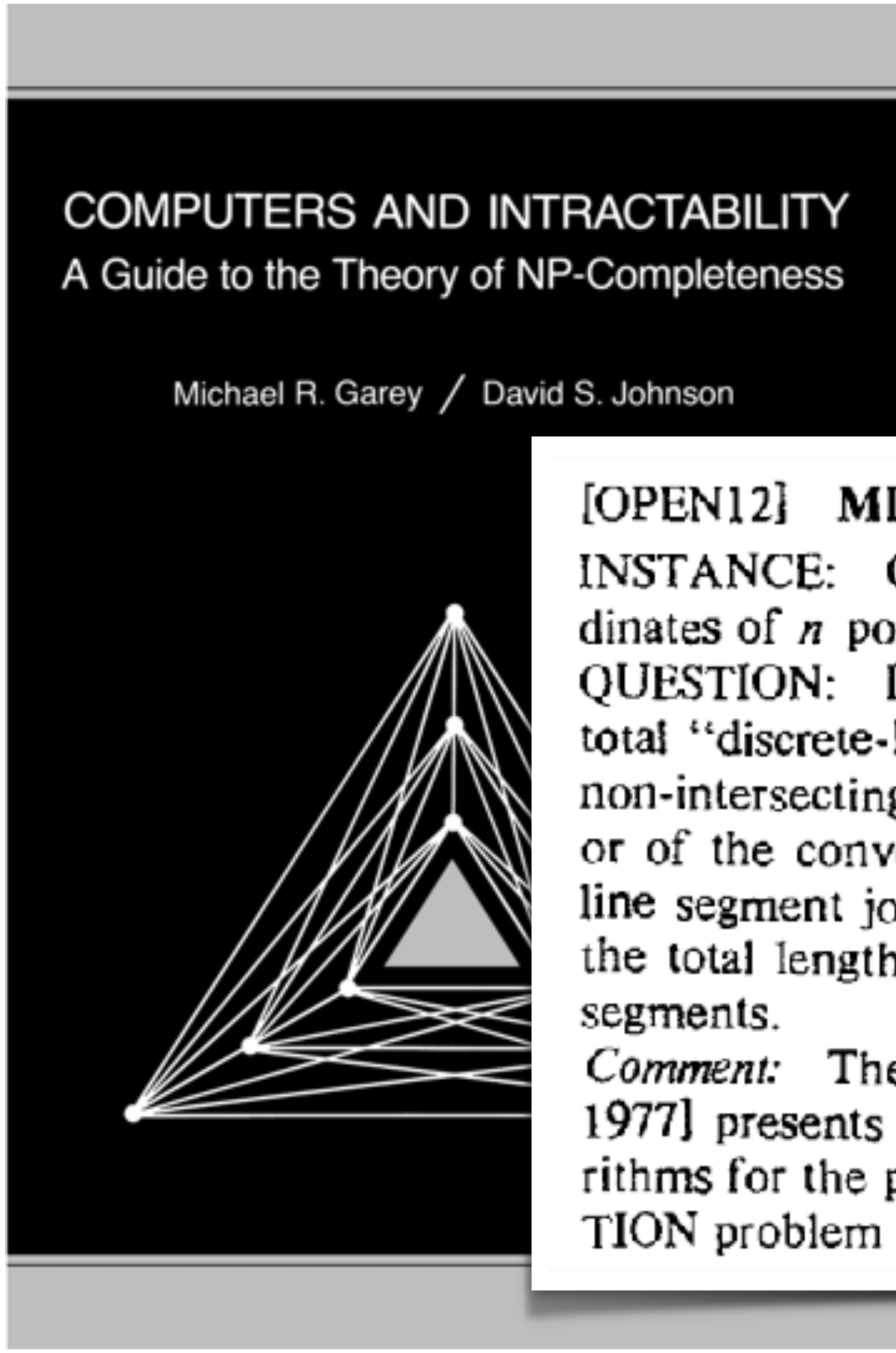
3 Minimum-weight triangulations


COMPUTERS AND INTRACTABILITY
A Guide to the Theory of NP-Completeness
Michael R. Garey / David S. Johnson



[OPEN12] MINIMUM LENGTH TRIANGULATION

INSTANCE: Collection $C=\{(a_i,b_i):1 \leqslant i \leqslant n\}$ of pairs of integers, giving the coordinates of n points in the plane, and a positive integer B.

QUESTION: Is there a triangulation of the set of points represented by C that has total "discrete-Euclidean" length B or less? Here a triangulation is a collection of non-intersecting line segments, each joining two points in C, that divides the interior of the convex hull into triangular regions. The discrete-Euclidean length of a line segment joining (a_i,b_i) and (a_j,b_j) is given by $\lceil ((a_i-a_j)^2+(b_i-b_j)^2)^{1/2} \rceil$, and the total length of a triangulation is the sum of the lengths of its constituent line segments.

Comment: The analogous problem for the rectilinear metric is also open. [Lloyd, 1977] presents counterexamples to a number of conjectured polynomial time algorithms for the problem and proves that the related CONSTRAINED TRIANGULATION problem is NP-complete.

MWT is NP-hard [Mulzer and Rote, 2008]

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Minimum-Weight Triangulation is NP-hard

WOLFGANG MULZER
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A triangulation of a planar point set S is a maximal plane straight-line graph with vertex set S. In the minimum-weight triangulation (MWT) problem, we are looking for a triangulation of a given point set that minimizes the sum of the edge lengths. We prove that the decision version of this problem is NP-hard, using a reduction from PLANAR 1-IN-3-SAT. The correct working of the gadgets is established with computer assistance, using dynamic programming on polygonal faces, as well as the β-skeleton heuristic to certify that certain edges belong to the minimum-weight triangulation.

Categories and Subject Descriptors: F.2.2 [**Nonnumerical Algorithms and Problems**]: Geometrical problems and computations; G.2.2 [**Graph Theory**]: Graph algorithms

General Terms: Algorithms, Theory

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1. INTRODUCTION

Given a set S of points in the Euclidean plane, a *triangulation* T of S is a maximal plane straight-line graph with vertex set S. The *weight* of T is defined as the total Euclidean length of all edges in T. A triangulation that achieves the minimum weight is called a *minimum-weight triangulation* (MWT) of S.

The problem of computing a triangulation for a given planar point set arises naturally in many applications such as stock cutting, finite element analysis, terrain modeling, and numerical approximation. The *minimum-weight* triangulation has attracted the attention of many researchers, mainly due to its natural definition of optimality, and not so much because it would be important for the mentioned applications. We show that computing a minimum-weight triangulation is NP-hard. Note that it is not known whether the MWT problem is in NP, because it is an open problem whether sums of radicals (namely, Euclidean distances) can be compared in polynomial time [Blömer 1991]. This problem is common to many geometric optimization problems, like, perhaps most famously, the Euclidean Traveling Salesperson Problem. To get a variant of the problem that is in NP, one can take the weight of an edge e as the rounded value $\lceil \|e\|_2 \rceil$. Our proof also shows that this variant is NP-complete.

Our proof uses a polynomial time reduction from POSITIVE PLANAR 1-IN-3-SAT, a variant of the well-known PLANAR 3-SAT problem, which is a standard tool for showing NP-hardness of geometric problems.

1.1 Optimal Triangulations

Usually, a planar point set has (exponentially) many different triangulations, and many applications call for triangulations with certain good properties (see Figure 1). Optimal triangulations under various optimality criteria were extensively surveyed

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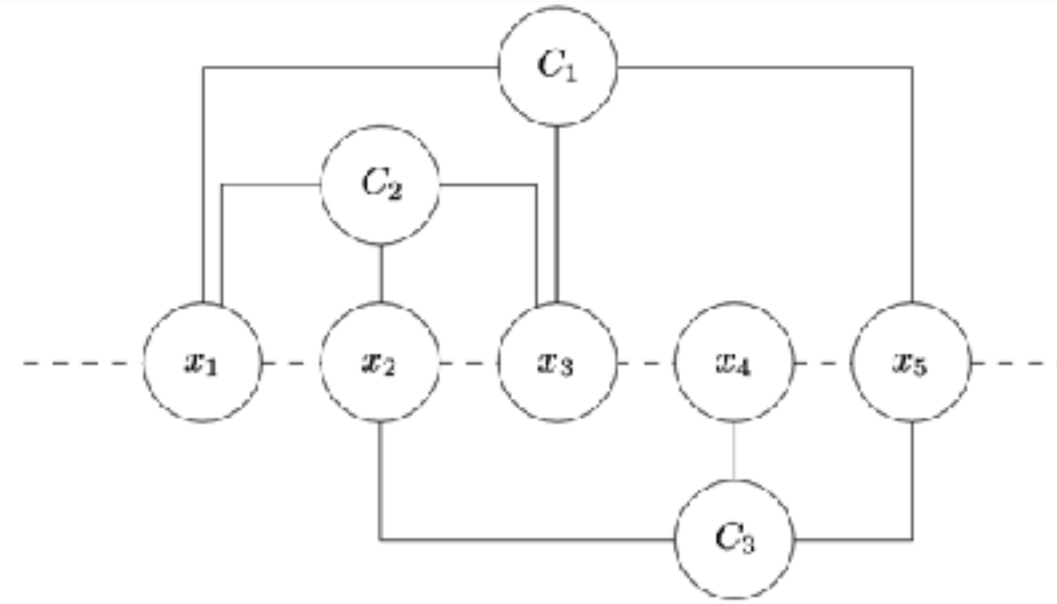

Fig. 5. A rectilinear embedding of graph that is associated with the Boolean formula $(x_1 \vee \neg x_3 \vee x_5) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_4 \vee \neg x_5)$.

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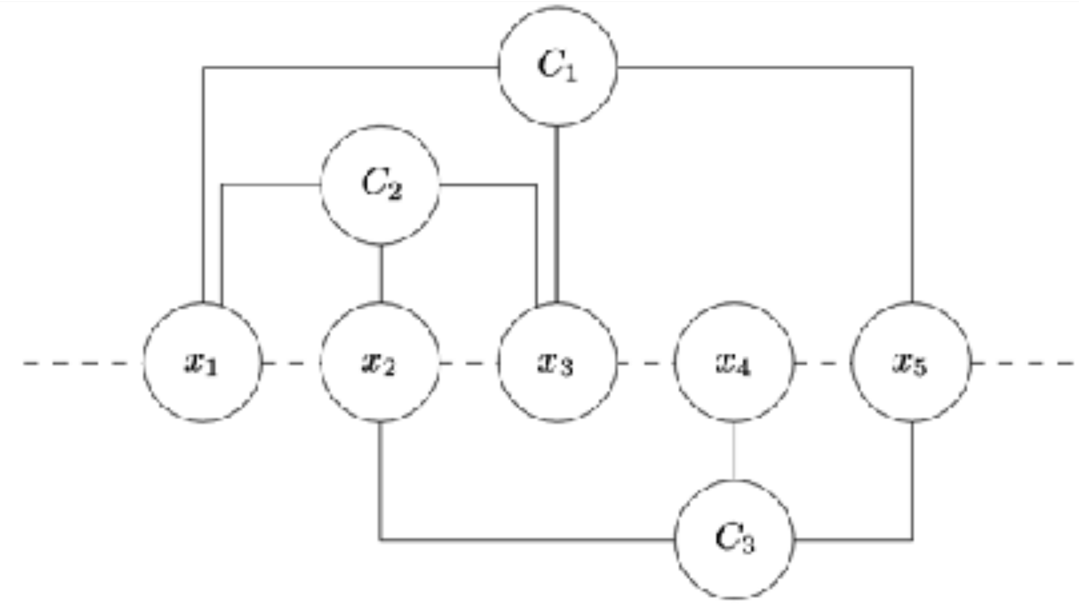

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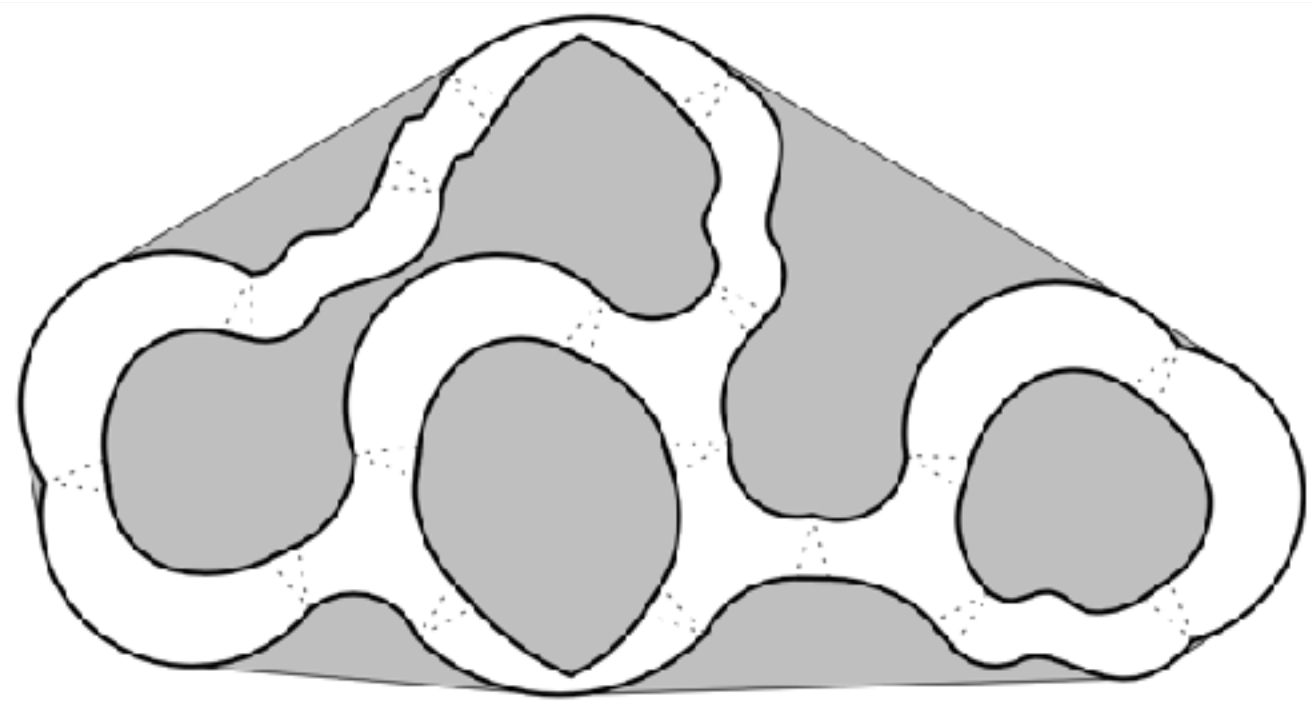

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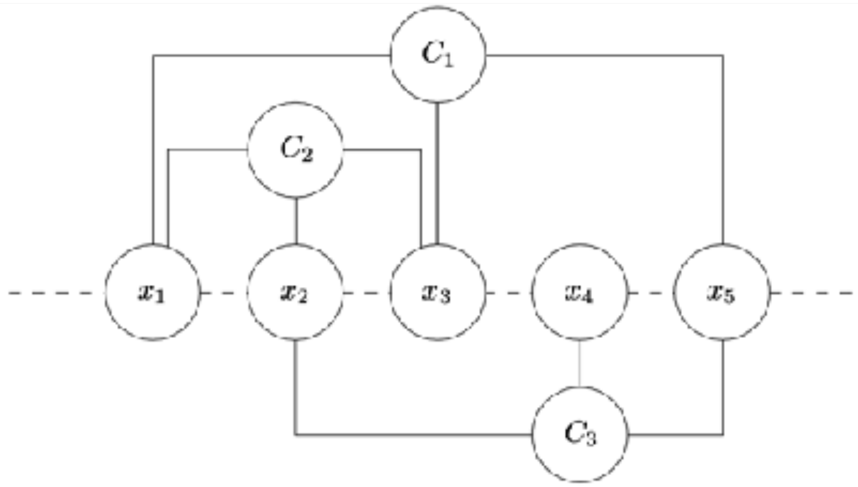

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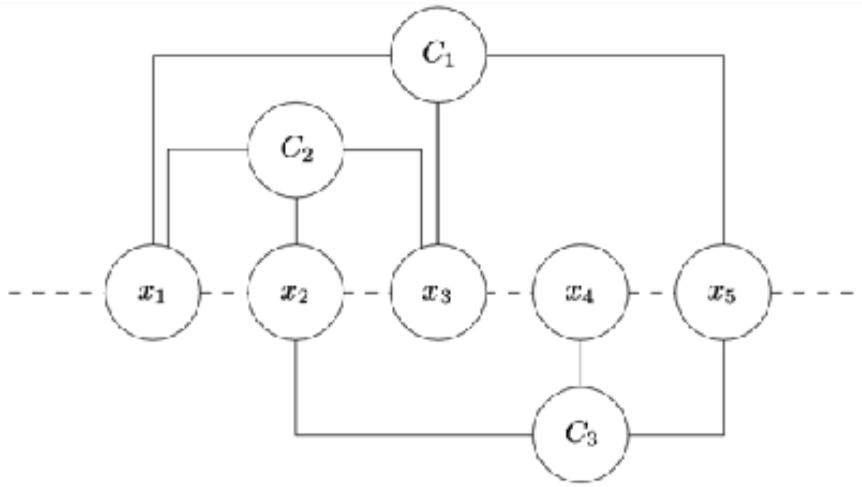

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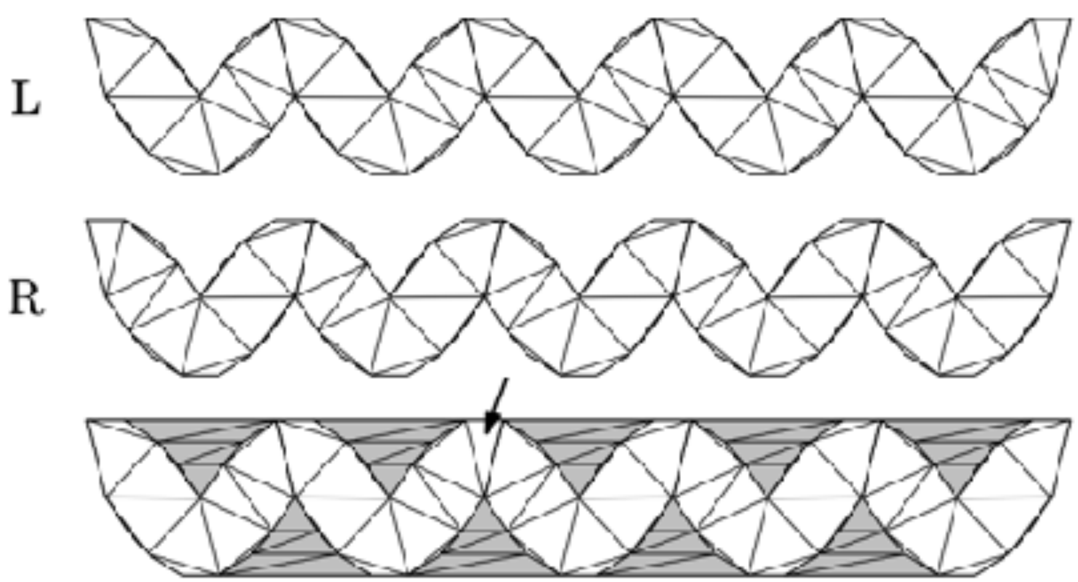

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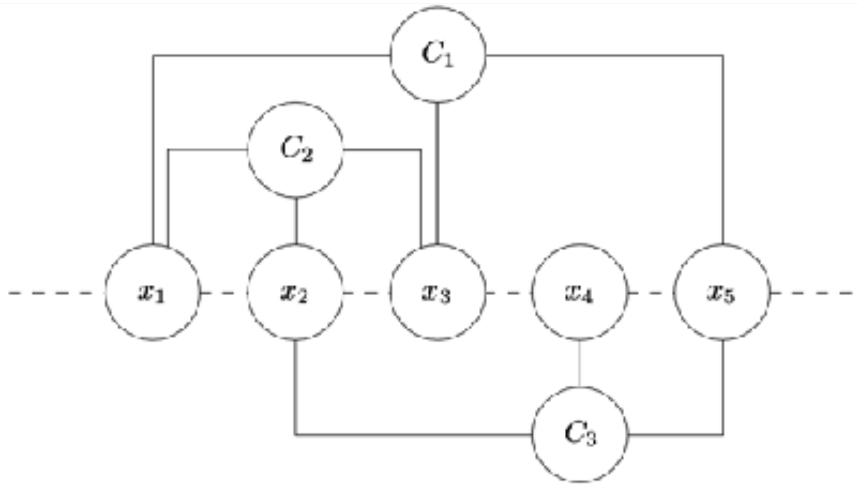

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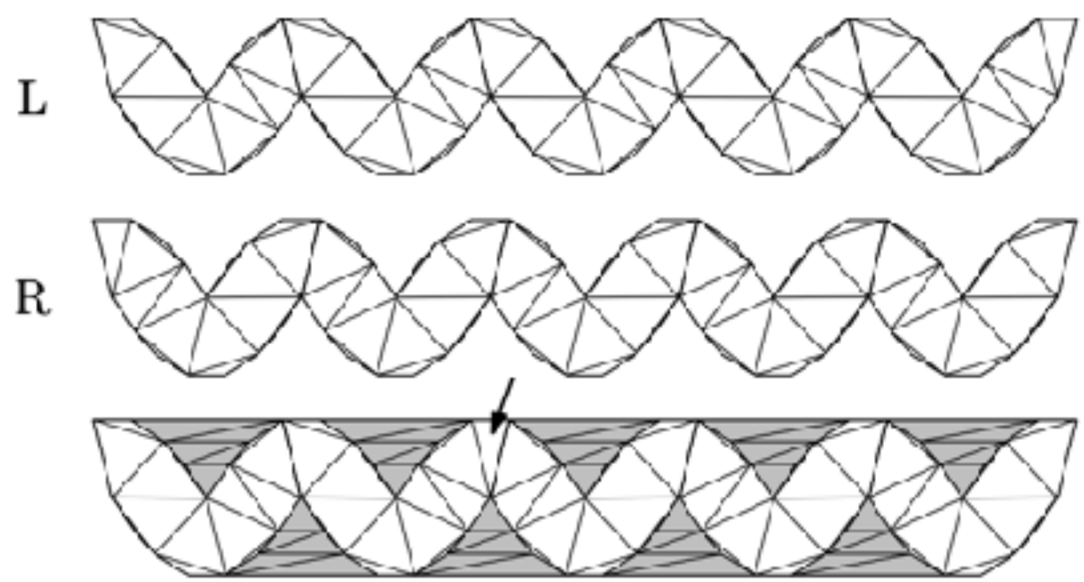

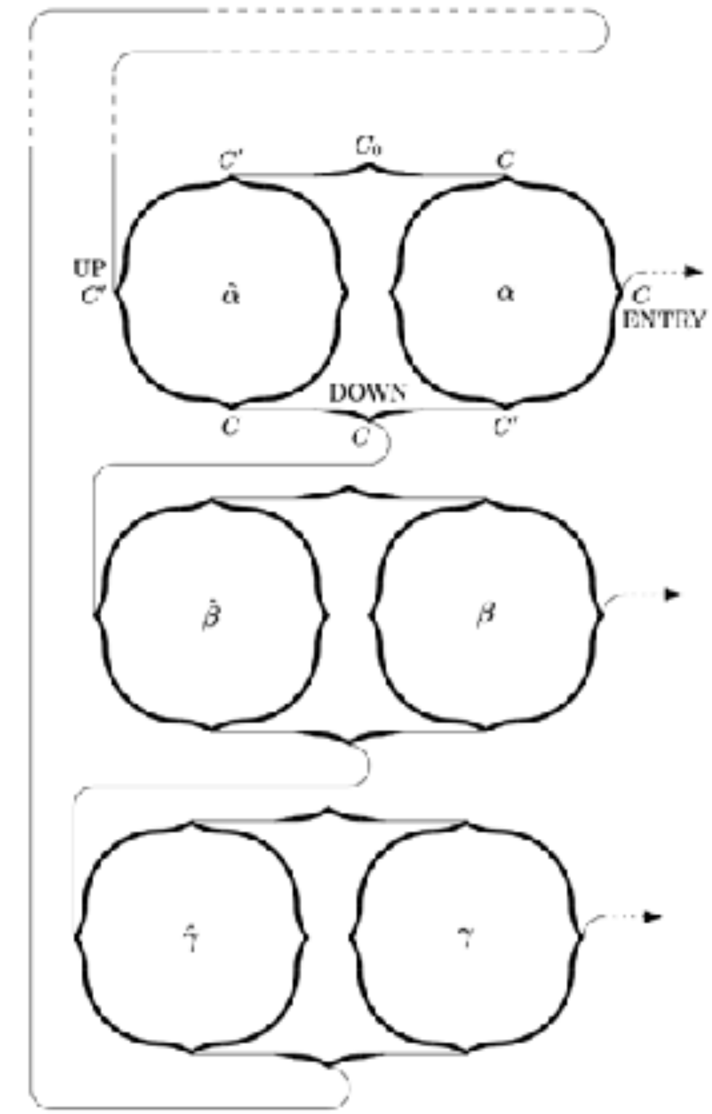

Fig. 19. Schematic view of the clause gadget. The dashed parts of the wires are long enough for sufficiently many copies of the wire-piece and the extended wire-piece to ensure that the wire can reach from the lowest C connection to the left C' connection of $\hat{\alpha}$ (Lemma 6.1).

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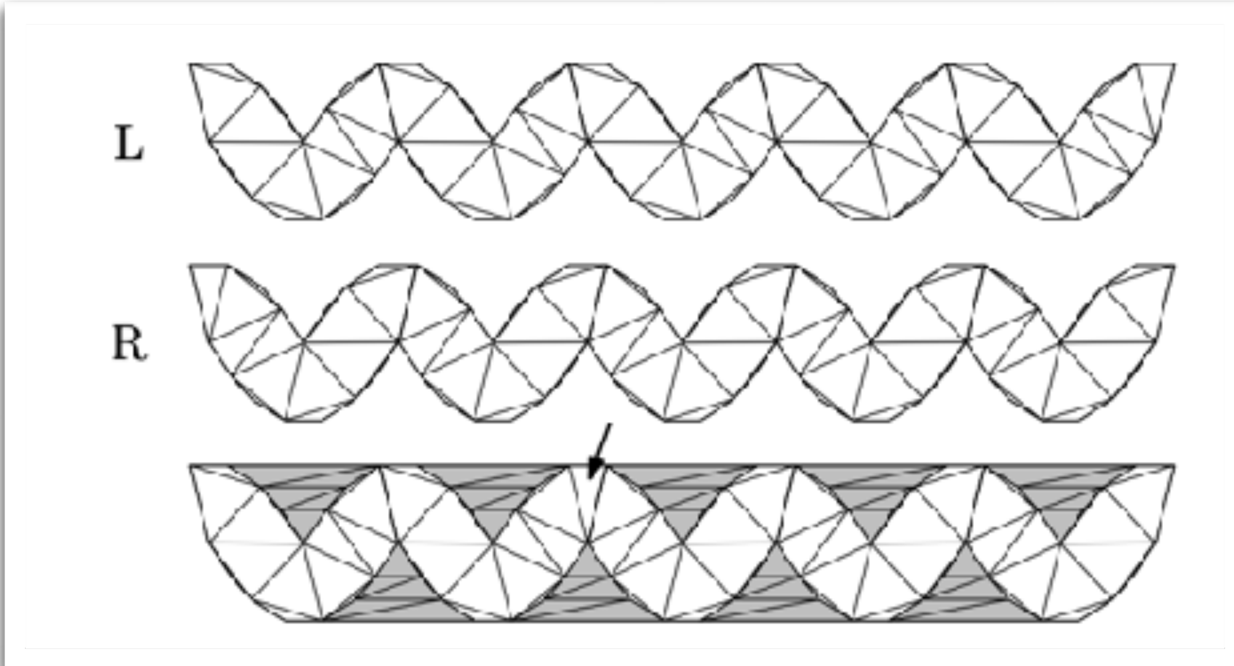

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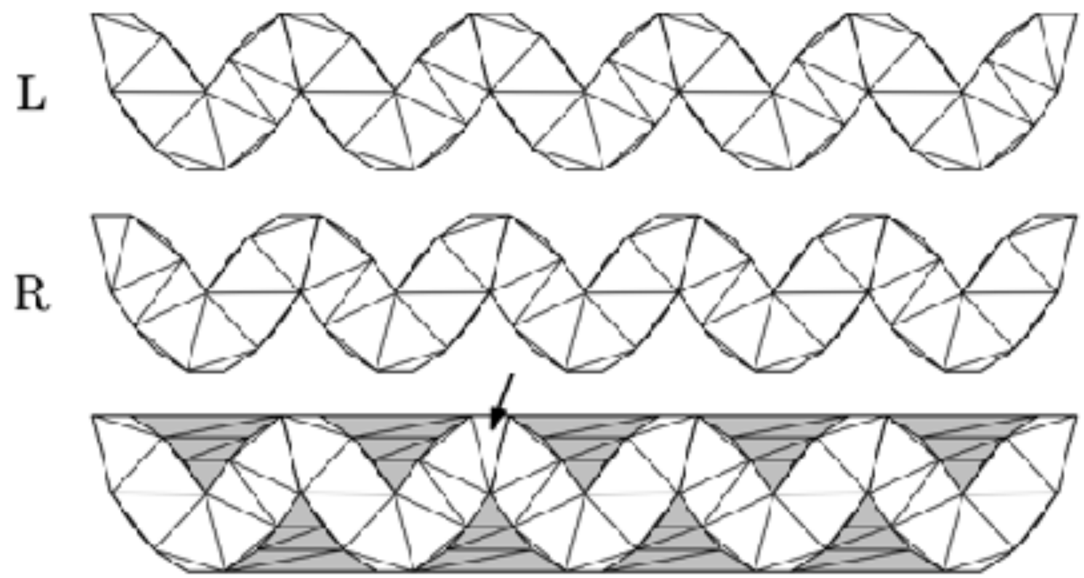

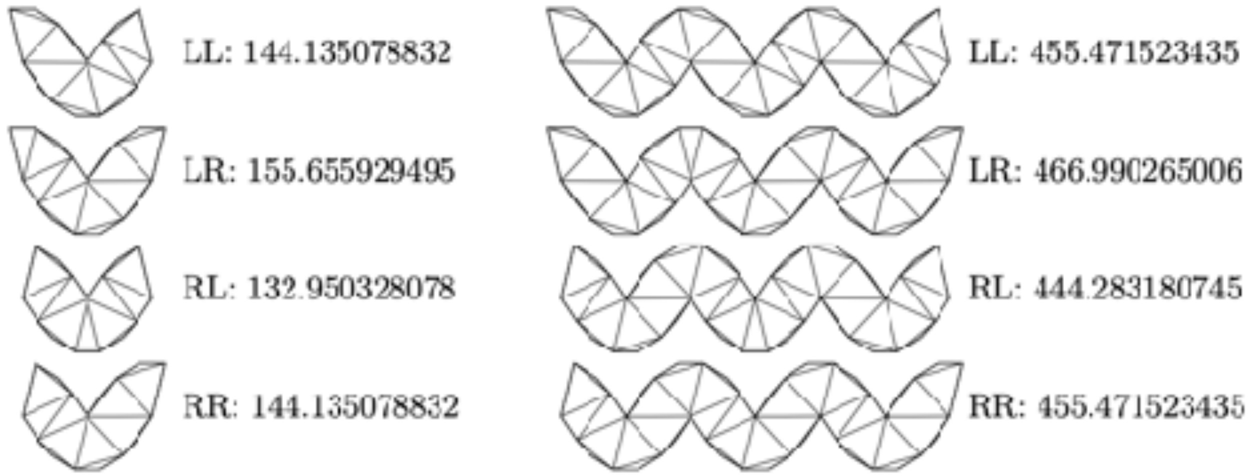

Fig. 17. Optimal solutions for all cases for the wire-piece and the extended wire-piece

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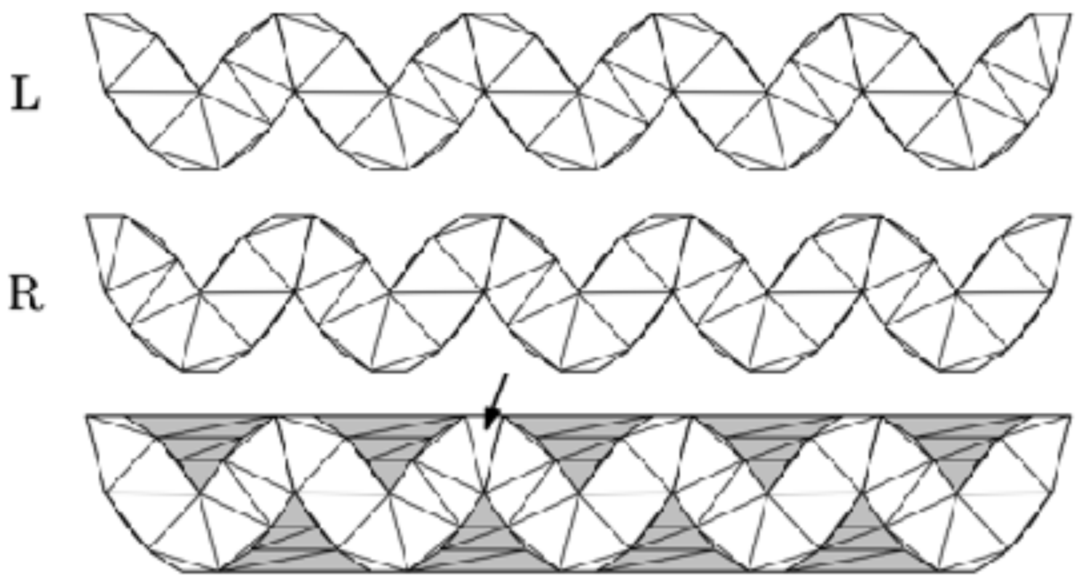

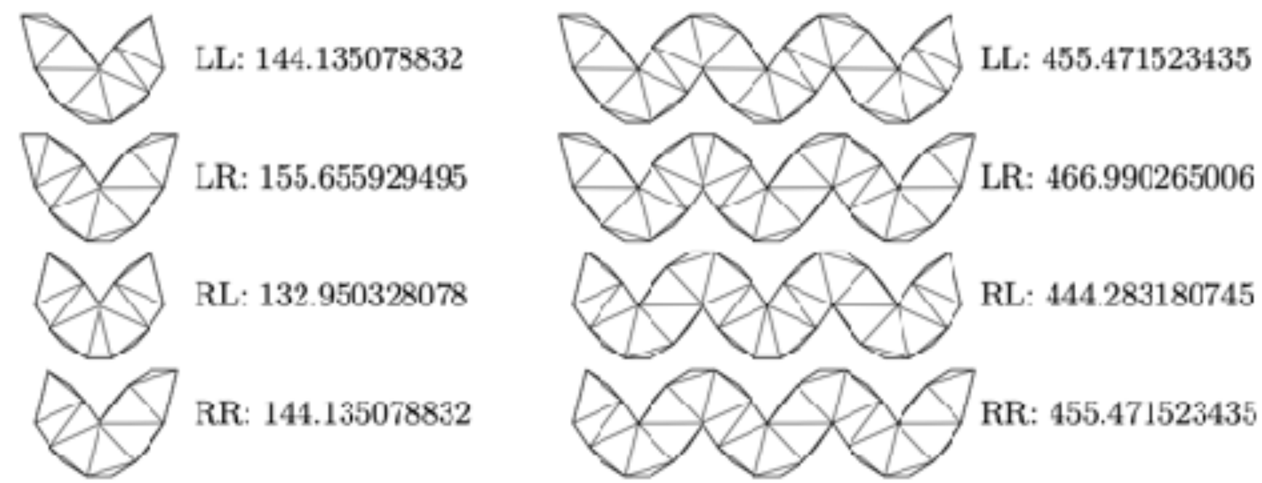

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pattern	multiplicity	internal cost c	reduced internal cost $\bar{c}$	relative reduced cost $\tilde{c}$
LL	1	455.471 523 435	455.471 523 435	0.000 000 000
LR	2	466.990 265 006	455.679 921 006	0.208 397 570
RL	1	444.283 180 745	455.593 524 745	0.122 001 310
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Table III. Analysis of the extended wire-piece

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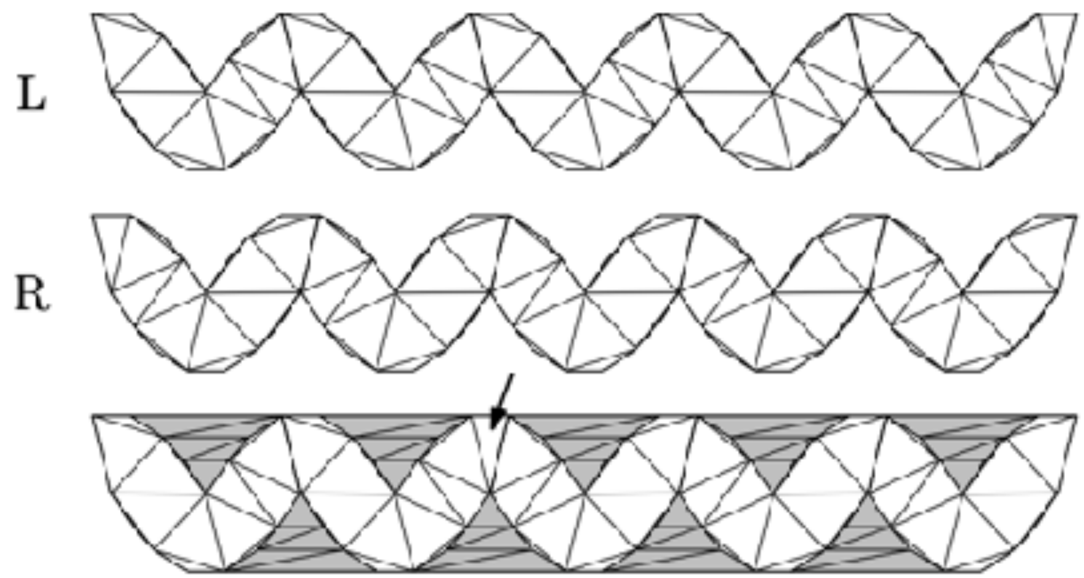

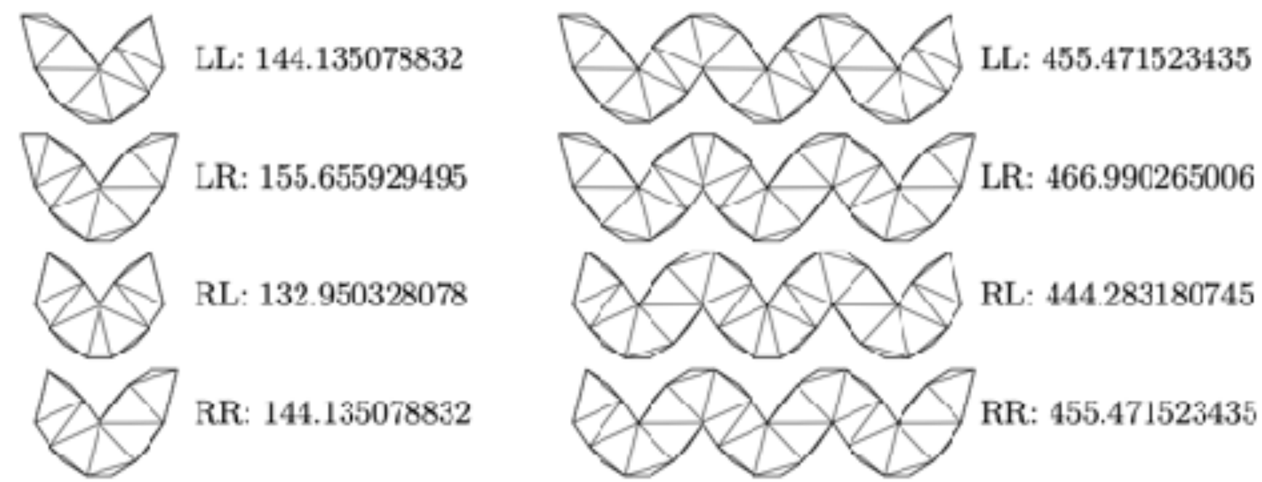

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LEMMA 6.1. *Consider two small vertical terminal triangles at the same height with a horizontal distance $d > 230\,000$ that is a multiple of 0.01. Then the two triangles can be connected by a sequence of wire-pieces and extended wire-pieces.*
An analogous statement holds for vertical connections.

PROOF. If the distance between two terminal triangles is $d = 0.01 \cdot z$ for some integer $z \geq 3 \cdot 2740 \cdot 2739 = 22\,514\,580$, they can be connected by concatenating $y := z \bmod 2740$ extended wire-pieces with $\lfloor z/2740 \rfloor - 3y$ wire-pieces. $\square$

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MWT is NP-hard [Mulzer and Rote, 2008]

	wire-piece	extended wire-piece	thickening adapter	thinning adapter
LL	0.000 000 000	0.000 000 000	0.000 051 402 . . . $= \varepsilon_1$	0.000 000 000
LR	0.210 506 663	0.208 397 570	0.018 887 246	0.018 887 246
RL	0.125 593 246	0.122 001 310	0.014 627 250	0.014 627 250
RR	0.000 000 000	0.000 000 000	0.000 000 000	0.000 051 402 . . . $= \varepsilon_1$

	C_0 connection	left bend	right bend	thick left bend
LL	0.000 000 000	0.000 000 000	0.000 000 000	0.000 000 000
LR	0.044 001 701	0.086 460 895	0.210 506 663	0.891 261 046
RL	0.020 571 757	0.125 593 246	0.125 593 246	0.020 571 757
RR	0.000 000 000	0.000 000 000	0.000 000 000	0.000 000 000

Table IV. The relative reduced costs $\tilde{c}$ for the pieces with two terminals

C		relative reduced cost	C'	
LLl		0.003 861 076 . . . $= \delta_1$		RRr
LLr		0.000 001 575 . . . $= \varepsilon_2$		RRl
LRl		0.029 994 716		LRr
LRr		0.026 135 215		LRl
RLl		0.020 571 757		RLr
RLr		0.020 573 332		RLl
RRl		0.000 000 000 $= 0$		LLr
RRr		0.004 630 878 . . . $= \delta_2$		LLl

	wire-piece	extended wire-piece	thickening adapter	thinning adapter
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LLr		$0.000\,001\,575\ldots = \varepsilon_2$		RRl
LRl		0.029 994 716		LRr
LRr		0.026 135 215		LRl
RLl		0.020 571 757		RLr
RLr		0.020 573 332		RLl
RRl		$0.000\,000\,000 = 0$		LLr
RRr		$0.004\,630\,878\ldots = \delta_2$		LLl

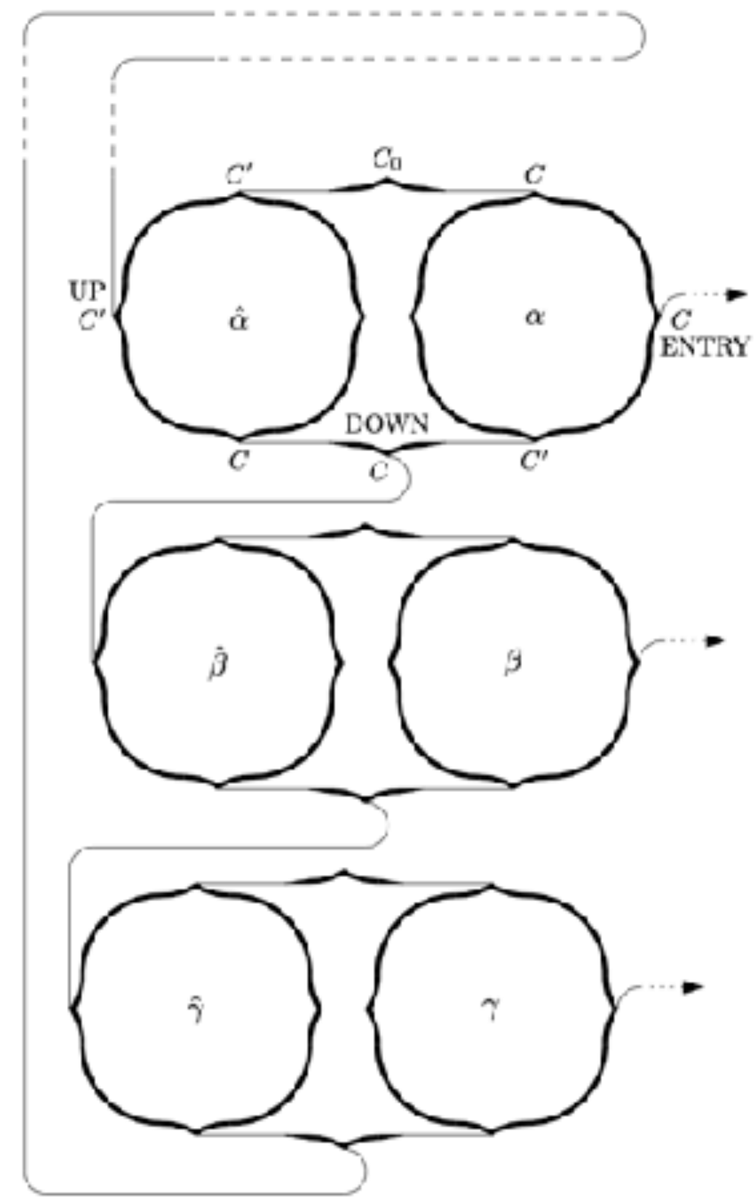

Fig. 19. Schematic view of the clause gadget. The dashed parts of the wires are long enough for sufficiently many copies of the wire-piece and the extended wire-piece to ensure that the wire can reach from the lowest C connection to the left C' connection of $\hat{\alpha}$ (Lemma 6.1).

MWT is practically solvable to optimality [Haas, 2018]

Solving Large-Scale Minimum-Weight Triangulation Instances to Provable Optimality

Andreas Haas[1]

1 Department of Computer Science, TU Braunschweig, 38106 Braunschweig, Germany. haas@ibr.cs.tu-bs.de

Abstract

We consider practical methods for the problem of finding a minimum-weight triangulation (MWT) of a planar point set, a classic problem of computational geometry with many applications. While Mulzer and Rote proved in 2006 that computing an MWT is NP-hard, Beirouti and Snoeyink showed in 1998 that computing provably optimal solutions for MWT instances of up to 80,000 uniformly distributed points is possible, making use of clever heuristics that are based on geometric insights. We show that these techniques can be refined and extended to instances of much bigger size and different type, based on an array of modifications and parallelizations in combination with more efficient geometric encodings and data structures. As a result, we are able to solve MWT instances with up to 30,000,000 uniformly distributed points in less than 4 minutes to provable optimality. Moreover, we can compute optimal solutions for a vast array of other benchmark instances that are not uniformly distributed, including normally distributed instances (up to 30,000,000 points), all point sets in the TSPLIB (up to 85,900 points), and VLSI instances with up to 744,710 points. This demonstrates that from a practical point of view, MWT instances can be handled quite well, despite their theoretical difficulty.

Digital Object Identifier 10.4230/LIPIcs.SoCG.2018.25

1 Introduction

Triangulating a set of points in the plane is a classic problem in computational geometry: given a planar point set S, find a maximal set of non-crossing line segments connecting the points in S. Triangulations have many real-world applications, for example in terrain modeling, finite element mesh generation and visualization. In general, a point set has exponentially many possible triangulations and a natural question is to ask for a triangulation that is optimal with respect to some optimality criterion. Well known and commonly used is the Delaunay triangulation, which optimizes several criteria at the same time: it maximizes the minimum angle and minimizes both the maximum circumcircle and the maximum smallest enclosing circle of all triangles. Another natural optimality criterion, and the one we are considering in this paper is minimizing the total weight of the resulting triangulation.

The minimum weight triangulation (MWT) is listed as one of the open problems in the famous book from 1979 by Garey and Johnson on NP-completeness [10]. The complexity status remained open for 27 years until Mulzer and Rote [19] finally resolved the question and showed NP-hardness of the MWT problem.

Independently, Gilbert [11] and Klincsek [17] showed that, when restricting it to simple polygons, the MWT problem can be solved in $O(n^3)$-time with dynamic programming. The dynamic programming approach can be generalized to polygons with k inner points. Hoffmann and Okamoto [16] showed how to obtain the MWT of such a point set in $O(6^k n^5 \log n)$-time. Their algorithm is based on a polygon decomposition through x-monotone paths. Grantson et al. [12] improved the algorithm to $O(n^4 4^k k)$ and showed another decomposition strategy based on cutting out triangles [13] which runs in $O(n^3 3^k k)$-time.

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34th International Symposium on Computational Geometry (SoCG 2018).
Editors: Bettina Speckmann and Csaba D. Tóth; Article No. 38; pp. 38:1–38:27
Leibniz International Proceedings in Informatics
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3.1 Diamond property

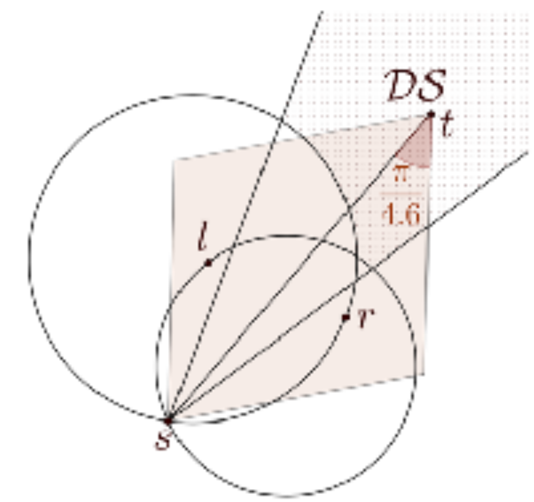

Figure 1 Points l and r induce a region DS such that all edges $e = st$ with $t \in DS$ fail the diamond test. DS is called a dead sector (dotted area).

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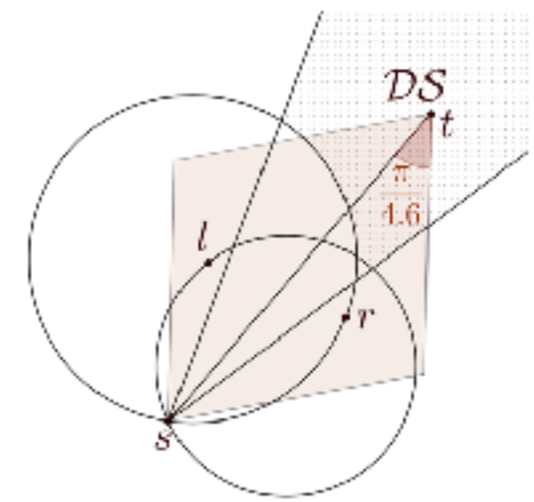
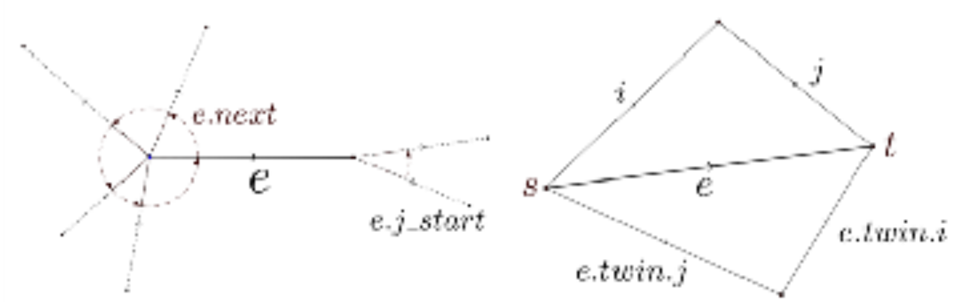


Figure 1 Points l and r induce a region DS such that all edges $e = st$ with $t \in DS$ fail the diamond test. DS is called a dead sector (dotted area).

3.2 LMT-skeleton

The LMT-skeleton was proposed by Dickerson et al. [8, 7], it is a subset of the minimum weight triangulation. The key observation is that MWT(S) is a locally minimal triangulation, i.e., no edge in MWT(S) can be flipped to reduce the total weight.

The LMT-skeleton algorithm eliminates all edges that have no certificate, i.e., for each edge e all pairs of empty triangles bordering e are examined until a certificate is found or no pairs are left. Eliminating e can invalidate previously found certificates. The remaining edges that are not intersected by any other remaining edge form the LMT-skeleton; they must be in all locally minimal triangulations.



(a) Half-edge e. **(b)** i and j point to the current certificate edges.

Figure 2 Representation of half-edge e.

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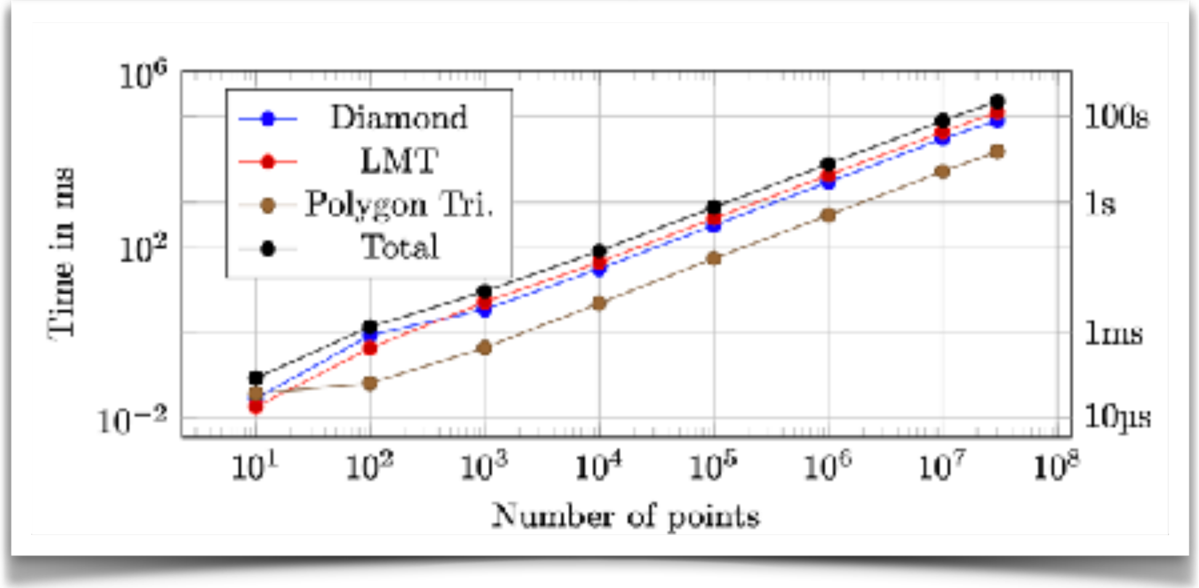

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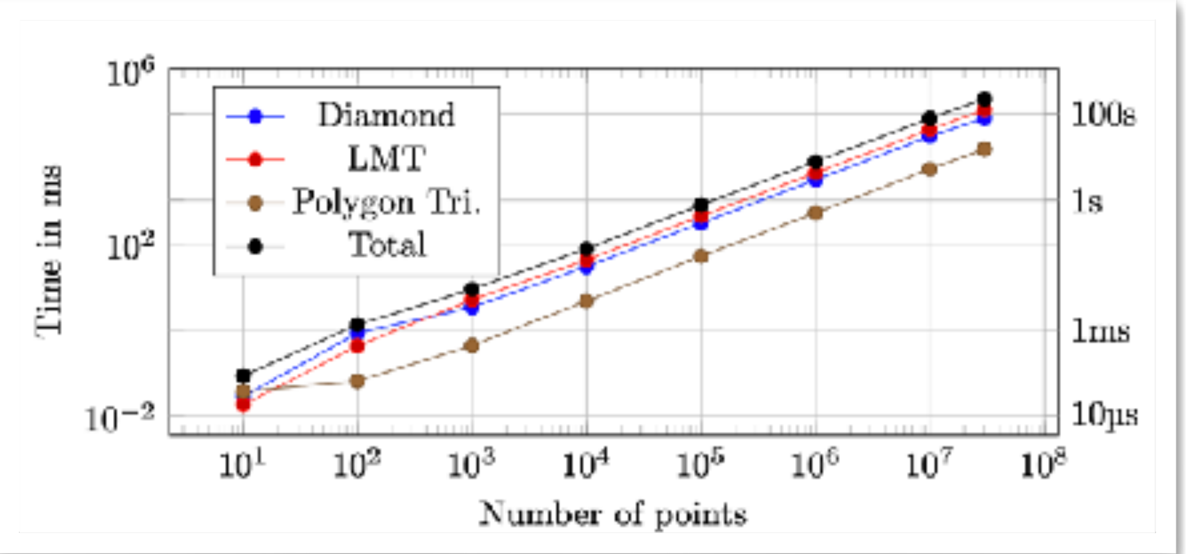

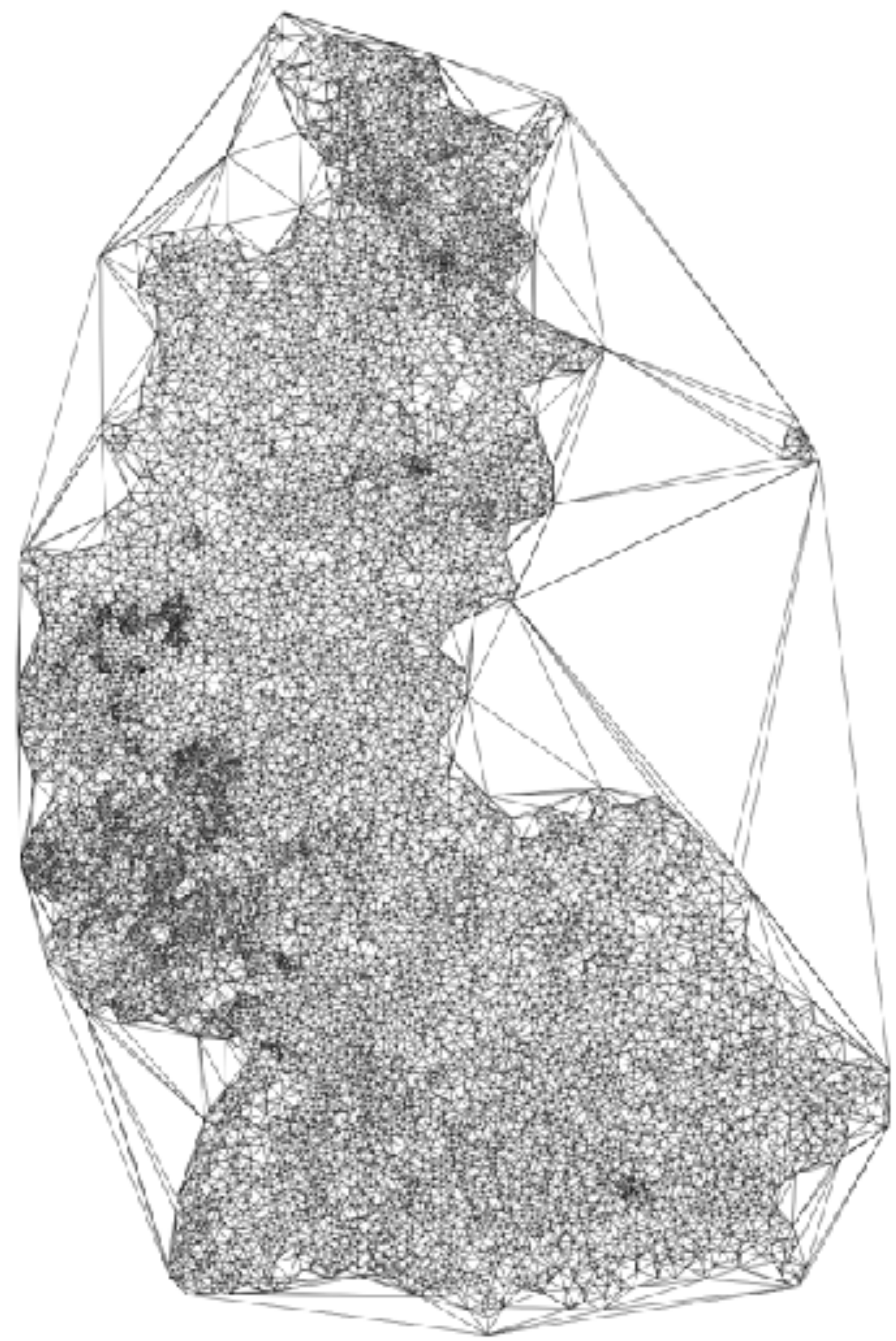

Figure 11 MWT of brd14501.

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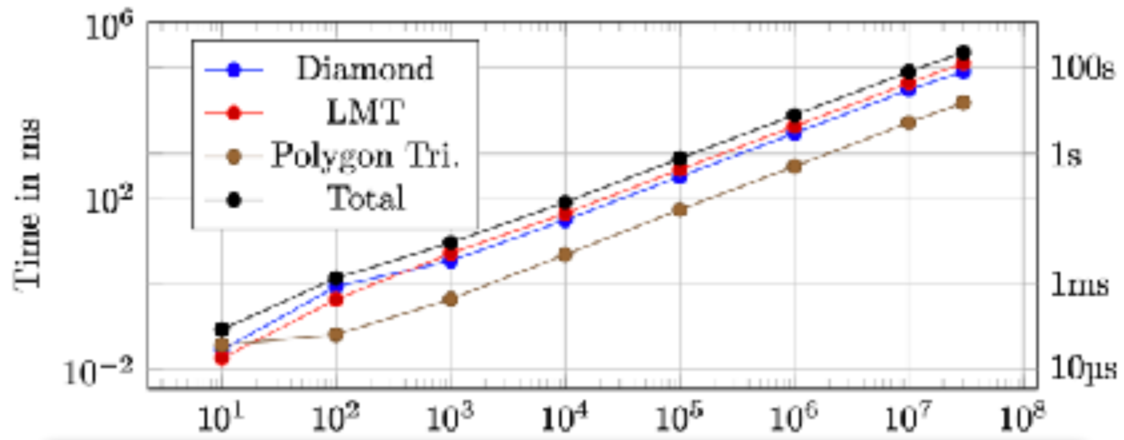

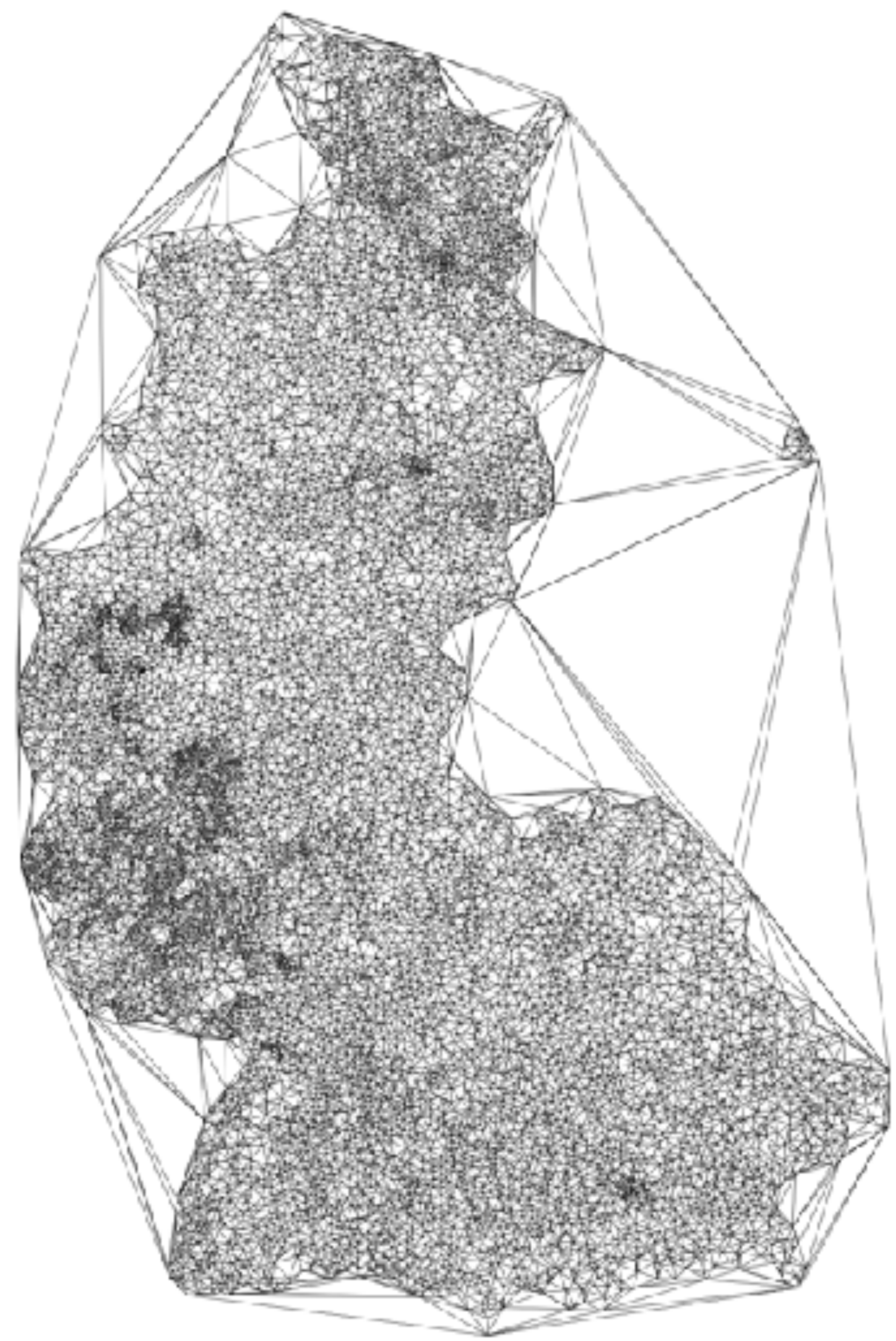

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4 Min-max triangulations

Minimizing maximum angle [Edelsbrunner, Tan 1993]

SIAM Journal on Computing, **22 (3)**, 527–551, (1993)

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A QUADRATIC TIME ALGORITHM FOR THE MINMAX LENGTH TRIANGULATION

HERBERT EDELSBRUNNER* AND TIOW SENG TAN†

Abstract. We show that a triangulation of a set of n points in the plane that minimizes the maximum edge length can be computed in time $O(n^2)$. The algorithm is reasonably easy to implement and is based on the theorem that there is a triangulation with minmax edge length that contains the relative neighborhood graph of the points as a subgraph. With minor modifications the algorithm works for arbitrary normed metrics.

Key words. Computational geometry, point sets, triangulations, two dimensions, minmax edge length, normed metrics

5 Max-min triangulations

A Long-Standing Open Problem

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... Open Problem

Journal of Computational Geometry jocg.org

COMPUTING MAXMIN EDGE LENGTH TRIANGULATIONS*

Sándor P. Fekete,[†] *Winfried Hellmann,*[†] *Michael Hemmer,*[†] *Arne Schmidt,*[†] *and Julian Troegel*[†]

Abstract. In 1991, Edelsbrunner and Tan gave an $O(n^2)$ algorithm for finding the MinMax Edge Length Triangulation of a set of points in the plane, but stated the complexity of finding a MaxMin Edge Length Triangulation (MELT) as a natural open problem. We resolve this long-standing problem by showing that computing a MELT is NP-complete. Moreover, we prove that (unless P=NP), there is no polynomial-time approximation algorithm that can approximate MELT within any polynomial factor.

While this may be taken as conclusive evidence from a *theoretical* point of view that the problem is hopelessly intractable, it still makes sense to consider powerful optimization methods, such as integer programming (IP), in order to obtain provably optimal solutions for intances of non-trivial size. A straightforward IP based on pairwise disjointness of the $\Theta(n^2)$ segments between the n points has $\Theta(n^4)$ constraints, making this IP hopelessly intractable from a *practical* point of view, even for relatively small n. The main algorithm engineering twist of this paper is to demonstrate how the combination of geometric insights with refined methods of combinatorial optimization can still help to put together an exact method capable of computing optimal MELT solutions for planar point sets up to $n = 200$. Our key idea is to exploit specific geometric properties in combination with more compact IP formulations, such that we are able to drastically reduce the number of constraints. On the practical side, we combine two of the most powerful software packages for the individual components: CGAL for carrying out the geometric computations, and CPLEX for solving the IPs. In addition, we discuss specific analytic aspects of the speedup for random point sets.

1 Introduction

Triangulating a set of points is one of the basic problems of Computational Geometry: given a set P of n points in the plane, connect them by a maximal set of non-crossing line segments. This implies that all bounded faces of the resulting planar arrangement are triangles, while the exterior face is the complement of the convex hull of P.

Triangulations are computed and used in a large variety of contexts, e.g., in mesh generation, but also as a stepping stone for other tasks. While it is not hard to compute some triangulation, most of these tasks require triangulations with special properties that

*A preliminary extended abstract appears in the Proceedings of ALENEX 2015 [16].

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LGORITHM FOR THE
TRIANGULATION

R* AND TIOW SENG TAN[†]

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me $O(n^2)$. The algorithm is reasonably easy to
is a triangulation with minmax edge length that
ints as a subgraph. With minor modifications the

ets, triangulations, two dimensions, minmax edge

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COMPUTING MAXMIN EDGE LENGTH TRIANGULATIONS*

Sándor P. Fekete,† *Winfried Hellmann,*† *Michael Hemmer,*† *Arne Schmidt,*†
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IP formulations, such that we are able to drastically reduce the number of constraints. On the practical side, we combine two of the most powerful software packages for the individual components: CGAL for carrying out the geometric computati… the IPs. In addition, we discuss specific analytic aspects of t… sets.

JoCG 9(1), 1–26, 2018

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1 Introduction

Triangulating a set of points is one of the basic problems of Computational Geometry: given a set P of n points in the plane, connect them by a maximal set of non-crossing line segments. This implies that all bounded faces of the resulting planar arrangement are triangles, while the exterior face is the complement of the convex hull of P.

Triangulations are computed and used in a large variety of contexts, e.g., in mesh generation, but also as a stepping stone for other tasks. While it is not hard to compute some triangulation, most of these tasks require triangulations with special properties that

*A preliminary extended abstract appears in the Proceedings of ALENEX 2015 [16].

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5.1: Complexity

An Open Problem Resolved

An Open Problem Resolved

THEOREM 3.1. *It is NP-hard to decide whether a set P of n points in the plane has a triangulation with smallest edge of length at least L, for some positive number L.*

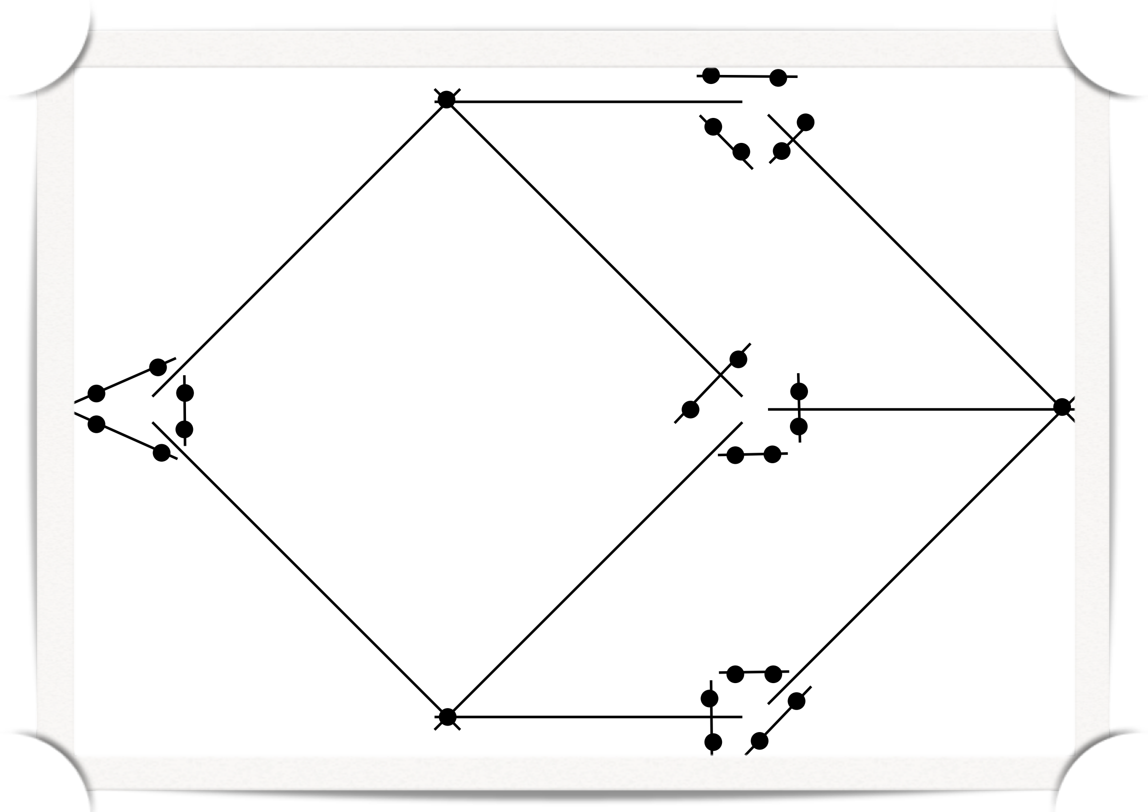
An Auxiliary Problem

An Auxiliary Problem

COVERING BY DISJOINT SEGMENTS

An Auxiliary Problem

Covering by Disjoint Segments

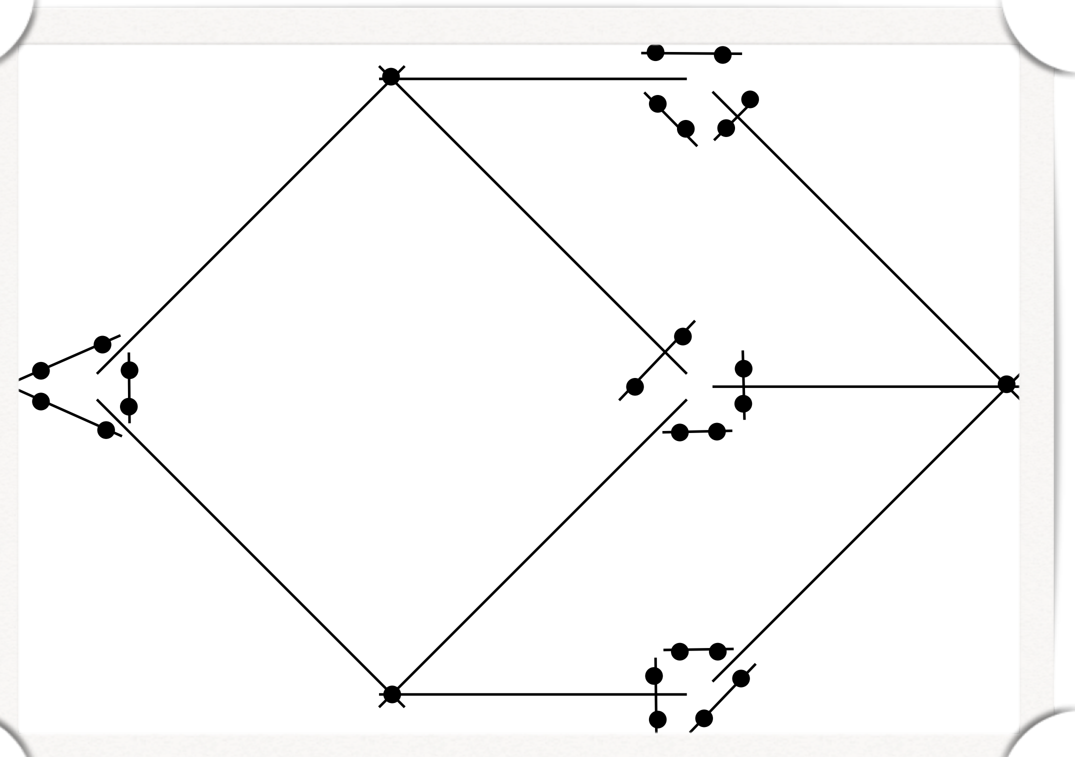


An Auxiliary Problem

COVERING BY DISJOINT SEGMENTS

Given: *A specified set S of line segments ("stabbers") in the Euclidean plane, and a subset T of their intersection points ("targets").*

Wanted: *A non-intersecting subset of the stabbers that covers all targets.*



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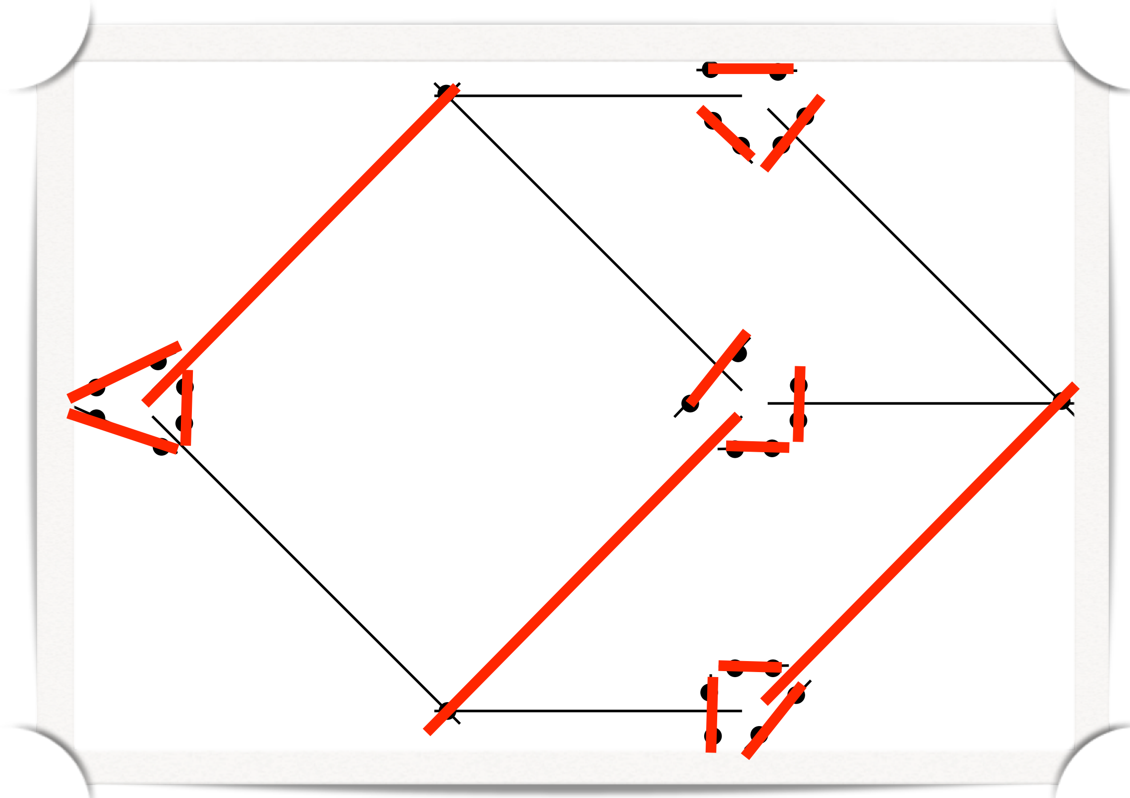

Proof of NP-Hardness

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Idea: Use reduction from ***Planar 3SAT***

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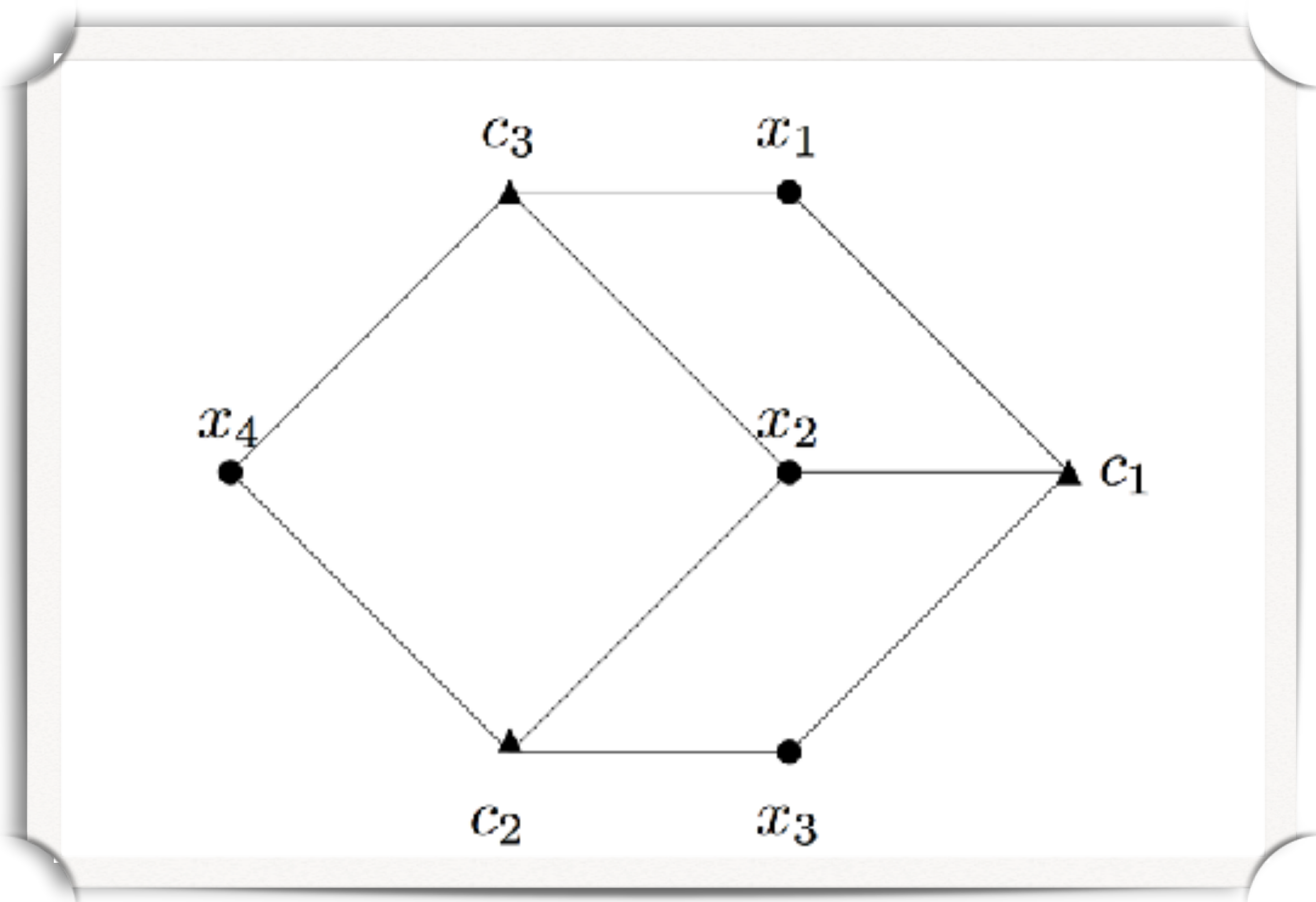

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Component 1: Variables

Proof of NP-Hardness

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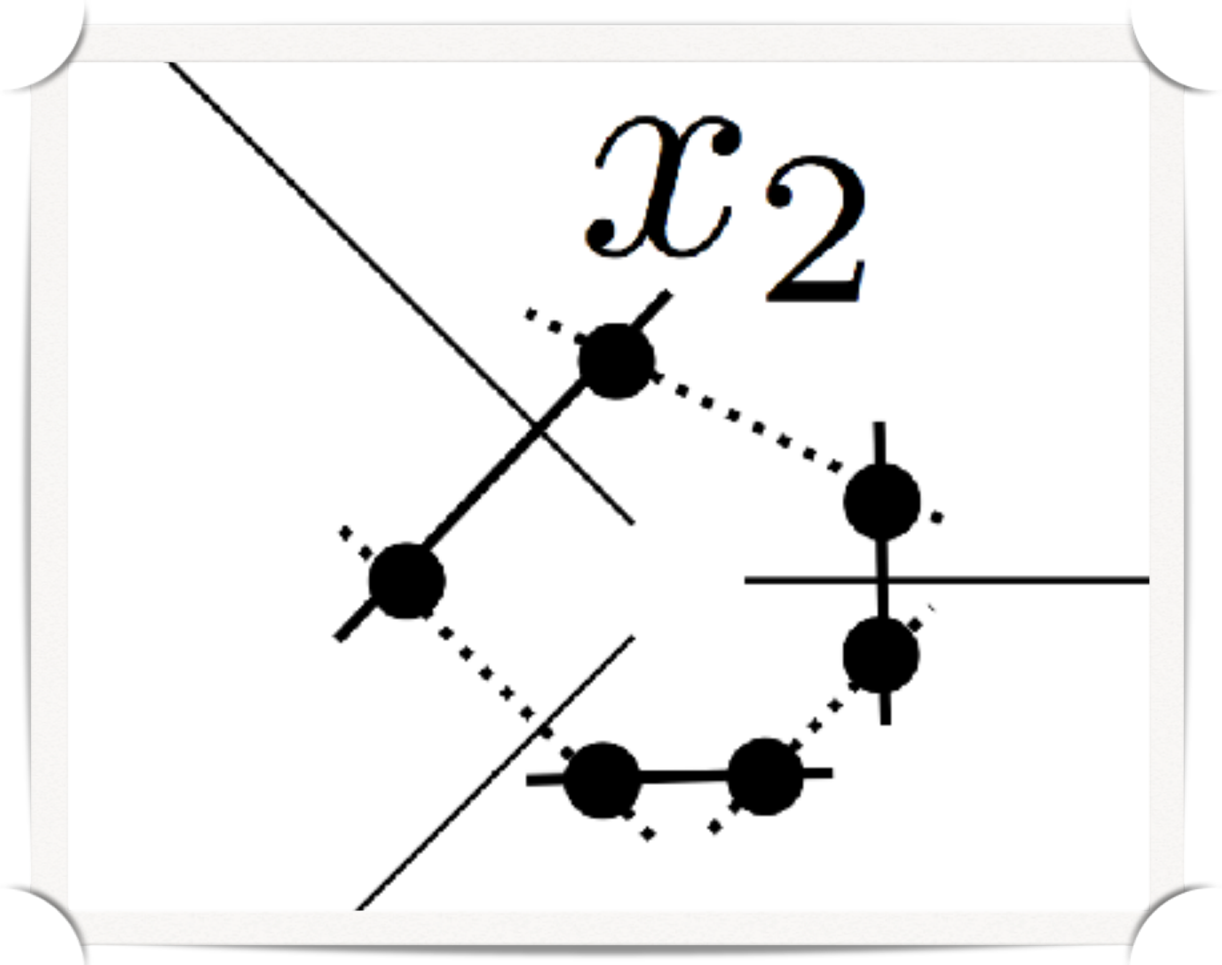

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Component 2: Clauses

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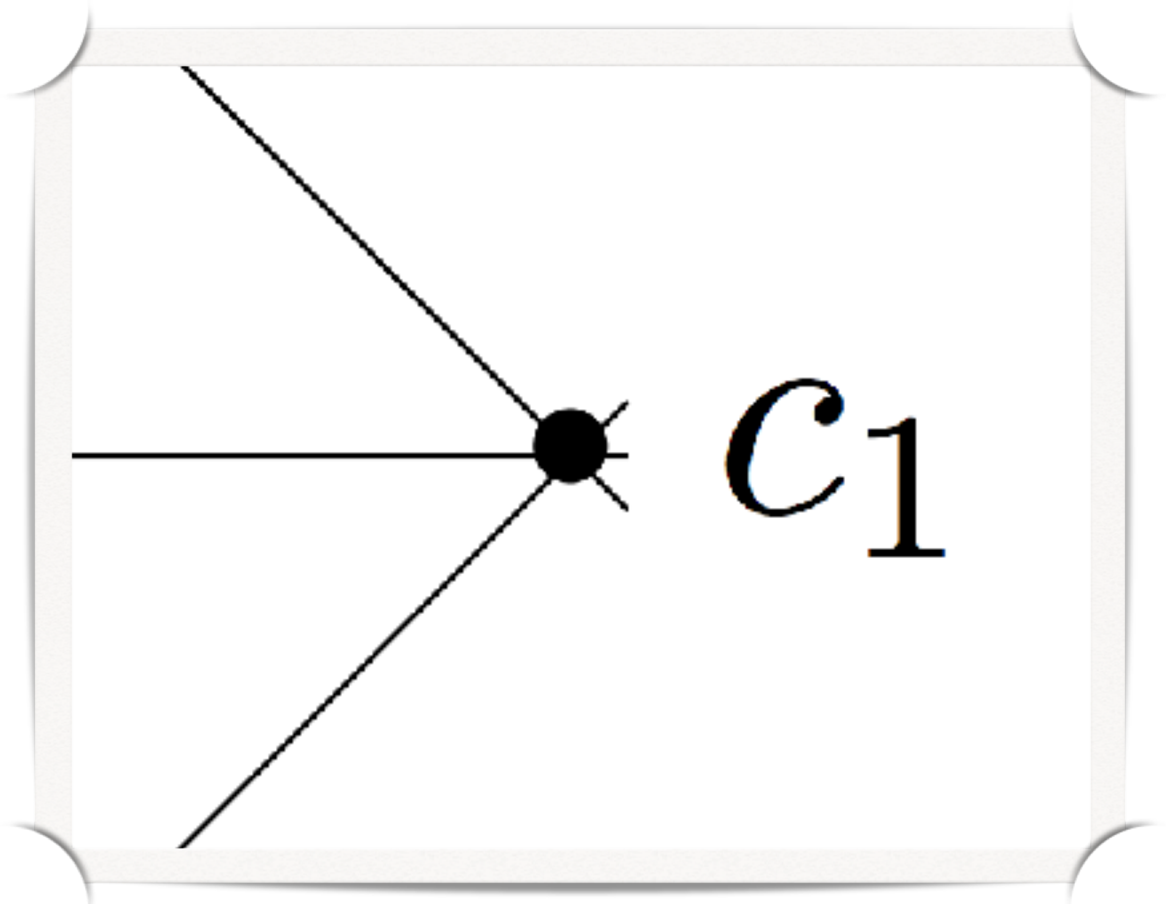

Proof of NP-Hardness

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Overall:

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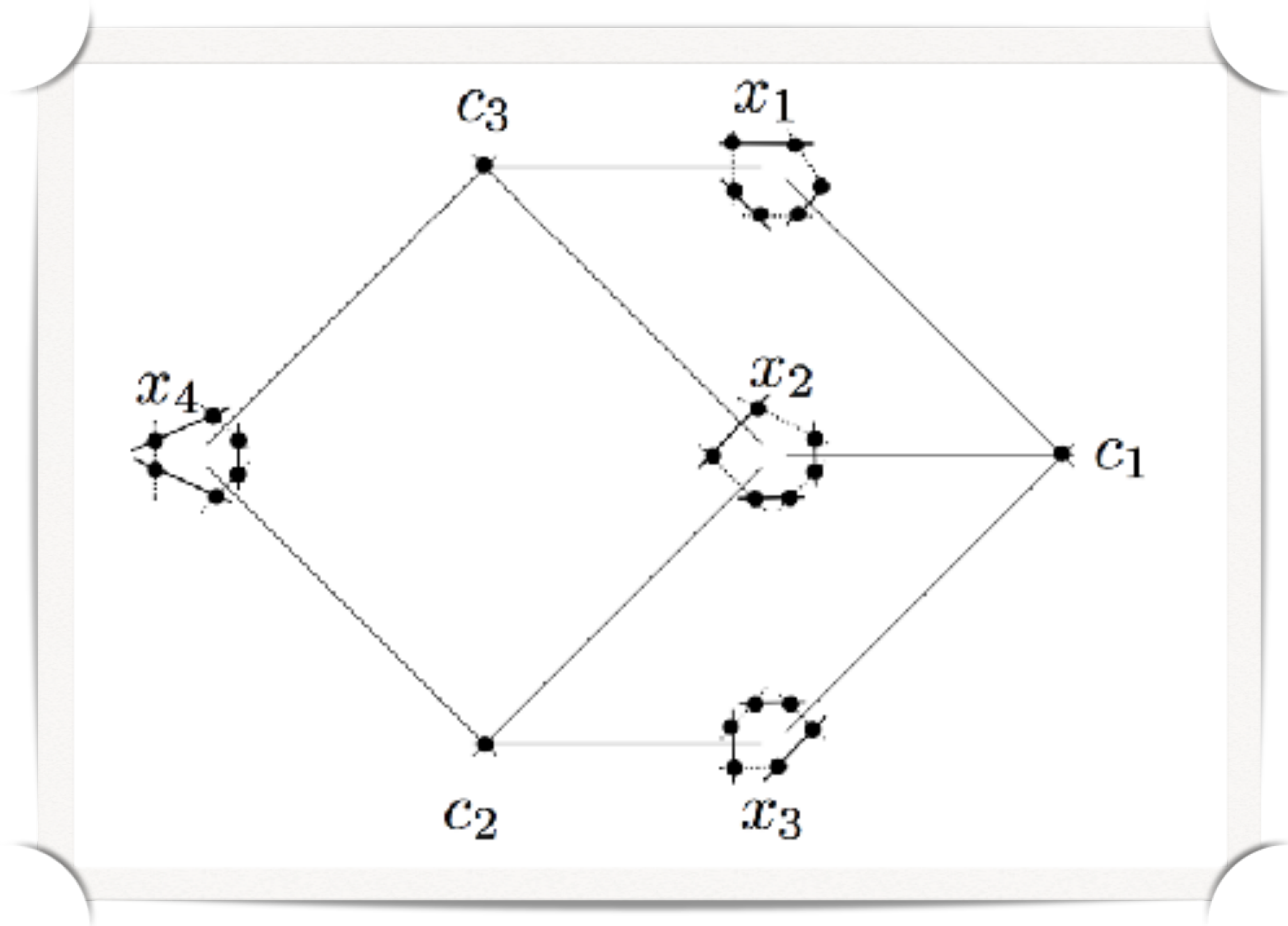

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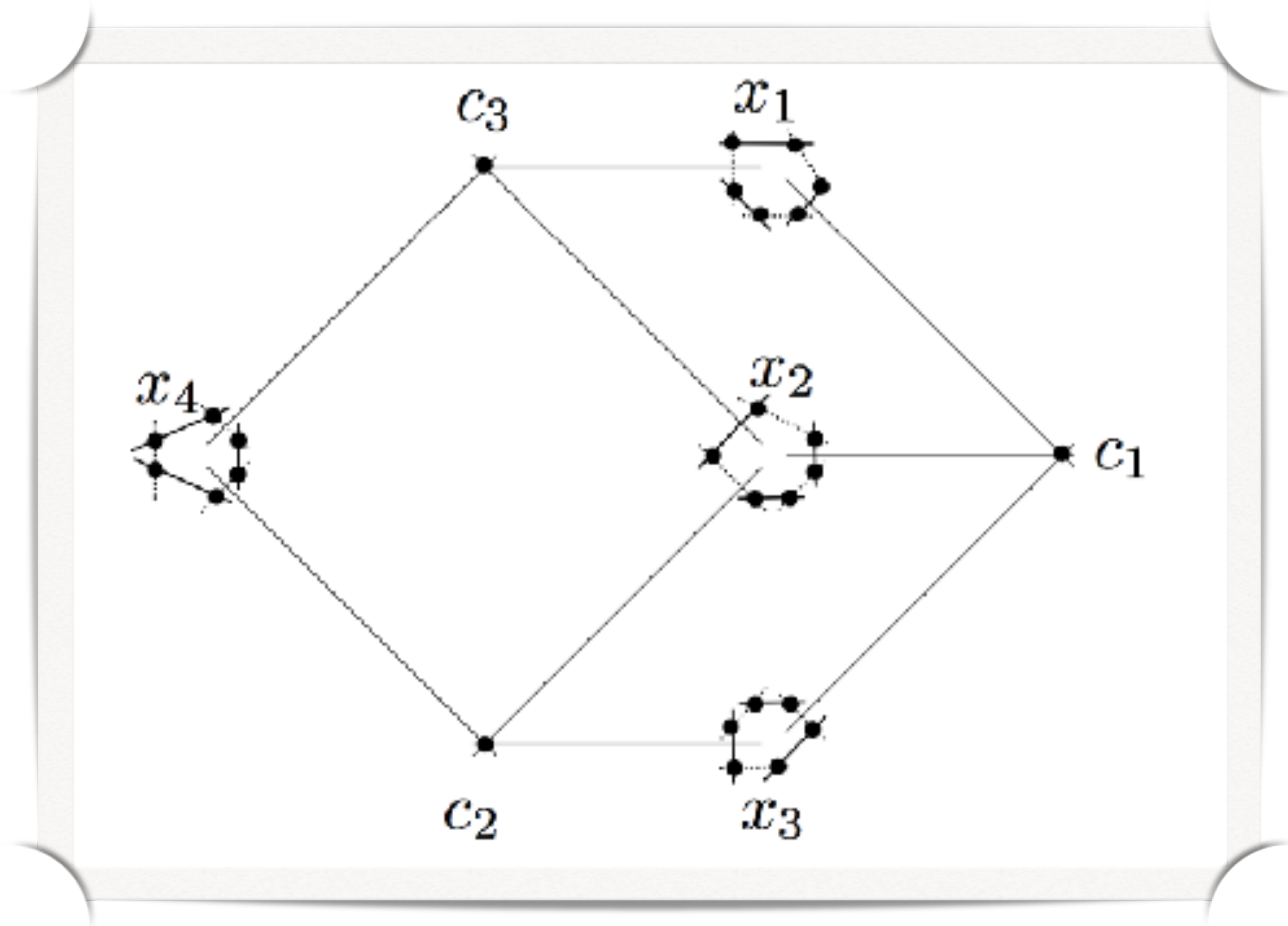

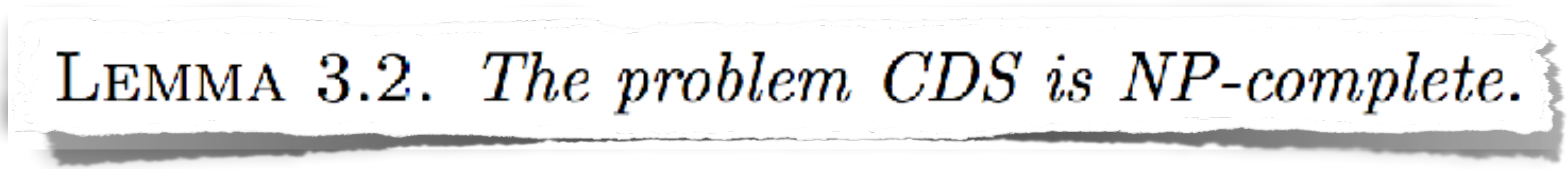

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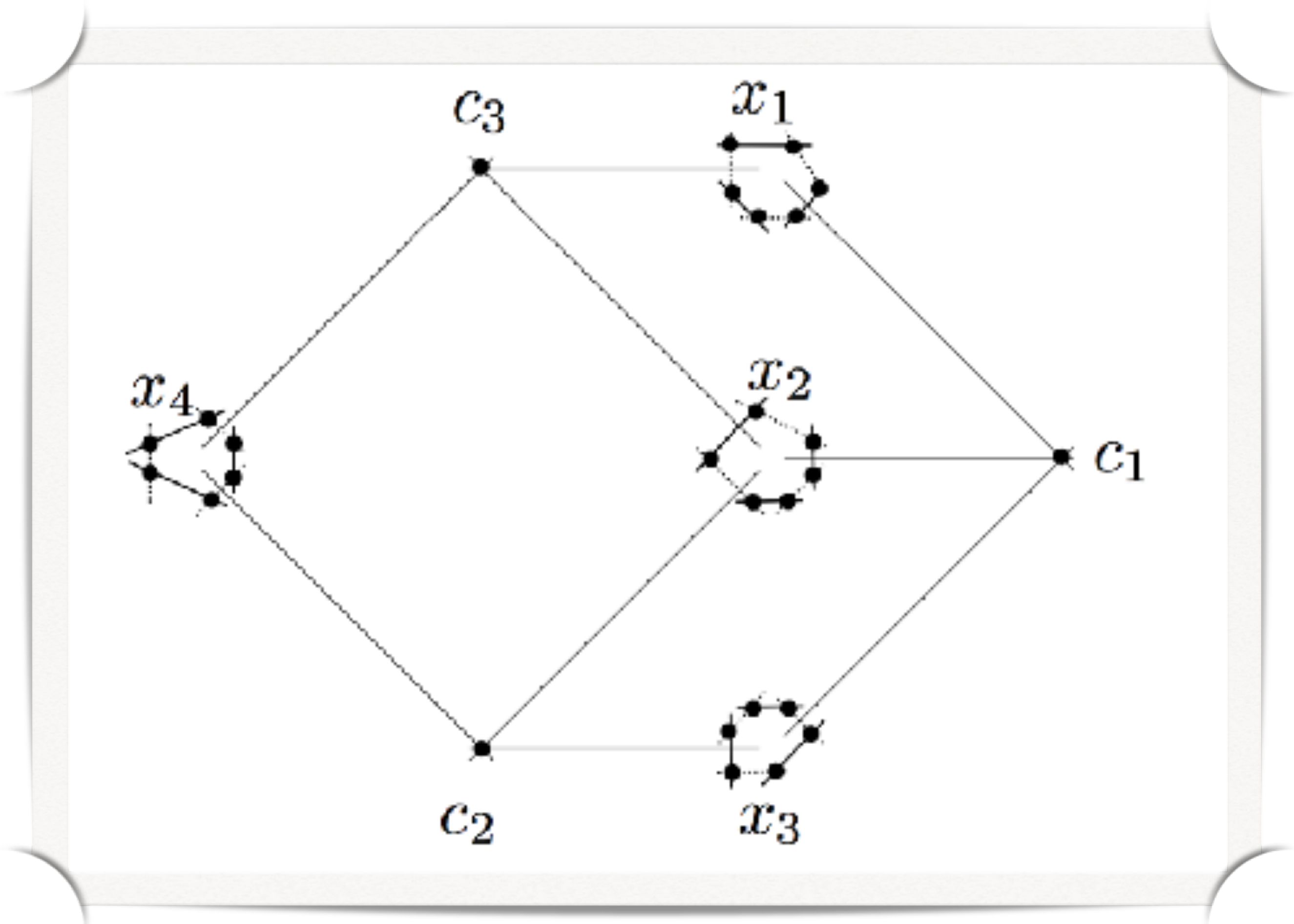

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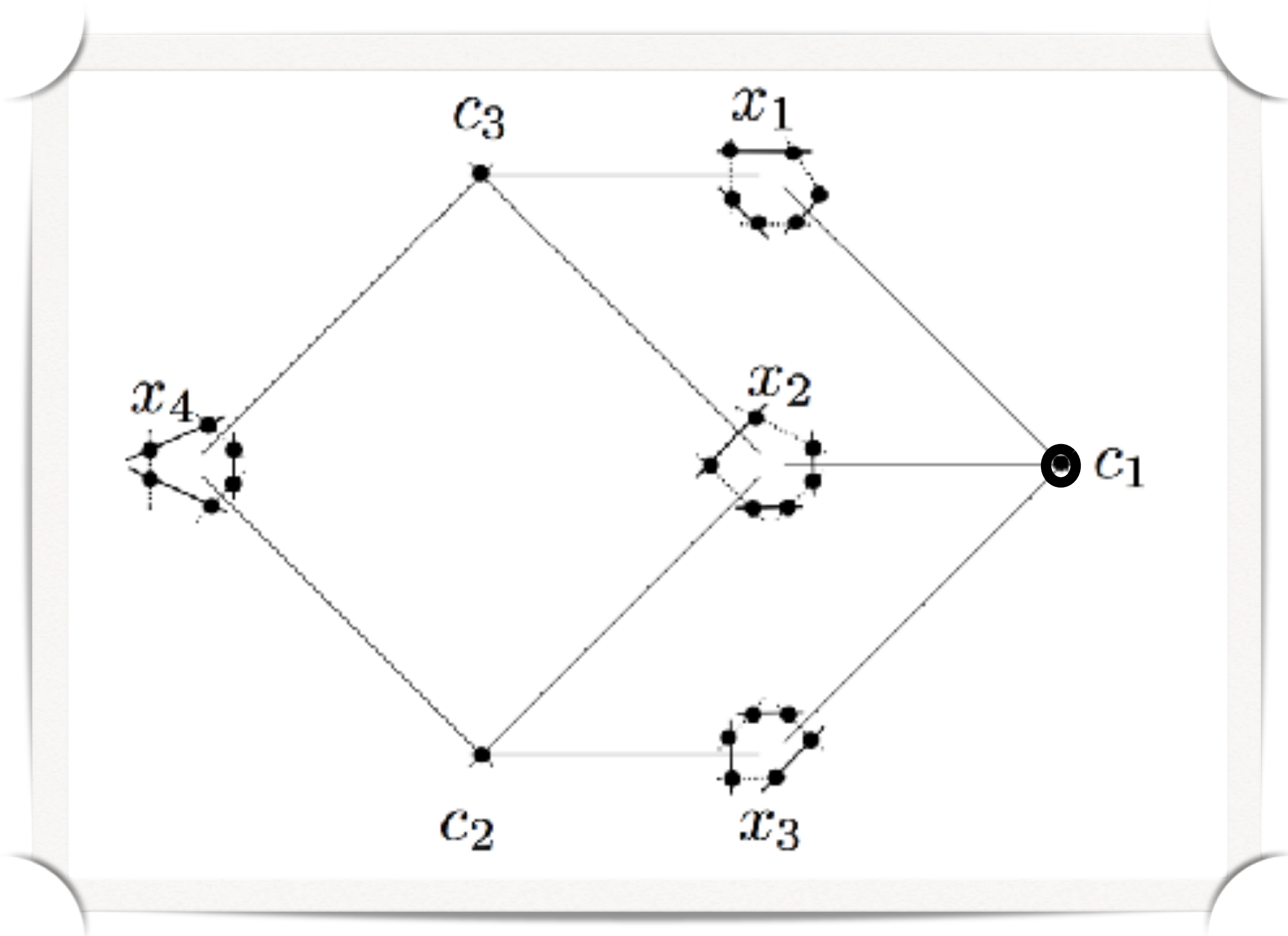

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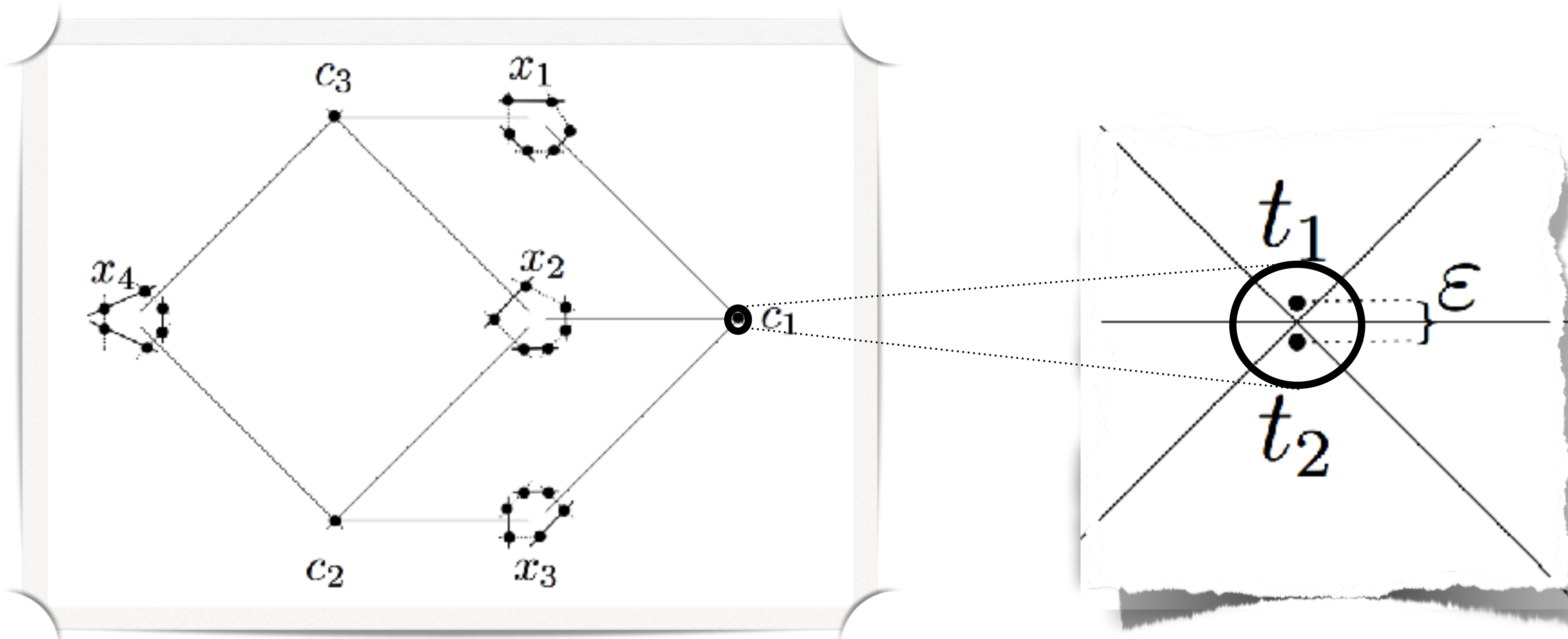

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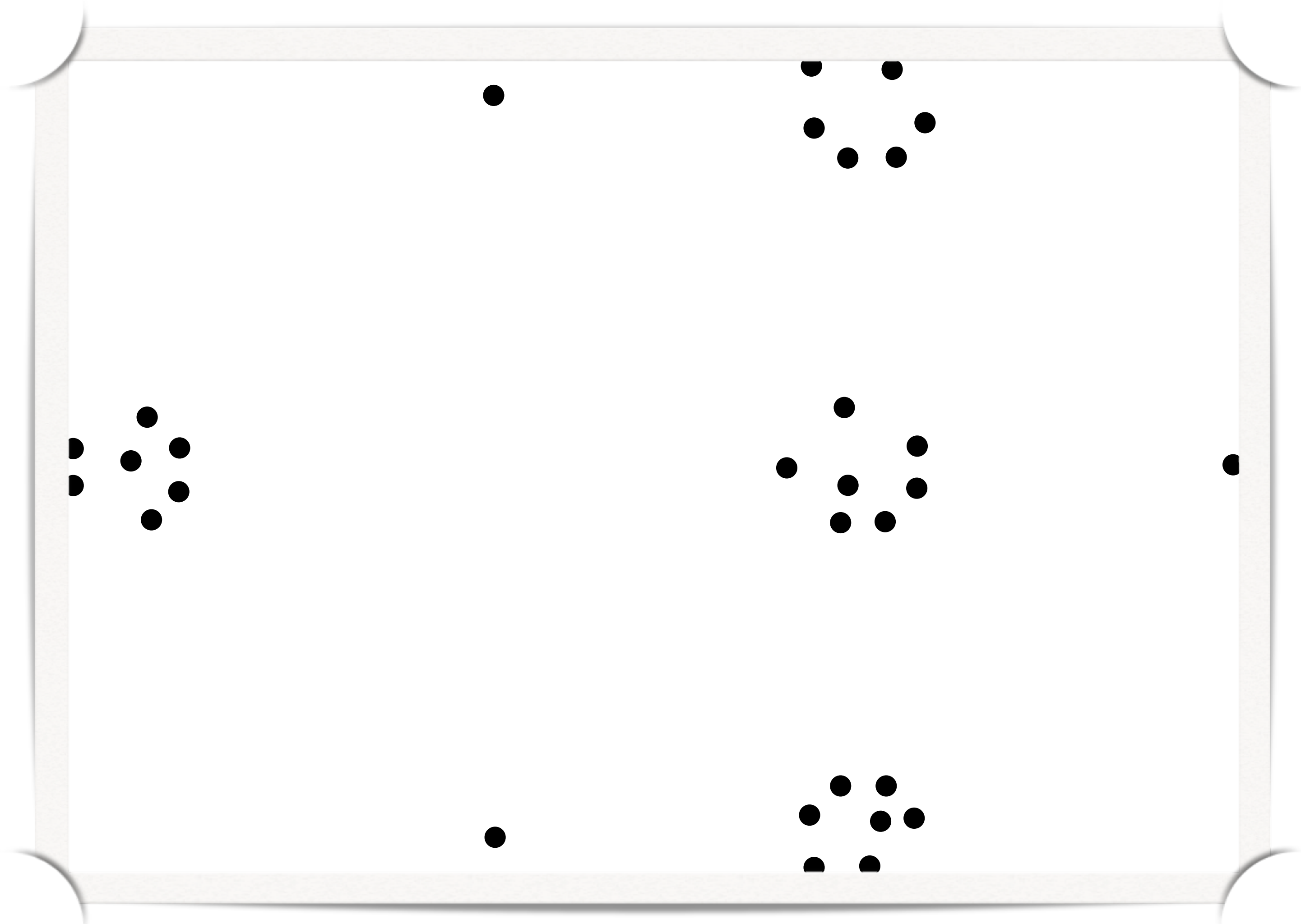

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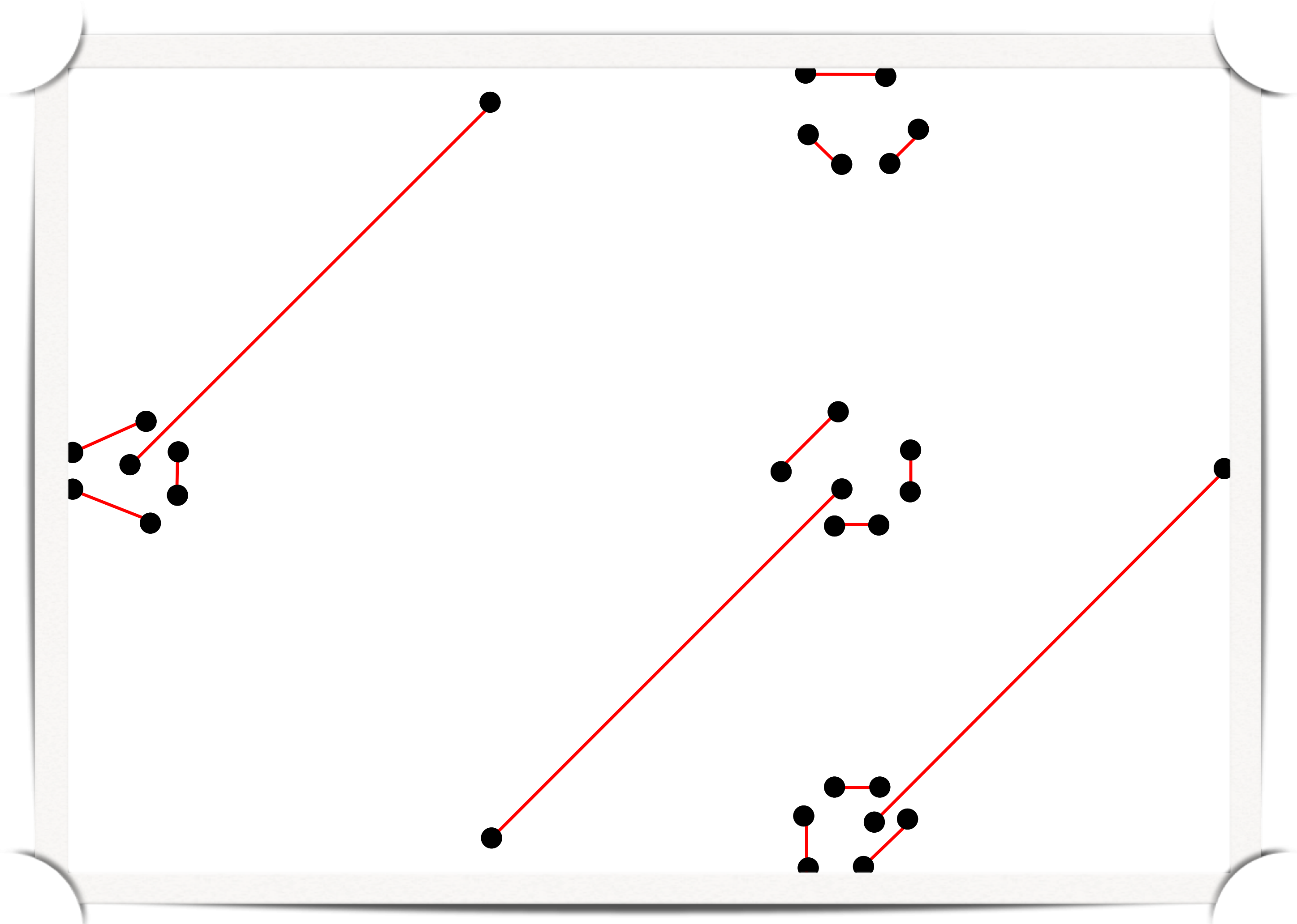

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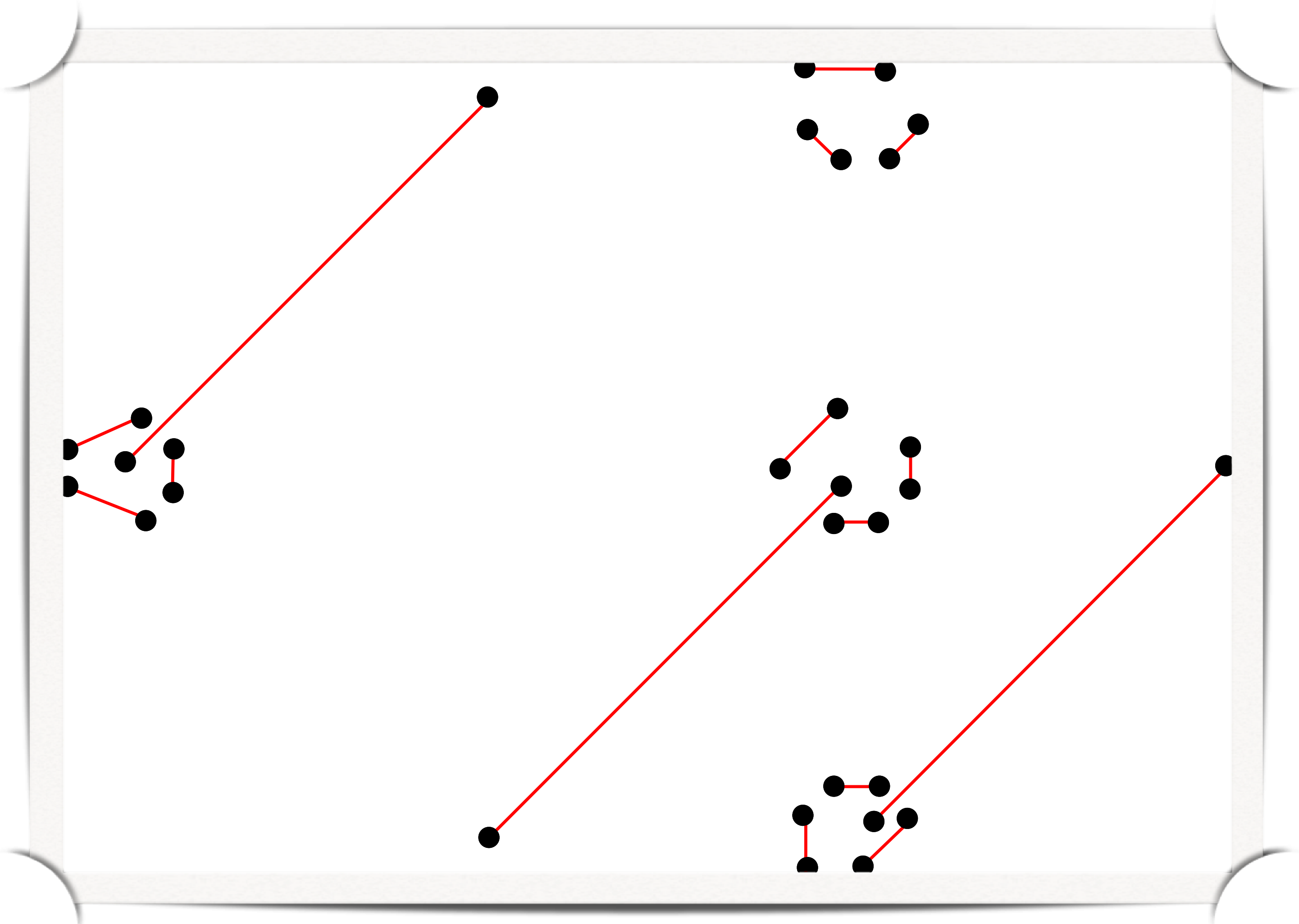

LEMMA 3.3. *Let P be a set of points in the plane, and let $p_i, p_j \in P$. A triangulation Δ contains the edge $\overline{p_i p_j}$, iff there is no edge in Δ that separates $\overline{p_i p_j}$.*

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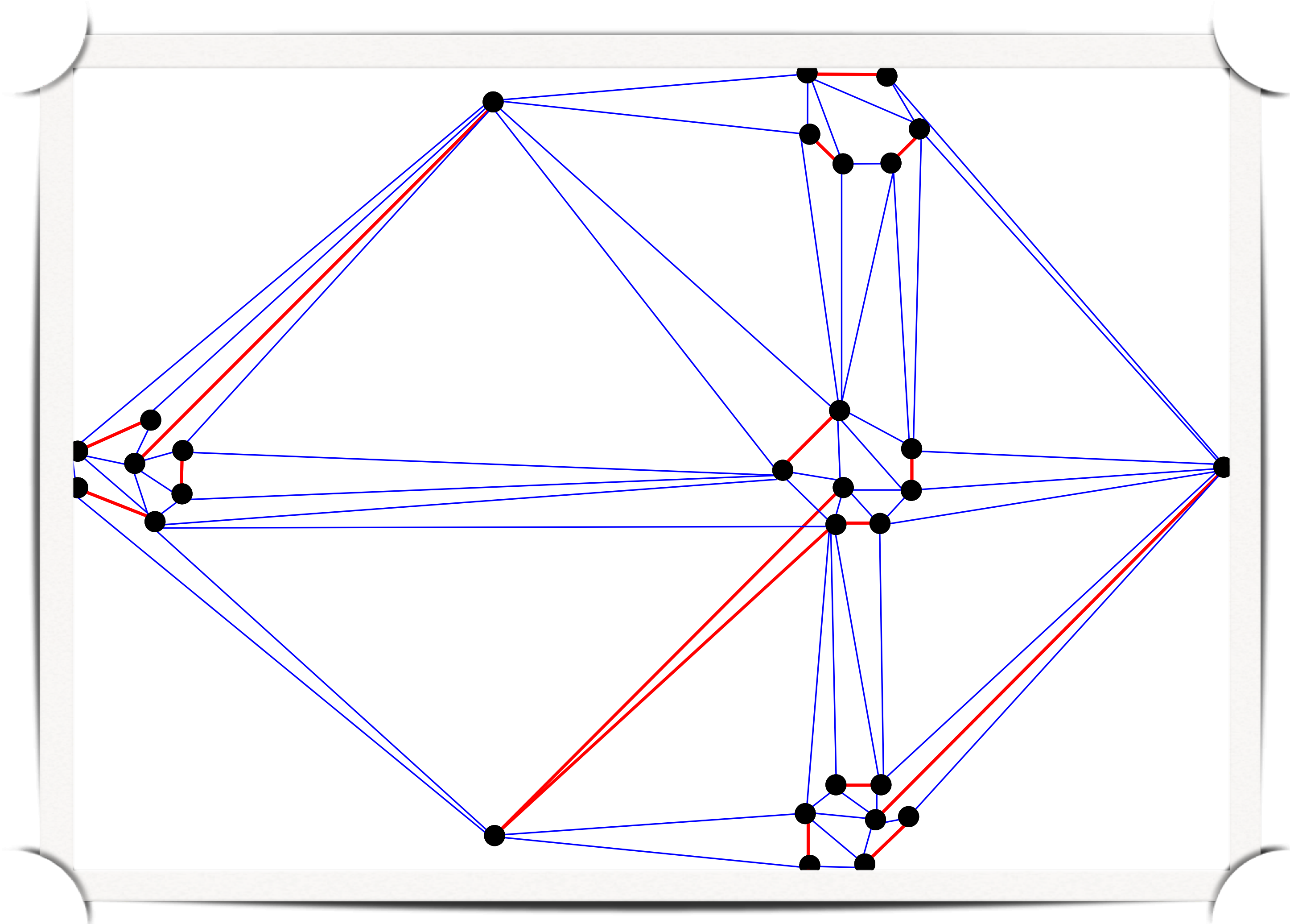

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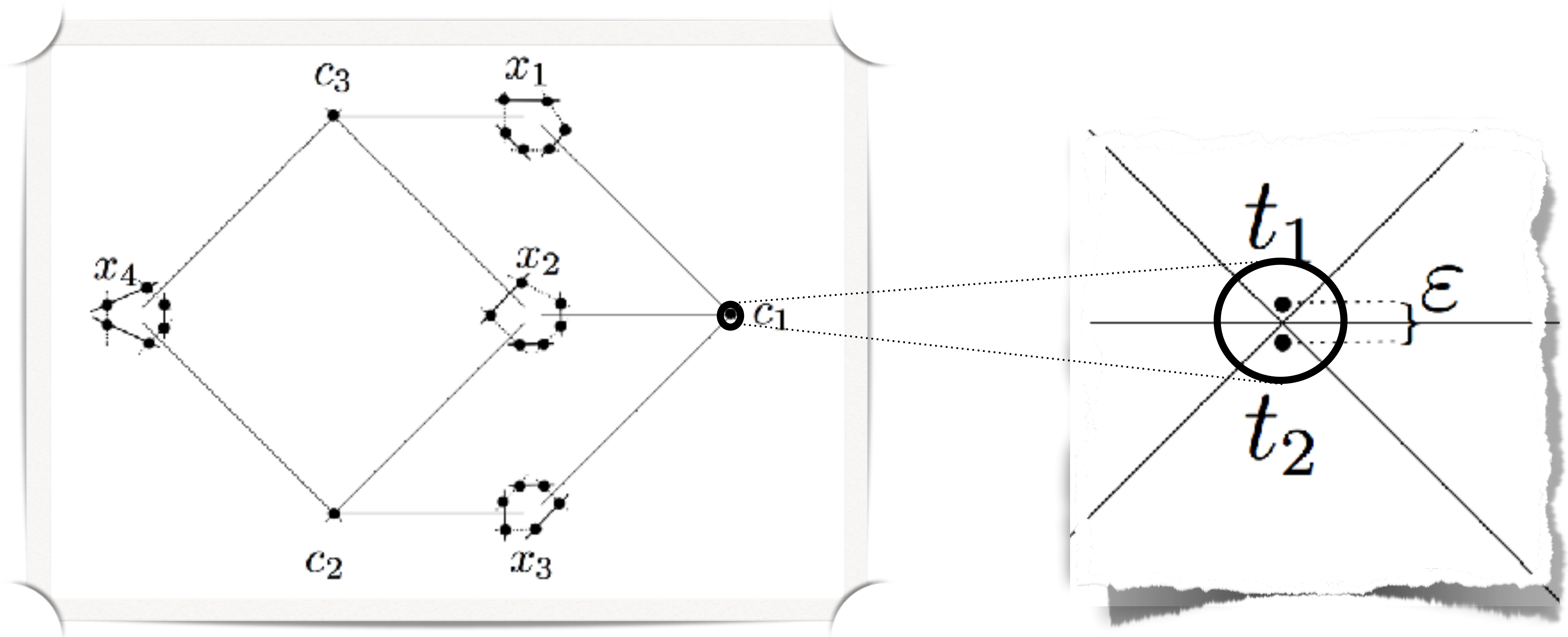

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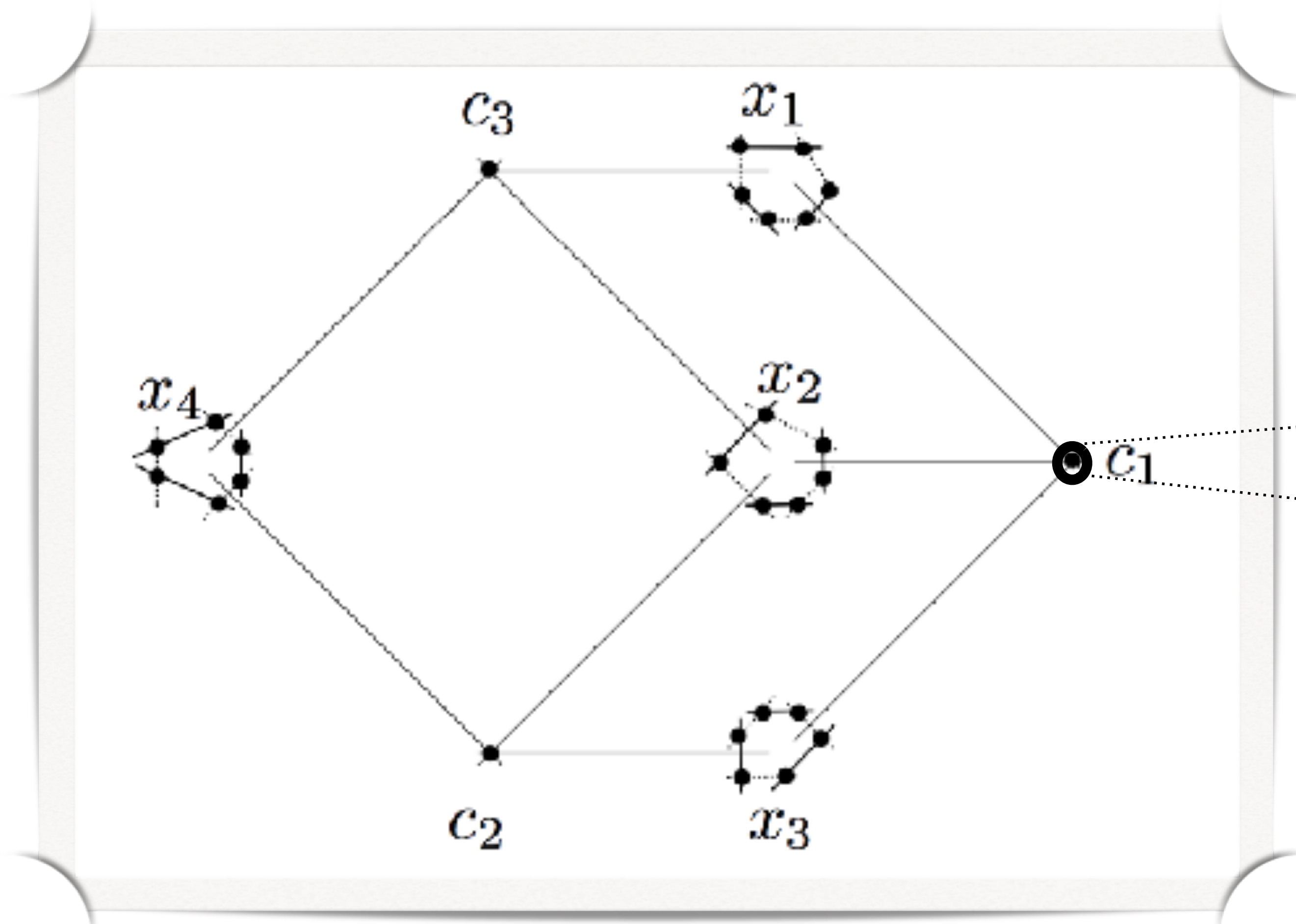

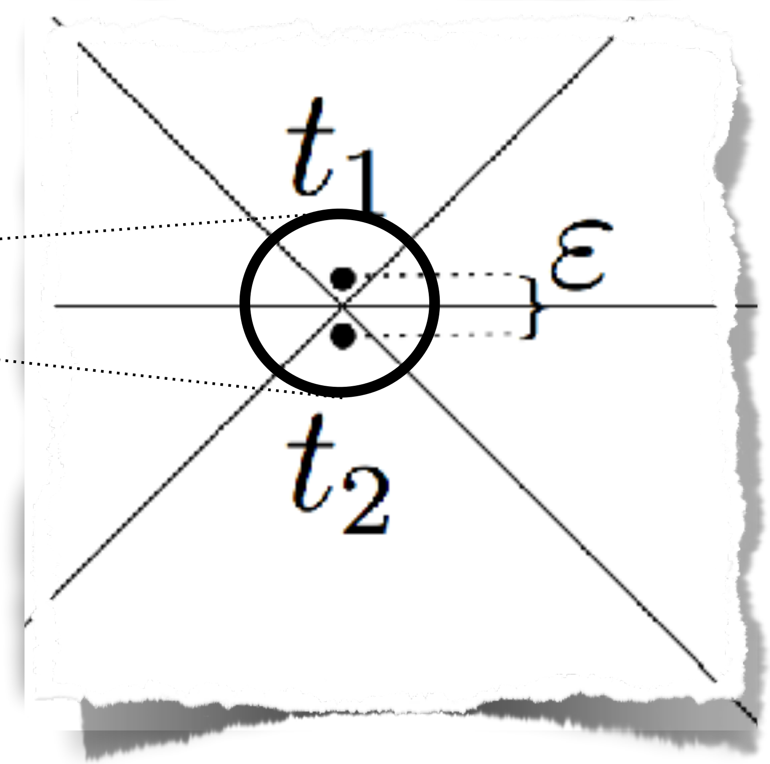

COROLLARY 3.4. *Let $p(x)$ be some polynomial. Then the existence of a polynomial-time algorithm that yields a $p(n)$-approximation for MELT implies P=NP.*

5.2: IP Models

Overall Goals

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★ Solve non-trivial instances to provable optimality!

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1. Model as Integer Linear Program (IP)

Overall Goals

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2.2. Solve IP

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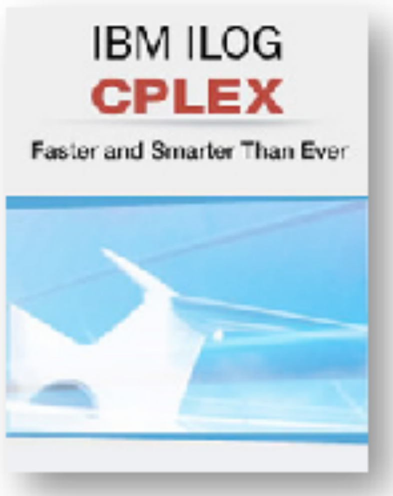
CGAL **meets**



Triangulation by Independent Set

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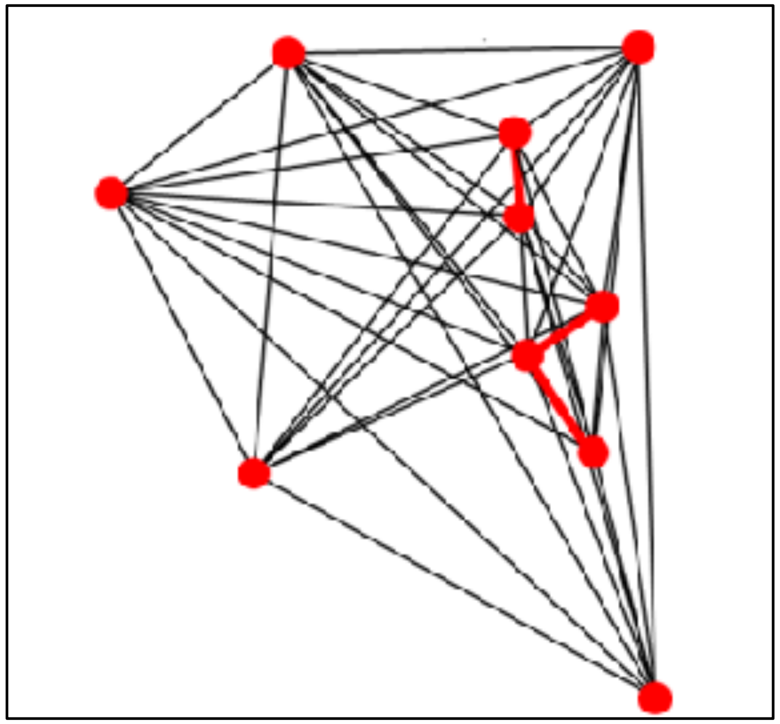

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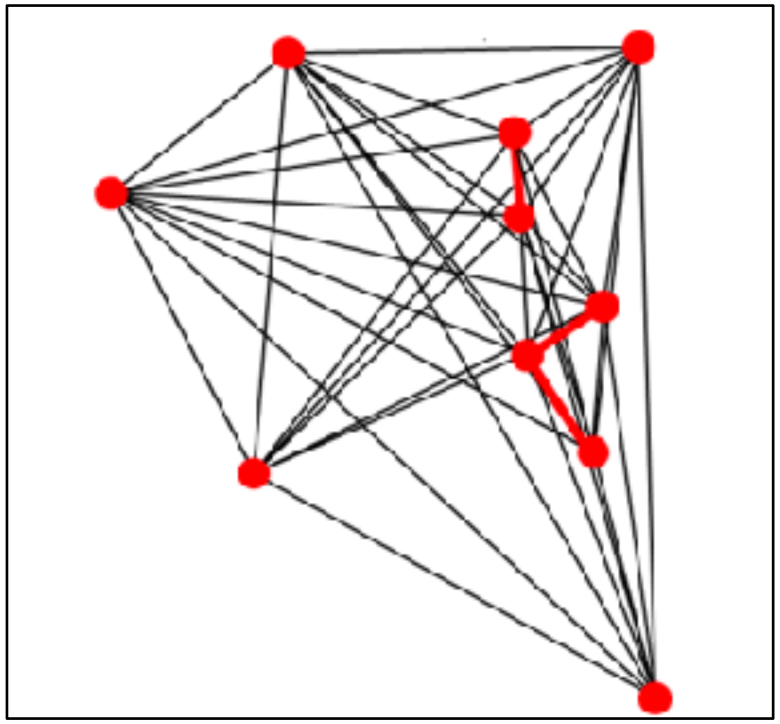

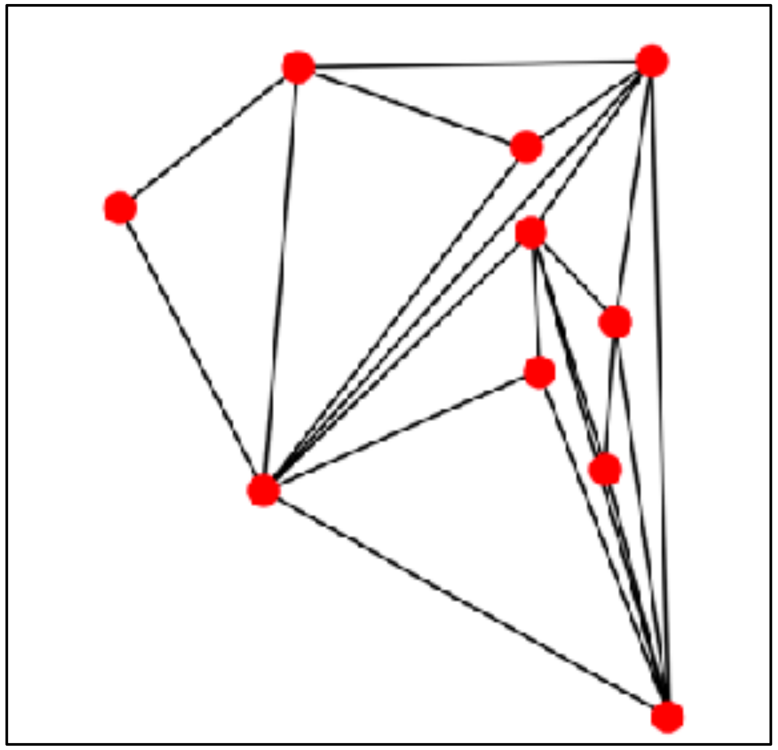

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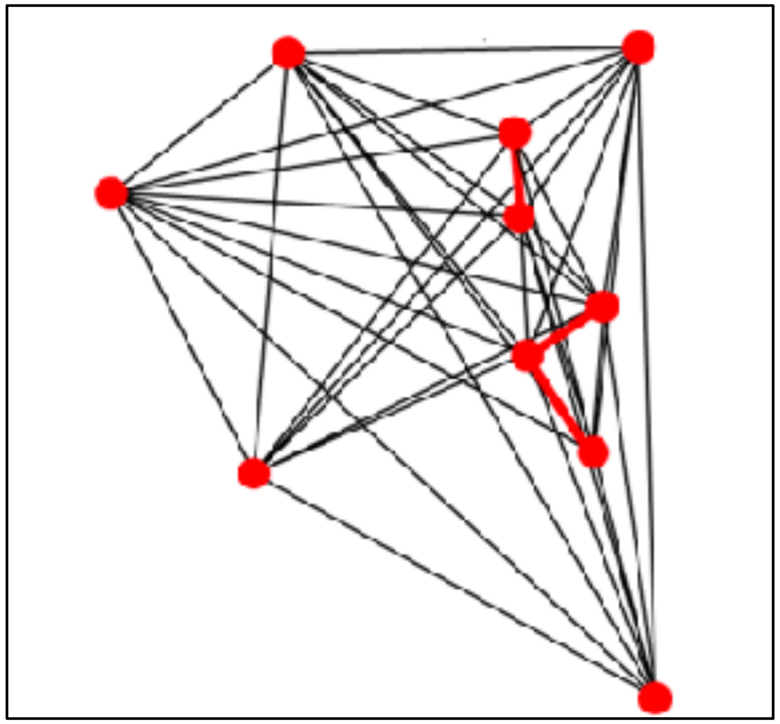

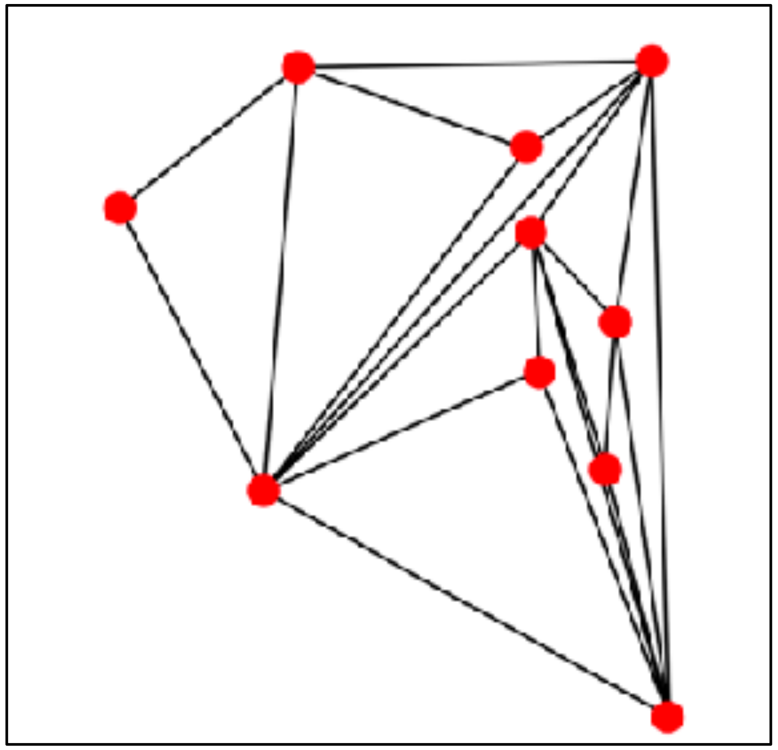

Integer Program 4.1.

(4.1)

$$\max \quad 0$$

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$$\text{s.t.} \quad x_{e_i} + x_{e_j} \leq 1 \qquad \forall \{e_i, e_j\} \in X_{\geq b}$$

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$$\sum_{e_i \in E} x_{e_i} = 3n - 3 - n_c \qquad \forall e_i \in E_{\geq b}$$

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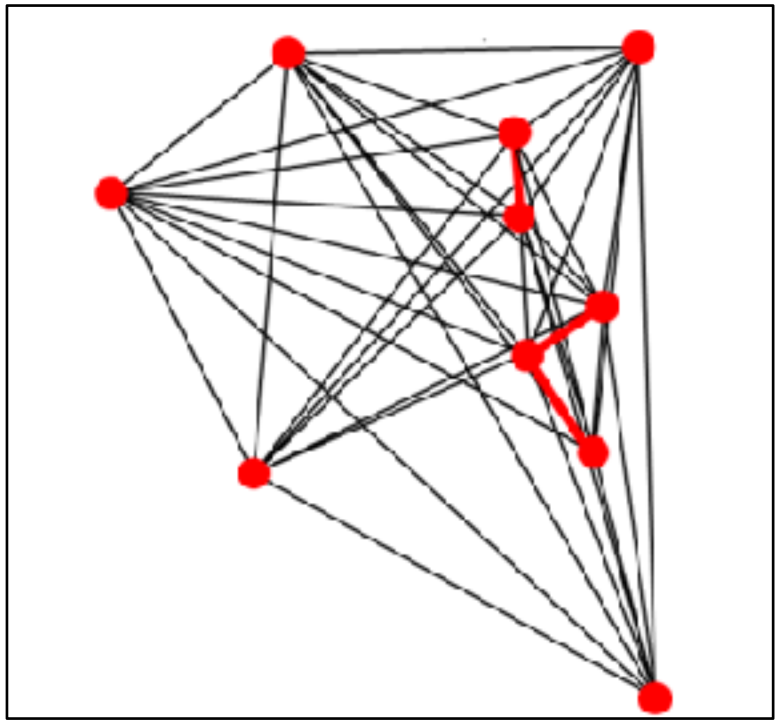

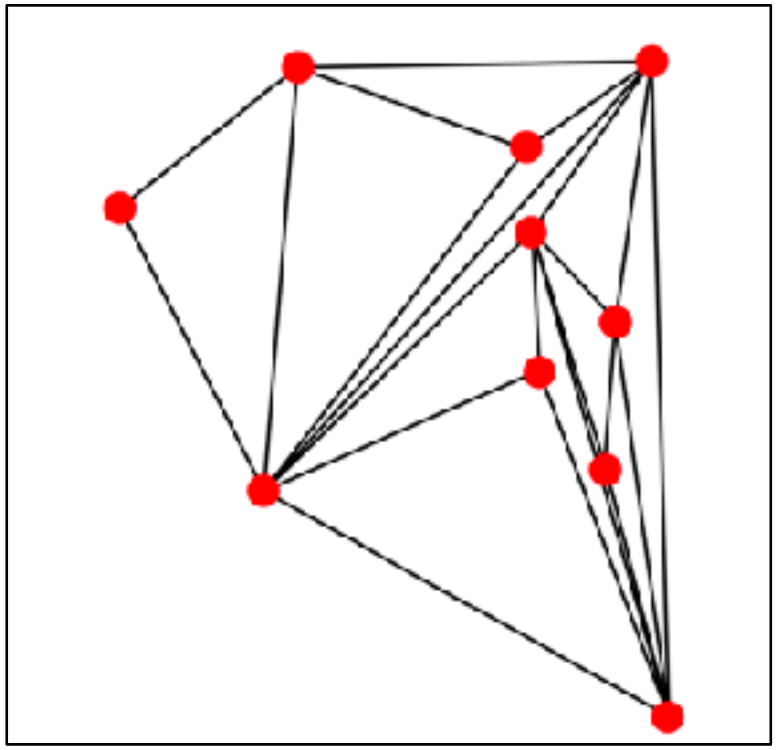

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THEOREM 4.2. *We can solve MELT by finding a maximum index b for which IP 4.1 is feasible.*

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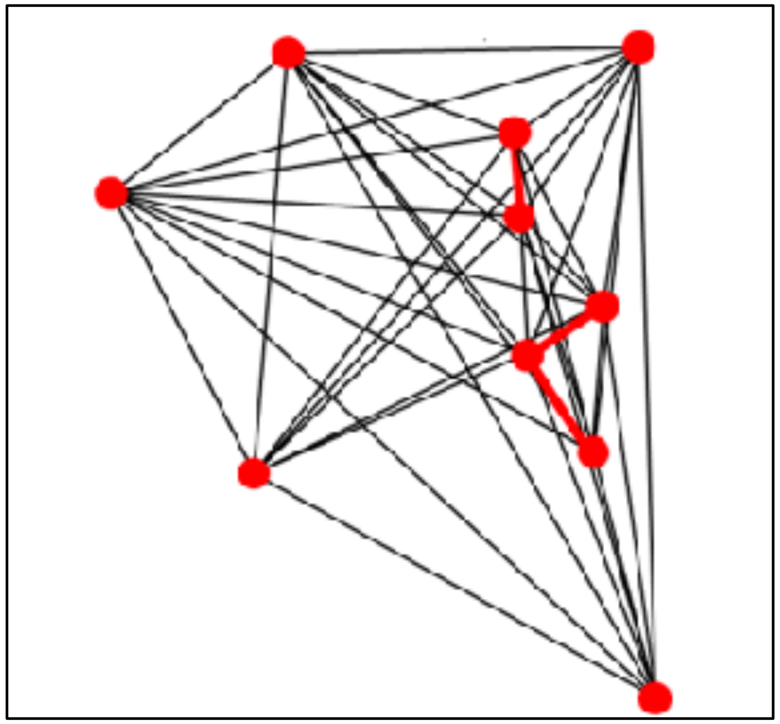

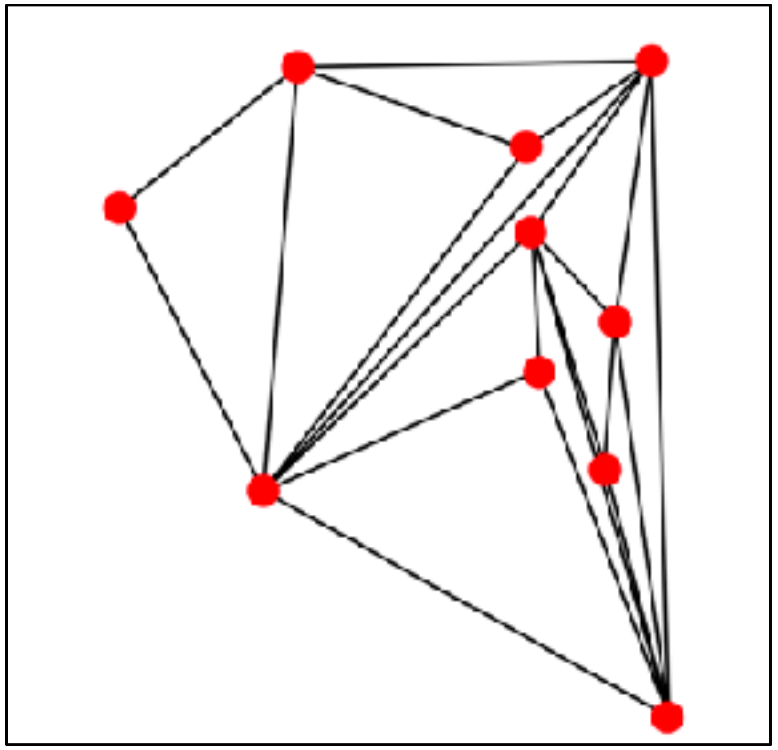

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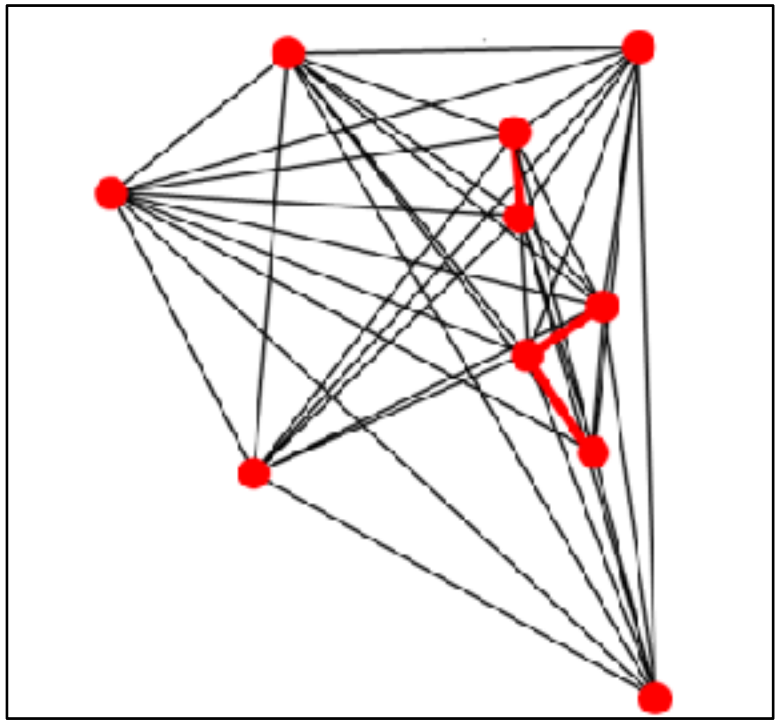

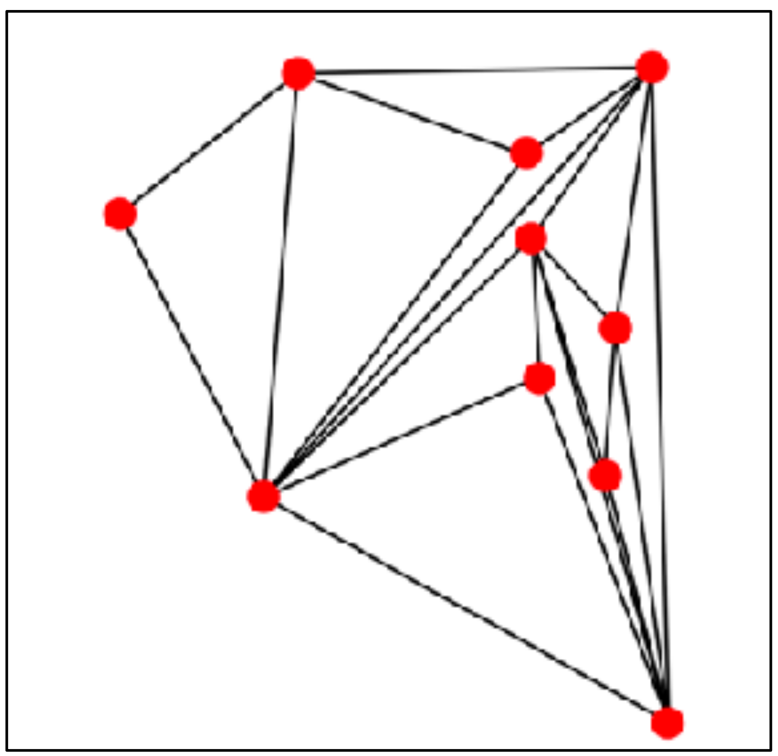

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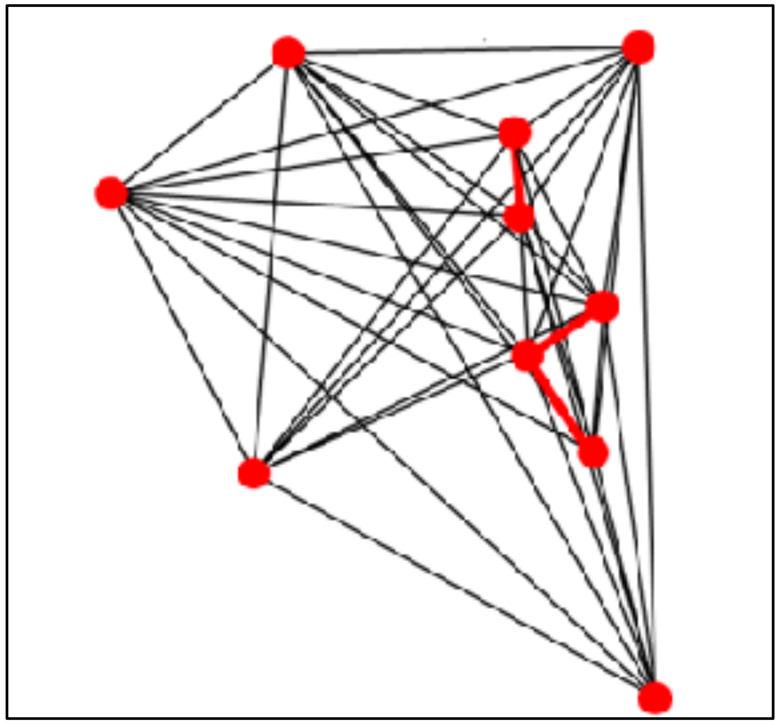
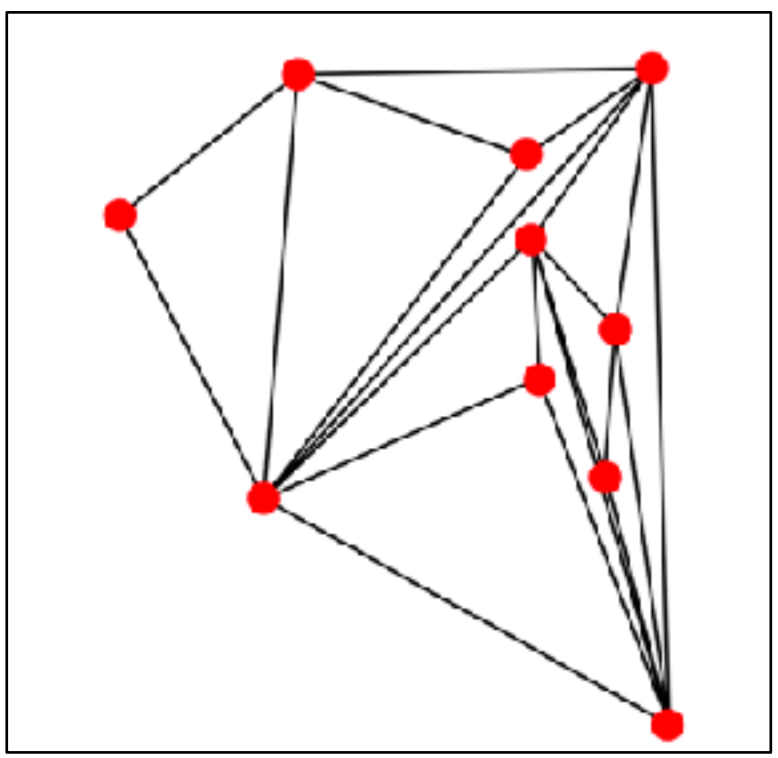
Best-case IP size:

Triangulation by Independent Set





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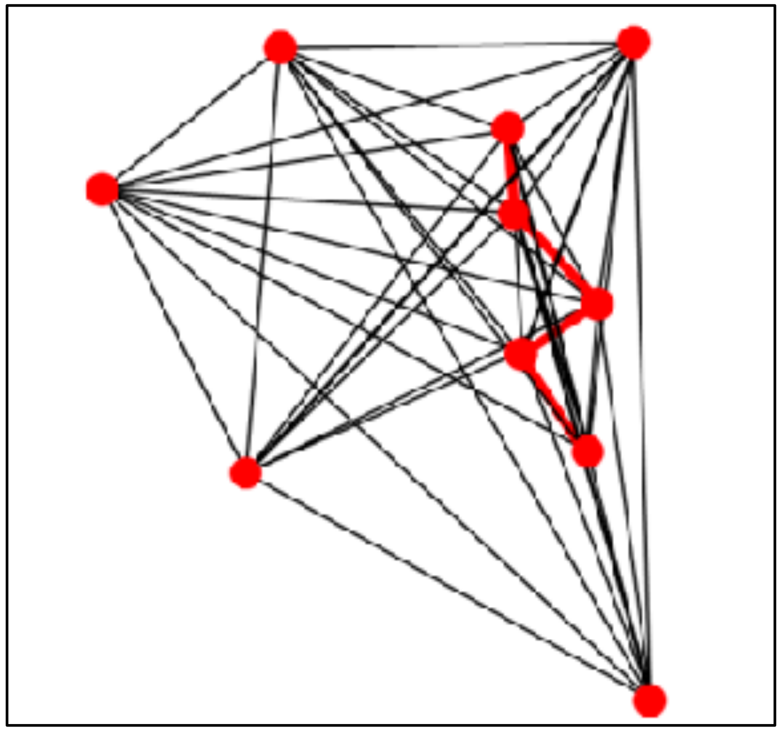
Best-case IP size: $\Omega(n^4)$

Triangulation by Hitting Set



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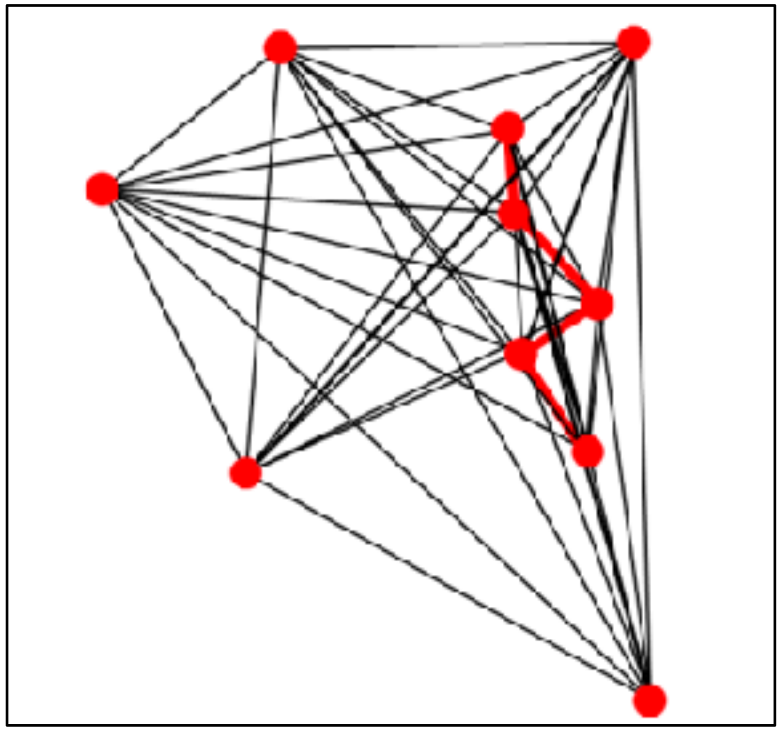

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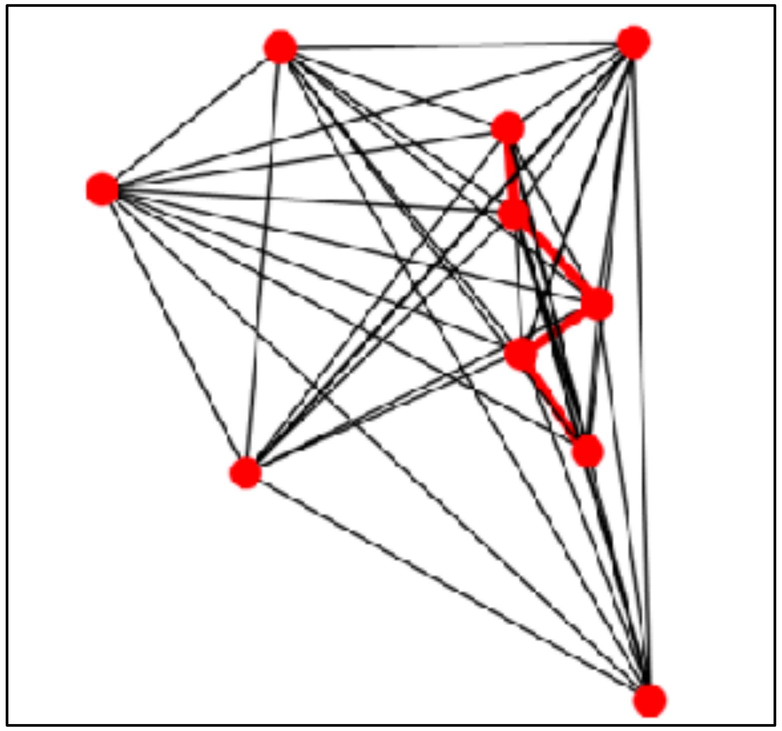

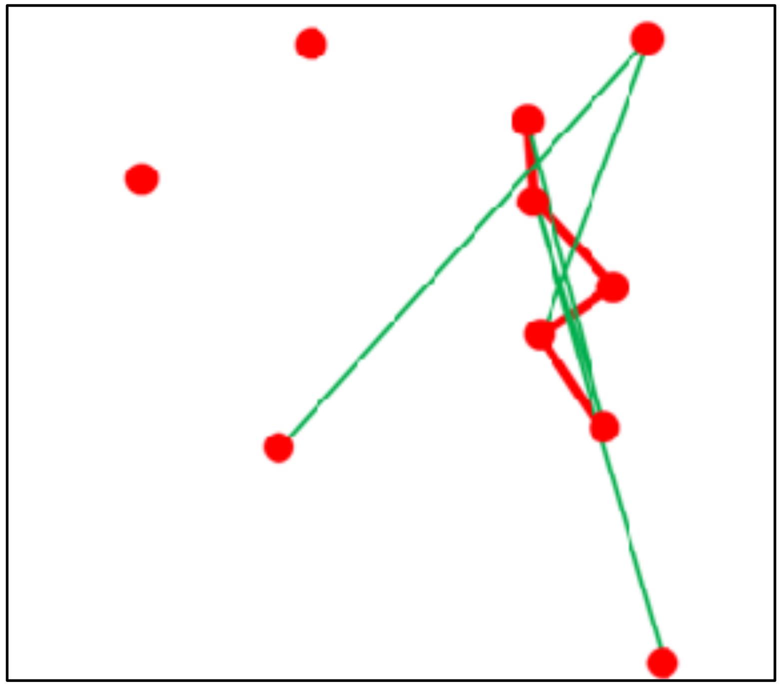

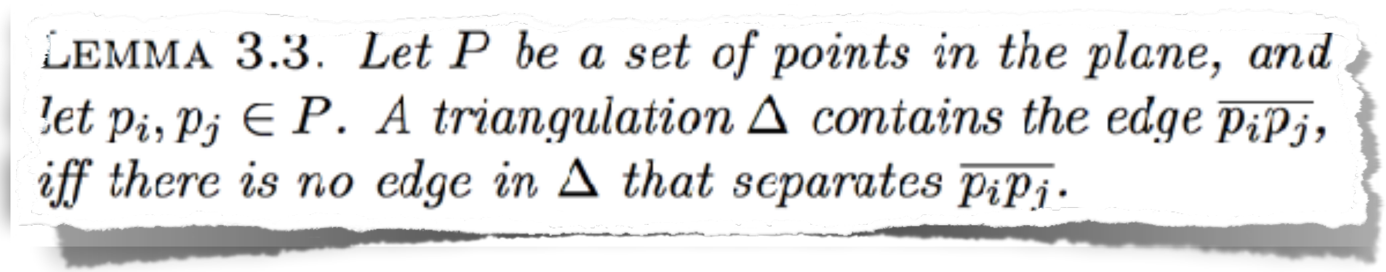
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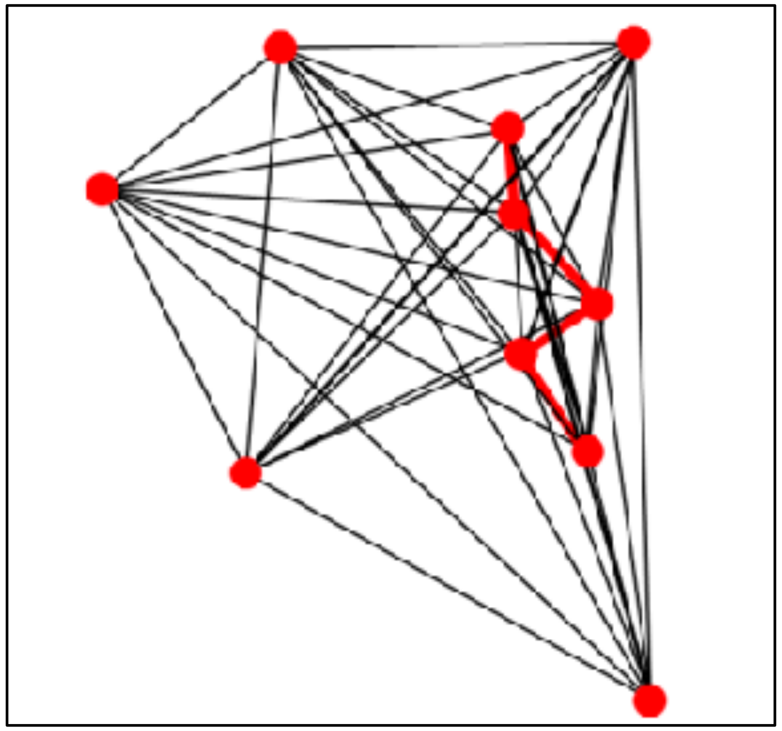

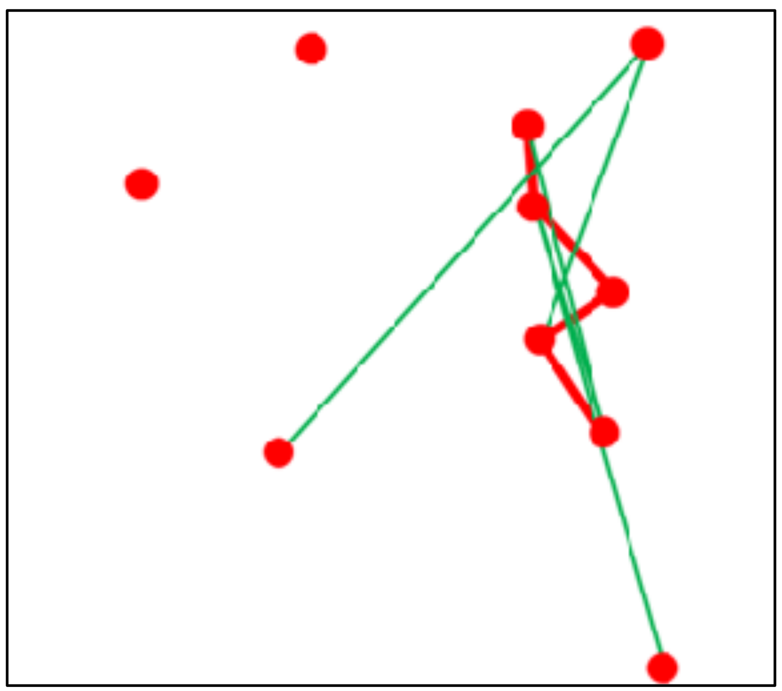

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(4.5)

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(4.6)

$$\text{s.t.} \quad x_{e_i} + x_{e_j} \leq 1 \qquad \forall \{e_i, e_j\} \in X^*_{\geq b}$$

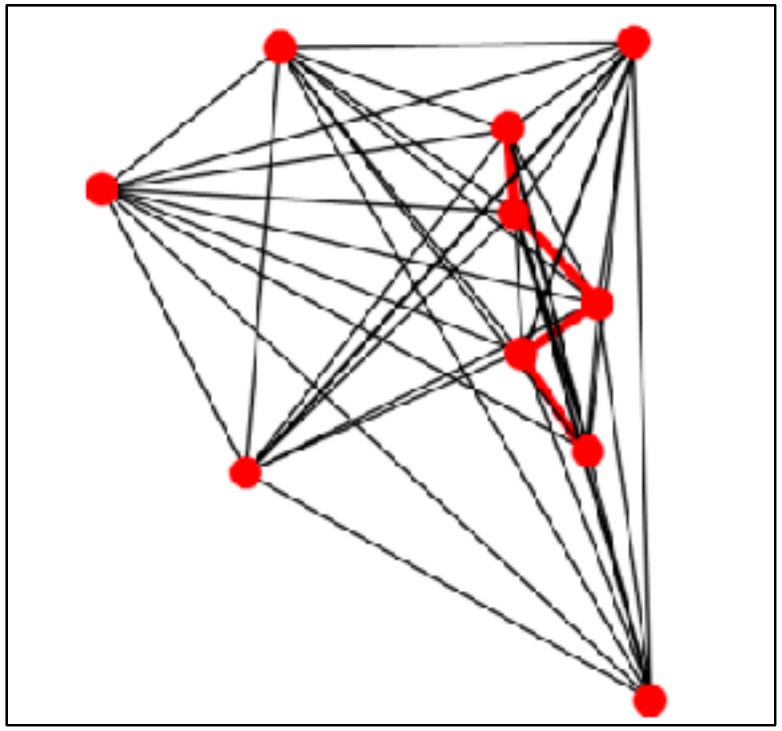
(4.7)

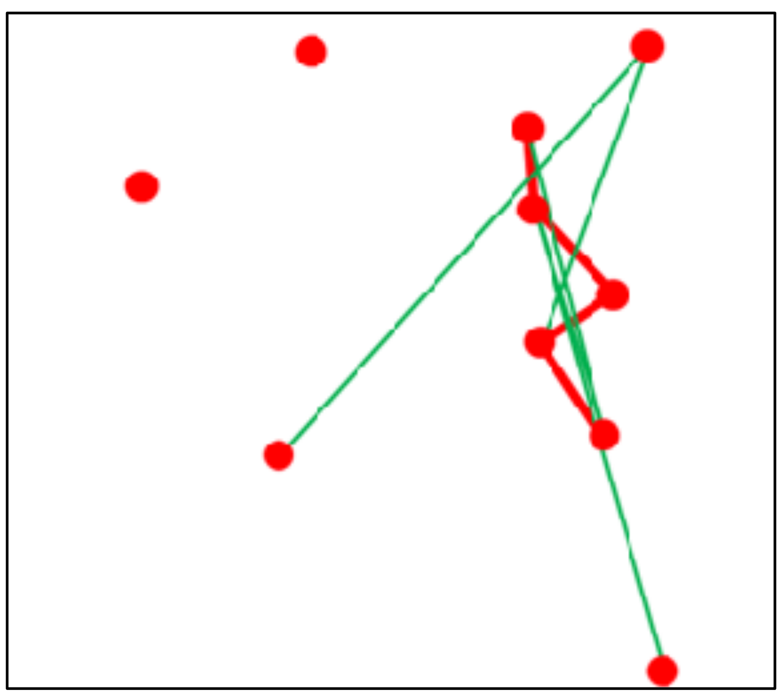
$$\sum_{e_i \in E^*_{\geq b}(e_j)} x_{e_i} \geq 1 \qquad \forall e_j \in E_{<b}$$

(4.8)

$$x_{e_i} \in \{0, 1\} \qquad \forall e_i \in E^*_{\geq b}$$

Triangulation by Hitting Set





LEMMA 3.3. *Let P be a set of points in the plane, and let $p_i, p_j \in P$. A triangulation Δ contains the edge $\overline{p_i p_j}$, iff there is no edge in Δ that separates $\overline{p_i p_j}$.*

INTEGER PROGRAM 4.3.

(4.5)

$$\max \quad 0$$

(4.6)

$$\text{s.t.} \quad x_{e_i} + x_{e_j} \quad \leq \quad 1 \qquad \forall \{e_i, e_j\} \in X^*_{\geq b}$$

(4.7)

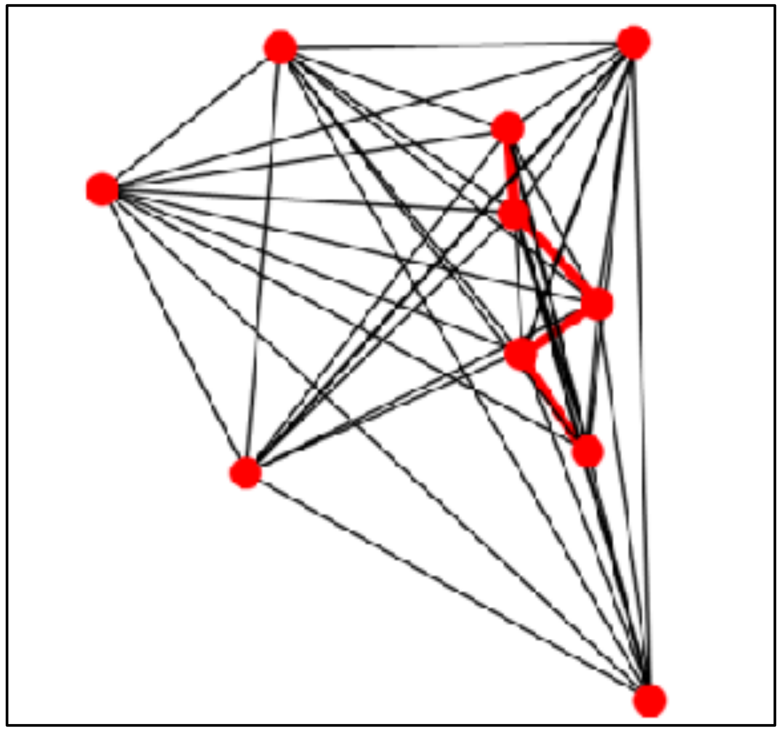
$$\sum_{e_i \in E^*_{\geq b}(e_j)} x_{e_i} \quad \geq \quad 1 \qquad \forall e_j \in E_{<b}$$

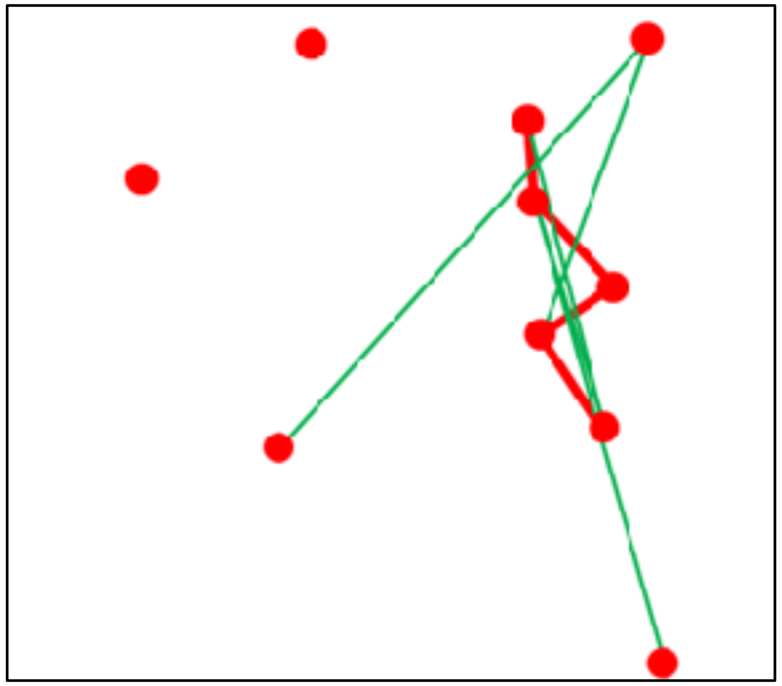
(4.8)

$$x_{e_i} \quad \in \quad \{0,1\} \qquad \forall e_i \in E^*_{\geq b}$$

THEOREM 4.4. *We can solve MELT by finding a maximum value b for which IP 4.3 is feasible.*

Triangulation by Hitting Set





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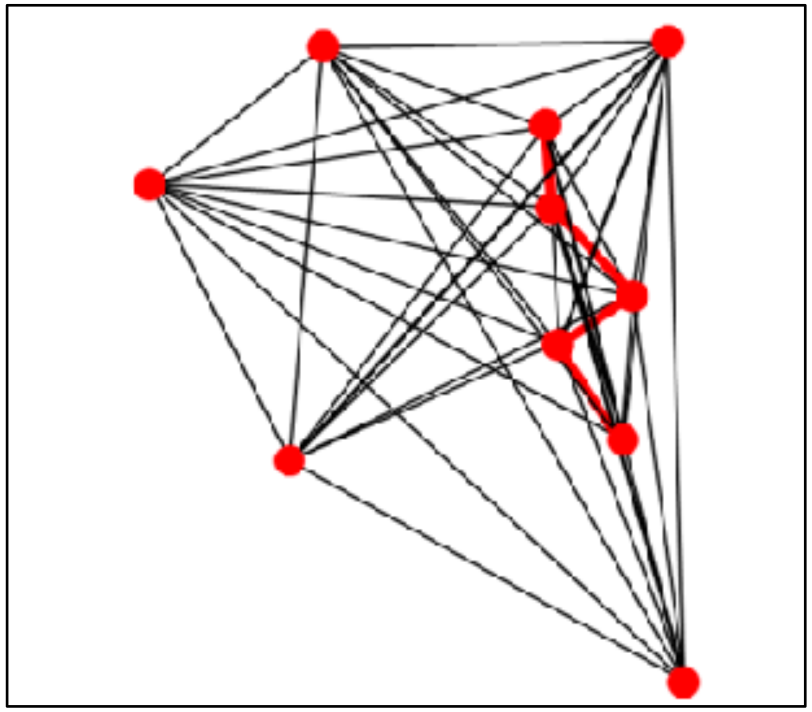
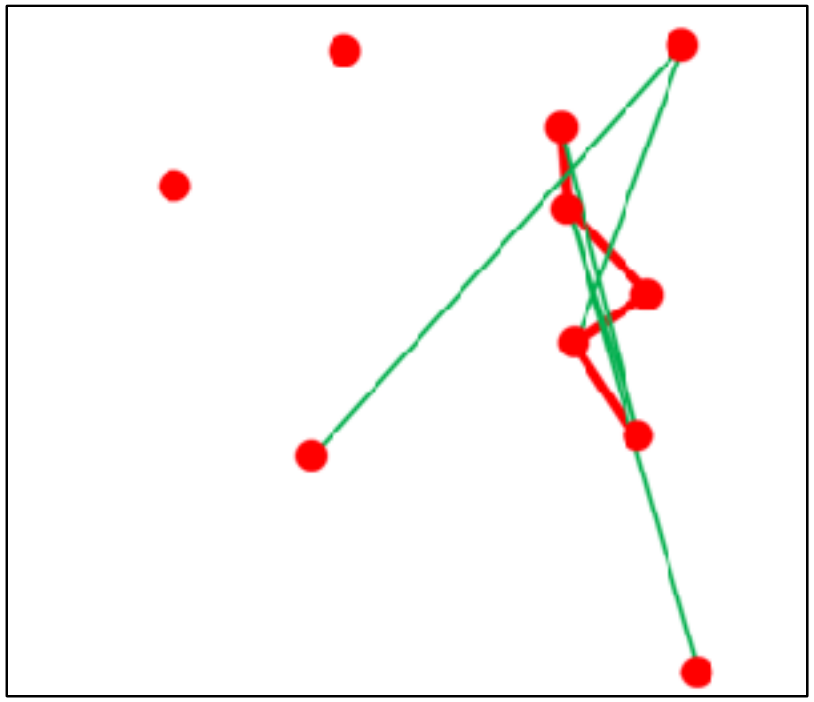
Triangulation by Hitting Set





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INTEGER PROGRAM 4.5.

(4.9)

$$\max \quad 0$$

(4.10)

$$\text{s.t.} \quad k_e \cdot x_e + \sum_{e_i \in E^*_{>b}(e)} x_{e_i} \quad \leq \quad k_e \qquad \forall e \in E^*_{>b}$$

(4.11)

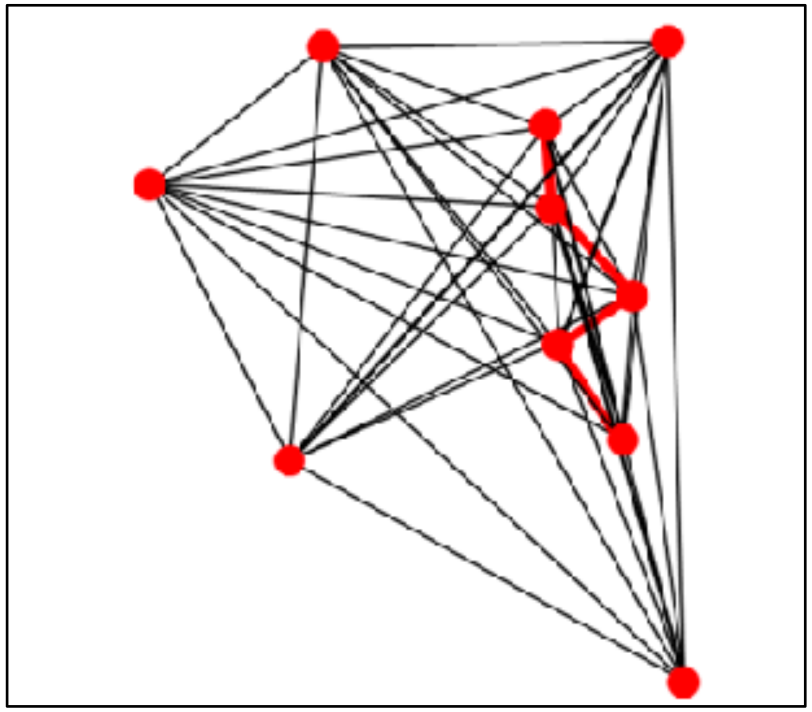
$$\sum_{e_i \in E^*_{\geq b}(e)} x_{e_i} \quad \geq \quad 1 \qquad \forall e \in E_{<a}$$

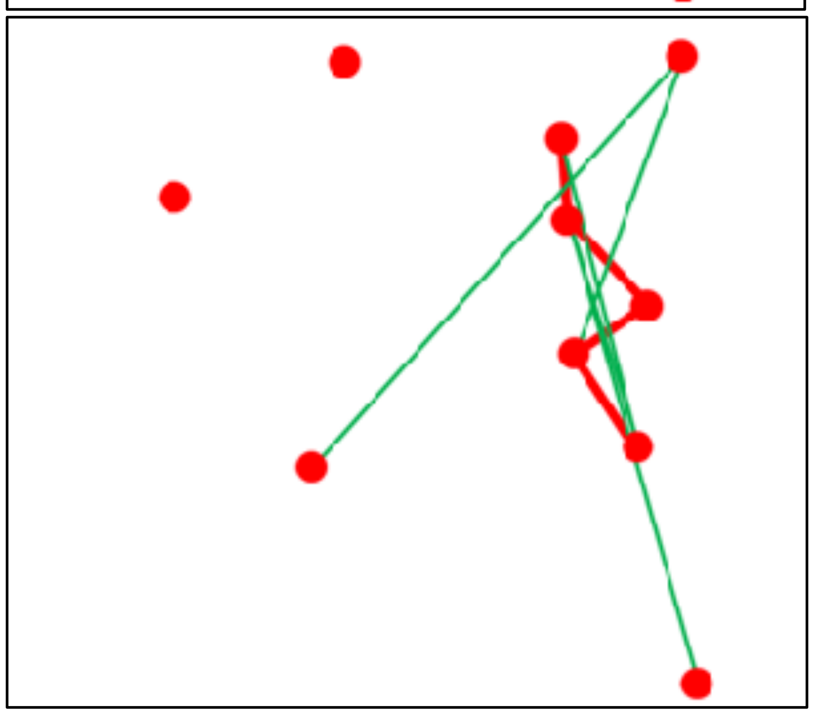
(4.12)

$$x_{e_i} \quad \in \quad \{0,1\} \qquad \forall e_i \in E^*_{\geq b}$$

THEOREM 4.6. *We can solve MELT by finding a maximum index b for which IP 4.5 is feasible.*

Finding the Critical Index





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INTEGER PROGRAM 4.7.

(4.13)

$$\max \quad b$$

(4.14)

$$\text{s.t.} \quad x_{e_i} \cdot i + (1 - x_{e_i}) \cdot c \geq b \qquad \forall i \in [a, c[$$

(4.15)

$$k_e \cdot x_e + \sum_{e_i \in E^*_{\geq a}(e)} x_{e_i} \leq k_e \qquad \forall e \in E^*_{\geq a}$$

(4.16)

$$x_e + \sum_{e_i \in E^*_{\geq a}(e)} x_{e_i} \geq 1 \qquad \forall e \in E_{[a,c[}$$

(4.17)

$$\sum_{e_i \in E^*_{\geq L}(e)} x_{e_i} \geq 1 \qquad \forall e \in E_{<a}$$

(4.18)

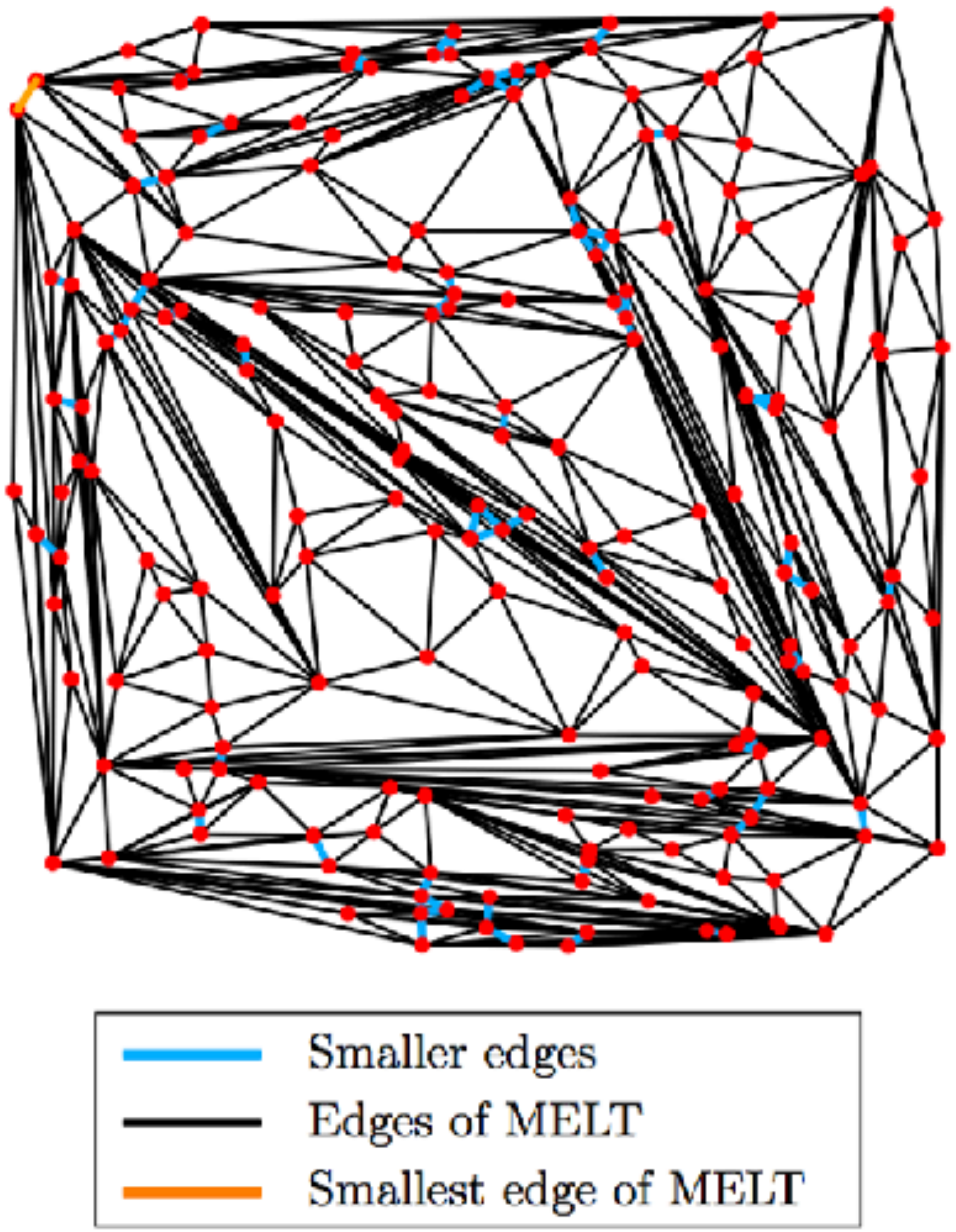
$$x_{e_i} \in \{0, 1\} \qquad \forall e_i \in E^*_{\geq a}$$

THEOREM 4.8. *If we know that the critical index $b \in [a, c[$, we can solve MELT by solving IP 4.7.*

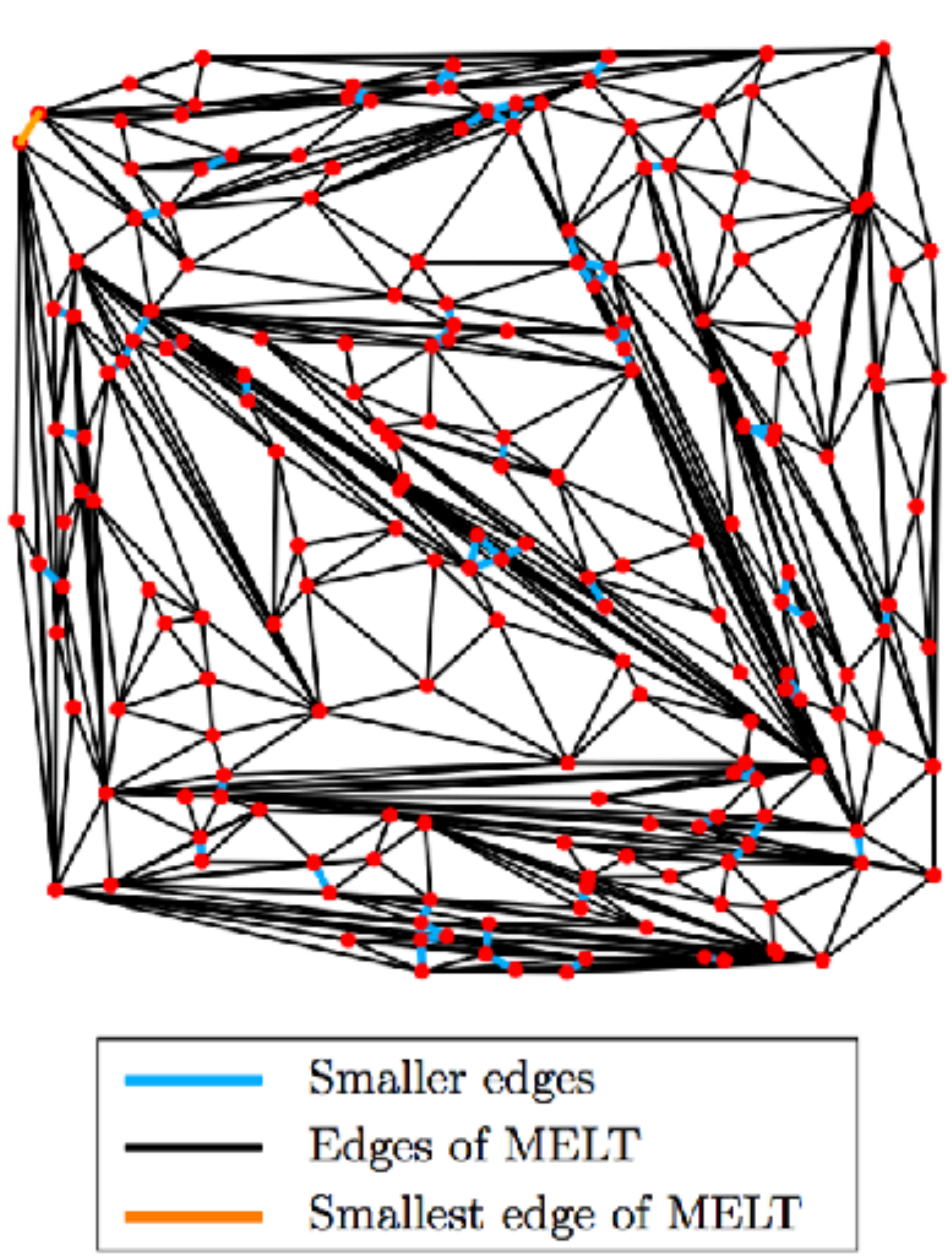
5.3: Experimental Results

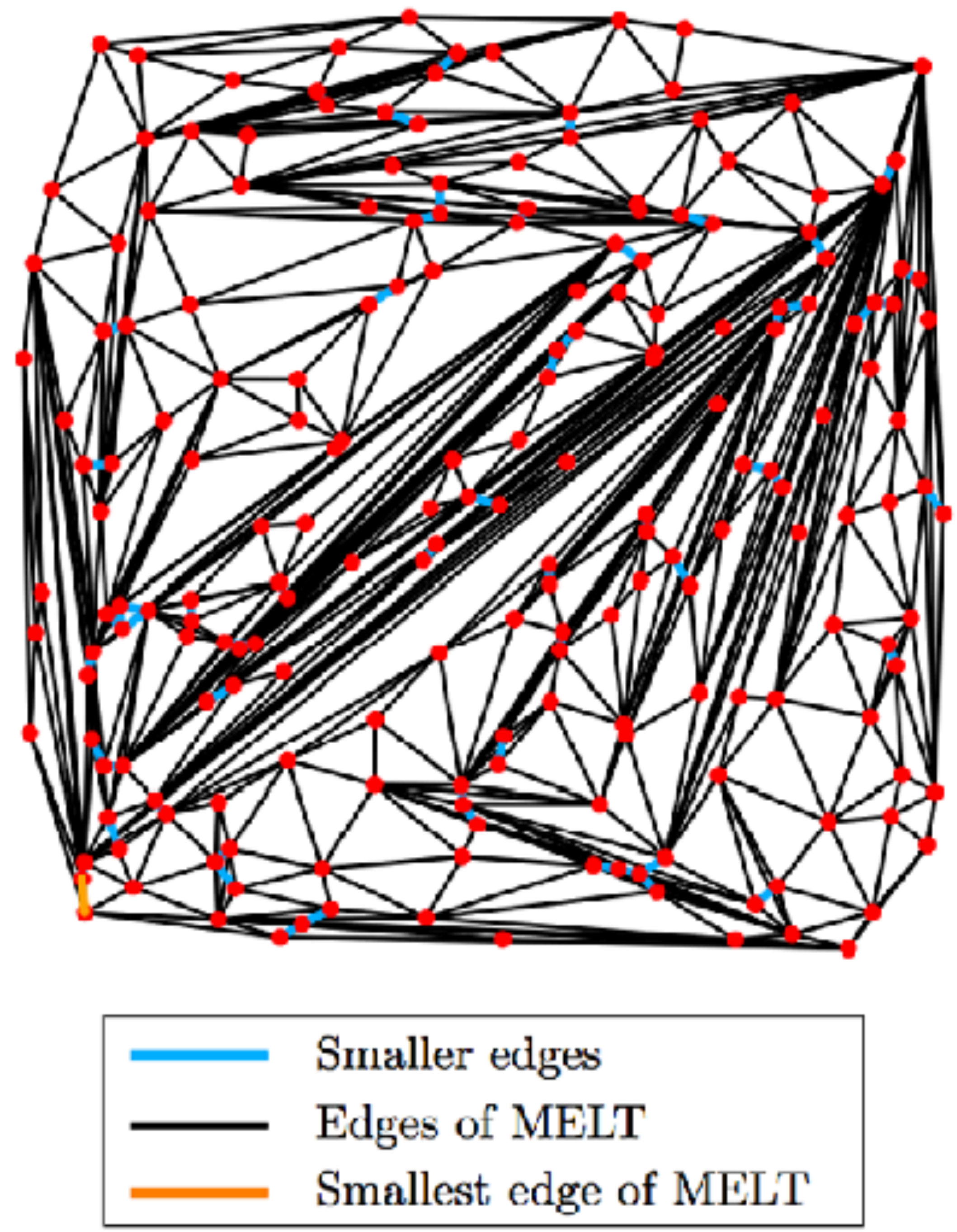
Looking at the Critical Index

Looking at the Critical Index



Looking at the Critical Index





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Upper Bounds in Comparison

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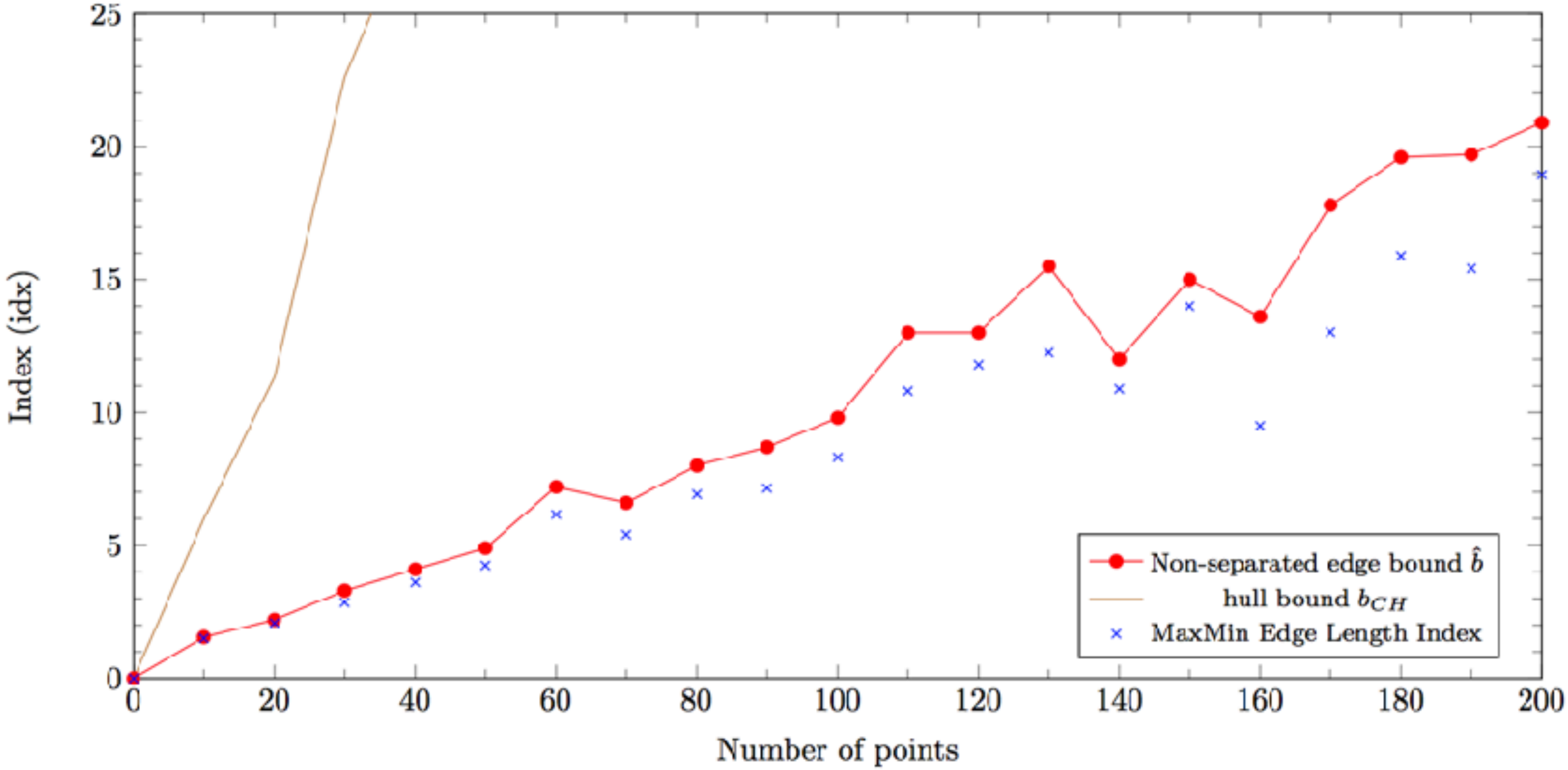

Figure 12: Average values for hull bound, non-separated edge bound, and optimal index for random instances.

Checking Intersections: AABB Trees

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Axis-aligned bounding box tree:
Data structure for speeding up intersection queries

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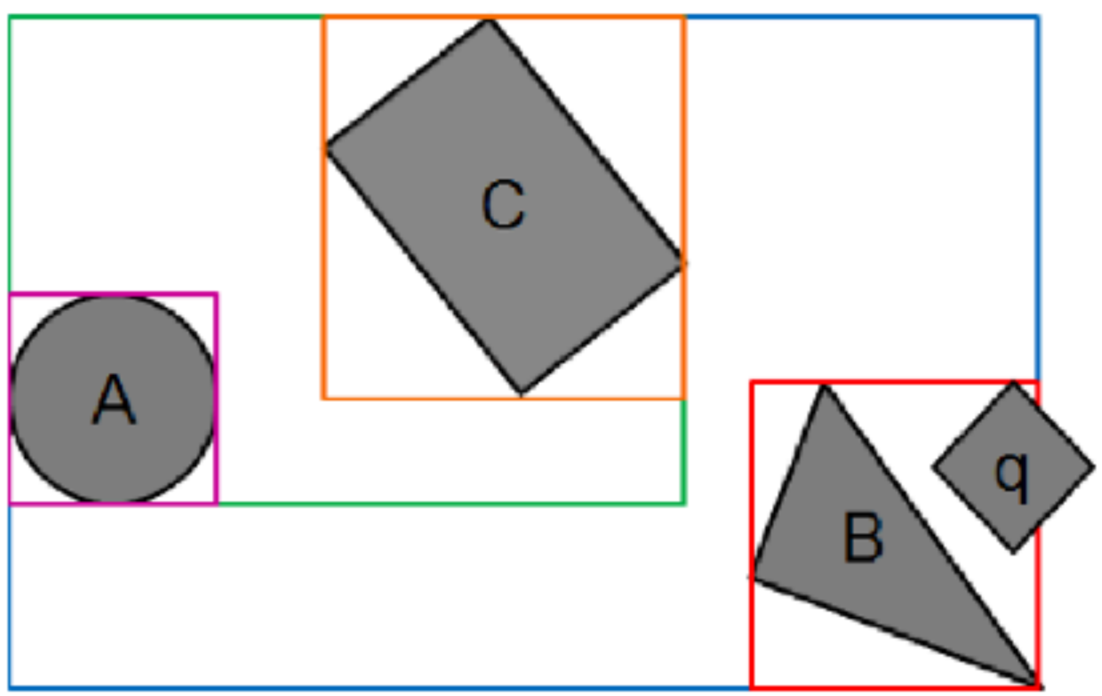

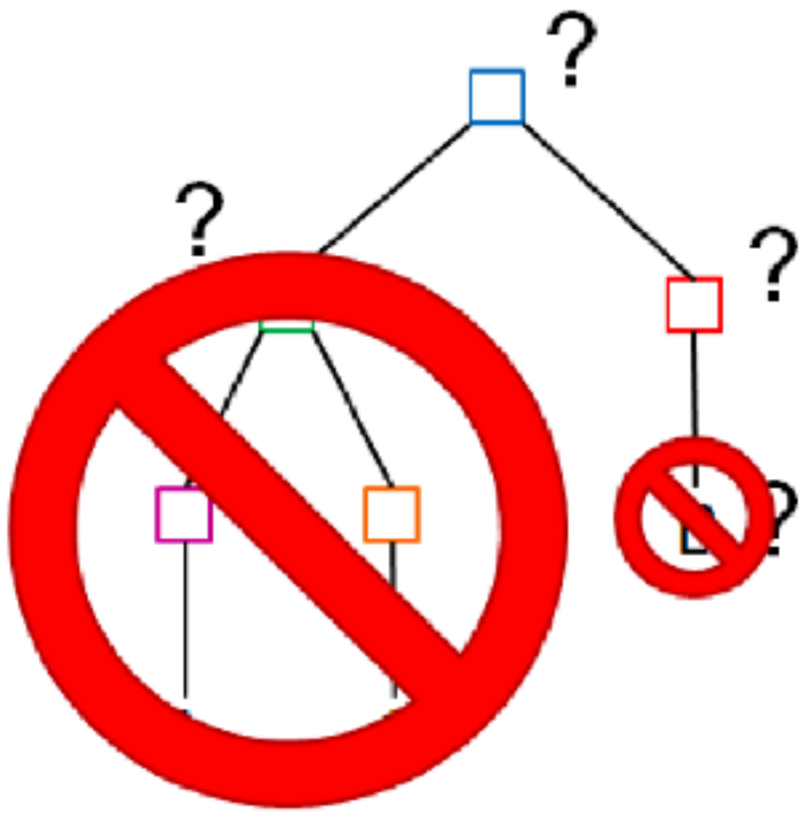

Using AABB Trees

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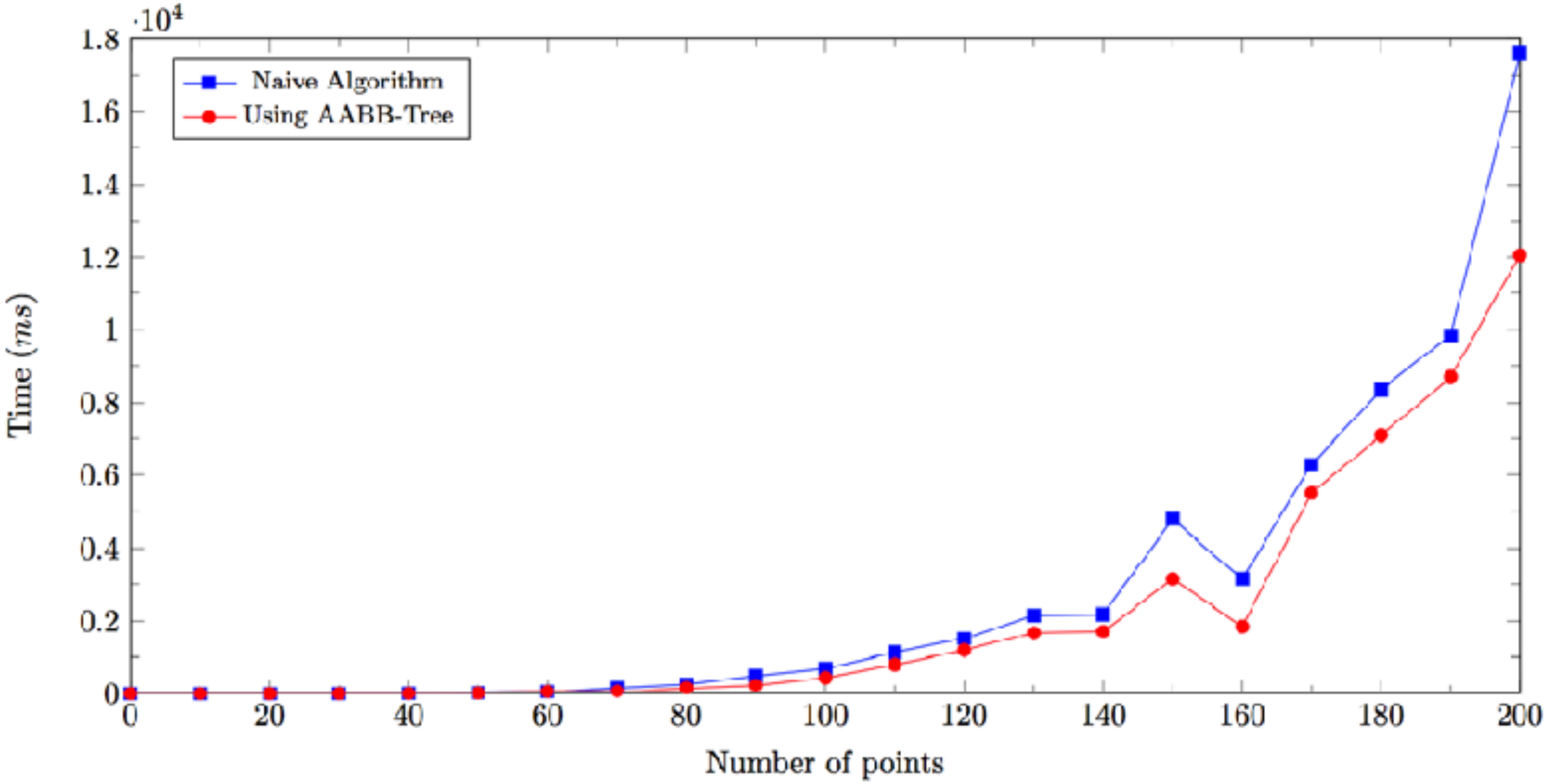

Figure 11: Average time for finding intersections in random instances.

Comparing IP 4.3 and IP 4.5 (1)

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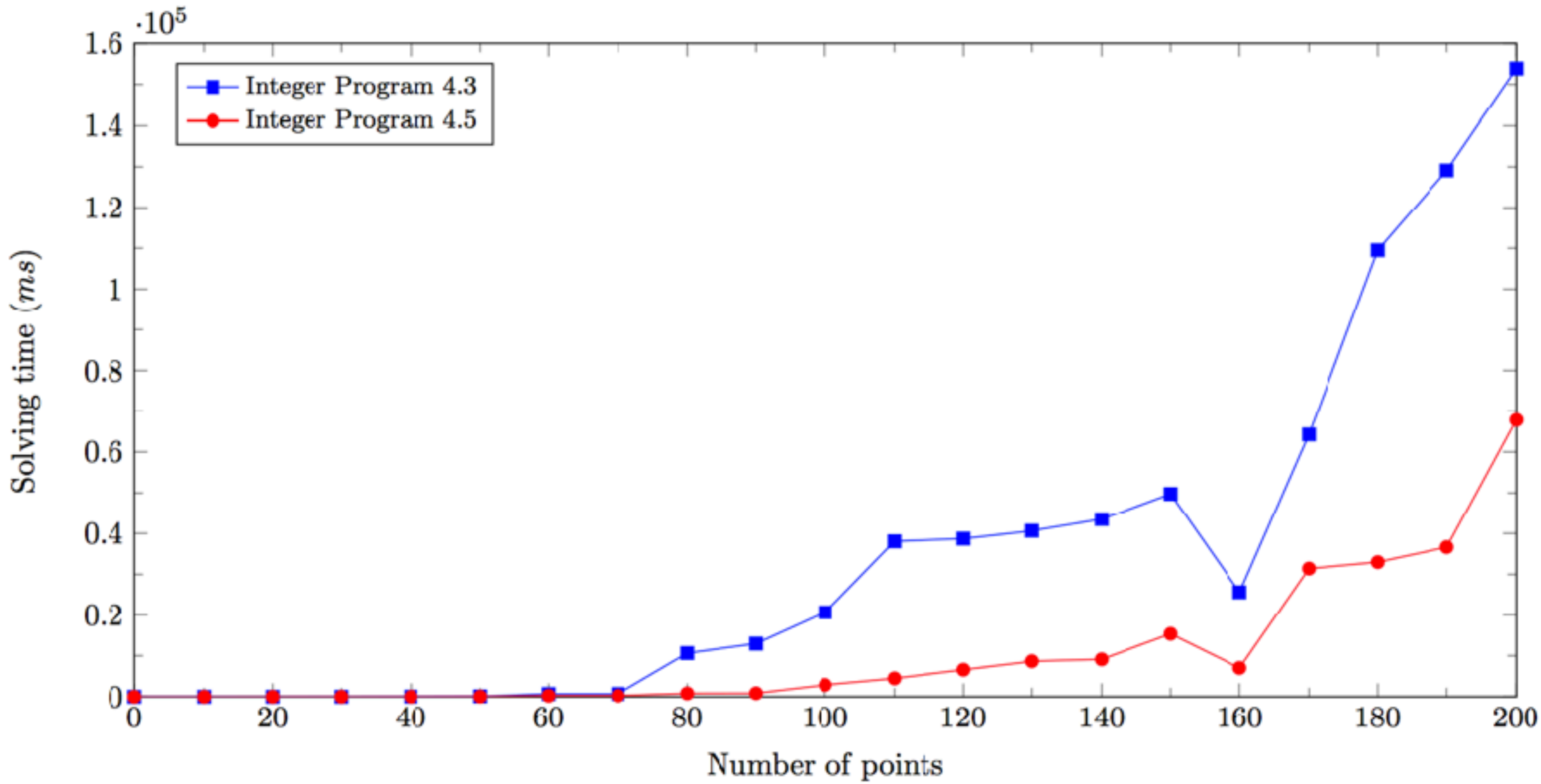

Figure 6: Average solution time of instances: solving time of the IPs.

Comparing IP 4.3 and IP 4.5 (2)

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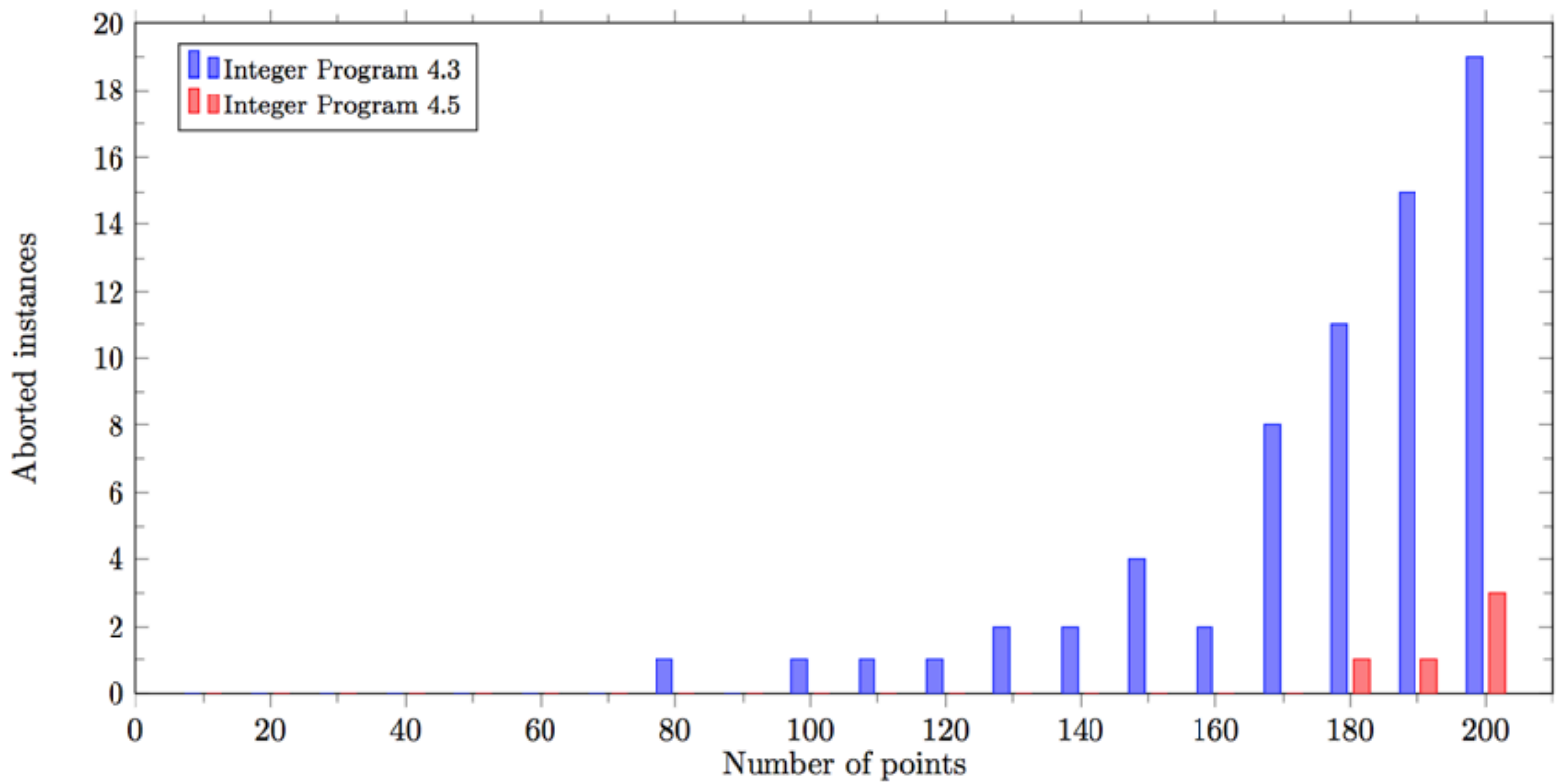

Figure 7: Aborted instances of the IPs, out of 100 instances.

Putting it together

Putting it together

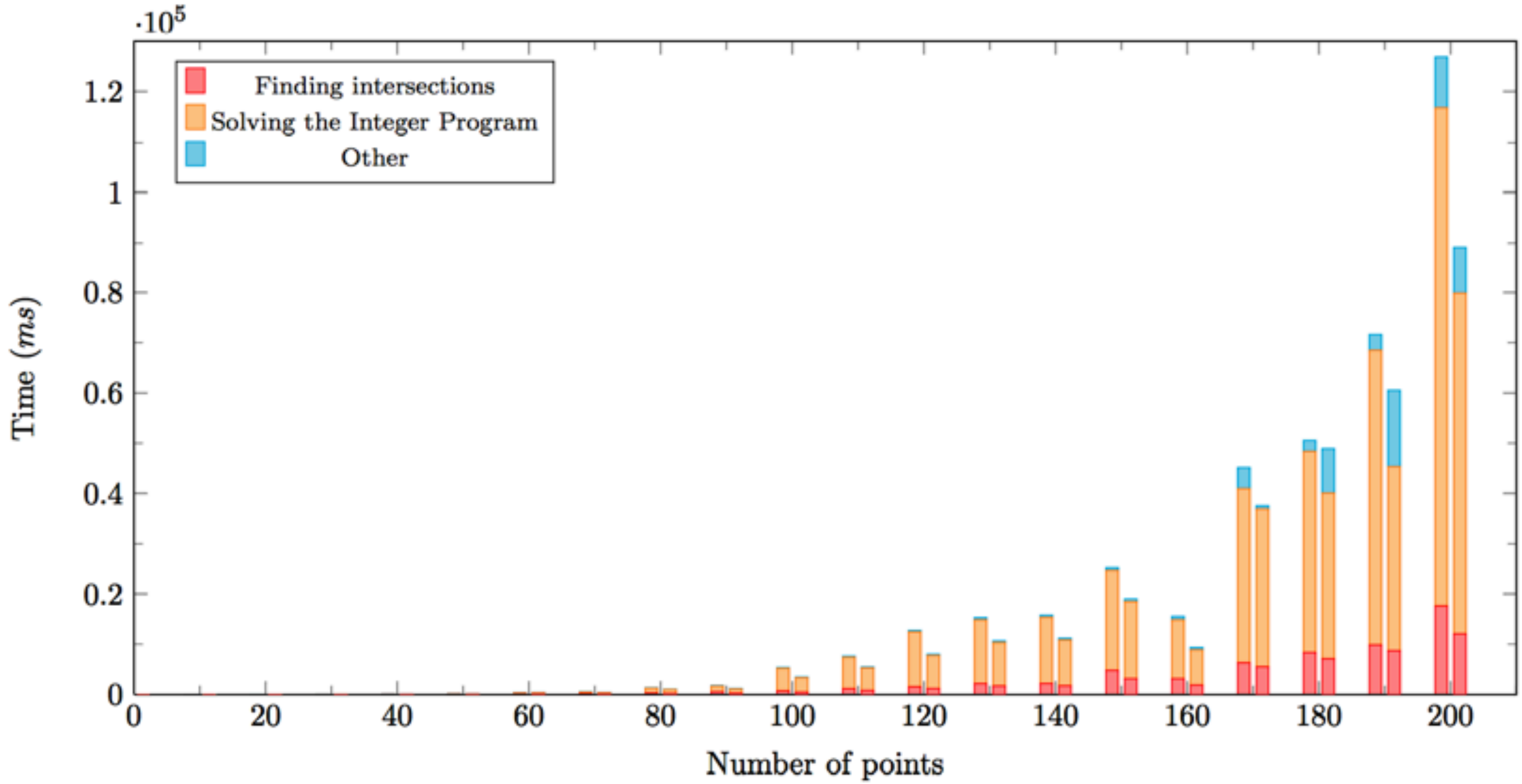

Figure 9: Average time (for random 100 random instances for each size) of the program components.
Left bars: Normal approach, based on IP 4.7
Right bars: Doubling approach, with AABB-Tree and IP 4.7

TSPLIB Instances

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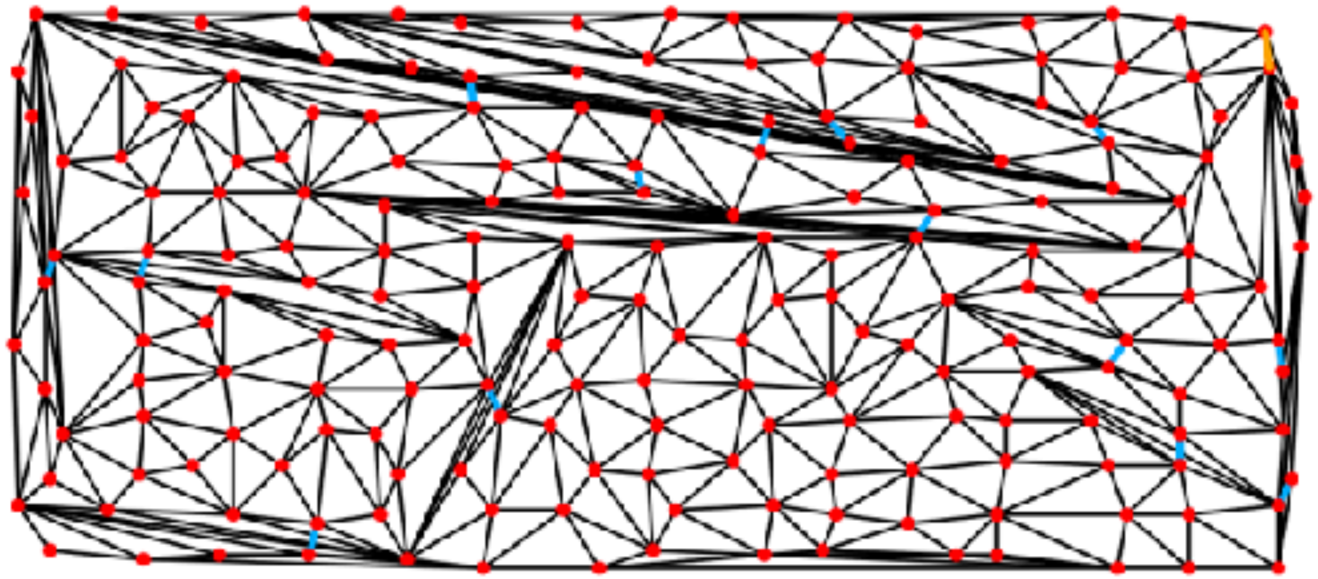
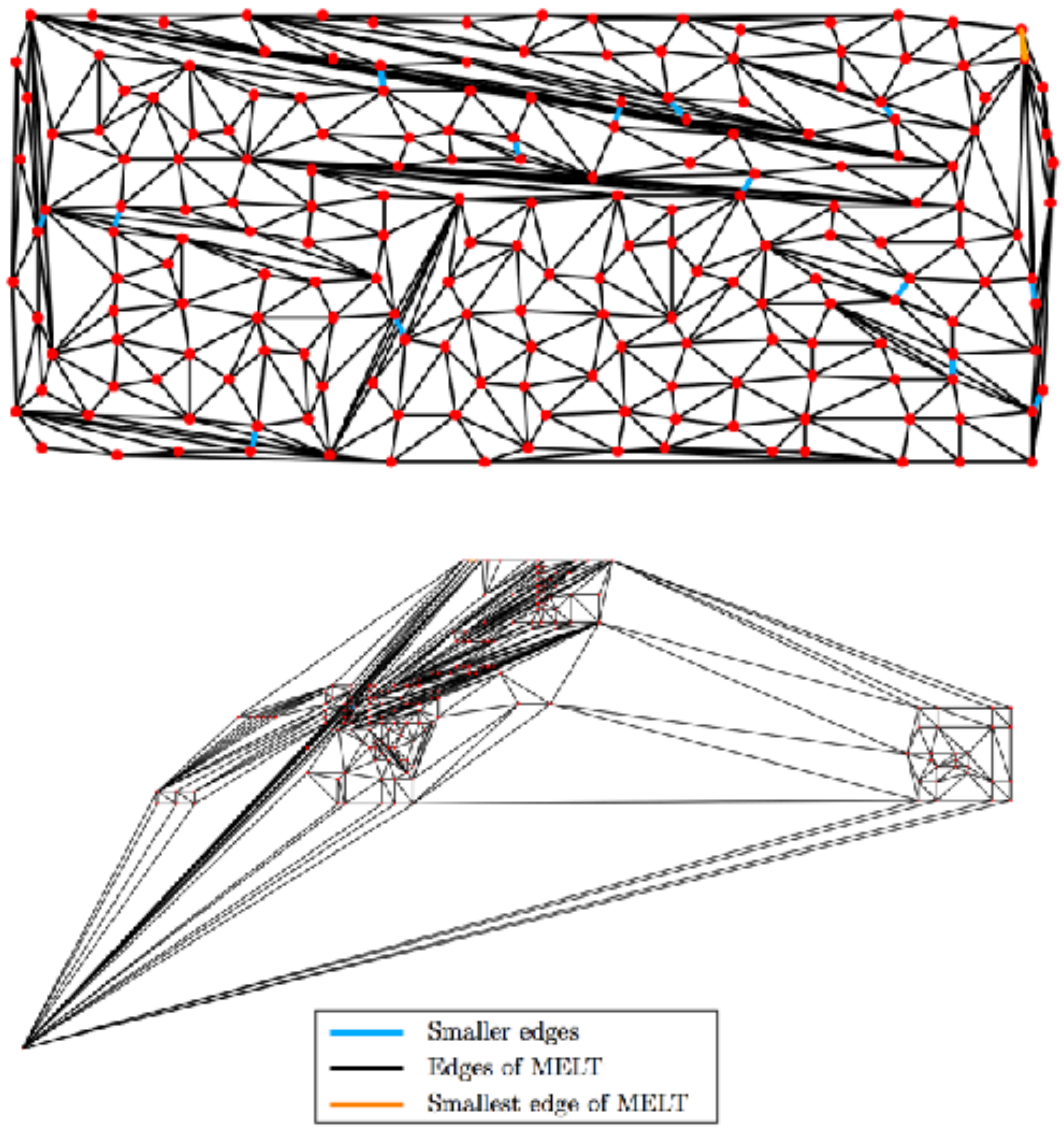


TSPLIB Instances



TSPLIB Instances (2)

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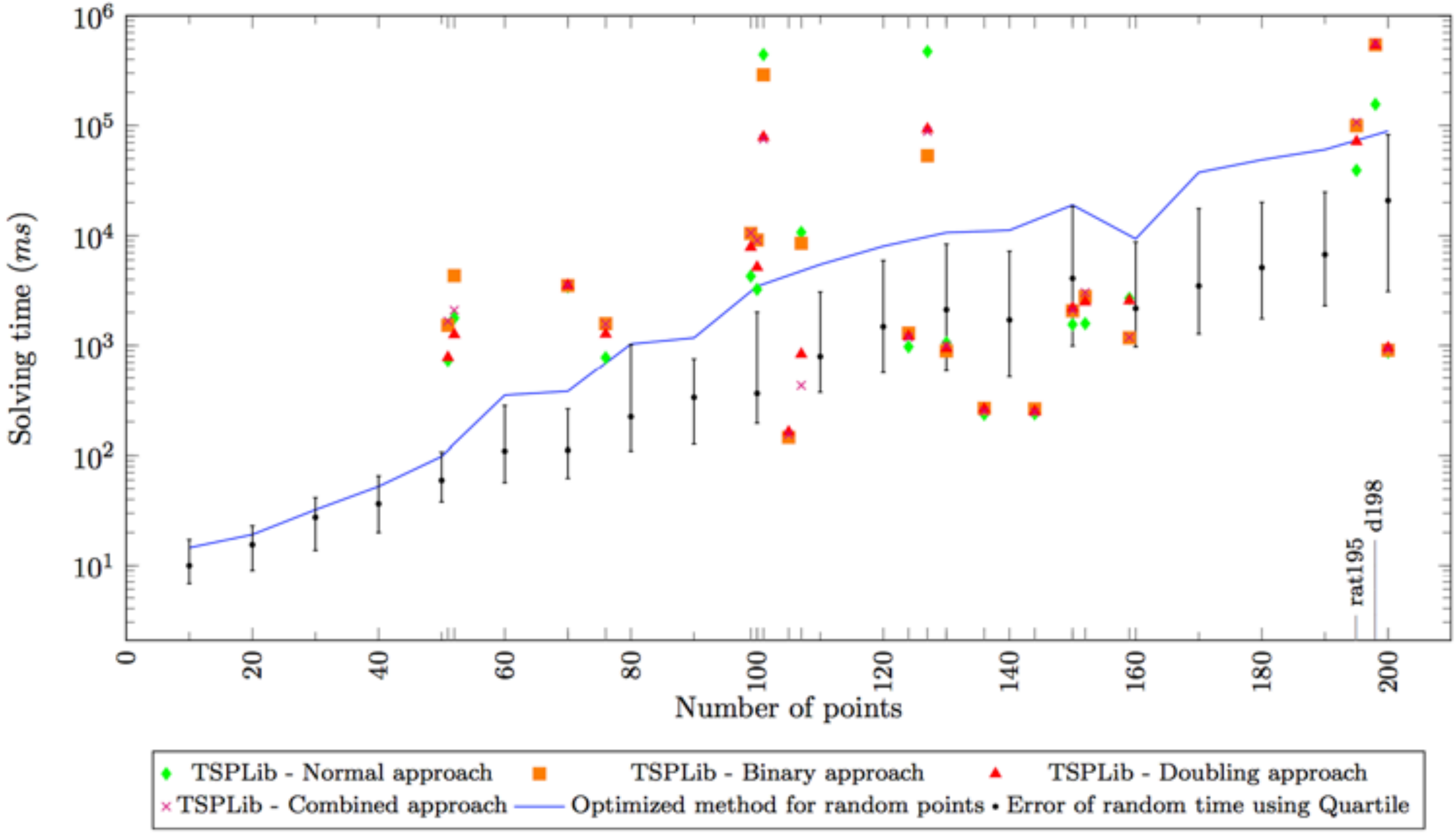

Figure 14: Comparison of TSPLIB instances with average times for random instances, using the doubling approach. (Error bars indicate percentiles 25, 50, 75.)

An Open Problem

An Open Problem

Conjecture: The length of the longest non-separated edge for n random points in a unit square grows with $O(1/n)$

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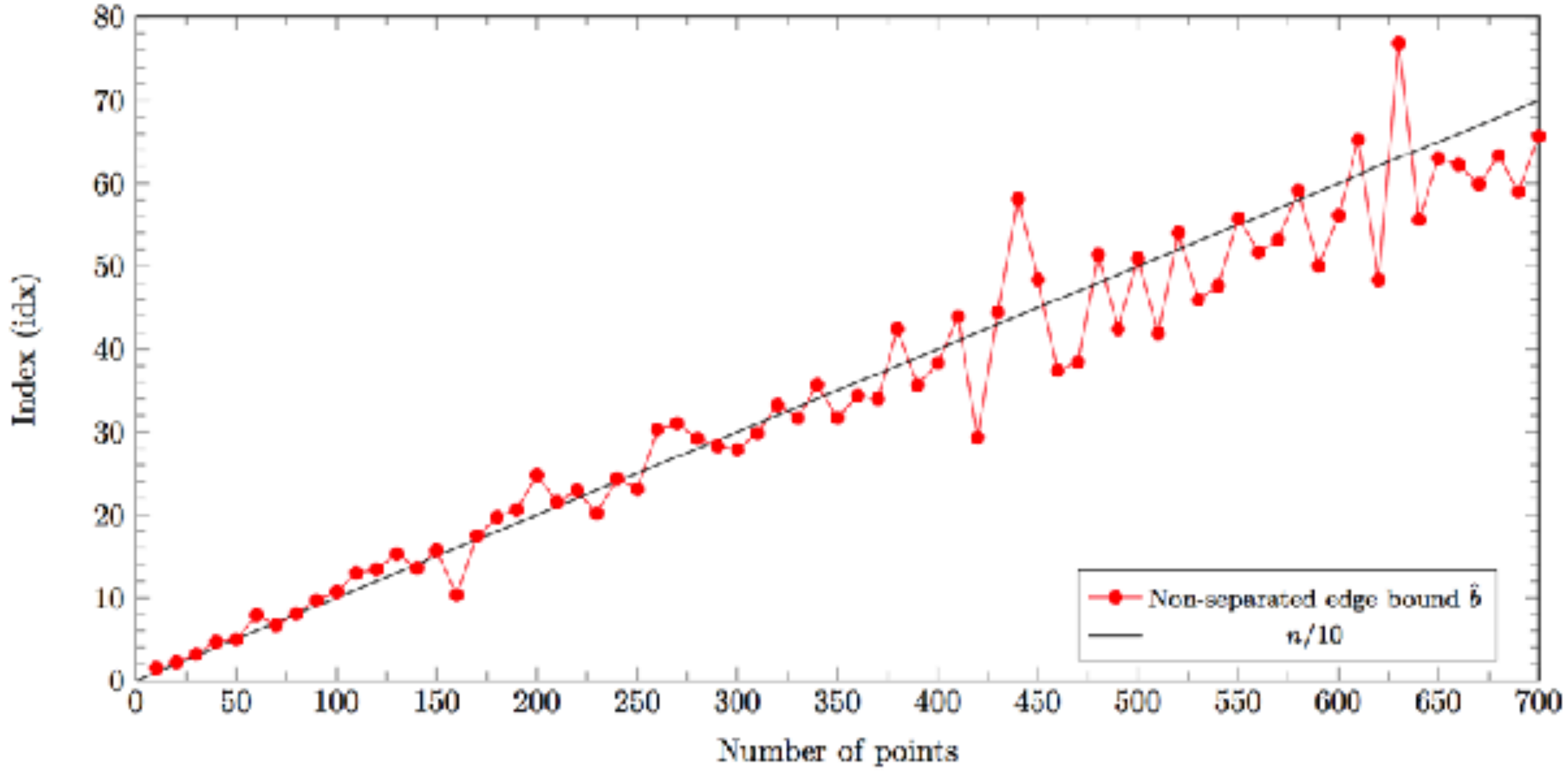

Figure 13: Average indices (for 100 random instances for each size) of the non-separated edge bound $\hat{b}$ for random sets with up to 700 points. (Error bars indicate one standard deviation.)

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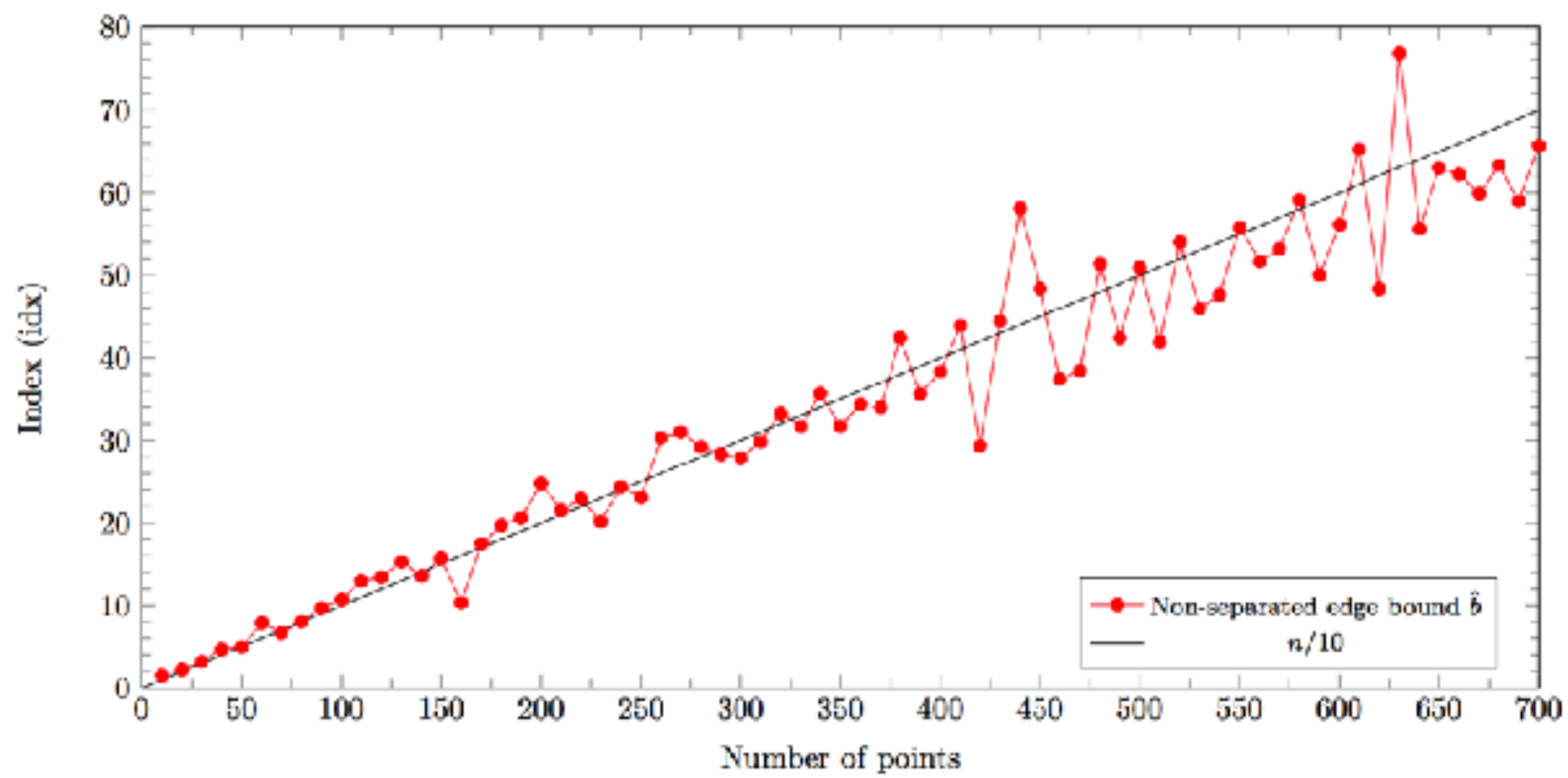
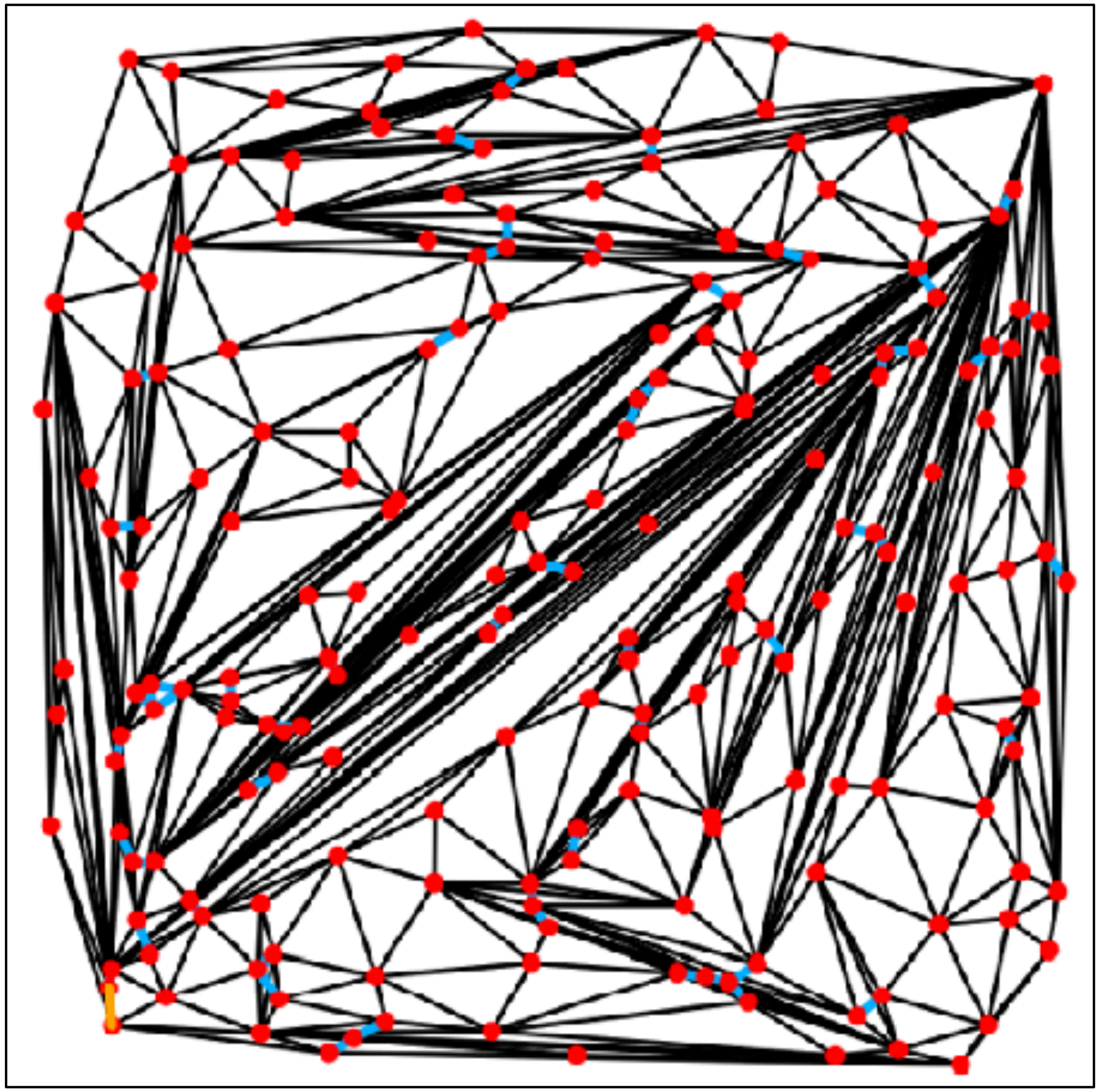


Figure 13: Average indices (for 100 random instances for each size) of the non-separated edge bound $\hat{b}$ for random sets with up to 700 points. (Error bars indicate one standard deviation.)

Consequence: Because the length of the shortest edge for uniform random n^2 points in a unit square grows with , the same $\Theta(1/n)$ would hold for the solution of MELT.

Thank you!



Thank you for today!