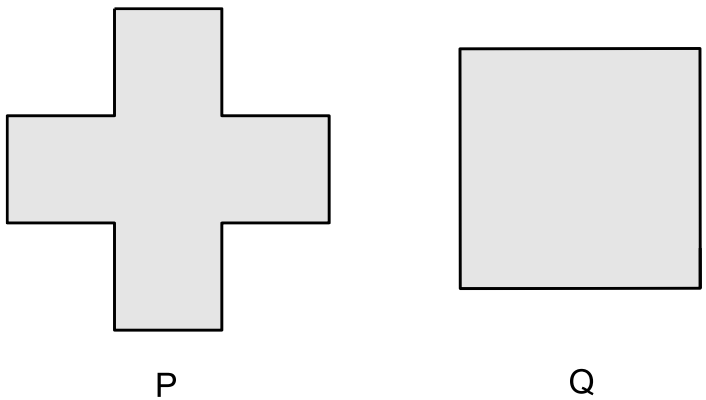
Technische Universität Braunschweig

Computational Geometry – Exercise Meeting #2

December 8th, 2021

Scissors congruence

Two simple polygons P and Q are called **scissors congruent**, if we can subdivide their area into sets of polygons $\{P_1, P_2, \ldots, P_l\}$ and $\{Q_1, Q_2, \ldots, Q_l\}$ such that each P_i is congruent to Q_i for each $i \in \{1, 2, \ldots, l\}$.



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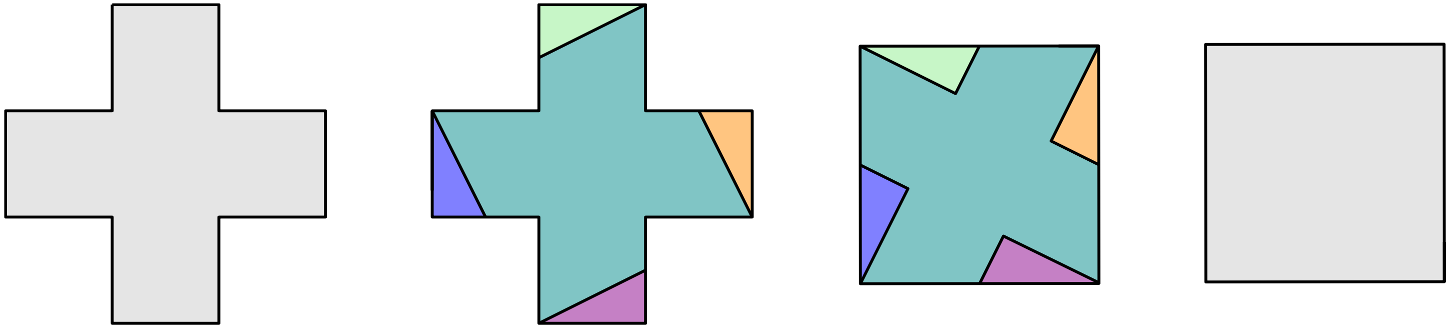
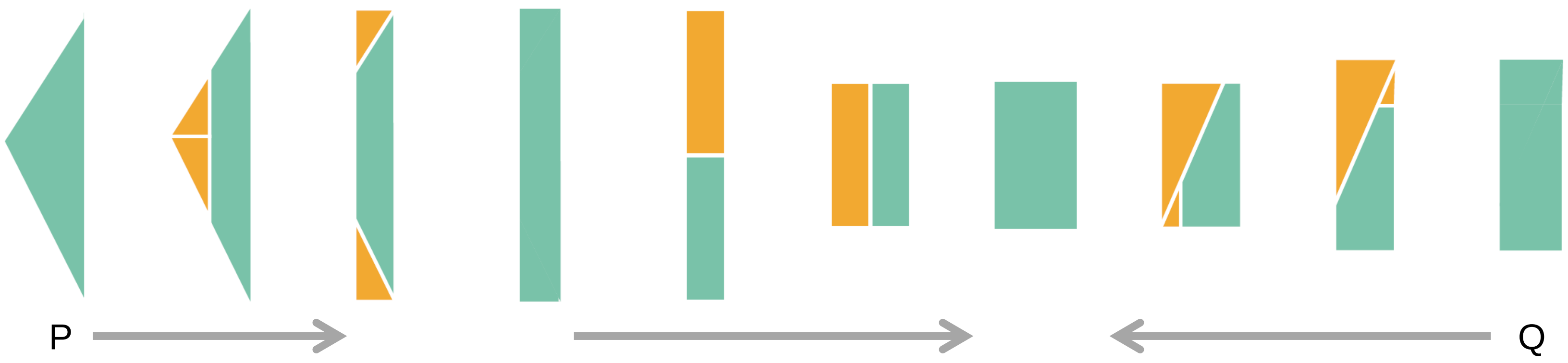


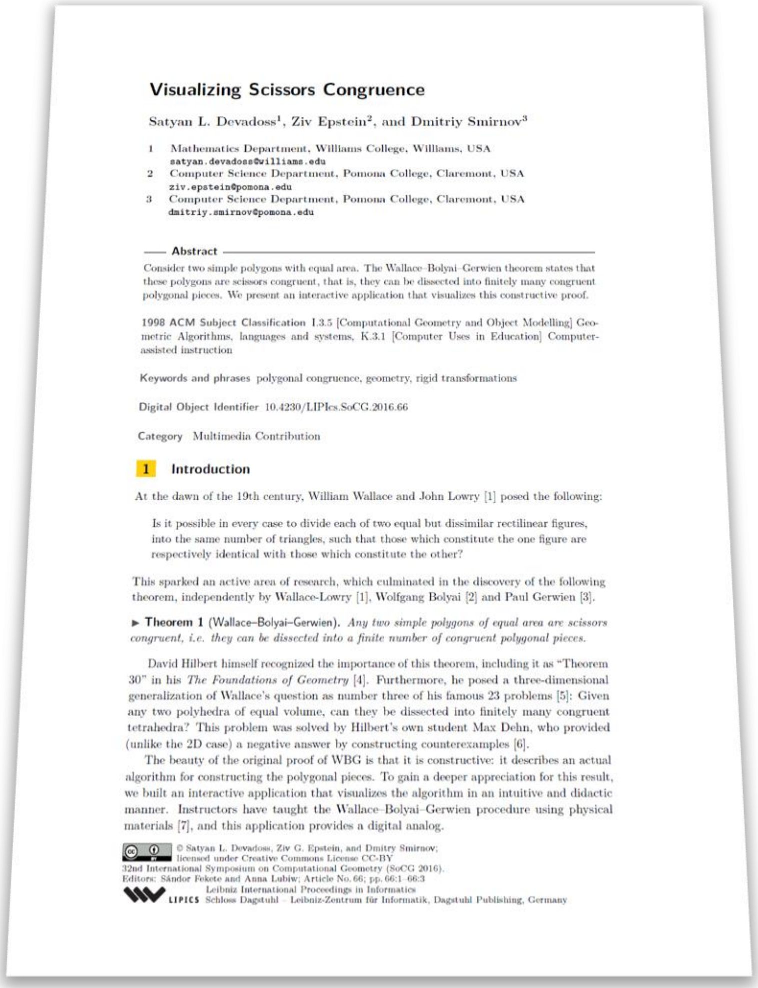
P P_1, P_2, P_3, P_4, P_5 Q_1, Q_2, Q_3, Q_4, Q_5 Q

Scissors congruence



Source: https://doi.org/10.4230/LIPIcs.SoCG.2016.66

Scissors congruence



Visualizing Scissors Congruence

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Abstract

Consider two simple polygons with equal area. The Wallace–Bolyai–Gerwien theorem states that these polygons are scissors congruent, that is, they can be dissected into finitely many congruent polygonal pieces. We present an interactive application that visualizes this constructive proof.

1998 ACM Subject Classification I.3.5 [Computational Geometry and Object Modelling] Geometric Algorithms, languages and systems, K.3.1 [Computer Uses in Education] Computer-assisted instruction

Keywords and phrases polygonal congruence, geometry, rigid transformations

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Category Multimedia Contribution

1 Introduction

At the dawn of the 19th century, William Wallace and John Lowry [1] posed the following:

> Is it possible in every case to divide each of two equal but dissimilar rectilinear figures, into the same number of triangles, such that those which constitute the one figure are respectively identical with those which constitute the other?

This sparked an active area of research, which culminated in the discovery of the following theorem, independently by Wallace-Lowry [1], Wolfgang Bolyai [2] and Paul Gerwien [3].

▶ **Theorem 1** (Wallace–Bolyai–Gerwien). *Any two simple polygons of equal area are scissors congruent, i.e. they can be dissected into a finite number of congruent polygonal pieces.*

David Hilbert himself recognized the importance of this theorem, including it as "Theorem 30" in his *The Foundations of Geometry* [4]. Furthermore, he posed a three-dimensional generalization of Wallace's question as number three of his famous 23 problems [5]: Given any two polyhedra of equal volume, can they be dissected into finitely many congruent tetrahedra? This problem was solved by Hilbert's own student Max Dehn, who provided (unlike the 2D case) a negative answer by constructing counterexamples [6].

The beauty of the original proof of WBG is that it is constructive: it describes an actual algorithm for constructing the polygonal pieces. To gain a deeper appreciation for this result, we built an interactive application that visualizes the algorithm in an intuitive and didactic manner. Instructors have taught the Wallace–Bolyai–Gerwien procedure using physical materials [7], and this application provides a digital analog.

32nd International Symposium on Computational Geometry (SoCG 2016).
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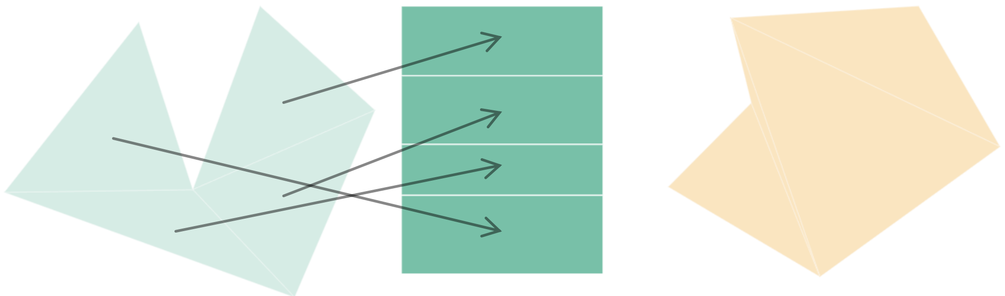
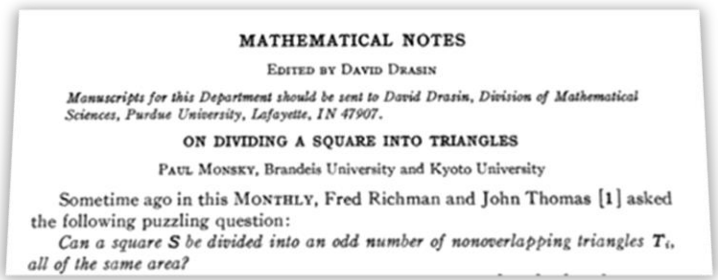
LIPICS Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



http://dmsm.github.io/scissors-congruence/

Source: https://doi.org/10.4230/LIPIcs.SoCG.2016.66

Scissors congruence – Notes and Open Problems



MATHEMATICAL NOTES

Edited by David Drasin

Manuscripts for this Department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

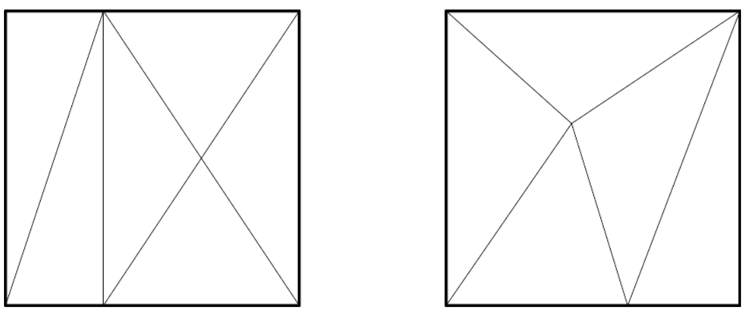
ON DIVIDING A SQUARE INTO TRIANGLES

Paul Monsky, Brandeis University and Kyoto University

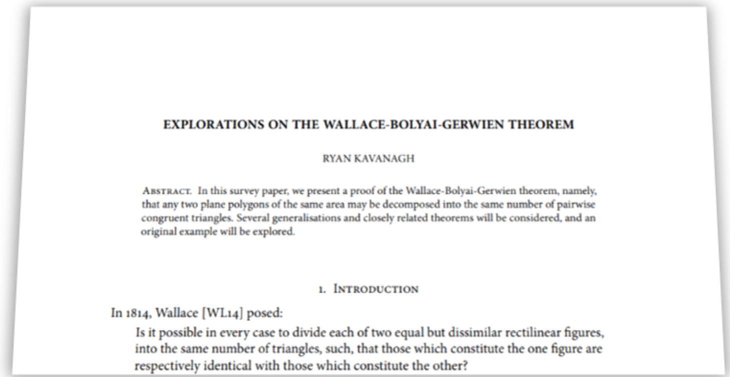
Sometime ago in this Monthly, Fred Richman and John Thomas [1] asked the following puzzling question:

Can a square S be divided into an odd number of nonoverlapping triangles T_i, all of the same area?

Monsky's Theorem (1970)
A square can never be divided into an odd number of non-overlapping triangles of equal area.



https://en.wikipedia.org/wiki/Monsky%27s_theorem



EXPLORATIONS ON THE WALLACE-BOLYAI-GERWIEN THEOREM

RYAN KAVANAGH

Abstract. In this survey paper, we present a proof of the Wallace-Bolyai-Gerwien theorem, namely, that any two plane polygons of the same area may be decomposed into the same number of pairwise congruent triangles. Several generalisations and closely related theorems will be considered, and an original example will be explored.

1. Introduction

In 1814, Wallace [WL14] posed:

Is it possible in every case to divide each of two equal but dissimilar rectilinear figures, into the same number of triangles, such, that those which constitute the one figure are respectively identical with those which constitute the other?

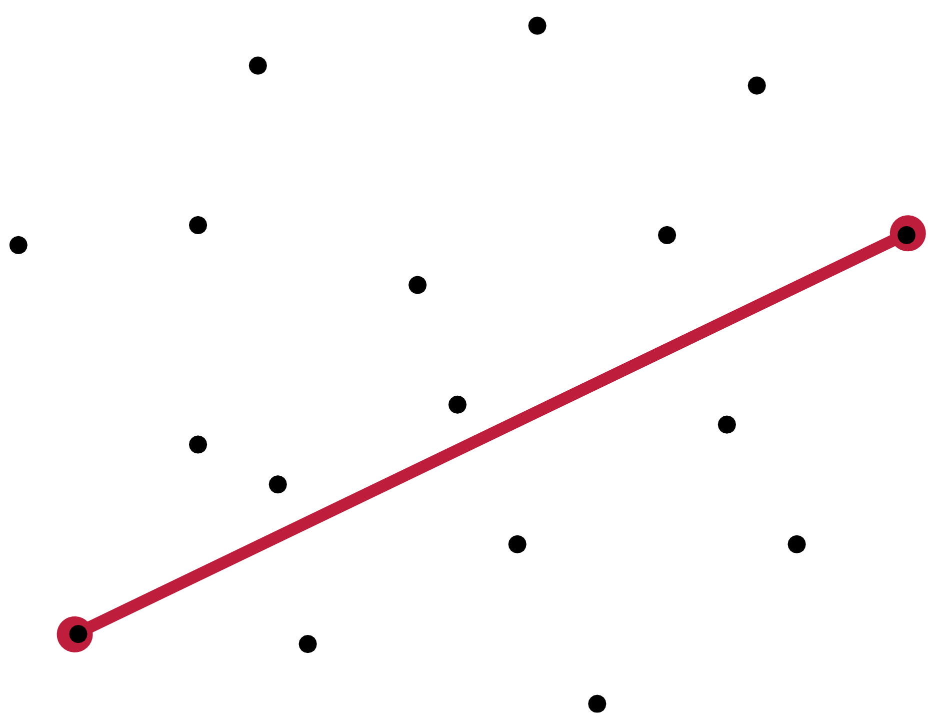
https://rak.ac/files/papers/wallace-bolyai-gerwien.pdf

Open Question #1
Can Monsky's Theorem be generalized for cubes of higher dimension?

Open Question #2
Is it possible to bound from below the number of cuts required to show that two polygons have the same area?

Farthest Pairs

Let $\mathcal{P}$ be a finite point set. Describe an $\mathcal{O}(n \log(n))$ algorithm which determines **two points $p, q \in \mathcal{P}$ of maximal distance in $\mathcal{P}$.**



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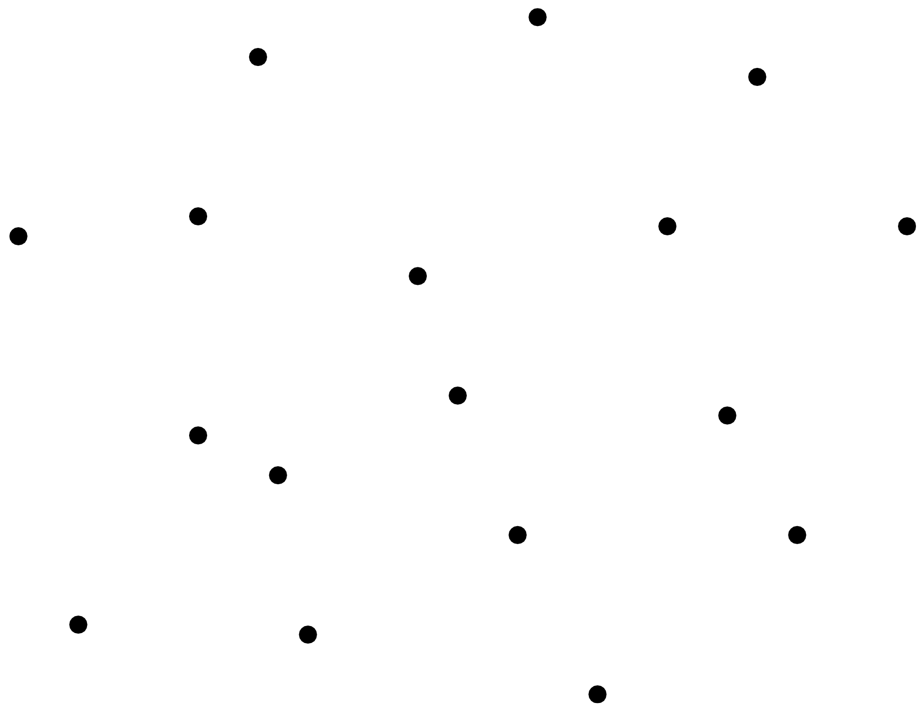
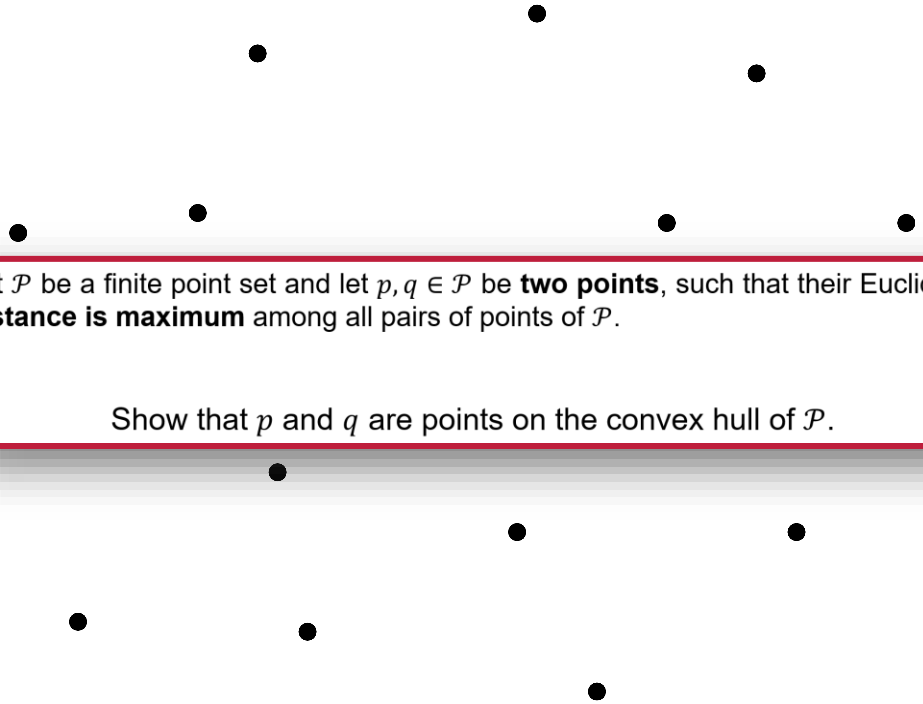


Farthest Pairs

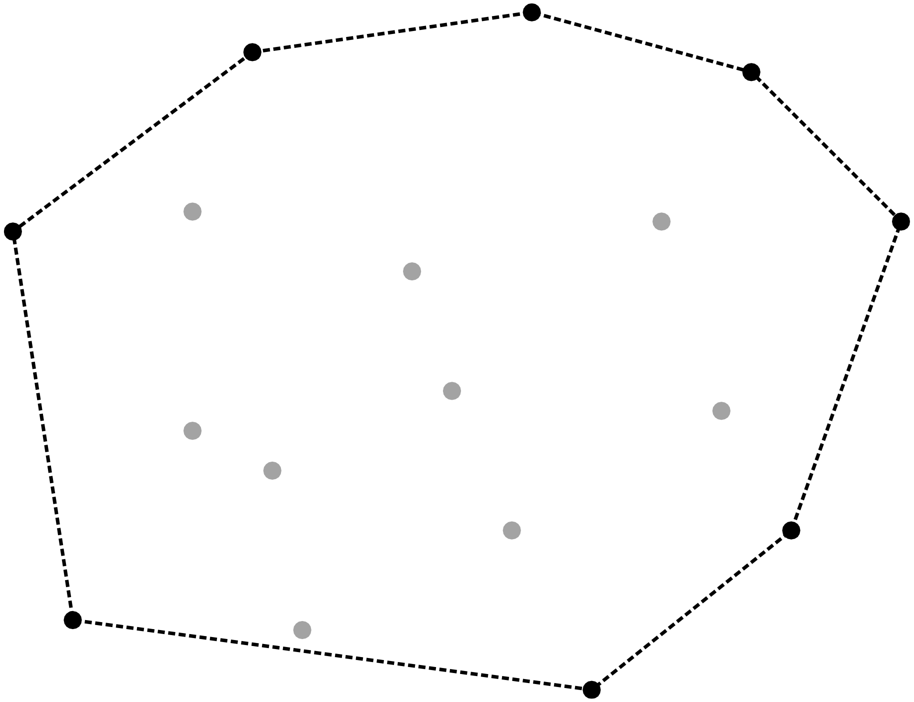


Let $\mathcal{P}$ be a finite point set and let $p, q \in \mathcal{P}$ be **two points**, such that their Euclidean **distance is maximum** among all pairs of points of $\mathcal{P}$.

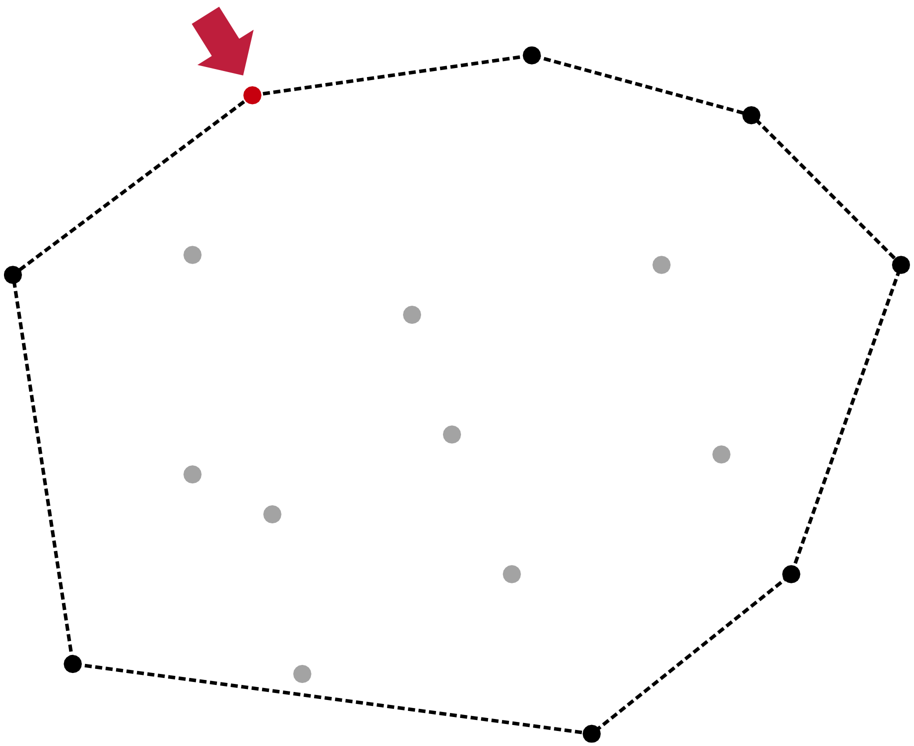
Show that p and q are points on the convex hull of $\mathcal{P}$.

Farthest Pairs

$$\mathcal{O}(n\log(n))$$

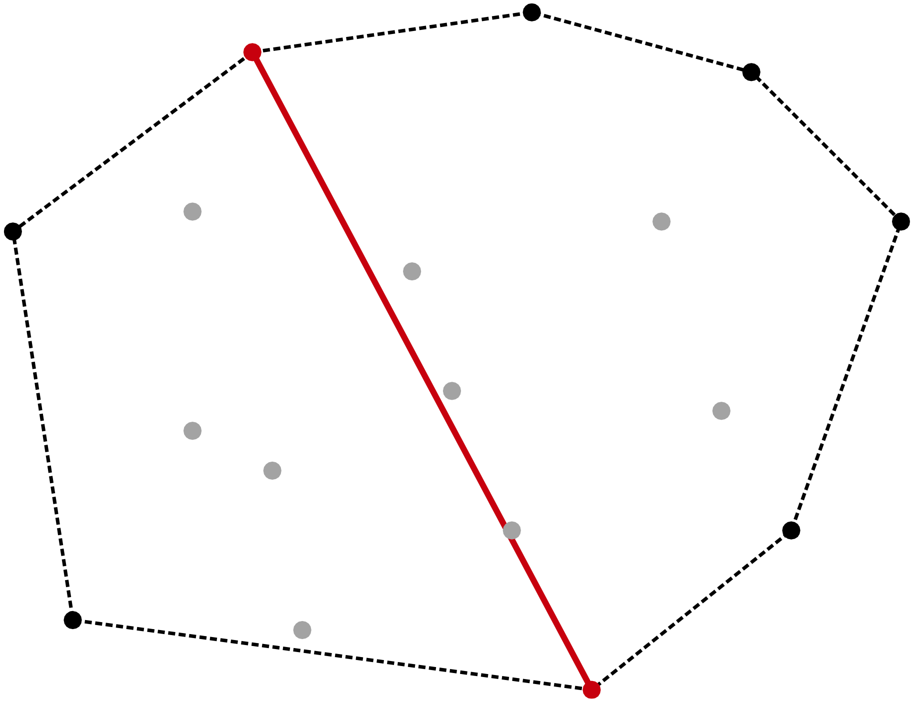


Farthest Pairs



Farthest Pairs

$\mathcal{O}(n)$



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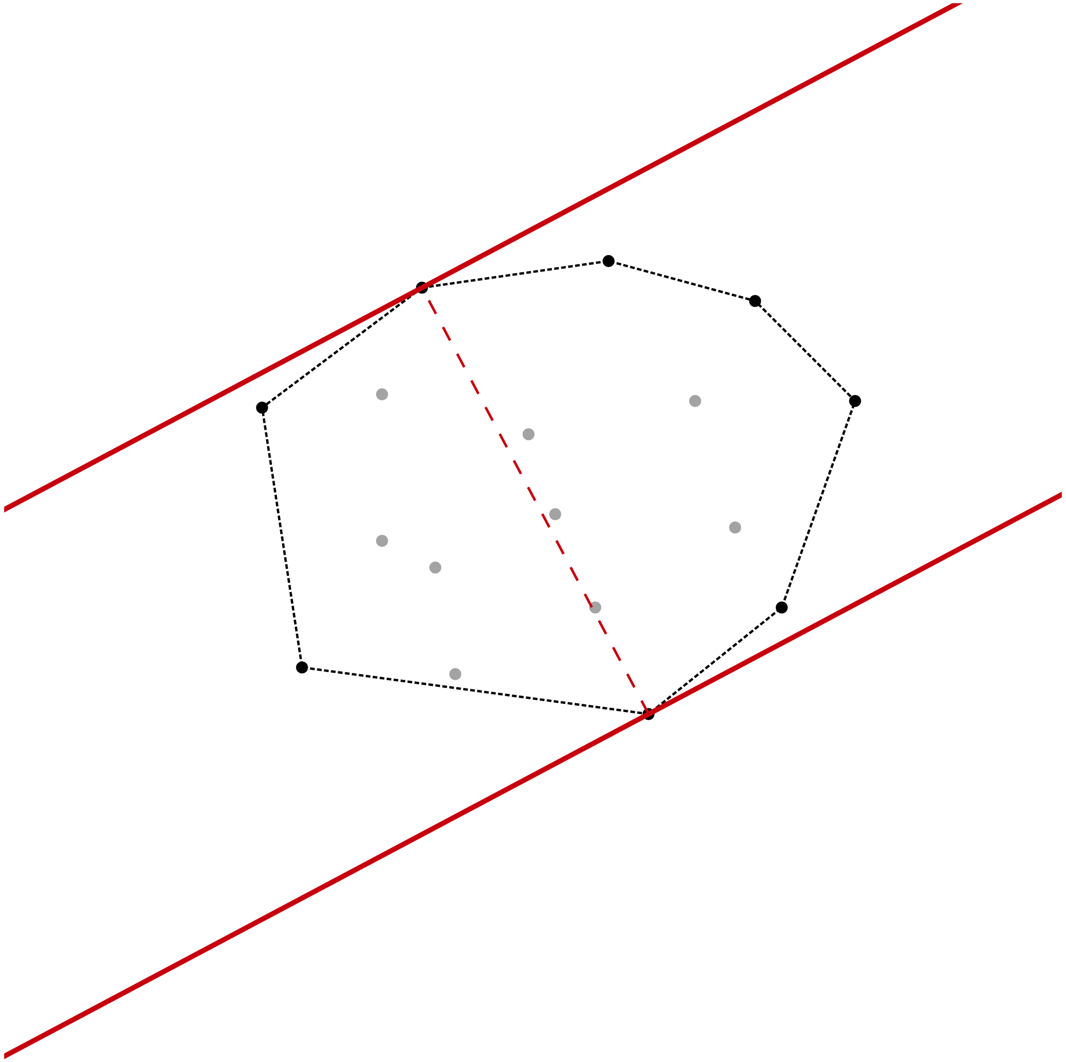

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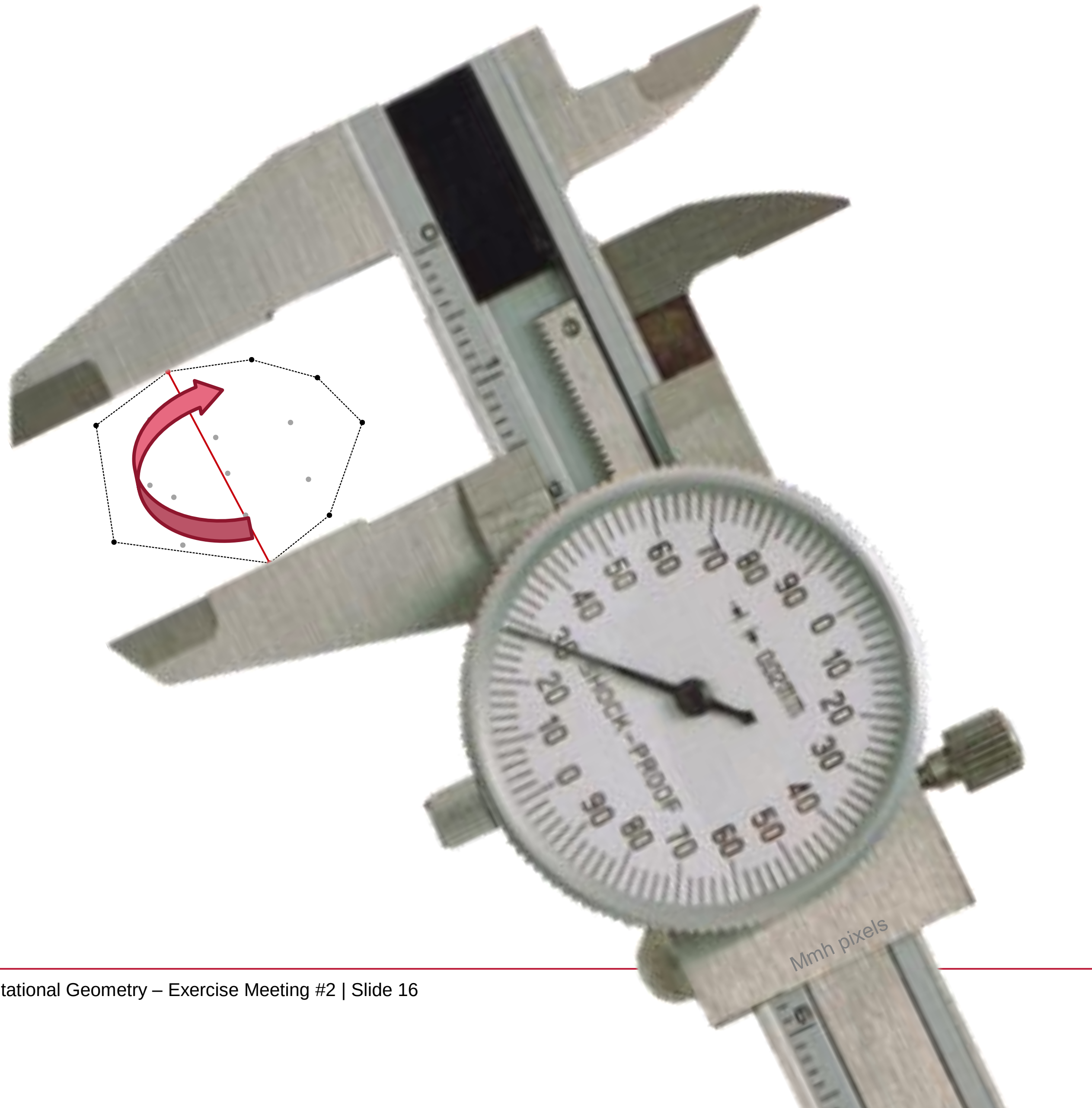
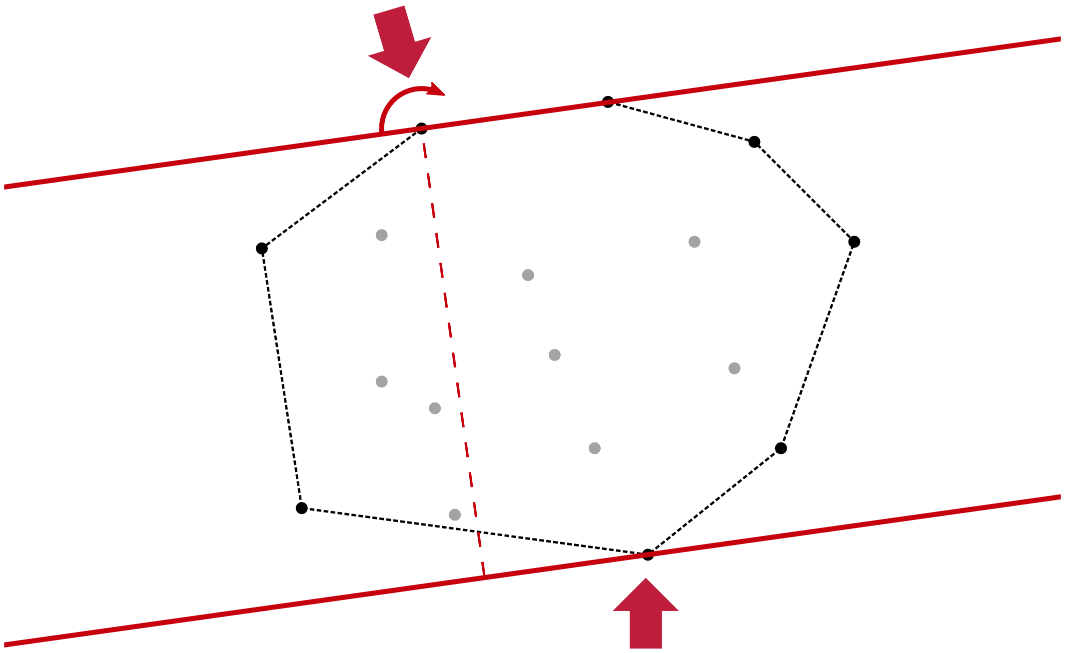


Source: www.richter-messzeuge.de

Farthest Pairs



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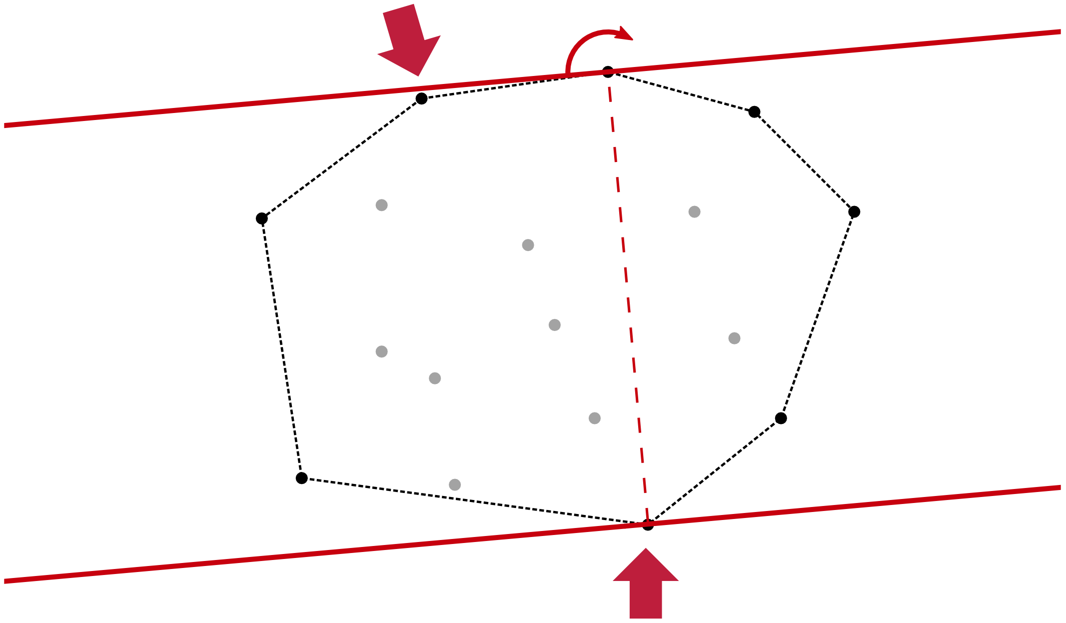

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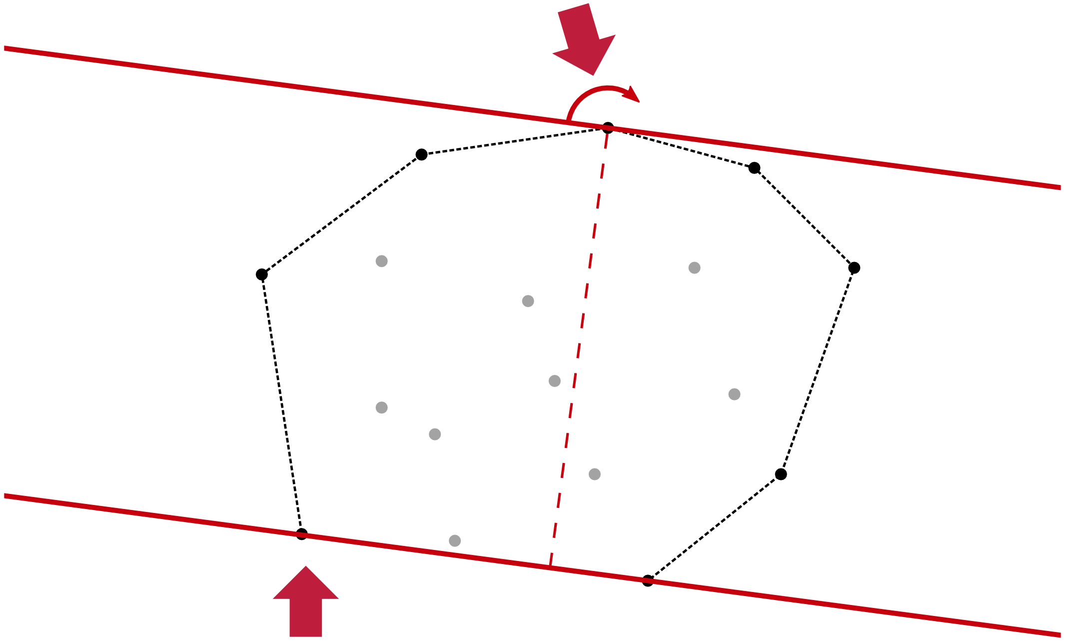

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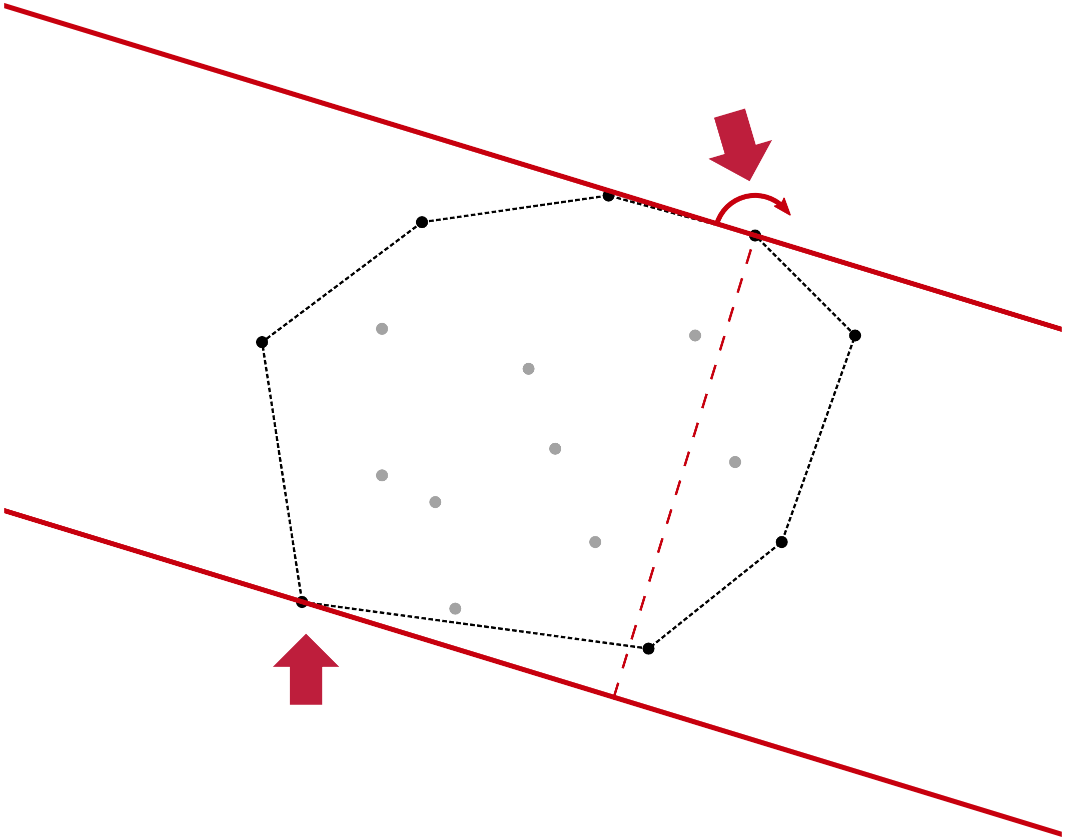

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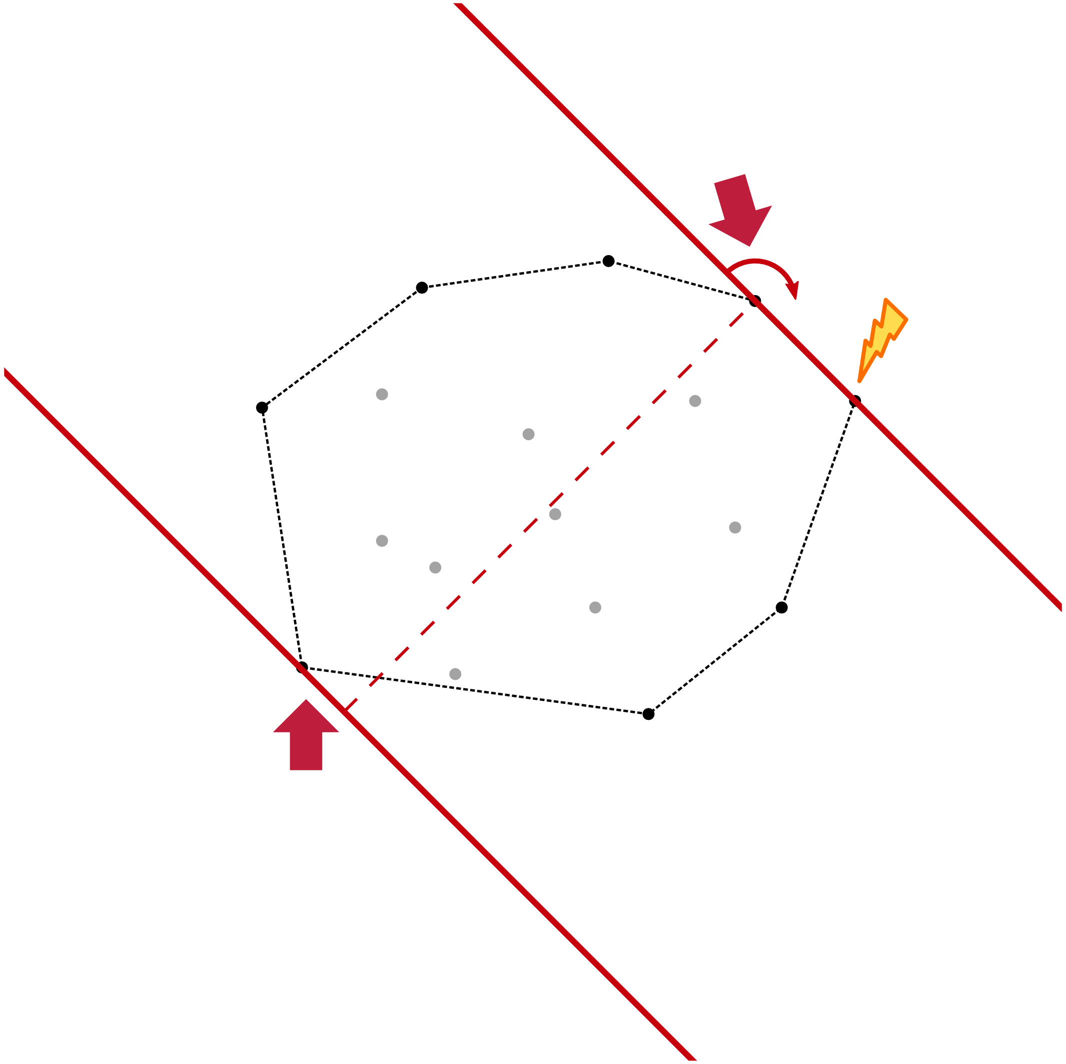

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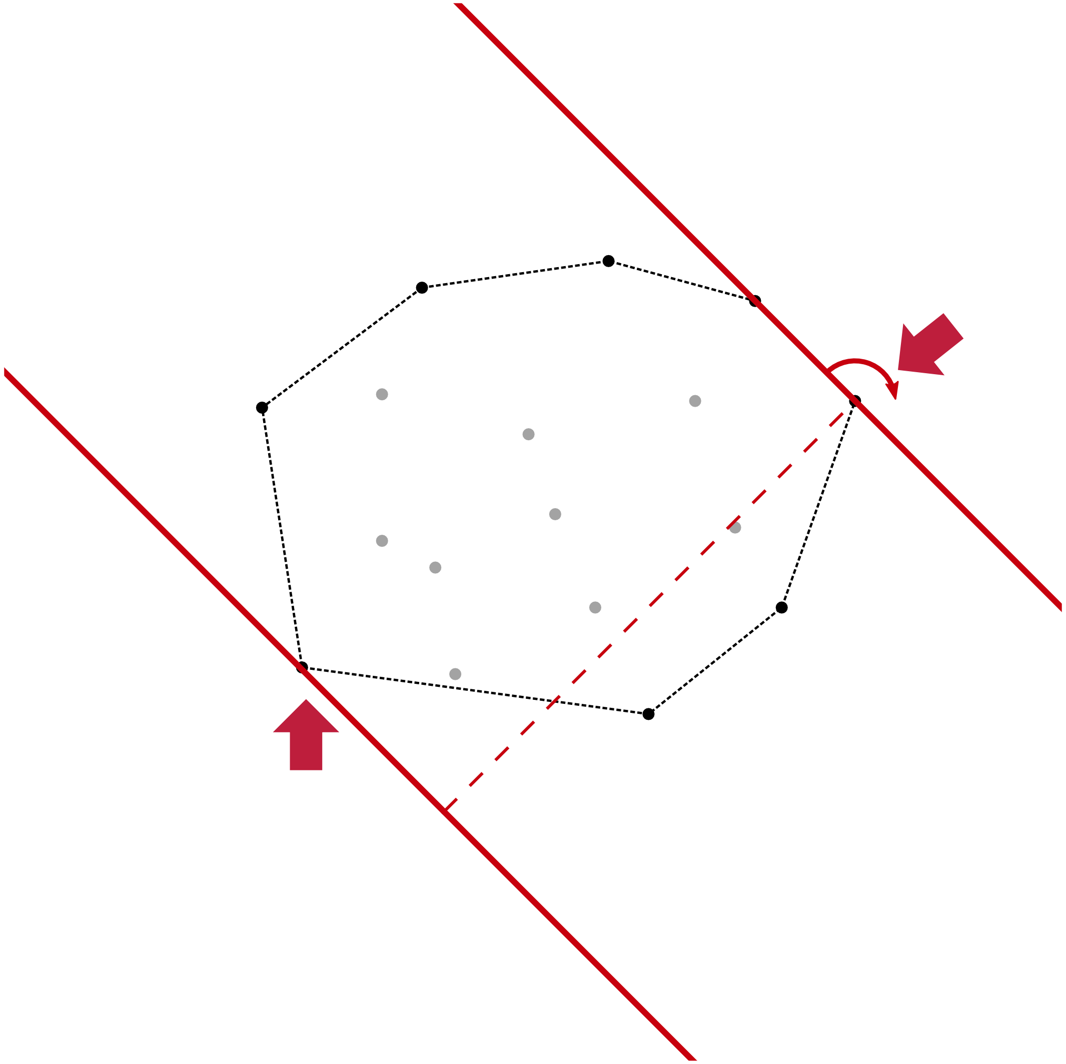

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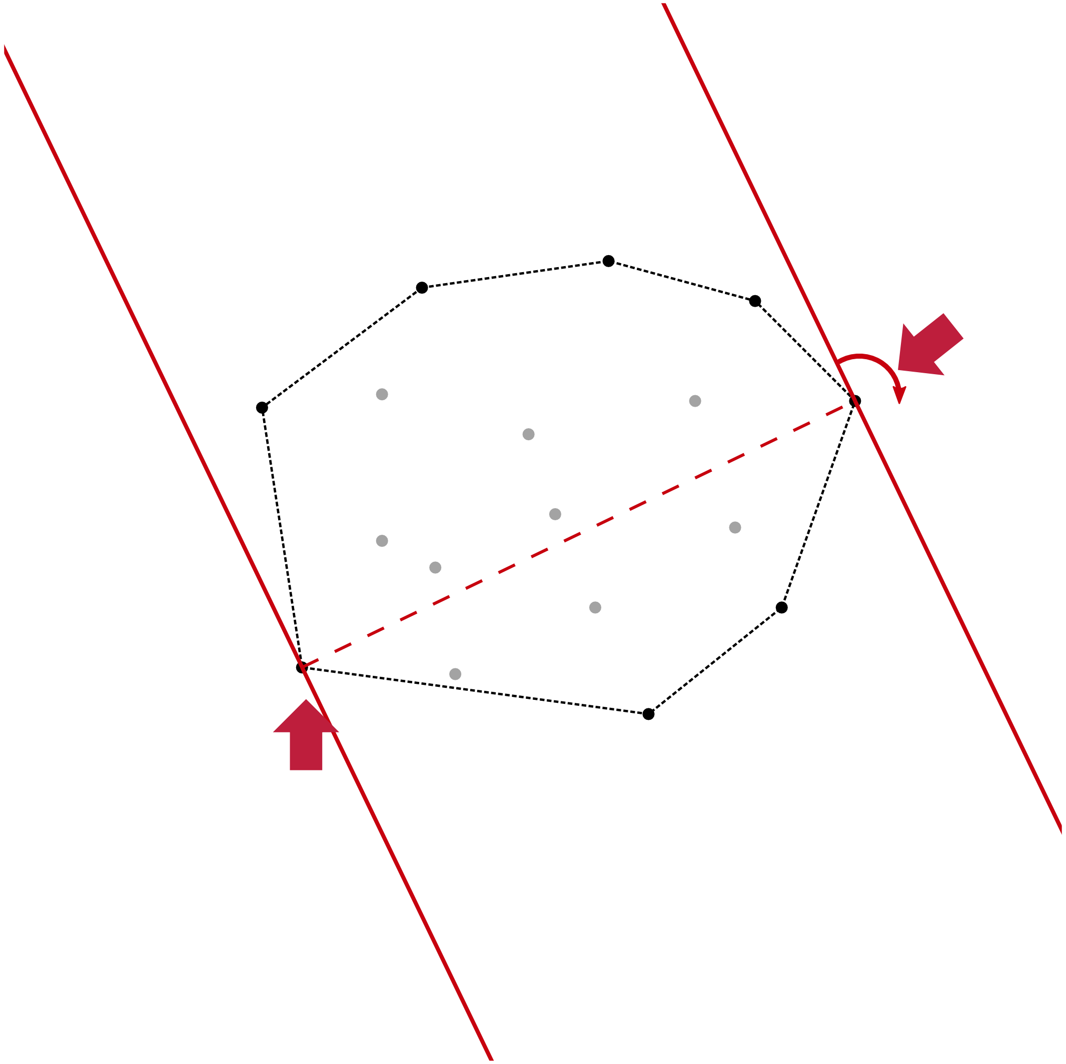

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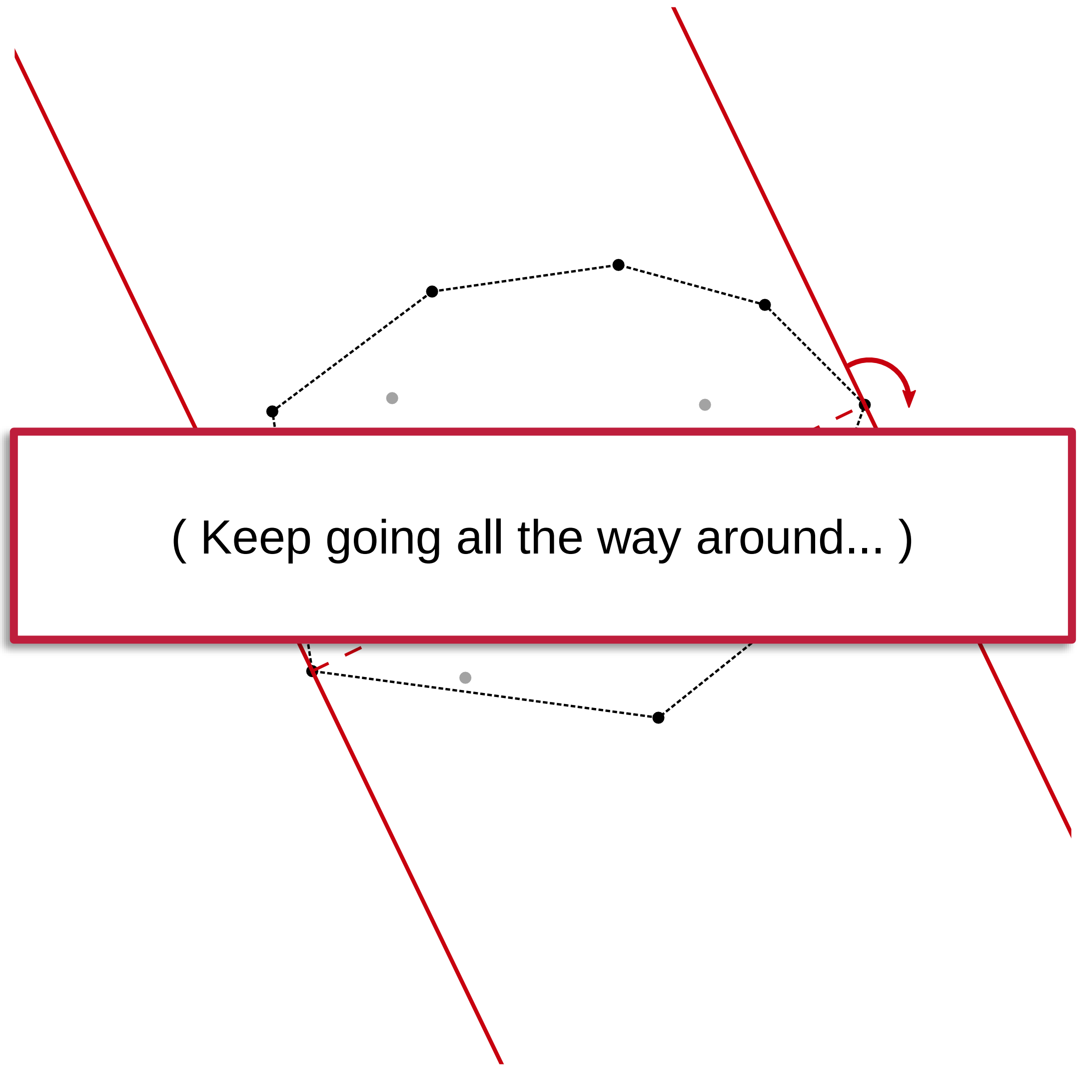
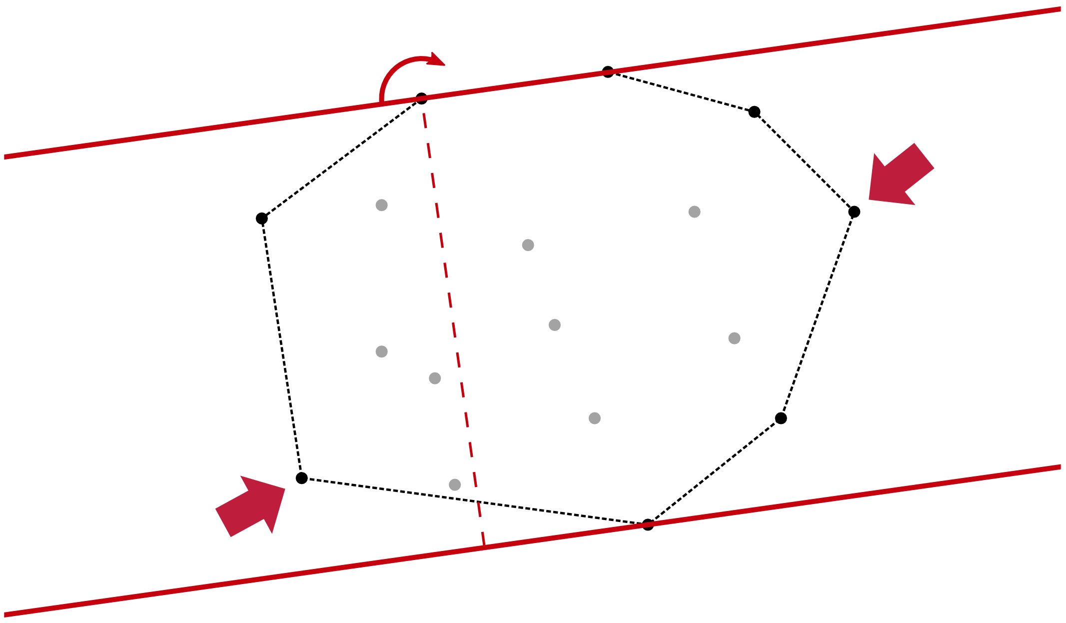


(Keep going all the way around...)

Farthest Pairs

$\mathcal{O}(n \log(n))$ total



Rotating Calipers – Other Applications

Distances [edit]

- Diameter (maximum width) of a convex polygon[6][7]
- Width (minimum width) of a convex polygon[8]
- Maximum distance between two convex polygons[9][10]
- Minimum distance between two convex polygons[11][12]
- Widest empty (or separating) strip between two convex polygons (a simplified low-dimensional variant of a problem arising in support vector machine based machine learning)
- Grenander distance between two convex polygons[13]
- Optimal strip separation (used in medical imaging and solid modeling)[14]

Bounding boxes [edit]

- Minimum area oriented bounding box
- Minimum perimeter oriented bounding box

Triangulations [edit]

- Onion triangulations
- Spiral triangulations
- Quadrangulation
- Nice triangulation
- Art gallery problem
- Wedge placement optimization problem[15]

Multi-polygon operations [edit]

- Union of two convex polygons
- Common tangents to two convex polygons
- Intersection of two convex polygons[16]
- Critical support lines of two convex polygons
- Vector sums (or Minkowski sum) of two convex polygons[17]
- Convex hull of two convex polygons

Traversals [edit]

- Shortest transversals[18][19]
- Thinnest-strip transversals[20]

Others [edit]

- Non parametric decision rules for machine learned classification[21]
- Aperture angle optimizations for visibility problems in computer vision[22]
- Finding longest cells in millions of biological cells[23]
- Comparing precision of two people at firing range
- Classify sections of brain from scan images

https://en.wikipedia.org/wiki/Rotating_calipers#Applications